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Limit theorems for Markov chains of finite rank

by

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Abstract. We consider a Markov chain with a general state space, but whose behaviour is governed by finite matrices. After a brief exposition of the basic properties of this chain its convenience as a model is illustrated by three limit theorems. The ergodic theorem, the central limit theorem and an extreme-value theorem are expressed in terms of dominant eigenvalues of finite matrices and proved by simple matrix theory.

Key words: Markov chains of finite rank, spectral decomposition of matrices, ergodic theorem, central limit theorem, extreme values.

1. Introduction

In 1960, Runnenburg [11] introduced a Markov chain with a (stationary) transition distribution function of the form

\[ P(y|x) = \sum_{j=1}^{r} a_j(x) B_j(y). \]
This chain, which Runnenburg used as a simple example of dependence rather close to independence, was studied more closely by Runnenburg and Steutel [12]. The chain was also considered by Kingman [5] as an example in his algebraic view on Markov chains; the term "of finite rank" is his. In his thesis, Hoekstra [3] will give a detailed account of the structure, properties and possible generalizations of Markov chains with transition distribution function of type (1.1).

In this paper, after a brief introduction to these Markov chains and some of their properties, we demonstrate the easy analysability of the model by proving three limit theorems, using only simple matrix theory. If the process is denoted by $X_0, X_1, \ldots$, we obtain the limit distributions of $X_n$, of $X_1 + \ldots + X_n$, and of $\max(X_1, \ldots, X_n)$. Of most proofs only outlines are given; for the details we refer to [3].

We emphasize that in particular finite Markov chains are of type (1.1), with $r$ equal to the number of states (or smaller), and that therefore all results hold for finite Markov chains as well.

Part of our theorems can, no doubt, be viewed as special cases of known results; the advantage of this particular model lies in the explicit nature of the results and the simplicity of the proofs.

2. Notation and some matrix theory

We shall use capitals for matrices (for random variables too, but confusion will be unlikely), $u$ will denote a column vector, and $u^\top$ its transpose. The vector $(1, \ldots, 1)^\top$ is denoted by $1$, the vector $(0, \ldots, 0)^\top$ by $0$, the unit matrix by $I$ and the zero matrix by $0$; dimensions will be clear from the context.
The following well-known result on spectral decomposition of matrices will play a central role in our proofs (for information we refer to [4] and [9]).

Lemma 2.1. Let $A$ be an arbitrary complex $r \times r$ matrix with distinct eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_s$, and let $\lambda_0, \ldots, \lambda_{d-1}$ have algebraic multiplicity one. Then for all $n \in \mathbb{N}$

$$A^n = \sum_{\ell=0}^{d-1} \lambda_\ell^n E_\ell + \sum_{\ell=d}^s \left( \sum_{k=0}^{n-1} \lambda_\ell^k F_\ell^{n-k} \right) E_\ell,$$

where $E_\ell E_k = E_\ell F_k = F_\ell E_k = F_\ell F_k = 0$ if $\ell \neq k$, $E_\ell F_k = F_\ell E_k = F_\ell F_k = E_\ell$ and

$$m_\ell = 0 \quad \text{for an } m_\ell \leq r \quad (\ell = d, \ldots, s).$$

Furthermore, if $u_\ell^T$ and $v_\ell$ are left and right eigenvectors of $A$ corresponding to $\lambda_\ell$ with $u_\ell^T v_\ell = 1$, then

$$E_\ell = \frac{v_\ell}{\lambda_\ell - \lambda_\ell} \quad (\ell = 0, 1, \ldots, d-1).$$

We note that (2.2) implies that the inner sum of (2.1) has less than $r$ terms.

3. Markov chains of finite rank

Finite kernels as in (1.1) are, of course, familiar in the contexts of integral equations and linear operators. They seem to be especially well suited for use in Markov chains. In this section we give a brief review of
the Markov chain corresponding to (1.1); for more details we refer to [3].

We first give a rather more general definition. We recall (cf. [4]) that a Markov chain is a sequence of random variables $X_0, X_1, \ldots$ such that

$$P(X_{n+1} \in E \mid X_0 = x_0, \ldots, X_n = x_n) = P(X_{n+1} \in E \mid X_n = x_n);$$

if this transition probability is independent of $n$ it is called stationary.

**Definition 3.1.** Let $(S, S)$ be a measurable space. A ($S$-valued) Markov chain $X_0, X_1, \ldots$ is said to be of finite rank if its transition probability $P(\cdot \mid x)$ is stationary and has the form

$$P(E \mid x) = P(X_n \in E \mid X_{n-1} = x) = \sum_{j=1}^{r} a_j(x) B_j(E),$$

for all $n \in \mathbb{N}$ and $E \in S$. Here the $a_j$ are $S$-measurable and the $B_j$ are finite signed measures.

Though a large part of our results can easily be made more general, we shall restrict ourselves to processes on $\mathbb{R}$ (or a subset of $\mathbb{R}$), i.e. we shall assume

$$S = \mathbb{R}, \quad S = \mathcal{B}(\mathbb{R}).$$

We shall also assume that $r$ is minimal and hence that the $a_j$ and $B_j$ are linearly independent over $\mathbb{R}$ and $\mathcal{B}(\mathbb{R})$, respectively; $r$ is called the rank of the Markov chain. Without loss of generality it may be assumed (cf. [3]) that the $B_j$ are probability measures, (this can be achieved by a linear transformation), and that the $a_j$ are real with, of course,

$$\sum_{j=1}^{r} a_j(x) = 1 \quad (x \in \mathbb{R}).$$
It can also easily be shown that the $a_j$ must be bounded; we do not, however, assume that the $a_j$ are nonnegative.

The following easily verified lemma, already in [11], is crucial; similar results will be proved in connection with the distributions of $X_1 + \ldots + X_n$ and of $\max(X_1, \ldots, X_n)$.

Lemma 3.1. Let $a^T = (a_1, \ldots, a_r)$ and $b^T = (b_1, \ldots, b_r)$, then for the $n$-step transition probability $P^n(\cdot | x)$ one has ($P^1 = P$)

$$P^n(E | x) = P(X_n \in E \mid X_0 = x) = \alpha^T(x)C^{n-1}b(E),$$

for $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $E \in B(\mathbb{R})$, and where the elements of the matrix $C$ are given by (when not otherwise indicated integration is over $S$)

$$c_{jk} = \int a_k(x)b_j(dx) \quad (j, k = 1, \ldots, r).$$

The following lemma shows that $C$, though not necessarily nonnegative, has a Perron-Frobenius eigenvalue. It generally behaves very much like a transition matrix (it need not be equivalent to one; for examples see [3]). In the case of a finite Markov chain, for $C$ we may take the transition matrix, if it has full rank. Properties of the "kernel" matrix $C$ are discussed in more detail in [3].

Lemma 3.2. Let $C$ be the matrix defined by (3.4). Then

(i) All eigenvalues of $C$ have modulus at most one.

(ii) $C1 = 1$, i.e. $C$ has an eigenvalue $\lambda_0 = 1$ with the strictly positive eigenvector $1$. 

(iii) If \( \lambda \) is an eigenvalue with \( |\lambda| = 1 \), then \( \lambda^d = 1 \) for an integer \( 1 \leq d \leq r \), and if \( d \) is chosen minimal, then all solutions of \( \lambda^d = 1 \) are eigenvalues of \( C \).

**Proof (outline).** The proof of (ii) is trivial (cf. (3.2)). The other proofs are quite analogous to those for finite Markov chains as given in [1, p. 15 ff.], if one introduces eigenfunctions corresponding to \( \lambda \) as follows: Let \( v \) be a vector, and define

\[
(3.5) \quad v(x) \equiv \sum_{j=1}^{r} v_j a_j(x); 
\]

it follows that \( v \) is an eigenfunction of \( P \), i.e.

\[
(3.6) \quad \int v(y) P(dy|x) = \lambda v(x), \]

if and only if \( Cv = \lambda v \). The properties (i) and (iii) now easily follow from (3.6).

**Remark.** From the proof it emerges that existence of eigenvalues \( \exp(2\pi ij/d) \) for \( j = 0,1,\ldots,d-1 \) is equivalent to the existence of \( d \) cyclically moving sets \( S_0, S_1,\ldots,S_{d-1}, S_d = S_0 \) such that the process moves from \( S_j \) to \( S_{j+1} \) with probability one. Similarly, the existence of a \( k \)-fold eigenvalue one is equivalent to the existence of \( k \) disjoint absorbing sets, i.e. to reducibility. If one has multiplicity one, then the same is true for all eigenvalues of modulus one.

**Assumption.** From here one we shall assume that the Markov chain is irreducible 1), i.e. that the distinct eigenvalues of \( C \) are as follows:

1) Here we deviate from the usual terminology by allowing transient states; this also affects our definition of ergodic in theorem 4.2.
\[ \lambda_{\ell} = e^{2\pi i \ell/d} \quad (\ell = 0, 1, \ldots, d-1), \]
(3.7)

\[ |\lambda_{\ell}| < 1 \quad (\ell = d, \ldots, s). \]

Lemma 2.1 now takes the following special form.

**Lemma 3.3.** If \( C, \) as defined by (3.4), satisfies (3.7), then

\[ C^n = \sum_{\ell=0}^{d-1} \lambda_{\ell}^n E_{\ell} + O(n) \]
(3.8)

for some \( \rho \) with \( 0 \leq \rho < 1, \) and with \( \lambda_{\ell} = \exp(2\pi i \ell/d). \) Furthermore \( E_0 = 1y^T \)

with \( y^T C = y^T \) and \( y^T 1 = 1, \) and

\[ E_0 1 = 1, \quad E_{\ell} 1 = 0 \quad (\ell = 1, \ldots, d-1). \]
(3.9)

4. The ergodic theorem

The behaviour of \( p^n(x | x) \), governed by (3.3), is very similar to the

behaviour of \( p^n_{jk} \) for a finite transition matrix \( P. \) We have, writing

\[ B(y) = B((-\infty, y]), \quad p^n(y | x) = p^n((-\infty, y] | x) : \]

**Theorem 4.1.** If a Markov chain of finite rank is irreducible, i.e. if it

satisfies (3.7), then

\[ n^{-1} \sum_{k=1}^{n} p^k(y | x) = G(y) + O(n^{-1}) \quad (n \to \infty), \]
(4.1)

where the distribution function \( G \) is given by

\[ G(y) = \sum_{j=1}^{r} \gamma_j B_j(y) \]
(4.2)
with $\gamma^T c = \gamma^T$. The distribution function $G$ is the unique stationary
distribution function, i.e. the unique distribution function satisfying

$$G(y) = \int P(y|x)G(dx).$$

**Proof.** Follows directly from (3.8) and the observation that we have
$E_0 = 1\gamma^T$ and that $a^T(x)1 = 1$ for all $x$. The statements about $G$ are easily
verified.

The ergodic case, i.e. the case where the chain is irreducible and non-
cyclic, is again similar to the finite Markov chain situation.

**Theorem 4.2.** If a Markov chain of finite rank is ergodic, i.e. if (3.7)
holds with $d = 1$ then

$$(4.3) \quad P^n(y|x) = G(y) + O(\rho^n) \quad (n \to \infty),$$

for some $\rho \in [0,1)$.

**Proof.** Follows directly from (3.8); take $\rho = \{\max(|\lambda_1|, \ldots, |\lambda_g|)\}^{\frac{1}{2}}$.

5. The central limit theorem

As in the case of independence we use characteristic functions and we
define (cf. (3.1))

$$(5.1) \quad \varphi_n(t|x) = E(e^{itS_n} | X_0 = x),$$

where

$$(5.2) \quad S_n = X_1 + \ldots + X_n.$$
The following lemma reduces most of the problem to matrix theory. Its proof is a simple exercise in mathematical induction. Here and elsewhere we refer to [3] for details.

Lemma 5.1. The characteristic function $\varphi_n$ is given by (see section 2 for notation)

$$\varphi_n(t|x) = a^T(x)C^{-1}(t)\delta(t),$$

where $\delta_j$ is the characteristic function of $B_j$, and the matrix $C(t)$ is defined by

$$c_{jk}(t) = \int a_k(x)e^{ix}B_j(dx) \quad (j,k = 1,\ldots,r).$$

As in the case of $C = C(0)$ (cf. lemma 3.2) it is not hard to see that the matrix $C(t)$ has the following properties.

Lemma 5.2. The, not necessarily distinct, eigenvalues $\lambda_0(t),\ldots,\lambda_{r-1}(t)$ of $C(t)$ (when properly identified) are continuous functions of $t$ with $|\lambda_j(t)| \leq 1$. For small $t$, the $\lambda_j(t)$ have the same multiplicity structure as the eigenvalues of $C$.

We shall further consider $C(t)$ for small $t$ only, and renumber the $\lambda_\ell(t)$ ($\ell = 0,1,\ldots,s$) in accordance with (3.7) for $\lambda_\ell = \lambda_\ell(0)$.

The following lemma is not surprising; we give it without its rather obvious proof.
Lemma 5.3. If 

\[(5.5) \int x^2 B_j(dx) < \infty \quad (j = 1, 2, \ldots, r),\]

then the eigenvalues \(\lambda_k(t)\) and their corresponding eigenvectors have continuous second derivatives (for sufficiently small \(t\)).

We are now ready for the central limit theorem.

Theorem 5.4. Let \(X_0, X_1, \ldots\) be an irreducible Markov chain of finite rank satisfying (5.5). Then for the characteristic function \(\varphi_n\) (cf. (5.1)) the following relation holds.

\[(5.6) \lim_{n \to \infty} \exp \left[ -\lambda_0'(0) t/\sqrt{n} \right] \varphi_n(t/\sqrt{n} | x) = \exp \left[ -\frac{1}{2} \left( (\lambda_0'(0))^2 - \lambda_0''(0) \right) t^2 \right].\]

Proof (outline). By (2.1), (3.7) and (5.3) (see also (3.8)) \(\varphi_n\) satisfies

\[(5.7) \varphi_n(t | x) = a^T(x) \sum_{j=0}^{d-1} \lambda_j^{n-1}(t) E_j(t) \mathcal{B}(t) + O(n) \quad (n \to \infty).\]

As by the orthogonality relations (3.9) we have \(E_j \mathcal{B}(0) = E_j 1 = 0\) for \(j = 1, \ldots, d-1\), and \(E_0 \mathcal{B}(0) = 1\), so \(a^T(x) E_0 \mathcal{B}(0) = 1\), from (5.7) we obtain for all \(x\) and \(t_n \to 0\) \((\lambda_0(0) = \lambda_0 = 1)\)

\[(5.8) \varphi_n(t_n | x) \sim \lambda_0^n(t_n) \quad (n \to \infty).\]

Differentiation of (5.7) yields (see remark 1 below)

\[(5.9) \varphi_n'(0 | x) \sim n \lambda_0'(0) \quad (n \to \infty),\]

and therefore \(i \lambda_0'(0)\) is real. It follows that \(\exp(-\lambda_0'(0)t) \varphi_n(t | x)\) is a characteristic function, and we obtain from (5.8) and (5.9) (compare lemma 5.3)
\[
\exp\left[ -\lambda_0(0) t/\sqrt{n} \right] \phi_n(t/\sqrt{n} \mid x) \sim \\
\sim \exp\left[ -\lambda_0(0) t/\sqrt{n} \right] \lambda_0(t/\sqrt{n}) + \exp\left[ -\frac{1}{2} \left( \lambda_0'(0) \right)^2 - \lambda_0''(0) \right] t^2
\]

as \( n \to \infty \).

Remark 1. It is not very hard to prove from (5.7) that actually (cf. (4.2))

\[
E(S_n \mid X_0 = x)/n = -i \phi'_n(0|x)/n + -i \lambda'_0(0) = \int y G(dy);
\]

with some more effort (see [3]) one obtains

\[
\operatorname{var}(S_n \mid X_0 = x)/n = \left\{ (\phi'_n(0|x))^2 - \phi''_n(0|x) \right\}/n + (\lambda'_0(0))^2 - \lambda''_0(0) .
\]

Remark 2. After completion of this paper we found that Onicescu and Mihoc [8] use the same technique for finite Markov chains. Romanovski [10] uses a similar method for finite Markov chains, but his emphasis is more on difference equations than on Matrix theory. In both instances not all arguments are quite clear.

6. Extreme values

The distribution function of

\[
M_n = \max(X_1, \ldots, X_n)
\]

can be treated in a similar way as the characteristic function of \( S_n \).

We have by an easy computation:
Lemma 6.1. Let \( X_0, X_1, \ldots \) be a Markov chain of finite rank, and let
\( M_n \equiv \max(X_1, \ldots, X_n) \). Define \( F_n \) by

\[
F_n(y|x) = P(M_n \leq y | X_0 = x) .
\]

Then (see section 2 for notation)

\[
F_n(y|x) = a^T(x) \tilde{C}^{n-1}(y) b(y) ,
\]

where the matrix \( \tilde{C}(y) \) is defined by

\[
\tilde{C}_{jk}(y) = \int_{-\infty}^{y} a_k(x) B_j(dx) .
\]

Clearly, contrary to the central limit situation, we cannot expect
\( F_n(y|x) \) to have a limit independent of \( x \); here the influence of transient
states does not disappear as \( n \to \infty \). Therefore we restrict the process to
its recurrent states, i.e. to the support of \( G \) (cf. theorem 4.1). We
define

\[
S = \text{supp}(G) ,
\]

and we assume that the \( a_j \) and \( B_j \) are linearly independent on \( S \); actually
the rank of the process on \( S \) may be less than that of the original process,
and the \( a_j, B_j \) and \( r \) may have to be redefined. Instead of restricting the
process to \( S \) one may consider the stationary process as is done by
Leadbetter [7]. It is not necessary to assume the mixing condition that
is used there.

It is not hard to prove the following analogue to lemma 5.2 and lemma
3.2 (i). For details we refer to [3].
Lemma 6.2. Let \( \sum_j a_j(x) B_j \) be the transition probability of an irreducible Markov chain of finite rank on the absorbing set \( S \). Then the eigenvalues \( \tilde{\lambda}_j(y) \) of \( \tilde{C}(y) \) satisfy

(i) \( |\tilde{\lambda}_j(y)| < 1 \) for \( j = 0, 1, \ldots, r-1 \) if \( G(y) < 1 \);

(ii) if \( y_n \) is such that \( G(y_n) \to 1 \) as \( n \to \infty \), then \( \tilde{C}(y_n) \to C \) and \( \tilde{\lambda}_\ell(y_n) \to \lambda_\ell \) (\( \ell = 0, 1, \ldots, s \)), the eigenvalues of \( C \).

We now state the main theorem of this section.

Theorem 6.3. Let \( X_0, X_1, \ldots \) be an irreducible Markov chain of finite rank on \( S \), let \( \tilde{\lambda}_\ell(y) \) be the eigenvalues of \( \tilde{C}(y) \), let \( F_n \) be defined by (6.2) and let \( x \in S \) be fixed. Let \( a_n \geq 0 \), \( b_n \in \mathbb{R} \), and a nondegenerate distribution function \( F \) exist such that, for continuity points of \( F \),

\[
\lim_{n \to \infty} F_n(a_n y + b_n | x) = F(y|x).
\]

Then (6.6) holds for all \( x \in S \), and \( F(y|x) \) is independent of \( x \). Furthermore, writing \( F(y) \) instead of \( F(y|x) \), we have

\[
F(y) = \lim_{n \to \infty} \tilde{\lambda}_0^n(a_n y + b_n).
\]

Proof (outline). For any \( y \) with \( F(y|x) > 0 \) we must have \( \lim \inf G(a_n y + b_n) = 1 \), since otherwise by lemma 6.2 (i) we would have \( \lim \inf |\tilde{\lambda}_\ell(a_n y + b_n)| < 1 \) for all \( \ell \), and so by (6.3) and (2.1) that \( \lim \inf F_n(a_n y + b_n | x) = 0 \), contradicting (6.6). It follows from lemma 6.2 (ii) that \( \tilde{\lambda}_\ell(a_n y + b_n) \to \lambda_\ell \) for \( \ell = 0, 1, \ldots, s \). Now (6.7) follows from (6.3) in the same way (5.8) follows from (5.3), by application of lemma 3.3.
Corollary 6.4. If \( F_n(a_n y + b_n) \to F(y) \), then \( F \) is of one of the three well-known types of distribution functions that occur as limits in classical (independent) extreme value theory.

**Proof.** This follows from (6.6) and (6.7) in exactly the same way as in the proof for the independent case (see e.g. [2]).

**Remark.** A result like theorem 6.3 is also in [7] with a rather more complicated proof. With some more difficulty, and along the same lines as the proof in [7] one obtains: \( G_n(a_n y + b_n) \to F(y) \) implies that \( F_n(a_n y + b_n | x) \to F(y) \); this agrees with corollary 6.4. For a proof see [3].

7. A simple example

We illustrate the limit theorems obtained in the previous sections by the following simple example. Let \( r = 2 \) and let \( P(\cdot | x) \) be absolutely continuous on \( S = (0, \infty) \) with density \( p(y|x) \) given by

\[
p(y|x) = e^{-x} e^{-y} + 2(1 - e^{-x}) e^{-2y} \quad (x, y > 0)
\]

This example is similar to the bivariate distributions considered by Gumbel and by Morgenstern, see e.g. [6]; in fact the model considered here can be viewed as a generalization of these distributions.

Simple calculations yield (cf. (3.4) and lemmas 2.1 and 3.3)

\[
c^n = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{3}{2} & \frac{1}{2}
\end{bmatrix}^n
= \begin{bmatrix}
\frac{4}{7} & \frac{3}{7} \\
\frac{4}{7} & \frac{3}{7}
\end{bmatrix}
+ (-6)^{-n}
\begin{bmatrix}
\frac{3}{7} & \frac{3}{7} \\
\frac{4}{7} & \frac{4}{7}
\end{bmatrix}
\]
and we obtain (cf. (4.2))

\[(7.3) \quad G(y) = 1 - \frac{4}{7} e^{-y} - \frac{3}{7} e^{-2y}.\]

With regard to the central limit theorem (cf. section 5) we find for \(C(t)\):

\[
C(t) = \begin{pmatrix}
(2-it)^{-1} & (1-it)^{-1} - (2-it)^{-1} \\
2(3-it)^{-1} & (2-it)^{-1} - 2(3-it)^{-1}
\end{pmatrix},
\]

and so the eigenvalues \(\lambda_0(t)\) and \(\lambda_1(t)\) follow from the equation

\[(7.4) \quad \lambda^2 - \left\{3(2-it)^{-1} - 2(3-it)^{-1}\right\} \lambda + 2(1-it)^{-2} - 2(1-it)^{-1}(3-it)^{-1} = 0.\]

From (7.4) we obtain with \(\lambda_0(0) = 1\)

\[
\lambda'(0) = \frac{11}{14} i = 0.786 i, \quad \lambda''(0) = -\frac{807}{686} = -1.176
\]

and hence, by theorem 5.4 and remark 1 that \(S_n\) is asymptotically normal with \(E S_n \sim 0.786 n\) and \(\text{var } S_n \sim 0.559 n\).

For the limit distribution of \(M_n = \max(X_1, \ldots, X_n)\) we need the eigenvalue \(\tilde{\lambda}_0(y)\) of \(\tilde{C}(y)\) (see (6.4)). The eigenvalues of \(C(y)\) are obtained from

\[
\det(\tilde{C}(y) - \lambda I) = 0, \text{ i.e. from}
\]

\[
\begin{vmatrix}
\frac{1}{2}(1-e^{-2y}) - \lambda & 1 - e^{-y} - \frac{1}{2}(1-e^{-2y}) \\
\frac{2}{3}(1-e^{-3y}) & 1 - e^{-2y} - \frac{2}{3}(1-e^{-3y}) - \lambda
\end{vmatrix} = 0.
\]
The resulting quadratic equation easily yields $\tilde{\lambda}_0(y) + 1$ as $y \to \infty$, and

$$\lambda_0(y + \log n) \sim 1 - \frac{4}{7} e^{-y/n} \quad (n \to \infty).$$

So by theorem 6.3 we have

$$F_n(y + \log n \mid x) \to e^{-4/7} e^{-y} \quad (n \to \infty; \ y \in \mathbb{R}, \ x > 0),$$

which agrees with $G(y + \log n) \to \exp(-\frac{4}{7} e^{-y})$ (cf. remark following corollary 6.4).

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