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Fixed modes
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Abstract

We consider the problem of noninteracting control with stability for a square affine nonlinear system, where its decoupling matrix is not necessarily invertible. The recent notion of quasi-static state feedback is used. The results of this paper show that the dimension of the fixed dynamics with respect to any quasi-static state feedback may be determined by using the so-called accessibility cospaces.

1 Introduction

The problem of noninteraction and stability of nonlinear systems by means of static state feedback has been considered by Isidori and Grizzle [11]. They have shown that there exists a fixed internal dynamics, called $P^\ast$ dynamics whose stability is a necessary condition to solve the noninteracting control problem with stability. In the case where the $P^\ast$ dynamics is unstable, Wagner in [16] has shown that there exists a well-defined dynamics, called $\Delta_{\text{mis}}$ dynamics, which cannot be eliminated by any regular dynamic feedback which renders the considered system noninteractive. The $\Delta_{\text{mis}}$ dynamics must then be asymptotically stable if noninteracting control and stability is achieved by means of dynamic state feedback. A sufficient condition to solve the problem of noninteracting control with stability by means of dynamic state feedback was given in (3),(4). Under some regularity assumptions, the problem of dynamic feedback noninteracting control with stability is solved if the $\Delta_{\text{mis}}$ dynamics is asymptotically stable and each decoupled subsystem is asymptotically stabilizable.

All the above results are valid under the assumption that the decoupling matrix $A(x)$ is non-singular. In the case where $A(x)$ is singular, and the system is square and invertible, Zhan et al. [18] introduced the so-called Canonical Dynamic Decoupling Algorithm to construct a canonical dynamic extension. They have shown that the dynamically decoupled system is stable only if the $\Delta_{\text{mis}}$ dynamics of the canonical dynamic extension is stable, which is an intrinsic property of the given system.

In this paper, we investigate the case where the decoupling matrix is not necessarily invertible and study the fixed modes in the noninteracting control problem with stability by means of quasi-static state feedback. The goal is to get intrinsic geometric conditions generalizing the above ones.

In Section 2, we briefly review the definition of quasi-static state feedback introduced in [5], and in Section 3 a definition of accessibility cospaces is given. Section 4 is devoted to characterize the fixed dynamics with respect to any quasi-static state feedback.
2 Quasi-static state feedback

Consider a nonlinear control system of the form

\[ \dot{x} = f(x) + g(x)u \]  

(1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and where the entries of \( f(x) \) and \( g(x) \) are meromorphic functions of \( x \). It is assumed that \( \text{rank} \ g(x) = m \) and that \( n \geq 1 \). We follow the notations and setting of [6]. Let \( \mathcal{K} \) denote the field of meromorphic functions of \( x, u, u_1, \ldots, u^{(n-1)} \). \( \mathcal{E} \) is the formal vector space spanned by \( \{dx, du, du_1, \ldots, du^{(n-1)} \} \) over \( \mathcal{K} \). The notation \( dx \) stands for \( \{dx_1, \ldots, dx_n \} \) and \( du^{(k)} \) for \( \{du^{(1)}, \ldots, du^{(k)} \} \). Let \( \mathcal{X} := \text{span}_\mathcal{K} \{dx\} \) and \( \mathcal{U} := \text{span}_\mathcal{K} \{du, du_1, \ldots, du^{(n-1)}\} \).

A generalized static state feedback for (1) is a feedback of the form

\[ u = \phi(x, u, \ldots, u^{(r)}) \]  

(2)

where \( v \in \mathbb{R}^m \) denotes the new controls. Let \( \mathcal{V}_v \) denote the field of meromorphic function of \( \{x, \{v^{(k)} | k \geq 0\}\} \) and define the formal vector space \( \mathcal{E}_v = \text{span}_\mathcal{K} \{dx | \xi \in \mathcal{K}_v\} \). We define the following filtrations of \( \mathcal{E}_v \):

\[ \mathcal{V}_{-1} = \text{span}_\mathcal{K}_v \{dx\} \]
\[ \mathcal{V}_k = \text{span}_\mathcal{K}_v \{dx, dv_1, \ldots, dv^{(k)}\} \] \( (k \geq 0) \)  
\[ \mathcal{U}_{-1} = \text{span}_\mathcal{K}_v \{dx\} \]
\[ \mathcal{U}_k = \text{span}_\mathcal{K}_v \{dx, d\phi_1, \ldots, d\phi^{(k)}\} \] \( (k \geq 0) \)

Definition 2.1 ([5]) \( u \) given by (2) is said to be a quasi-static state feedback for (1) if the filtrations \( \mathcal{U}_k \) and \( \mathcal{V}_k \) have bounded difference, i.e., if there exists an \( s \in \mathbb{N} \) such that for all \( k \geq -1 \), we have \( \mathcal{U}_k \subset \mathcal{V}_{k+s} \), \( \mathcal{V}_s \subset \mathcal{U}_{k+s} \).

This definition yields that there exists a function \( \varphi(x, u, \ldots, u^{(r)}) \) such that locally (2) is equivalent to

\[ v = \varphi(x, u, \ldots, u^{(r)}) \]

(5)

3 Accessibility cospaces

Let us first review the definition of controlled invariance by means of quasi-static state feedback introduced in ([7],[8]).

3.1 Definition of controlled invariance

Consider the control system (1) together with a quasi-static state feedback (2) and define \( \mathcal{V} := \text{span}_\mathcal{K}_v \{du^{(k)} | k \geq 0\} \). In the rest of the paper, we denote \( \Theta^{(k)} \) as the time derivative of order \( k \) of \( \Theta \) along the trajectories of the system (1), and \( \Theta^{(k)} \) as the time derivative of order \( k \) of \( \Theta \) along the trajectories of the closed loop system (1) fed back with (2). We will write simply \( \Theta \) for \( \Theta^{(1)} \).

Definition 3.1 ([7],[8]) A subspace \( \Omega \subset \mathcal{X} \) is said to be controlled invariant for (1) if there exists a quasi-static state feedback (2) such that for (1),(2) one has

\[ \Omega^{[1]} \subset \Omega + \mathcal{V} \]

(6)

In this section, accessibility cospaces are studied under quasi-static state feedbacks as a special class of controlled invariant subspaces previously defined. These accessibility cospaces are related to the dual of dynamic controllability distributions (see [15]).
3.2 Definition of accessibility cospaces

Accessibility cospaces consist of vectors which are autonomous after applying a certain quasi-static state feedback \( u = \phi(x,v,\ldots,v^{(r)}) \) and zeroing certain input channels \( v_j \), where \( j \in \mathcal{J} \subset \{1, \ldots, m\} \). Such nonregular transformations are not defined for every element in \( \mathcal{K} \). One possibility to circumvent this problem is to consider the module \( \text{span}_A \{dx\} \) over the ring of analytic functions rather than the linear space over the field of meromorphic functions. Another way is chosen here; it consists in taking a particular basis of a given subspace of \( \text{span}_K \{dx\} \) so that its time derivative is well defined when applying nonregular feedback. Such a basis always exists. More precisely, let \( \Theta \subset \mathcal{X} \) be a subspace which admits a basis \( \theta_1, \ldots, \theta_d \) with

\[
\theta_i = \sum_{k=1}^{n} \alpha_{ik}(x,v,\ldots,v^{(k)}) dx_i,
\]

where \( \alpha_{ik} \) and \( \beta_{ik} \) are in \( A \), the ring of all analytic functions of \( \{x,\{v^{(k)} \mid k \geq 0\}\} \). Obviously, we can choose another basis for \( \Theta, \tilde{\theta}_1, \ldots, \tilde{\theta}_d \), in the module spanned by \( \{dx\} \) over the ring \( A \) by taking

\[
\tilde{\theta}_i = \left( \prod_{k=1}^{n} \beta_{ik} \right) \theta_i
\]

Definition 3.2 A subspace \( C \subset \mathcal{X} \) is said to be an accessibility cospace for (1) if there exists a quasi-static state feedback (2) and a set of integers \( J \subset \{1, \ldots, m\} \) such that for (1),(2) one has

\[
C = \max \{ \Theta \subset \mathcal{X} \mid \text{span}_A \{\tilde{\theta}_i^{(1)} \mid v_j = 0, j \in J \} \subset \Theta \}
\]

where \( \tilde{\theta}_i \) is defined as above.

This means that \( C \) is the largest autonomous subspace in \( \mathcal{X} \) of the closed loop system.

3.3 The smallest accessibility cospace containing the differential of the output

In general, the intersection of two accessibility cospaces is not an accessibility cospace. Thus it is unclear whether there exists a smallest accessibility cospace containing some given subspace. However, if an output \( h \) is given (i.e. an exact subspace \( \text{span}_K \{dh(z)\} \)), then there exists a smallest one containing the differential of the output.

We consider the system (1) together with the output

\[
y = h(z), \ y \in \mathbb{R}^p
\]

From ([7],[8]) we know that the smallest controlled invariant subspace containing \( \text{span}_K \{dh(z)\} \) is \( \Omega^* = \mathcal{X} \cap \mathcal{Y} \), where \( \mathcal{Y} = \text{span}_K \{dy, \ldots, dy^{(n-1)}\} \). The next theorem will relate \( \Omega^* \) to the smallest accessibility cospace containing \( \text{span}_K \{dh(z)\} \).

Theorem 3.3 Define the sequence \( C_\mu \) according to

\[
C_0 = \mathcal{X}, \ : \ : \ : \ C_\mu = \text{span}_K \{\omega \in C_{\mu-1} \mid \dot{\omega} \in C_{\mu-1} + \dot{\Omega}^*\}
\]

Then \( C^* = \lim C_\mu \) is the smallest accessibility cospace containing \( \text{span}_K \{dh(z)\} \).

Remark 3.4 When specialized to linear systems, the sequence \( C_\mu \) (9) turns out to be equal to the dual of the sequence \( \mathcal{R}_\mu \) (the sequence computing the maximal controllability subspace in kernel of the output mapping).
4 Fixed modes by quasi-static state feedback

Let us consider a square invertible nonlinear affine system \((\Sigma)\) defined by:

\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad x \in \mathbb{R}^n, \quad u_i \in \mathbb{R}
\]

\[
y_i = h_i(x), \quad i = 1, \ldots, m, \quad y_i \in \mathbb{R}
\]  \(\text{(10)}\)

Let \(\{n'_i\}\) be the set of orders of zeros at infinity, where \(n'_1 > n'_2 > \cdots > n'_m\). Permute if necessary \(y_i\) such that the corresponding order of zero at infinity is \(n'_i\). Let \(C^*_i\) be the smallest accessibility cospace containing \(\text{span}_{\mathbb{R}}\{dh_i(x)\}\). A first result is the following:

**Lemma 4.1** Suppose that the system \((10)\) is decouplable by a quasi-static state feedback \(u = \psi(x, v, \ldots, v^{(i)})\). Then, there always exist coordinates \(\xi = (\xi_0, \xi_1, \ldots, \xi_m, \xi)\) such that the system \((10)\) is presented in the following form:

\[
\begin{align*}
\dot{\xi}_0 &= f_0(\xi_0) \\
\dot{\xi}_1 &= f_1(\xi_0, \xi_1, v_1) \\
\vdots \\
\dot{\xi}_m &= f_m(\xi_0, \xi_m, v_m) \\
\dot{\xi} &= f(\xi, v, v, \ldots, v^{(i)}) \\
y_i &= h_i(\xi_0, \xi_i)
\end{align*}
\]  \(\text{(11)}\)

**Proof** For a scalar output \(y_i, C^*_i\) is an exact subspace. Thus, \(C^*_i\) is exact too as well as \(\sum_{j=0}^{n_i'-1} C_i^{(j)}\).

Let \(C_0\) as the uncontrollable subspace of \((\Sigma)\) which is the subspace \(\mathcal{H}_{\infty}\) introduced in [1]. It is obvious that for each \(i = 1, \ldots, m\)

\[
C_0 = \sum_{j=0}^{n_i'-1} C_i^{(j)} \cap \sum_{k \neq i} \sum_{j=0}^{n_k'-1} C_k^{(j)}
\]

Let \(d\xi_0\) be a basis of \(C_0\), thus \(\dot{\xi}_0 = f_0(\xi_0)\). For an invertible system, we can construct a quasi-static state feedback which decouples the system \(\Sigma\) by taking \(v_i = \psi^{(i)}\). For \(i = 1, \ldots, m\), then choose \(d\xi_i\) such that \(\{d\xi_0, d\xi_i\}\) is a basis of \(\sum_{j=0}^{n_i'-1} C_i^{(j)}\). Then one has

\[
\dot{\xi}_i = f_i(\xi_0, \xi_i, v_i)
\]

Complete the new coordinates by choosing \(\xi\), such that \(\{d\xi_0, d\xi_1, \ldots, d\xi_m, d\xi\}\) is linearly independent. Thus, one has

\[
\dot{\xi} = f(\xi, v, v, \ldots, v^{(i)}),
\]

and \((11)\) is established. \(\blacksquare\)

Using this Lemma, the following corollary is obtained.

**Corollary 4.2** The dimension of the fixed dynamics with respect to any quasi-static state feedback which decouples the system, is

\[
n - \dim(\sum_{i=1}^{m} \sum_{j=0}^{n_i'-1} C_i^{(j)})
\]  \(\text{(12)}\)
Remark 4.3 It is easy to see that

\[
\dim\left(\sum_{i=1}^{m} \sum_{j=0}^{n_i-1} \mathcal{C}_{i+j}^{(0)}\right) = \dim(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} \mathcal{C}_{i+j}^{(0)})
\]  

(13)

From Remark 4.3 and Corollary 4.2, the following theorem is derived.

**Theorem 4.4** For a square invertible nonlinear system, the dimension of fixed dynamics with respect to any quasi-static state feedback is

\[
n - \dim(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} \mathcal{C}_{i+j}^{(0)})
\]

The above theorem allows to characterize the fixed dynamics by computing only \( \mathcal{C}_i \). The next example illustrates the above theorem.

**Example 4.5** Let us consider a nonlinear system given by:

\[
\begin{align*}
\dot{x}_1 &= u_1, \dot{x}_2 = x_4 + x_3 u_1, \dot{x}_3 = x_3 + x_4, \dot{x}_4 = u_2, \dot{x}_5 = x_1 + x_2 \\
y_1 &= x_1, y_2 &= x_2
\end{align*}
\]

We have \( \{n_i\} = \{2, 1\} \). Permute then \( y_1 \), and thus \( \mathcal{C}_1 = \{dx_2\} \) and \( \mathcal{C}_2 = \{dx_1\} \). The quasi-static feedback which decouples the system is \( u_1 = v_1 \) and \( u_2 = v_2 - (x_3 + x_4)v_1 - x_3v_1 \), where \( (v_1, v_2) \) is a new input vector. It is clear that \( \mathcal{C}_0 = 0 \). We choose \( \{d\xi_1 = \{dx_2, dx_3, x_4 u_1\}\} \) as a basis of \( \{\mathcal{C}_1 + \mathcal{C}_2\} \), and thus

\[
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{pmatrix} = \begin{pmatrix}
\dot{\xi}_{11} \\
\dot{\xi}_{12}
\end{pmatrix} = \begin{pmatrix}
\xi_{12} \\
v_2
\end{pmatrix}
\]  

(14)

Choose now \( \{d\xi_2\} = \{dx_1\} \) as a basis of \( \mathcal{C}_2 \), and one has

\[
\dot{\xi}_2 = v_1
\]  

(15)

Complete the coordinates transformation by taking \( \tilde{\xi} = \begin{pmatrix}
\tilde{\xi}_1 \\
\tilde{\xi}_2
\end{pmatrix} = \begin{pmatrix}
x_4 \\
x_5
\end{pmatrix} \). So in the new coordinates \((\xi_1, \xi_2, \tilde{\xi})\), the considered system becomes

\[
\begin{align*}
\dot{\xi}_{11} &= \xi_{12}, \dot{\xi}_{12} = v_2, \dot{\xi}_2 = v_1, \\
\dot{\xi}_1 &= v_2 - (\xi_{12} - \xi_{11}) - \xi_{11} v_1 + (\xi_{12} - \xi_{11}) v_1, \dot{\tilde{\xi}}_2 = \xi_2 + \xi_{11}
\end{align*}
\]

and

\[
y_1 = \xi_2, y_2 = \xi_{11}
\]

Clearly \( \dim(\tilde{\xi}) = 2 \). This dimension equals \( n - \dim(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} \mathcal{C}_{i+j}^{(0)}) \) \( n - \dim(dx_1, dx_2, dx_4 + u_1 dx_3) \). Thus, the dimension of the fixed dynamics is two.

Similarly to Wagner and Battilotti's results, in the case where no quasi-static state feedback can render the system simultaneously noninteractive and stable, a suitable dynamic feedback may still solve the problem. This reduces to the results in Zhan et al. [18].

Finally, we can then summarize the existing results related to the dimension of the fixed dynamics of a nonlinear square decoupled system in the following table:

<table>
<thead>
<tr>
<th>( V )</th>
<th>( \mathcal{C}_i )</th>
<th>( \mathcal{C}_j )</th>
<th>( \mathcal{C}_{i+j} )</th>
<th>( \mathcal{C}_i^{(0)} )</th>
<th>( \mathcal{C}_j^{(0)} )</th>
<th>( \mathcal{C}_{i+j}^{(0)} )</th>
<th>( \dim(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} \mathcal{C}_{i+j}^{(0)}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( n )</td>
<td>( n )</td>
<td>( n )</td>
<td>( n )</td>
<td>( n )</td>
<td>( n )</td>
<td>( n )</td>
</tr>
<tr>
<td>Feedback</td>
<td>$A(z)$ invertible</td>
<td>$A(z)$ non-invertible</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>----------</td>
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<td></td>
</tr>
<tr>
<td>(Quasi) Static</td>
<td>$\dim (\mathcal{P}^*)$</td>
<td>$n - \dim {X \cap (\sum_{i=1}^{m} \sum_{j \geq 0} \mathcal{C}_i^{(j)})}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dynamic</td>
<td>$\dim (\Delta_{mix})$ (Wagner, Battilotti)</td>
<td>$\dim (\Delta_{mix}(\Sigma_p))$ (Zhan et al.)</td>
<td></td>
<td></td>
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</tbody>
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References


