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Citation for published version (APA):

Document status and date:
Published: 01/01/1986

Publisher Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

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Memorandum COSOR 86-17

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Eindhoven, the Netherlands
November 1986
A PROBABILISTIC ANALYSIS OF THE DUAL BIN PACKING PROBLEM

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ABSTRACT

In the dual bin packing problem, the objective is to assign items of given size to the largest possible number of bins, subject to the constraint that the total size of the items assigned to any bin is at least equal to 1. We carry out a probabilistic analysis of this problem under the assumption that the items are drawn independently from the uniform distribution on [0,1], and reveal the connections between this problem and the classical bin packing problem as well as to renewal theory.

KEYWORDS

Bin packing, probabilistic analysis of algorithms, next fit heuristic.
1. INTRODUCTION

Given $n$ items of size $a_1, \ldots, a_n$ ($a_i \in (0,1)$, $i=1, \ldots, n$), the classical bin packing problem is to assign the items to the smallest possible number of bins, subject to the constraint that the total size of the items assigned to any bin is at most equal to 1. The dual bin packing problem, which is the subject of this paper, is to assign the items to the largest possible number of bins, subject to the constraint that the total size of the items assigned to any bin is at least equal to 1. The problem could also appropriately be called the bin covering problem. While superficially similar to its traditional counterpart, the problem poses a challenge of its own: as in the case of more general packing and covering problems, a result for one problem occasionally carries over immediately to the other, but generally the differences between them are as pronounced as their common traits.

We shall see examples of both phenomena as we carry out a first exploration of the dual bin packing problem. We shall do so from a probabilistic point of view, i.e., we shall assume that the item sizes $a_1, a_2, \ldots$ are drawn independently from the uniform distribution on $(0,1)$. Many results will in fact be seen to hold under more general assumptions, but the uniform distribution provides a traditional starting point for this type of enquiry.

In addition to the bin packing problem, we shall also consider a two-dimensional analogue, the dual vector packing problem. Here, we are given $n$ pairs $(a_1, b_1), \ldots, (a_n, b_n)$ that have to be assigned to the largest possible number of bins subject to the constraint that both the sum of the $a$-coordinates and the sum of the $b$-coordinates of the pairs in any bin are at least equal to 1. Many of our results can in fact be extended to the obvious $m$-dimensional version of this problem, for any fixed $m$.

In Section 2, we consider the optimal solution value $\text{OPT}(n)$ to the dual bin packing problem and prove that

$$\lim_{\sup_{n \to \infty}} \frac{E(\text{OPT}(n)) - n/2}{n^{1/2}} \leq - (32\pi)^{-1/2} \quad (1)$$

i.e., for $n$ large enough, $E(\text{OPT}(n)) = n/2 - \Omega(n^{1/2})$. In Section 3, we demonstrate that this estimate is the best possible one up to a multiplicative
constant by demonstrating that a simple heuristic, the Pairing Heuristic, produces a value \( PA(n) \) satisfying

\[
E(PA(n)) \geq \frac{n}{2} - \left(\frac{n}{2\pi}\right)^{1/2} - \alpha
\]

for some constant \( \alpha \). This heuristic can be adapted to show that the expected optimal solution value of the dual vector packing problem is also asymptotic to \( n/2 \).

These two results have their counterparts in the classical bin packing problem, where an upper bound of \( n/2 + O(n^{1/2}) \) on the optimal solution value can be proved to be best possible in a similar fashion [Knödel 1981, Lueker 1983]. (Actually, our technique yields an improvement on the best known upper bound on the multiplicative constant for this case.) The result in Section 4, where we analyze the expected performance of a suitably adapted version of the Next Fit Heuristic, has a different flavor. Using techniques from renewal theory that do not carry over to the classical case, we establish the strong result that the solution value \( NF(n) \) satisfies

\[
\lim_{n \to \infty} \left( \frac{E(NF(n))}{E(OPT(n))} - 1 \right) = -\frac{2}{e} - 1 = -0.2642\ldots
\]

Hence, the expected relative error \( (E(OPT(n)) - E(NF(n)))/E(OPT(n)) \) converges to \( 1 - 2/e \). A similar strong result is obtained for an appropriately modified version of Next Fit, applied to the dual vector packing problem. Both result can be easily extended to distributions other than the uniform one.

In Section 5, we present a probabilistic analysis of the Next Fit Decreasing heuristic, which can again be easily adapted to our model. Surprisingly, its performance is inferior to that of Next Fit, in remarkable contrast to their behaviour on the classical bin packing model.

Some concluding remarks are contained in Section 6.
2. THE EXPECTED OPTIMAL SOLUTION VALUE

In deriving an upper bound on the optimal solution value to the dual bin packing problem OPT(n), we shall find it convenient to assume that n is even or, equivalently, to focus on OPT(2n).

To obtain an upper bound on the expected value of this random variable, we start by defining \( b_{2n} \) to be the number of big items (i.e., those with size greater than 1/2). Since, with probability 1, each bin must contain at least two items in any feasible solution, we always have that \( \text{OPT}(2n) \leq n \). If, however, we know that \( b_{2n} < n \), then the best that we can hope for is to pair each big item with a small item to cover a bin, and to divide the remaining small items in groups of 3 of which each covers an additional bin. Hence, in this case \( \text{OPT}(2n) \leq b_{2n} + (2n - 2b_{2n})/3 = 2n/3 + b_{2n}/3 \).

Since obviously also \( \text{OPT}(2n) \leq \sum_{i=1}^{2n} a_i \), we have that

\[
E(\text{OPT}(2n)) \leq \sum_{k=0}^{2n} E(\min\{\sum_{i=1}^{2n} a_i, n\} | b_{2n} = k) \binom{2n}{k} 2^{-2n} + \\
+ \sum_{k=0}^{n-1} E(\min\{\sum_{i=1}^{2n} a_i, \frac{2n}{3} + \frac{k}{3}\} | b_{2n} = k) \binom{2n}{k} 2^{-2n}
\]

(4)

The first term in (4) is clearly bounded from above by

\[
\sum_{k=n}^{2n} \binom{2n}{k} 2^{-2n},
\]

(5)

which is equal to

\[
2^{-2n} \sum_{k=n}^{2n} \binom{2n}{k} = 2^{-2n} \sum_{k=0}^{n} \binom{2n}{k} = \frac{n}{2} + n 2^{-2n-1}\binom{2n}{n}
\]

(cf. [Riordan 1968, p. 34]).

If we define
\[ d_i = \begin{cases} a_i & (0 \leq a_i \leq 1/2) \\ 1-a_i & (1/2 < a_i \leq 1) \end{cases} \quad (7) \]

we may observe by the exchangeability of \((a_1,\ldots,a_n)\) and the independence of \(b_{2n}\) and \(d_i \) for every \(k\)

\[ E(\min \{ \sum_{i=1}^{2n} a_i, \frac{2n}{3} + \frac{k}{3} \mid b_{2n} = k \}) = E(\min \{ \sum_{i=1}^k (1-d_i) + \sum_{i=k+1}^{2n} d_i, \frac{2n}{3} + \frac{k}{3} \}) \quad (8) \]

Hence, the second term in (4) equals

\[ \sum_{k=0}^{n-1} k^{2n} 2^{-2n} + \sum_{k=0}^{n-1} E(\min \{ \sum_{i=k+1}^{2n} d_i - \sum_{i=1}^{k} d_i, \frac{2n}{3} (n-k) \}) (k^{2n} 2^{-2n} \quad (9) \]

The first term is equal to \(n/2 - 2n 2^{-2n-1}(2n)\) [Riordan 1968, p. 34]. We bound the minimum in the second term by \((2n - 2k) Ed_i = (n - k)/2\) to obtain (cf. [Riordan 1968, p. 34])

\[ \frac{1}{2} \sum_{k=0}^{n-1} (n-k) (\frac{2n}{k}) 2^{-2n} = \]

\[ = \frac{1}{2} n \frac{n}{k=0} (\frac{2n}{k}) 2^{-2n} - \frac{1}{2} \sum_{k=0}^{n-1} (\frac{2n}{k}) 2^{-2n} = \]

\[ = \frac{1}{2} n \left( \frac{1}{2} + 2^{-2n-1}(\frac{2n}{n}) \right) - \frac{1}{2} \cdot \frac{n}{2} = \]

\[ = \frac{1}{2} n 2^{-2n-1}(\frac{2n}{n}). \quad (10) \]

Summing up the various components, we conclude that

\[ E(\text{OPT}(2n)) \leq n - \frac{1}{2} n 2^{-2n-1}(\frac{2n}{n}) \quad (11) \]

Since, for large \(n\), \((\frac{2n}{n})\) is asymptotic to \((mn)^{-1/2} 2^{2n}\), we obtain the desired result (1):

\[ \lim_{n \to \infty} \sup \frac{E(\text{OPT}(2n)) - n}{(2n)^{1/2}} \leq - (32n)^{-1/2} \quad (12) \]
Inequality (12) is essentially valid for a much larger class of distributions than the uniform one; all that turns out to be required is symmetry around 1/2. For the optimal solution value to the classical bin packing problem, the above technique yields an asymptotic lower bound equal to 
\[ n/2 + (32\pi)^{-1/2} n^{1/2}, \]
which is a slight improvement over the result in [Lueker 1983].

3. THE PAIRING HEURISTIC

In this section, we demonstrate that the upper bound (12) is sharp by showing that a certain heuristic for the dual bin packing produces a solution value that is equal to \( n/2 - O(n^{1/2}) \) in expectation.

For this purpose, we adapt the binary pairing heuristic for the classical bin packing problem ([Lueker 1983, Knödel 1981]) to obtain a Pairing Heuristic (PA) for dual bin packing. In this heuristic, the largest unassigned item is always combined with the smallest unassigned item such that together they can cover a bin (i.e., such that the sum of their sizes exceeds 1). If no such item exists, all items then remaining are added to the bin most recently opened, and the algorithm terminates.

We analyze this heuristic along the lines of [Karp 1984], using the random variables \( d_i \) defined in (5). If we label \( d_i \) by '+1' and call it big if \( a_i > 1/2 \), and label it by '-1' and call it small if \( a_i < 1/2 \), and consider the labeled sequence \( d_i \) in \((0,1/2)\) in increasing order, then the PA heuristic amounts to matching each successive '+1' to the unassigned '-1' that is closest to its right. If there are no unassigned '-1' 's to its right, match it with a '+1' closest to its right. If such a '+1' also does not exist, then put it in the bin most recently opened. If \( u_n \) is the number of unmatched small \( d_i \), then clearly 
\[ E(PA(n)) \geq E((n - u_n)/2) = n/2 - E(u_n/2). \]

To compute \( E_u_n \), we first observe that the sequence of '+1' 's and '-1' 's can be viewed as a realization of a Bernoulli process [Cinlar 1975], defined by a sequence \( e_j \) (j=1,2,...) of i.i.d. random variables \( e_j \), with
\[ \Pr(e_j = +1) = \Pr(e_j = -1) = 1/2. \] (This is a nontrivial statement; we leave the proof to the reader [Frenk 1986a].) We have that \( u_n = \max_{1 \leq k \leq n} (-s_k) \),
with \( s_k = \sum_{j=1}^{k} e_j \). Actually, \( s_k^d = s_k \), so that it suffices to compute the expectation of \( \max_{1 \leq k \leq n} \{ s_k \} \).

According to the theory of \textit{fluctuations} (cf. [Chung 1974]), we know that (assuming \( n \) is even):

\[
E u_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{k} E s_k^+ = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2k} E s_{2k}^+ + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2k-1} E s_{2k-1}^+ \tag{13}
\]

where generally \( x^+ = \max\{x, 0\} \).

Now, using [Riordan 1968, p. 34], we find that

\[
E s_{2k}^+ = 2^k \sum_{p=1}^{k} \Pr(s_{2k} = 2p)p = 2^k \sum_{p=1}^{k} \binom{2k}{2p} \tag{14}
\]

Similarly,

\[
E s_{2k-1}^+ = 2^{-2k+2} \sum_{p=1}^{k} \binom{2k-1}{2p-1} (2k-1) (2k-1) \tag{15}
\]

Now, \( 2^{-2k} \binom{2k}{k} = (-1)^k \binom{k}{k} (1/2)^{k-1} \), with \( (-1)^k \binom{k}{k} (1/2)^{k-1} \) defined as \( (-1/2)(-1/2-1) \ldots (-1/2-k+1)/k! \), so that (cf. [Feller 1966])

\[
\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2k} E s_{2k}^+ = \frac{1}{2} \sum_{k=0}^{n/2} (-1)^{k} \binom{k}{k} (1/2)^{k-1} - \frac{1}{2} = \frac{1}{2} (-1)^{n/2} \binom{-3/2}{n/2} - \frac{1}{2} \tag{16}
\]
A similar manipulation with respect to \( \sum_{k=1}^{n/2} \frac{1}{(2k-1)/2} \) yields as a final exact result:

\[
E\{u_n\} = (-1)^{n/2}(-3/2) - \frac{1}{2}(-1)^{n/2}(-1/2) - \frac{1}{2}
\]  

(17)

A refinement of Stirling's formula [Feller 1966] then produces as an approximation that

\[
\left| E\{u_n\} - \left(2n/\pi\right)^{1/2} \right| \leq \alpha
\]

(18)

for some constant \( \alpha \). Hence,

\[
E(PA(n)) \geq \frac{n}{2} - E\{u_n/2\} \geq \frac{n}{2} - \left(\frac{n}{2\pi}\right)^{1/2} - \alpha,
\]

(19)

as was to be proved (cf (2)).

We observe that (19) is valid under much more general conditions than imposed here. Rather than independence, all that turns out to be needed is exchangeability and a symmetry condition on the joint distribution of the item sizes. We do not pursue this generalization in detail here.

For the classical bin packing problem, the expected solution value of the binary pairing heuristic is given by \( n/2 + E\{u_n/2\} \). Thus, (17) provides an exact expression for this value, improving on the asymptotic estimates that have appeared in the literature.

A variation on the binary pairing heuristic can be used to analyze the optimal solution value \( OPT(n) \) of the dual vector packing problem, under the assumption that \( a_i \) and \( b_i \) are independently uniformly distributed on \([0,1]\). To describe this heuristic, divide \([0,1] \times [0,1]\) into the regions \( A_i \) and \( B_i \) (\( i=0,1,\ldots,m-1 \)) where \( A_i = [0,1/2] \times [\frac{1}{m}, \frac{i+1}{m}] \) and \( B_i = [1/2,1] \times [\frac{1}{m}, \frac{i+1}{m}] \) (\( m \) arbitrary, but fixed) and label the stochastic pairs \( (a_i, b_i)(i=1,\ldots,n) \) as follows:
(i) = k \quad \text{if } (a_i, b_i) \in A_k \cup B_{m-k} \quad (k=1, \ldots, m-1) \quad (20)

\quad (i) = 0 \quad \text{if } (a_i, b_i) \in A_0 \cup B_0

Now it is easy to check that, conditional on (i) = k, $a_i$ is still uniformly distributed for $k = 1, \ldots, m-1$. Consider now all the pairs $(a_i, b_i)$ with (i) equal to $k$ ($k=1, \ldots, m-1$), and apply the pairing heuristic for the one-dimensional dual binpacking problem to their first coordinates.

The number of filled bins then equals at least $1/2(w_k - u_{k,n})$ where $w_k$ is the number of items $(a_i, b_i)$ with (i) = $k$ and $u_{k,n}$ is the number of unmatched small items among the elements with label (i) = $k$.

Hence the total number of filled bins $PAV(n)$ satisfies

$$E(PAV(n)) \geq \frac{1}{2} \left( \sum_{k=1}^{m-1} (w_k - u_{k,n}) / 2 \right) = \frac{n}{2} \cdot \frac{m-1}{m} - \frac{1}{2} \sum_{k=1}^{m-1} E(u_{k,n} / 2),$$

where we use the fact that $w_k$ is binomially distributed with parameters $n$ and $1/m$.

We know that (cf. (18)) $E(u_{k,n} | w_k = p) \leq C/p$ for some constant $C$, and hence by Jensen's inequality

$$E(u_{k,n}) \leq C \cdot E(\sqrt{w_k}) \leq C \cdot \sqrt{E}w_k = C \cdot \sqrt{n/m} \quad (21)$$

Thus, $E(PAV(n)) \geq n(m-1)/2m - C/\sqrt{nm}$ and hence

$$\lim \inf_{n \to \infty} \frac{E(PAV(n))}{n} \geq \frac{m-1}{2m} \quad (23)$$

so that

$$\lim \inf_{n \to \infty} \frac{E(OPTV(n))}{n} \geq 1/2. \quad (24)$$

Since it is obvious that $\lim \sup_{n \to \infty} E(OPTV(n)/n \leq 1/2$, we obtain
4. THE NEXT FIT HEURISTIC

A simple and natural solution method for the dual bin packing problem is given by an adaptation of the well known next fit heuristic for classical bin packing.

In a Next Fit Heuristic (NF) for dual bin packing, one assigns items in arbitrary order to a bin until the sum of their sizes exceeds 1 and the bin is covered. A new bin is then opened and the process repeats itself.

The number of items $v_1$ assigned to the first bin is equal to $(t + 1)$ where $t = \sup \{ k \geq 1 \mid \sum_{i=1}^{k} a_i < 1 \}$. The NF heuristic is such that the same applies to the number of items $v_j$ assigned to the $j$-th bin, for any $j$.

Thus, the random solution value $\text{NF}(n)$ is related to the renewal process $R_n$, associated with the sequence $v_j$ and defined by

$$R_n = \sup \{ m \geq 0 \mid \sum_{j=1}^{m} v_j \leq n, v_0 = 0 \},$$

in that $\text{NF}(n) = R_n$. To compute $E(\text{NF}(n))$, it suffices to compute the discrete renewal function $ER_n$.

We first observe that $E v_j = E(t+1) = \sum_{k=0}^{\infty} 1/k! = e$, that $E v_j^2 = \sum_{k=1}^{\infty} k \cdot \Pr\{ t + 1 \geq k \} - e = 3e < \infty$, and that the distribution of $t + 1$ satisfies the property that g.c.d. $\{ n \mid n > 0, \Pr\{ t + 1 = n \} > 0 \} = 1$. Hence, the weak renewal theorem [Karlin & Taylor 1975] immediately yields that

$$\lim_{n \to \infty} \frac{E(\text{NF}(n))}{n} = \frac{1}{e}.$$  \hspace{1cm} (26)

We obtain a much stronger result by considering $\lim_{n \to \infty} (E(\text{NF}(n)) - n/e)$. The strong renewal theorem yields (cf. [Feller, 1949])

$$\lim_{n \to \infty} (E(\text{NF}(n)) - \frac{n}{e}) = \frac{2}{e} - 1$$  \hspace{1cm} (27)

In fact, convergence in (27) can be shown to be exponentially fast [Frenk 1986].
In view of the result from Section 3, (27) implies that the expected relative error of the NF heuristic converges to \(1 - 2/e\).

The weaker result (26) can be generalized to the case in which the item sizes are distributed uniformly over the interval \([0, u]\) \((u \in (0, 1))\). In that case, the right hand side of (26) has to be replaced by \(1/\mu\), with

\[
u = \frac{\bar{k}}{L_{z=0}(-1)^{z} \frac{1}{z!} \left( -1 \right)^z \exp \left( \frac{1}{u} - z \right)},
\]

\[
\bar{k} = \left\lfloor \frac{1}{\mu} \right\rfloor.
\]

The derivation of this result is based on the result [Feller 1966] that in this case

\[
\Pr \left( \sum_{i=1}^{n} a_i \leq x \right) = \frac{1}{u^n n!} \sum_{z=0}^{n} (-1)^z \left( \frac{z}{u} \right)^n \left( \max\{x - zu, 0\} \right)^n
\]

and will not be presented here in full detail.

Again, this analysis is valid for many distributions other than the uniform one. In view of the general applicability of renewal theory, this should not come as a surprise.

The obvious extension of the Next Fit heuristic to the dual vector packing problem can be analyzed similarly. Let us assume again that \(a_i\) and \(b_i\) are independently uniformly distributed on \([0, 1]\).

We now have two random variables

\[
t_a = \inf \{k \geq 1 : \sum_{i=1}^{k} a_i > 1\},
\]

\[
t_b = \inf \{k \geq 1 : \sum_{i=1}^{k} b_i > 1\},
\]

and the number of items packed in an arbitrary bin equals \(\max\{t_a, t_b\}\). Note that by the independence of \(t_a\) and \(t_b\) we have

\[
\Pr\{\max\{t_a, t_b\} \leq t\} = \Pr\{t_a \leq t\} \Pr\{t_b \leq t\}
\]
\[ \text{Hence} \]
\[ \sum_{t=0}^{\infty} P \{ \max(t_a, t_b) \leq t \} z^t = \frac{1}{1-z} - 2e^z + \sum_{t=0}^{\infty} \frac{z^t}{(t!)^2} \]
\[ \text{and this implies, with } \hat{F}(z) = \sum_{t=0}^{\infty} \Pr[\max(t_a, t_b) = t] z^t, \text{ that} \]
\[ \frac{1 - \hat{F}(z)}{1 - z} = 2e^z - \sum_{t=0}^{\infty} \frac{z^t}{(t!)^2} \]
Now the number of bins used for \( n \) items is given by
\[ \text{NFV}(n) = \sup \{ m \geq 0 \mid \sum_{j=0}^{m} v_j \leq n \} \]
where \( v_j (j \geq 1) \) are independent and identically distributed random variables \( (v_0 = 0), \text{with } v_1 = \max(t_a, t_b). \)

Hence, by the weak renewal theorem,
\[ \lim_{n \to \infty} \frac{\text{E}(\text{NFV}(n))}{n} = \frac{1}{\text{E}(\max(t_a, t_b))} = \]
\[ = \lim_{z \to 1} \frac{1}{2e^z - \sum_{t=0}^{\infty} \frac{z^t}{(t!)^2}} = \]
\[ = \frac{1}{2e - \sum_{t=0}^{\infty} \frac{1}{(t!)^2}} \]
Because the probability distribution of \( \max(t_a, t_b) \) is lattice with span equal to 1, the strong renewal theorem yields that
\[ \lim_{n \to \infty} \left( \text{E}(\text{NFV}(n)) - \frac{n}{E_1} \right) = \frac{E_2 + E_1}{2E_1^2} \]
with \( E_1 = \text{E}(\max(t_a, t_b)) = 2e - \sum_{t=0}^{\infty} \frac{1}{(t!)^2} = 3.1567... \) and
\[ E_2 = \text{E}((\max(t_a, t_b))^2). \]

Note that from (34)
\[ \sum_{t=0}^{\infty} t \Pr[\max(t_a, t_b) > t] = \]
\[
\lim_{z \to 1} \frac{d}{dz} \left( \frac{1 - F(z)}{1 - z} \right) = \\
= \lim_{z \to 1} \left( 2e^z - \sum_{t=1}^{\infty} \frac{z^{t-1}}{t!(t-1)!} \right) = \\
= 2e - \sum_{t=0}^{\infty} \frac{1}{(t+1)!t!} 
\] (37)

It is also easy to prove that
\[
\sum_{t=0}^{\infty} t P\{\max(t_a, t_b) > t\} = 1/2 E_2 - 1/2 E_1 
\] (38)

and hence
\[
E_2 = 6e - \sum_{t=0}^{\infty} \frac{1}{(t!)^2} - 2 \sum_{t=0}^{\infty} \frac{1}{(t+1)!t!} = 10.8488 
\] (39)

Thus, the right hand side of (36) is equal to -0.2974...

5. THE NEXT FIT DECREASING HEURISTIC

In this section we adapt and analyze the Next Fit Decreasing Heuristic (NFD) to our model.

Given a list of n items of size \(a_1, a_2, \ldots, a_n\) \((0 \leq a_i \leq 1)\), the NFD heuristic for the dual bin packing problem first reindexes the elements in decreasing order and then applies the NF heuristic to this new list. To analyze the behaviour of the expected solution value \(E(\text{NFD}(n))\), we approximate the performance of the NFD heuristic by that of the sliced NFD heuristic with parameter \(r\) (SNFD\(_r\)), in which first items larger than \(1/r\) are packed according to the NFD heuristic, the last opened bin is completed by adding elements of decreasing size smaller than \(1/r\) and any remaining items are packed in groups of size \(r + 1\).

The number of bins used by this heuristic on n items is denoted by SNFD\(_r\)(n). It is clear that
\[
\text{SNFD}_r(n) \geq \text{NFD}(n) \quad (r > 1) 
\] (40)
and

$$\lim_{r \to \infty} \text{SNFD}_r(n) = \text{NFD}(n) \quad (\text{a.s.}) \quad (41)$$

Let $k_i$ be the number of items whose size falls in the interval $(1/(i+1), 1/i]$, $(i \geq 1)$ and let $K_i = k_1 + k_2 + \ldots$

Then, for any $r > 1$,

$$\text{SNFD}_r(n) \leq \frac{k_1}{2} + \frac{k_2}{3} + \ldots + \frac{k_{r-1}}{r} + \frac{K_r}{r+1} + r \quad (42)$$

where the last term is included to allow for rounding errors.

Since $a_i$ are uniformly distributed and independent, we obtain

$$E(k_i) = n/(i(i+1))$$

and

$$E(K_i) = n/i.$$ 

Hence

$$E(\text{SNFD}_r(n)) \leq n \sum_{i=1}^{r-1} \frac{1}{i(i+1)^2} + \frac{n}{r(r+1)} + r \quad (43)$$

and this implies that

$$\limsup_{n \to \infty} \frac{E(\text{NFD}(n))}{n} \leq \sum_{i=1}^{\infty} \frac{1}{i(i+1)^2}. \quad (44)$$

Moreover,

$$\text{NFD}(n) \geq \left(\frac{k_1}{2} - 1\right) + \left(\frac{k_2}{3} - 1\right) + \ldots + \left(\frac{k_{r-1}}{r} - 1\right) \quad (45)$$

and, by choosing $r$ as a suitable function of $n$, we find that

$$\liminf_{n \to \infty} \frac{E(\text{NFD}(n))}{n} \geq \sum_{i=1}^{\infty} \frac{1}{i(i+1)^2}. \quad (46)$$

Since

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)^2} = 2 - \sum_{i=1}^{\infty} \frac{1}{i^2} = 2 - \frac{\pi^2}{6},$$

we obtain from (45) and (46) that

$$\lim_{n \to \infty} \frac{E(\text{NFD}(n))}{n} = 2 - \frac{\pi^2}{6} = 0.3551\ldots \quad (47)$$

We note that $\lim_{n \to \infty} E(\text{NFD}(n))/n = 1/e = 0.3679\ldots$, so that the expected
performance of the NF heuristic is better than the (expected) performance of the NFD heuristic. For the classical binpacking problem, exactly the reverse is true! We have no satisfying intuitive explanation for this phenomenon.

5. CONCLUDING REMARKS

The probabilistic analysis of the dual bin packing problem, carried out in the preceding sections, reveals its connections to the classical bin packing problem and, surprisingly, to renewal theory. It also leaves several open questions of interest. Perhaps most prominent among these would be the challenge to find an on-line heuristic for this problem with better expected relative error than the NF heuristic discussed in Section 4.
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