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ON THE SOLUTION OF AN INTEGRAL EQUATION ARISING IN POTENTIAL PROBLEMS FOR CIRCULAR AND ELLIPTIC DISKS*

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Abstract. This paper deals with a two-dimensional Fredholm integral equation of the first kind over the circular disk $S$, with a kernel of the form $g(\theta)/|r - r'|$, $r, r' \in S$, where $\theta$ is the angle between $r - r'$ and some reference direction. By expansions in Fourier series and in series involving Legendre functions, and by use of a new closed-form result for a Legendre-function integral, the integral equation is reduced to a system of linear equations for the expansion coefficients. It is shown that the system has a unique solution because of the Toeplitz structure of the system matrix. As an application, the electrostatic potential problem for a charged elliptic disk is discussed.

Key words. two-dimensional integral equation, Legendre functions, electrostatics, elliptic disk, potential theory

AMS(MOS) subject classifications. 45B05, 33C45, 78A30, 31B10

1. Introduction. This paper concerns the solution of the two-dimensional integral equation

\begin{equation}
\int \int_S \frac{g(\theta)u(r')}{|r - r'|} \, dS' = f(r), \quad r \in S,
\end{equation}

over the circular disk $S$ of unit radius. In Cartesian coordinates, $r = (x, y, 0)$, $r' = (x', y', 0)$, $S$ is described by $x^2 + y^2 \leq 1$, and $\theta$ is the angle between the vector $r - r'$ and the positive $x$-axis. In (1.1), $u(r')$ is the unknown function, while $f(r)$ and $g(\theta)$ are given functions with $g(\theta)$ real-valued and periodic of period $\pi$. The integral equation arises, for example, in Danicki's [3] treatment of the scattering of a surface acoustic wave by a conducting disk placed on top of a piezoelectric half-space.

We introduce polar coordinates $(\rho, \varphi)$, $(\rho', \varphi')$, $(r, \theta)$, specified by

\begin{equation}
x + iy = \rho e^{i\varphi}, \quad x' + iy' = \rho' e^{i\varphi'}, \quad \rho e^{i\varphi} - \rho'e^{i\varphi'} = re^{i\theta},
\end{equation}

then the integral equation (1.1) takes the form

\begin{equation}
\int_0^{2\pi} \int_0^1 \frac{g(\theta)}{r} u(\rho', \varphi')\rho' \, d\rho' \, d\varphi' = f(\rho, \varphi), \quad 0 \leq \rho \leq 1, \quad 0 \leq \varphi \leq 2\pi.
\end{equation}

In the special case of constant $g(\theta)$, the latter equation has been studied by Wolfe [11]. He showed that the solution for $u(\rho', \varphi')$ can be determined by expansion in a series of "eigenfunctions"

\begin{equation}
P_n^{m}(\sqrt{1 - \rho^2})e^{im\varphi'}/\sqrt{1 - \rho'^2}, \quad n = 0, 1, 2, \ldots, \quad m = -n(2)n;
\end{equation}

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here $P^m_n$ is the associated Legendre function of the first kind [4, Chap. 3], and the notation $m = -n(2)n$ means that $m$ varies from $-n$ to $n$ with step 2, so that $m + n$ is even. The basic ingredient in Wolfe’s method is the key integral

$$
\int_0^{2\pi} \int_0^1 \frac{P^m_n(\sqrt{1 - \rho^2})e^{im\varphi}}{\sqrt{1 - \rho^2}} \rho' \, dp' \, d\varphi'
$$

(1.5) 

$$
= \pi \frac{\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2})\Gamma(\frac{1}{2} n - \frac{1}{2} m + \frac{1}{2})}{\Gamma(\frac{1}{2} n - \frac{1}{2} m + 1)\Gamma(\frac{1}{2} n + \frac{1}{2} m + 1)} P^m_n(\sqrt{1 - \rho^2})e^{im\varphi},
$$

$$
0 \leq \rho \leq 1, \quad 0 \leq \varphi \leq 2\pi,
$$

which is proved in [11]. This integral relation explains why the functions (1.4) are called “eigenfunctions.”

It is shown in this paper that Wolfe’s approach can be extended to solve the integral equation (1.3) with general $g(\theta)$. To that end, the solution $u(\rho', \varphi')$ is represented by a series of functions (1.4), and the function $g(\theta)$ is expanded in a Fourier series

$$
g(\theta) = \sum_{l \text{ even}} g_l e^{il\theta}.
$$

(1.6)

Only terms with $l$ even appear in the latter series, since $g(\theta)$ has period $\pi$. On inserting these expansions into (1.3), we must evaluate the key integral

$$
I_{m,n,l}(\rho, \varphi) = \int_0^{2\pi} \int_0^1 \frac{e^{il\theta} P^m_n(\sqrt{1 - \rho^2})e^{im\varphi}}{\sqrt{1 - \rho^2}} \rho' \, dp' \, d\varphi',
$$

(1.7)

in which $n = 0, 1, 2, \ldots, m = -n(2)n, l$ even, $0 \leq \rho \leq 1, 0 \leq \varphi \leq 2\pi$. The evaluation is carried out in § 2 by methods adopted from Wolfe [11] and Bouwkamp [2]. A closed-form result is obtained, reading

$$
I_{m,n,l}(\rho, \varphi) = 0 \quad \text{if} \quad |m + l| > n,
$$

(1.8)

$$
I_{m,n,l}(\rho, \varphi) = C_{m,n,l} P^{m+1}_n(\sqrt{1 - \rho^2})e^{i(m+l)\varphi} \quad \text{if} \quad |m + l| \leq n,
$$

(1.9)

where

$$
C_{m,n,l} = 2^{-l}\pi \frac{\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2})\Gamma(\frac{1}{2} n - \frac{1}{2} m - \frac{1}{2} l + \frac{1}{2})}{\Gamma(\frac{1}{2} n - \frac{1}{2} m + 1)\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2} l + 1)}.
$$

(1.10)

This result reduces to (1.5) if $l = 0$. Note that the function on the right of (1.9) belongs to the set (1.4) (apart from a division by $\sqrt{1 - \rho^2}$), since $m$ and $m + l$ have the same parity. With these preparations, the integral equation (1.3) is reduced in § 3 to a sequence of mutually independent, finite systems of linear equations for the coefficients in the expansion of $u(\rho', \varphi')$. The system matrices prove to be related to the Toeplitz matrix associated with the function $g(\theta)$. By means of a known spectral property of Toeplitz matrices [5], it is shown that the systems have a unique solution under the sufficient condition that $g(\theta)$ is continuous and nonzero. As an application, the electrostatic potential problem for a charged elliptic disk is discussed in § 4.
2. Evaluation of the key integral (1.7). We start with some preliminaries. From
\[4, 3.2(20), 3.4(17)\], we quote the following expressions for \( P_m(n) \) in terms of
a hypergeometric function \( F \) of argument \( \sqrt{1 - \rho^2} \):
\[
(2.1) \quad P_m(n) = (-1)^m \frac{(n + m)!}{(n - m)!} \frac{(-1)^m m!}{(n + m)!} \rho^m F\left( \frac{1}{2}, \frac{1}{2} + \frac{1}{2} m \right)
\]
valid for integral \( m \) and \( n \) subject to \( 0 \leq m \leq n \). For the Bessel function \( J_m(\rho t) \), we have
the following expansion as a series of Bessel functions:
\[
(2.2) \quad J_m(\rho t) = (-1)^m \frac{2m + 1}{m+1/2} \frac{1}{\sqrt{m+1/2}} \sum_{\nu = \max(0, -m)}^{\infty} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + m + 1)} \rho^{m+2s} \left( \sqrt{1 - \rho^2} \right)^{m+2s+1/2} \left( \int_0^1 P_m(\rho t) e^{im\phi} d\phi \right)
\]
valid for integral \( m \). For \( m \geq 0 \), the expansion follows from \[9, 5.21(3)\]; the latter
formula involves a hypergeometric function, which is expressed as \( P_m(\sqrt{1 - \rho^2}) \) by
the use of (2.1). For \( m < 0 \), expansion (2.2) is reducible to the expansion of \( (-1)^m J_{-m}(\rho t) \),
which is equal to \( J_{m}(\rho t) \).

We now come to the evaluation of the key integral \( I_{m,n}(\rho, \varphi) \), introduced in (1.7).
Following Wolfe \[11\], who considered the special case where \( l = 0 \), we note that the
integral (1.7) is the convolution of the two functions \( F(\rho, \varphi) \) and \( G(\rho, \varphi) \), given by
\[
(2.3) \quad F(\rho, \varphi) = \frac{e^{il/\rho}}{\rho}, \quad G(\rho, \varphi) = \begin{cases} \frac{P_m(\sqrt{1 - \rho^2}) e^{im\phi}}{\sqrt{1 - \rho^2}}, & \rho < 1, \\ 0, & \rho \geq 1. \end{cases}
\]
This convolution is evaluated by the method of Fourier transforms. Using polar coordinates,
we define the two-dimensional Fourier transform of a function \( f(\rho, \varphi) \) by
\[
(2.4) \quad (\mathcal{F} f)(\rho, \varphi) = \int_0^{2\pi} \int_0^\infty f(\rho', \varphi') \exp(i\rho' \cos(\varphi - \varphi')) \rho' d\rho' d\varphi'.
\]
By means of \[9, 2.2(1), 13.24(1)\], the Fourier transform of \( F(\rho, \varphi) \) is found to be
\[
(2.5) \quad (\mathcal{F} F)(\rho, \varphi) = \int_0^\infty \left[ \rho^l e^{il/\rho} \right] \int_0^{2\pi} \exp[i(l+1)\rho' \cos(\varphi - \varphi')] \rho' d\rho' d\varphi'
\]
valid for \( l \) even, so that \( J_l(t) = J_{-l}(t) \). Here the analysis is admittedly formal but could
be made rigorous in the context of the theory of distributions. The final result in (2.5)
should be considered as the distributional Fourier transform of \( F(\rho, \varphi) \).

Next, we determine the Fourier transform of \( G(\rho, \varphi) \), as follows:
\[
(2.6) \quad (\mathcal{F} G)(\rho, \varphi) = \frac{1}{(2\pi)^2} \int_0^\infty \left[ \rho^l e^{il/\rho} \right] \frac{P_n(\sqrt{1 - \rho^2})}{\sqrt{1 - \rho^2}} \int_0^{2\pi} \exp[i(l+1)\rho' \cos(\varphi - \varphi')] \rho' d\rho' d\varphi'
\]
valid for \( l \) even, so that \( J_l(t) = J_{-l}(t) \). Here the analysis is admittedly formal but could
be made rigorous in the context of the theory of distributions. The final result in (2.5)
should be considered as the distributional Fourier transform of \( F(\rho, \varphi) \).
again by use of Bessel’s integral [9, 2.2(1)]. To evaluate the final integral in (2.6), we replace $J_m(\rho')$ by expansion (2.2) and integrate term by term. All but one of the resulting integrals will vanish, since it follows from the orthogonality relation [4, 3.12(19), (21)] for Legendre functions that

\begin{equation}
J_{m+1/2}(\rho') = \frac{(-1)^m}{\sqrt{2\pi}} \left( \frac{1}{\rho'} \right)^{1/2} \left( \frac{\rho}{\rho'} \right)^{1/2} \sum_{n=-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}m + 1\right)} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}m + 1\right)} \rho^{-1/2} J_{m+1/2}(\rho) e^{im\rho'.}
\end{equation}

Here it is recalled that $m = -n(2)n$, so that $n$ and $m + 2n$ have the same parity. By properly combining these results and substituting into (2.6), we obtain

\begin{equation}
(\mathcal{F}G)(\rho, \varphi) = (-1)^m 2^{m+1/2} \pi e^{im\rho/2} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}m + 1\right)} \rho^{-1/2} J_{n+1/2}(\rho) e^{im\varphi}.
\end{equation}

The Fourier transform of $I_{m,n,l}(\rho, \varphi)$ is equal to the product of the Fourier transforms (2.5) and (2.8), as follows:

\begin{equation}
(\mathcal{F} I_{m,n,l})(\rho, \varphi) = (-1)^m 2^{m+3/2} \pi^2 e^{im\rho/2} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}m + 1\right)} \rho^{-3/2} J_{n+1/2}(\rho) e^{im\varphi},
\end{equation}

valid for $n = 0, 1, 2, \ldots, m = -n(2)n$, $l$ even. By means of the inversion formula

\begin{equation}
I_{m,n,l}(\rho, \varphi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} (\mathcal{F} I_{m,n,l})(\rho', \varphi') \exp \left[ -i\rho \varphi' \cos (\varphi - \varphi') \right] d\rho' d\varphi',
\end{equation}

we then find that

\begin{equation}
I_{m,n,l}(\rho, \varphi) = (-1)^m 2^{m-1/2} \pi e^{im\rho/2} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}m + 1\right)} \cdot \int_0^{\infty} (\rho')^{-1/2} J_{n+1/2}(\rho') d\rho'.
\end{equation}

\begin{equation}
\cdot \int_0^{2\pi} \exp \left[ i(m + l)\varphi' - i\rho \varphi' \cos (\varphi - \varphi') \right] d\varphi' = (-1)^m 2^{m+1/2} \pi \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}m + 1\right)} e^{im\varphi} \cdot \int_0^{\infty} (\rho')^{-1/2} J_{n+1/2}(\rho') J_{m+1}(\rho \rho') d\rho',
\end{equation}

in which $0 \leq \rho \leq 1$, $0 \leq \varphi \leq 2\pi$. 
The final Bessel-function integral in (2.11) is evaluated in two different ways. First, we replace $J_{m+\ell}(\rho \rho')$ by expansion (2.2) and integrate term by term to obtain

$$
\int_0^\infty (\rho')^{-1/2} J_{n+1/2}(\rho') J_{m+\ell}(\rho \rho') \, d\rho'
$$

(2.12)

$$
= (-1)^{m+\ell} 2^{-m-l+1/2} \sum_{\nu = \max(0,-m-l)}^\infty \left( m + l + 2\nu + \frac{1}{2} \right) \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + m + l + 1)} \cdot P_{m+\ell}^{l+1/2}(\sqrt{1 - \rho^2}) \int_0^\infty (\rho')^{-1/2} J_{n+1/2}(\rho') J_{m+\ell+2\nu+1/2}(\rho') \, d\rho'.
$$

Note that $n$ and $m + l$ have the same parity. From [9, 13.41(7)], we then have

$$
\int_0^\infty (\rho')^{-1/2} J_{n+1/2}(\rho') J_{m+\ell+2\nu+1/2}(\rho') \, d\rho' = \frac{1}{2n+1} \delta_{n,m+l+2\nu}.
$$

(2.13)

Thus all terms of the series (2.12) vanish, except for the term with $\nu = \frac{1}{2} n - \frac{1}{2} m - \frac{1}{2} l$. This term is contained in the series only if $\frac{1}{2} n - \frac{1}{2} m - \frac{1}{2} l \geq \max(0,-m-l)$, that is, if $|m + l| \geq n$. By substitution of (2.13) into (2.12), we find that

$$
\int_0^\infty (\rho')^{-1/2} J_{n+1/2}(\rho') J_{m+\ell}(\rho \rho') \, d\rho'
$$

(2.14)

$$
= \begin{cases} (-1)^{m+\ell} 2^{-m-l+1/2} \frac{\Gamma(\frac{1}{2} n - \frac{1}{2} m - \frac{1}{2} l + \frac{1}{2})}{\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2} l + 1)} P_{m+\ell}^{l+1/2}(\sqrt{1 - \rho^2}) & \text{if } |m + l| \leq n, \\
0 & \text{if } |m + l| > n. 
\end{cases}
$$

Here we have set $(-1)^{m+\ell} = (-1)^m$, since $\ell$ is even. A second evaluation of the integral (2.14) by use of [9, 13.4(2)] leads to an expression involving a hypergeometric function, viz.,

$$
\int_0^\infty (\rho')^{-1/2} J_{n+1/2}(\rho') J_{m+\ell}(\rho \rho') \, d\rho'
$$

(2.15)

$$
= 2^{-1/2} \frac{\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2} l + \frac{1}{2})}{\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2} l + 1)} \cdot \rho^{m+\ell} F\left( \frac{1}{2} n + \frac{1}{2} m + \frac{1}{2} l + \frac{1}{2}, -\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2} l; m + l + 1; \rho^2 \right),
$$

valid for $m + l \geq 0$, $0 \leq \rho \leq 1$. Because of the denominator factor $\Gamma\left(\frac{1}{2} n - \frac{1}{2} m - \frac{1}{2} l + 1\right)$, the integral vanishes if $m + l > n$. For $0 \leq m + l \leq n$, the hypergeometric function can be expressed as $P_{m+\ell}^{l+1/2}(\sqrt{1 - \rho^2})$ by means of (2.1), and the result (2.14) is recovered. The case of $m + l < 0$ can be reduced to the previous case, because $J_{m+\ell}(\rho \rho') = (-1)^{m+\ell} J_{-m-\ell}(\rho \rho')$.

Finally, by inserting (2.14) into (2.11), we obtain the closed-form results

$$
I_{m,n,l}(\rho, \varphi) = \begin{cases} 0 & \text{if } |m + l| > n, \\
2^{-1/2} \pi \frac{\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2} l)}{\Gamma(\frac{1}{2} n - \frac{1}{2} m + \frac{1}{2} l + 1)} \times P_{m+\ell}^{l+1/2}(\sqrt{1 - \rho^2}) e^{i(m+l)\varphi} & \text{if } |m + l| \leq n, 
\end{cases}
$$

(2.16) (2.17)
valid for $n = 0, 1, 2, \ldots$, $m = -n(2)n$, $l$ even, $0 \leq \rho \leq 1$, $0 \leq \varphi \leq 2\pi$, and announced in (1.8)-(1.10).

We briefly discuss an alternative approach to the evaluation of the key integral (1.7), inspired by Bouwkamp [2]. The main idea is to replace the factor $e^{i\theta}/r$ in (1.7) by

$$e^{i\theta} e^{-i\varphi'} \frac{p_1 e^{-i(\varphi - \varphi')}}{r} \int_0^\infty J_l(t \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}) \, dt. \tag{2.18}$$

This representation was found by means of (1.2) and [9, 13.24(1)]. In the resulting triple integral the $\varphi'$-integration is carried out by application of Graf’s addition theorem [9, 11.3(1)] for Bessel functions, yielding

$$e^{i\theta} e^{-i\varphi'} \frac{p_1 e^{-i(\varphi - \varphi')}}{r} J_l(t \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}) \, d\varphi' \tag{2.19} = 2\pi J_{m+1}(\rho t)J_m(\rho' t) e^{im\theta}.$$

Thus the triple integral reduces to

$$I_{m,n,l}(\rho, \varphi) = 2\pi e^{i(m+1)\theta} \int_0^\infty J_{m+1}(\rho t) dt \int_0^1 \frac{P_n^m(\sqrt{1 - \rho^2})}{\sqrt{1 - \rho'^2}} J_m(\rho' t) \rho' \, dp'. \tag{2.20}$$

Here the inner integral has been evaluated in the text following (2.6). By inserting its value, implicit in (2.8), we arrive at a representation for $I_{m,n,l}(\rho, \varphi)$ that is identical to (2.11).

3. Solution of the integral equation (1.3). To solve the integral equation (1.3), we insert the following expansions for $g(\theta)$ and $u(\rho', \varphi')$. The function $g(\theta)$ is replaced by its Fourier series (1.6), while the unknown function $u(\rho', \varphi')$ is represented by a series of functions (1.4), viz.,

$$u(\rho', \varphi') = \sum_{n=0}^{\infty} \sum_{m=-n(2)}^{n} a_{m,n} P_n^m(\sqrt{1 - \rho'^2}) e^{im\varphi'}, \tag{3.1}$$

in which the coefficients $a_{m,n}$ are to be determined. In (3.1), the summation index $m$ runs from $-n$ to $n$ with step 2. Then, through a term-by-term integration, where we need the values (1.8) and (1.9) of $I_{m,n,l}(\rho, \varphi)$, the left-hand side of (1.3) reduces to

$$= \sum_{n=0}^{\infty} \sum_{m=-n(2)}^{n} \sum_{|m|+|l| \leq n} g_{l} a_{m,n} C_{n,m,l} P_n^{m+l}(\sqrt{1 - \rho^2}) e^{im\varphi}. \tag{3.2}$$

Next, we expand the known function $f(\rho, \varphi)$ of (1.3) in a series of functions $P_n^m(\sqrt{1 - \rho^2}) e^{im\varphi}$, $n = 0, 1, 2, \ldots$, $m = -n(2)n$ as observed by Wolfe [11], the latter functions form a complete orthogonal set on the unit disk, with weight function $(1 - \rho^2)^{-1/2}$ and squared norm

$$\int_0^1 \int_0^{2\pi} \frac{[P_n^m(\sqrt{1 - \rho^2})]^2}{\sqrt{1 - \rho^2}} \rho \, d\rho \, d\varphi = 2\pi \int_0^1 [P_n^m(t)]^2 \, dt = \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!}. \tag{3.3}$$
Thus the required expansion is readily found to be

\[ f(\rho, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n(2)} f_{m,n} P_n^m(\sqrt{1-\rho^2}) e^{im\varphi}, \]

with expansion coefficients given by

\[ f_{m,n} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{0}^{2\pi} \int_{0}^{1} f(\rho, \varphi) \frac{P_n^m(\sqrt{1-\rho^2}) e^{-im\varphi}}{\sqrt{1-\rho^2}} \rho \, d\rho \, d\varphi. \]

Identifying corresponding terms in expansions (3.2) and (3.4), we conclude that the integral equation (1.3) reduces to the set of linear equations

\[ \sum_{l=-n(2)}^{n} g_{m-l} c_{l,n,m-l} a_{l,n} = f_{m,n}, \quad n = 0, 1, 2, \ldots, \quad m = -n(2)n. \]

For fixed \( n = 0, 1, 2, \ldots \), this is a system of \( n + 1 \) linear equations for the coefficients \( a_{m,n}, m = -n(2)n \). Using the known value of \( c_{l,n,m-l} \) from (1.10), we rescale the coefficients \( a_{m,n} \) and \( f_{m,n} \) according to

\[ 2^m \frac{\Gamma(\frac{1}{2} n + \frac{1}{2} m + \frac{1}{2})}{\Gamma(\frac{1}{2} n - \frac{1}{2} m + \frac{1}{2})} a_{m,n} = \tilde{a}_{m,n}, \quad 2^m \frac{\Gamma(\frac{1}{2} n + \frac{1}{2} m + 1)}{\Gamma(\frac{1}{2} n - \frac{1}{2} m + \frac{1}{2})} f_{m,n} = \tilde{f}_{m,n}, \]

whereupon system (3.6) simplifies to

\[ \sum_{l=-n(2)}^{n} g_{m-l} \tilde{a}_{l,n} = \tilde{f}_{m,n}, \quad n = 0, 1, 2, \ldots, \quad m = -n(2)n. \]

Here, the system matrix \( (g_{m-l}) \) is recognized to be the \( (n+1) \times (n+1) \) Toeplitz matrix associated with the function \( g(\theta) \) [5, p. 17]. Since \( g(\theta) \) is real-valued, it follows that \( g_{-l} = \overline{g_l} \) and the matrix \( (g_{m-l}) \) is Hermitian. From [5, p. 64], we take the following spectral property: Let \( M_1 \leq g(\theta) \leq M_2 \); then the eigenvalues of the Toeplitz matrix \( (g_{m-l}) \) lie in the interval \([M_1, M_2]\). To use this property, we impose the sufficient condition that \( g(\theta) \) be continuous and nonzero. Then the matrix \( (g_{m-l}) \) has no zero-eigenvalues, and \( \det (g_{m-l}) \neq 0 \). Consequently, for each \( n = 0, 1, 2, \ldots \), system (3.8) has a unique solution. Furthermore, if \( f_{m,n} = 0 \) for some specific \( n \) and \( m = -n(2)n \), then also \( a_{m,n} = 0 \) for the same value of \( n \) and \( m = -n(2)n \). This completes the solution of the integral equation (1.3).

4. Electrostatic problems for an elliptic disk. As an application, we consider the integral equation for the electrostatic potential due to a charged elliptic disk, viz.,

\[ \frac{1}{4\pi} \int \int_{S_e} \frac{\sigma(r')}{|r-r'|} dS' = V(r), \quad r \in S_e. \]

In Cartesian coordinates, \( r = (x, y, 0), r' = (x', y', 0) \); furthermore, \( S_e \) is the elliptic disk described by \( x^2/a^2 + y^2/b^2 \leq 1 \), where \( a \geq b \). In (4.1), \( V(r) \) is the prescribed potential on the disk and \( \sigma(r') \) is the charge density to be determined. We apply the transformation of variables

\[ x = a\rho \cos \varphi, \quad y = b\rho \sin \varphi, \quad x' = a\rho' \cos \varphi', \quad y' = b\rho' \sin \varphi', \]

and we introduce polar coordinates \((r, \theta)\) specified by

\[ \rho e^{i\varphi} - \rho' e^{i\varphi'} = re^{i\theta} \]
as in (1.2). Then the distance $|r - r'|$ transforms into
\begin{equation}
|r - r'| = r(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{1/2},
\end{equation}
and the integral equation (4.1) takes the form (1.3) with
\begin{equation}
g(\theta) = \frac{ab}{4\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{-1/2}.
\end{equation}

Clearly, $g(\theta)$ has the required properties for the integral equation to be solvable by the method of § 3. For later use, we determine the Fourier coefficients $g_0$ and $g_{\pm 2}$ of $g(\theta)$, as follows:
\begin{align}
g_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \, d\theta = \frac{ab}{2\pi^2} \int_0^{\pi/2} \frac{d\theta}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{1/2}} = \frac{b}{2\pi^2} K(k), \\
g_{\pm 2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{\pm 2i\theta} \, d\theta = \frac{ab}{2\pi^2} \int_0^{\pi/2} \frac{\cos(2\theta) \, d\theta}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{1/2}} \\
&= \frac{b}{\pi^2 k^2} \left[ E(k) - \left( 1 - \frac{1}{2} k^2 \right) K(k) \right],
\end{align}
where $K(k)$ and $E(k)$ are the complete elliptic integrals of first and second kinds, of modulus $k = (1 - b^2/a^2)^{1/2}$ (equal to the eccentricity of the elliptic disk).

Two examples are discussed. First, we consider the conducting elliptic disk at unit potential $V = 1$, where the charge density $\sigma_0$ is to be determined from the integral equation
\begin{equation}
\int_0^{2\pi} \int_0^1 \frac{g(\theta)}{r} \sigma_0(\rho', \varphi') \rho' \, d\rho' \, d\varphi' = 1, \quad 0 \leq \rho \leq 1, \quad 0 \leq \varphi \leq 2\pi.
\end{equation}

Expansion (3.4) of the right-hand side simply becomes $1 = P_0(\sqrt{1 - \rho^2})$. Accordingly, we may restrict the expansion (3.1) for $\sigma_0(\rho', \varphi')$ to the $n = 0$ term, i.e.,
\begin{equation}
\sigma_0(\rho', \varphi') = \frac{a_{0,0}}{\sqrt{1 - \rho'^2}}.
\end{equation}
Here, the coefficient $a_{0,0}$ is to be determined from system (3.6), which reduces to the single equation
\begin{equation}
g_0C_{0,0,0}a_{0,0} = 1,
\end{equation}
with the solution
\begin{equation}
a_{0,0} = \frac{1}{g_0C_{0,0,0}} = \frac{2}{bK(k)}.
\end{equation}

In terms of the original coordinates $r = (x, y, 0)$ of the elliptic disk, the charge density $\sigma_0$ is found to be
\begin{equation}
\sigma_0(x, y) = \frac{2}{bK(k)} \frac{1}{\sqrt{1 - x^2/a^2 - y^2/b^2}}.
\end{equation}
Since the disk is at unit potential, the capacity $C_0$ of the elliptic disk is given by
\begin{equation}
C_0 = \int_{S_2} \sigma_0(x, y) \, dx \, dy = \frac{2}{bK(k)} \int_{S_2} dx \, dy \frac{1}{\sqrt{1 - x^2/a^2 - y^2/b^2}} = \frac{4\pi a}{K(k)},
\end{equation}
in accordance with results of Smythe [7, p. 124] and Szegö [8, p. 342].
Our second example deals with the conducting elliptic disk in a uniform electric field of potential \( V(r) = -Ax - By \), where \( A \) and \( B \) are constants. The integral equation for the induced charge density \( \sigma_1 \) now reads

\[
\int_0^{2\pi} \int_0^1 \frac{g(\theta)}{r} \sigma_1(\rho', \phi') \rho' \, d\rho' \, d\phi' = \rho(Aa \cos \phi + Bb \sin \phi), \quad 0 \leq \rho \leq 1, \quad 0 \leq \phi \leq 2\pi.
\]

Using the expressions

\[
P_{1}^{-1}(\sqrt{1 - \rho^2}) = \frac{1}{2} \rho,
\]

\[
P_{1}'(\sqrt{1 - \rho^2}) = -\rho,
\]

known from (2.1), we rewrite the right-hand side of (4.14) as

\[
\rho(Aa \cos \phi + Bb \sin \phi) = (Aa + iBb)P_{1}^{-1}(\sqrt{1 - \rho^2})e^{-i\phi} - \frac{1}{2}(Aa - iBb)P_{1}'(\sqrt{1 - \rho^2})e^{i\phi}.
\]

Accordingly, expansion (3.1) for \( \sigma_1(\rho', \phi') \) may be restricted to the \( n = 1 \) terms, i.e.,

\[
\sigma_1(\rho', \phi') = \left[ a_{-1,1}P_{1}^{-1}(\sqrt{1 - \rho^2})e^{-i\phi'} + a_{1,1}P_{1}'(\sqrt{1 - \rho^2})e^{i\phi'} \right]\sqrt{1 - \rho^2}
\]

\[
= \left( \frac{1}{2} a_{-1,1} - a_{1,1} \right) \frac{\rho' \cos \phi'}{\sqrt{1 - \rho^2}} - i \left( \frac{1}{2} a_{-1,1} + a_{1,1} \right) \frac{\rho' \sin \phi'}{\sqrt{1 - \rho^2}}.
\]

Here, the coefficients \( a_{-1,1} \) and \( a_{1,1} \) are determined from system (3.6) with \( n = 1 \). Setting \( g_{-2} = g_2 \) and using (1.10) to evaluate \( C_{1,1,m,-1} \), we have the two linear equations

\[
\begin{align*}
\frac{1}{2} g_0 a_{-1,1} + g_2 a_{1,1} &= \pi^{-2}(Aa + iBb), \\
\frac{1}{2} g_2 a_{-1,1} + g_0 a_{1,1} &= -\pi^{-2}(Aa - iBb).
\end{align*}
\]

By suitably combining these equations, we obtain the solution

\[
\begin{align*}
\frac{1}{2} a_{-1,1} - a_{1,1} &= \frac{2Aa}{\pi^2(g_0 - g_2)} = \frac{2A k^2 a}{b[K(k) - E(k)]}, \\
\frac{1}{2} a_{-1,1} + a_{1,1} &= \frac{2iBb}{\pi^2(g_0 + g_2)} = \frac{2iB k^2}{E(k) - (1 - k^2)K(k)},
\end{align*}
\]

which is to be inserted into (4.17). In terms of the original coordinates \( r = (x, y, 0) \) of the elliptic disk, the charge density \( \sigma_1 \) is represented by

\[
\sigma_1(x, y) = A\sigma_x(x, y) + B\sigma_y(x, y),
\]

where

\[
\begin{align*}
\sigma_x(x, y) &= \frac{2k^2}{b[K(k) - E(k)]} \frac{x}{\sqrt{1 - x^2/a^2 - y^2/b^2}}, \\
\sigma_y(x, y) &= \frac{2k^2}{b[E(k) - (1 - k^2)K(k)]} \frac{y}{\sqrt{1 - x^2/a^2 - y^2/b^2}}.
\end{align*}
\]
are the induced charge densities due to a uniform electric field of unit strength parallel to the x- or y-axis, respectively. The associated dipole moments are found to be

\begin{align}
M_x &= \int \int_{S_e} \sigma_x(x, y) x \, dx \, dy = \frac{4\pi}{3} \frac{a^3 k^2}{K(k) - E(k)}, \\
M_y &= \int \int_{S_e} \sigma_y(x, y) y \, dx \, dy = \frac{4\pi}{3} \frac{ab^2 k^2}{E(k) - (1 - k^2) K(k)}.
\end{align}

In the special case when \( a = b \), the present results can be shown to reduce to the known results [7, pp. 172–175] for the charge density and the dipole moment induced in a circular disk of radius \( a \). The integral equations (4.8) and (4.14), rewritten in the original form (4.1), have also been solved by Roy and Sabina [6] by a different method.

The integral equation (4.1) also arises in the analysis of low-frequency acoustic diffraction by a soft elliptic disk [6] or through an elliptic aperture in a rigid screen [1], [10]. For a plane wave incident on an aperture \( S \) in a rigid screen, Van Bladel [1, (32)] determined the first two terms in the low-frequency expansion of the transmission coefficient \( t \), as follows:

\begin{equation}
t = \frac{2L_0}{\pi} \left[ 1 - \frac{k^2 L_0^2}{\pi^2} + \frac{k^2 L_0 L_1}{\pi} - \frac{k^2 L_2^2}{\pi} \left( \frac{1}{3} + \cos^2 \theta_x \right) - \frac{k^2 L_y^2}{\pi} \left( \frac{1}{3} + \cos^2 \theta_y \right) \right].
\end{equation}

Here, \( \kappa \) is the wavenumber, \( \theta_x \) and \( \theta_y \) are the angles between the direction of incidence and the x- and y-axes, while \( L_0, L_1, L_x, L_y \) are expressible as integrals involving the solution of one electrostatic problem, namely, that of the conducting disk \( S \) at unit potential. In the special case of the elliptic aperture \( S_e \), we have, in terms of the solution \( \sigma_0 \) from (4.12),

\begin{align}
L_0 &= \frac{1}{4} \int \int_{S_e} \sigma_0(x, y) \, dx \, dy = \frac{1}{4} C_0 = \frac{\pi a}{K(k)}, \\
L_1 &= \frac{1}{16L_0} \int \int_{S_e} \int \int_{S_e} \sigma_0(x, y)| \mathbf{r} - \mathbf{r}' | \sigma_0(x', y') \, dx \, dy \, dx' \, dy' = \frac{2a}{3} E(k), \\
L_x^2 &= \frac{1}{4L_0} \int \int_{S_e} \sigma_0(x, y) x^2 \, dx \, dy = \frac{a^2}{3}, \\
L_y^2 &= \frac{1}{4L_0} \int \int_{S_e} \sigma_0(x, y) y^2 \, dx \, dy = \frac{b^2}{3}.
\end{align}

The latter two integrals are elementary, while in the evaluation of \( L_1 \) we used the key integral (1.7) as evaluated in (1.8)–(1.10). Inserting the values (4.26)–(4.28) into (4.25), we obtain the low-frequency expansion up to second order

\begin{equation}
t = \frac{2\pi a^2}{K^2(k)} \left[ 1 + \frac{k^2 a^2}{3} \left( \frac{2E(k)}{K(k)} - \frac{3}{K^2(k)} - \frac{1}{3} - \cos^2 \theta_x \right) - \frac{k^2 b^2}{3} \left( \frac{1}{3} + \cos^2 \theta_y \right) \right],
\end{equation}

in accordance with the expression found by Williams [10, p. 42].

REFERENCES