Electromagnetic Forming by Distributed Forces in Magnetic and Nonmagnetic Materials
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Abstract—In this paper, we discuss the electromechanical force densities associated with pulsed electromagnetic fields in inhomogeneous, linear media with conductive losses, in the context of a process of shaping metal objects. We show that the conductivity and the gradients in permittivity and in permeability lead to volume forces, while jump discontinuities in permittivity and permeability lead to surface forces. These electromagnetic forces are assumed to act as volume (body) source densities in the elastodynamic equations and as surface source densities in the corresponding boundary conditions that govern the elastic motion of deformable matter. As an example, we apply the theory to the calculation of the elastic field in a hollow cylindrical object made of a conducting magnetic or nonmagnetic material. We compare the numerical results with those for the classical theory of elasticity with concentrated forces on the boundaries of the material as the source of the elastodynamic field.

Index Terms—Conducting magnetic materials, elastic field, electromechanical forces.

I. INTRODUCTION

THE literature pertaining to forces in electromagnetic bodies is vast. An extensive review on more fundamental aspects has been given by Penfield and Haus [1] while the expressions of the force distribution in magnetized material have been discussed by many other authors like Carpenter [2], [3], Byrne [4], Carter [5], Reyne et al. [6], and Bobbio [12]. In their book, Penfield and Haus [1] consider the complete physical system that describes the motion of a continuum under the influence of electromagnetic field. It consists of three mutually coupled subsystems: a mechanical subsystem describing the mechanics of the moving material masses; an electromagnetic subsystem describing the dynamics of the electromagnetic fields; and a thermodynamic subsystem taking into account the internal energy and the generation of heat and its flow. However, for the engineering application of electromagnetic forming where materials bodies are shaped in intense, pulsed electromagnetic fields, a number of observations and assumptions related to forces in electromagnetic bodies can be made. The force expression that follows from the general theory contains certain terms that are too small to be of engineering importance. In particular, we assume that piezoelectric, magnetoelectric, and magnetoelastic effects are higher order effects and in the first instance can be neglected. The deformation velocities $|v| \approx v$ of all points in the material body are supposed to be small in comparison with the velocity of light $c$. In electromagnetic forming the particle deformation velocities are typically in the order of $v \approx 100-300$ m/s. Thus, terms of first and higher order in $(v/c)$ may be neglected.

With these assumptions and the theory of Penfield and Haus [1] in mind, the simplified physical system for deriving force densities in electromagnetic forming devices consists of an electromagnetic system and a mechanical system only, coupled through electromechanical force densities. Within a linear approximation of the model of electromagnetic forming, the electromagnetic force densities are assumed to act as volume (body) source densities in the elastodynamic equations and as surface source densities in the corresponding boundary conditions that govern the elastic motion of deformable matter.

For the derivation of the distributed electromechanical force in this simplified system, we start with a macroscopic model approach, where the force that impressed, external currents and charges exert on the electromagnetic field in the configuration follows from a balance of electromagnetic momentum. Most of the standard considerations on the subject focus on the relevant forces in static or quasi-static electric and magnetic fields (see, e.g., Stratton [7], Moon [8], Fano [9], Penfield and Haus [1]), or on the case of continuously differentiable spatial variations in the constitutive properties (see, e.g., Stratton [7] and Landau [10]).

In the present paper, we follow the macroscopic approach along the line of the analysis presented by Stratton [7]. His analysis will be extended to the case of piecewise inhomogeneous, isotropic media, thus allowing for interfaces across which the constitutive parameters jump by finite amounts. We will show that finite gradients in permittivity and permeability lead to distributed volume forces and interface conditions lead to distributed surface forces. The pulsed field behavior introduces a distributed volume force that is associated with the time derivative of the electromagnetic momentum of the field. Further, the distributed volume and surface forces lead to an elastodynamic wave field in the relevant medium. The mechanical stress associated with the elastic wave motion then determines the amount of mechanical deformation that the medium undergoes. In many papers [11], [13]–[15], [20], [17], [18] on electromagnetic forming, the present problem is dealt with by the use of equivalent surface source accounting for the
II. ELECTROMECHANICAL VOLUME AND SURFACE FORCES

We consider a configuration consisting of a pulsed current loop with external current density \( \mathbf{J}^{\text{ext}}(\mathbf{r}, t) \) and external charge density \( \rho^{\text{ext}}(\mathbf{r}, t) \) that irradiates a smooth inhomogeneous, isotropic object with constitutive parameters \( \{\varepsilon, \sigma, \mu\} \) located in vacuum as shown in Fig. 1. The electromagnetic field equations are given by

\[
\begin{align*}
\nabla \times \mathbf{H} - \varepsilon \partial_t \mathbf{E} - \sigma \mathbf{E} &= \mathbf{J}^{\text{ext}} \\
\nabla \times \mathbf{E} + \mu \partial_t \mathbf{H} &= 0.
\end{align*}
\]

The compatibility relations are

\[
\begin{align*}
\nabla \cdot (\varepsilon \partial_t \mathbf{E} + \sigma \mathbf{E}) &= -\nabla \cdot \mathbf{J}^{\text{ext}} \\
\nabla \cdot (\mu \partial_t \mathbf{H}) &= 0.
\end{align*}
\]

Further, we introduce

\[
\rho^{\text{ext}} = -\int_{t'=0}^{t} \nabla \cdot \mathbf{J}^{\text{ext}} \, dt'
\]

as volume density of external electric charge. Across a source-free interface of jump discontinuity in \( \varepsilon, \mu, \) and \( \sigma, \) the field quantities satisfy the continuity conditions

\[
\begin{align*}
\nu \times \mathbf{H} &= \text{continuous across interface} \\
\nu \times \mathbf{E} &= \text{continuous across interface}
\end{align*}
\]

and

\[
\begin{align*}
\nu \cdot (\varepsilon \partial_t \mathbf{E} + \sigma \mathbf{E}) &= \text{continuous across interface} \\
\nu \cdot (\mu \partial_t \mathbf{H}) &= \text{continuous across interface}
\end{align*}
\]

where \( \nu \) is the normal vector of the interface.

total electromagnetic force exerted on the workpiece during the deformation process. We investigate for a simple forming problem the differences between this approach and the present method in which both the volume force and surface force densities as local source distributions generating the elastic wave motion are taken into account appropriately.

We take as physical interpretation that the volume source densities of external currents and electric charges in domain \( \mathcal{D}_e \) exert on the field the volume force, cf. Stratton [7]

\[
F_V^{\text{ext}} = \int_{\mathcal{D}} \mathbf{f}_V^{\text{ext}} \, dV = -\int_{\mathcal{D}} [\rho^{\text{ext}} \mathbf{E} + \mathbf{J}^{\text{ext}} \times \mathbf{B}] \, dV. \tag{6}
\]

To arrive at an expression that shows how the electromagnetic field transmits this force to other parts of the configuration (e.g., outside \( \mathcal{D}_e \) in Fig. 1), we use (1)–(3), to end up with

\[
F_V^{\text{ext}} = \int_{\mathcal{D}} \left[ \rho^{\text{ind}} \mathbf{E} + \sigma \mathbf{E} \times \mu \mathbf{H} - \frac{1}{2} (\nabla \varepsilon)(\mathbf{E} \cdot \mathbf{E}) - \frac{1}{2} (\nabla \mu)(\mathbf{H} \cdot \mathbf{H}) \right] \, dV
\]

\[
+ \int_{\mathcal{D}} \nabla \cdot T^M \, dV. \tag{7}
\]

In (7), we have

\[
\rho^{\text{ind}} = -\int_{t'=0}^{t} \nabla \cdot J^{\text{ind}} \, dt'
\]

where

\[
J^{\text{ind}} = \sigma \mathbf{E} \tag{9}
\]

which is the volume density of induced electric current. Further

\[
T^M = \frac{1}{2} \varepsilon (\mathbf{E} \cdot \mathbf{E}) I - \varepsilon \mathbf{E} \mathbf{E} + \frac{1}{2} \mu (\mathbf{H} \cdot \mathbf{H}) I - \mu \mathbf{H} \mathbf{H} \tag{10}
\]

is the Maxwell stress tensor with \( I \) the unit tensor of rank two. Using Gauss’ divergence theorem, the last term of (7) can be rewritten as

\[
\int_{\mathcal{D}} \nabla \cdot T^M \, dV = \int_{\partial \mathcal{D}_1} \mathbf{v} \cdot T^M \, dA + \int_{\partial \mathcal{D}_2} f_S^M \, dA \tag{11}
\]

where

\[
f_s^M = -\lim_{h \to 0} [\mathbf{v} \cdot T^M (\mathbf{r} + h \mathbf{v}) - \mathbf{v} \cdot T^M (\mathbf{r} - h \mathbf{v})] \tag{12}
\]

with \( \mathbf{r} \in \partial \mathcal{D}_1 \) as in Fig. 2. The results presented in (7) and (11) can be rewritten as the balance of electromagnetic momentum for the configuration, viz.,

\[
F_V^{\text{ext}} = F_V^{\text{ind}} + F_V^{\text{grad}} + F_S^M + F_S^{\text{ext}} + \partial_i G. \tag{13}
\]
In this expression\[ F_{V}^{\text{ind}} = \int_{D} F_{V}^{\text{ind}} dV = \int_{D} \left( \varepsilon_{0} \varepsilon \mathbf{E} + \sigma \mathbf{E} \times \mathbf{H} \right) dV \] is the volume force that the electromagnetic field exerts on the induced volume density of electric current\[ F_{V}^{\text{grad}} = \int_{D} F_{V}^{\text{grad}} dV = \int_{D} \left[ -\frac{1}{2}(\nabla \varepsilon)(\mathbf{E} \cdot \mathbf{E}) - \frac{1}{2}(\nabla \mu)(\mathbf{H} \cdot \mathbf{H}) \right] dV \] is the volume force that the field exerts on gradients of permittivity and permeability. Further\[ F_{S}^{M} = \int_{\partial D_{1}} F_{S}^{M} dA \] is a force due to a jump in the Maxwell stress tensor at the boundary of $D_{1}$. This jump is conjectured to act as a surface force at $\partial D_{1}$ for the elastodynamic wavefield in the configuration. At the boundary of $D$, a force is exerted that equals\[ F_{S}^{M \infty} = \lim_{\Delta \to \infty} \int_{S_{\Delta}} F_{S}^{M \infty} dA = \lim_{\Delta \to \infty} \int_{S_{\Delta}} \mathbf{v} \cdot \mathbf{T}^{M} dA. \] For its interpretation, take for $D$ the ball $B = \{ \mathbf{r} \in \mathbb{R}^{3}; 0 < |\mathbf{r}| < \Delta \}$ of radius $\Delta$ and center at the origin. In the far-field region, the behavior of the field radiated by the sources in $D_{c}$ guarantees that $F_{S}^{M \infty}$ exists as $\Delta \to \infty$. Similarly, also $\partial_{\mathbf{r}} G$ exists as $\Delta \to \infty$. The corresponding term $\mathbf{v} \cdot \mathbf{T}^{M}$ can be interpreted as the “radiation pressure” that the field, by carrying its momentum from the exciting sources to $S_{\Delta}$, exerts on the “sphere at infinity.” Note that exerting this pressure is compatible with the property that this sphere absorbs the power radiated to it. This radiation pressure is irreversibly lost to the electromagnetic momentum. Finally\[ \mathbf{G} = \int_{D} \mathbf{g} dV = \int_{D} (\varepsilon \mathbf{E} \times \mathbf{H}) dV \] is the electromagnetic momentum carried by the field.

With the interpretation of the radiation pressure on the sphere at infinity in mind, also the terms $F_{V}^{\text{ind}}$ and $F_{V}^{\text{grad}}$ have the structure of an irreversible loss of momentum. It is noted that, when no matter is present in $D_{1}$, both terms will vanish. As a consequence, in $D_{1}$\[ F_{V}^{\text{ind}} = \sigma \mathbf{E} \times \mathbf{H} \] and\[ F_{V}^{\text{grad}} = -\frac{1}{2} \nabla \varepsilon (\mathbf{E} \cdot \mathbf{E}) - \frac{1}{2} \nabla \mu (\mathbf{H} \cdot \mathbf{H}) \] can be conjectured to be the driving volume source densities of body force in the elastodynamic field equations that govern the behavior of dynamic stress and particle velocity in mechanically deformable matter. In conclusion, we have found that in an inhomogeneous medium, the total volume force consists of two forces; see (19)–(20). Later, we present some numerical results for these forces in a piecewise homogeneous material where these two forces still exist due to the jump in permittivity and permeability.

III. FORCES IN A HOMOGENEOUS MAGNETIC OBJECT

From the discussion in Section II we observe that in a homogeneous, highly conducting magnetic object with medium parameters $\sigma, \mu$ located in vacuum the volume density of induced charge $f_{V}^{\text{ind}}$ vanishes and in $D_{1}$ we find\[ F_{V}^{\text{ind}} = \sigma \mathbf{E} \times \mathbf{H} \] (21)\[ F_{V}^{\text{grad}} = 0. \] (22) Since in the electromechanical system for electromagnetic forming, the electromagnetic field may be considered transient diffusive, the derivative with respect to time of the electromagnetic momentum is always negligibly small in comparison to other terms in the balance of electromagnetic momentum. To conclude, the force exerted on $D_{1}$ is in the present case\[ F_{V}^{\text{ext}} = F_{V}^{\text{ind}} + F_{S}^{M}. \] (23) In this case, the volume density of force $F_{V}^{\text{ext}}$ inside $D_{1}$ is given by\[ F_{V}^{\text{ext}} = F_{V}^{\text{ind}} = \sigma \mu (\mathbf{E} \times \mathbf{H}) \] (24) while the surface density of force at $\partial D_{1}$ follows immediately from (11) and (12).

IV. ELASTODYNAMIC WAVE MOTION GENERATED BY VOLUME FORCES AND SURFACE FORCES

Here, the subscript notation and summation convention will be used. The elastodynamic wavefield is characterized by its tensorial dynamic stress $\tau_{pq} = \tau_{pq}(\mathbf{r}, t)$, together with its vectorial particle velocity $v_{r} = v_{r}(\mathbf{r}, t)$. These quantities satisfy the equation of motion\[ \partial_{t} \rho_{m} \dot{v}_{k} - \rho_{m} \partial_{r} \mathbf{u}_{k} = -f_{V}^{\text{ext}} \] (25) in which $\rho_{m} = \rho_{m}(\mathbf{r})$ is the volume density of mass of the medium, and the constitutive equation\[ \tau_{pq} = C_{pqk} \varepsilon_{k}^{ij} \] (26) in which $C_{pqk}^{ij}$ is the elastic stiffness tensor and $\varepsilon_{k}^{ij} = \varepsilon_{kij}$ is the strain. The latter quantity satisfies the strain-displacement relation\[ \varepsilon_{kij} = \frac{1}{2} (\partial_{i} u_{j} + \partial_{j} u_{i}) \] (27) in which $u_{j}$ is the particle displacement. The relation between particle velocity and particle displacement is\[ v_{r} = \partial_{r} u_{r}. \] (28) In our isotropic object (see Fig. 3), we have\[ C_{pqk}^{ij} = \lambda_{L} \delta_{p}^{ij} \delta_{k}^{ij} + \mu_{L} (\delta_{p}^{k} \delta_{q}^{ij} + \delta_{q}^{k} \delta_{p}^{ij}) \] (29)
in which $\lambda_L$ and $\mu_L$ are the Lamé coefficients of the medium. Combination of (26)–(29) results in the deformation rate equation

$$\partial_t \tau_{pq} = \lambda_L (\partial_r v_r) \delta_{pq} + \mu_L (\partial_p v_q + \partial_q v_p).$$

(30)

Across the surface discontinuity in elastodynamic properties $\partial D_1$, the boundary conditions are

$$\left\{ \begin{array}{l}
\lim_{t \to 0^+} \left[ \nu_{r} \tau_{rjk}(r + h\nu) - \nu_{r} \tau_{rjk}(r - h\nu) \right] = -\int_{\Sigma_r^k} f_{S_r^k}, \\
\lim_{t \to 0^+} \left[ \nu_{r} \tau_{rjk}(r + h\nu) - \nu_{r} \tau_{rjk}(r - h\nu) \right] = 0.
\end{array} \right.$$

(31)

The surface density of force $f_{S_r^k}$ in (31) follows from (11) and (12).

V. APPLICATION TO A SIMPLE CONFIGURATION

The configuration in which the electromagnetic field and then the elastic field will be calculated is presented in Fig. 4. This configuration models the case of electromagnetic compression of hollow cylindrical objects, where the object to be deformed (the workpiece) is placed inside a forming coil. The configuration can be divided into four cylindrical subdomains, namely, the inner space of the workpiece with $r \in [0, a]$, the workpiece domain $D$ with $r \in [a, b]$, the air gap in between the workpiece and the forming coil with $r \in (b, c)$, and the space outside the forming coil with $r \in (c, \infty)$.

A. Electromagnetic Solution

The medium of the workpiece is homogeneous, linear, time invariant, locally and instantaneously reacting, and isotropic in its electromagnetic behavior, with permeability $\mu$ and electrical conductivity $\sigma$. The other media are assumed to be vacuum with permeability $\mu_0$ and zero conductivity. The configuration is excited by a single sheet antenna, located at radial position $r = c$, carrying an electric current in the positive $\varphi$ direction. This infinitesimal thin sheet antenna models the forming coil in the real electromagnetic forming system. The configuration under investigation is assumed to have infinite length in the $z$ direction. This fact, combined with the axial symmetry of the configuration, gives that all quantities related to this configuration depend on the radius $r$ and time $t$. Only, in the cylindrical sheet at $r = c$, an electric current $I_S(t)$ per unit length along the $z$ direction is present. The electromagnetic field is causally related to the action of this electric-current source. The given source generates a one-dimensional, $r$-dependent field of which $E_z = E_z(r, t)$ and $H_z = H_z(r, t)$ differ from zero. We assume that the time variation is such that the displacement current can be neglected and the field components satisfy the diffusive electromagnetic field equations

$$\begin{cases}
\partial_t H_z + \sigma E_z = 0, \\
\frac{1}{r} \partial_r (r E_z) + \mu_0 H_z = 0,
\end{cases}$$

(32)

for $a < r < b$.

The simplest way to construct solutions that satisfy these equations together with the boundary and excitation conditions and ensure causality is to use the Laplace transformation with respect to time. To illustrate the notation, let

$$\tilde{I}_S(s) = \int_{t=0}^{\infty} \exp(-st) I_S(t) \, dt$$

(33)

where it has been assumed that the electric current source starts to act at the instant $t = 0$. The complex transform parameter $s$ (also denoted as the complex frequency) is taken in the right half of the complex $s$-plane. The Laplace transformed quantities are denoted with a hat symbol and we omit the explicit $s$-dependence in our notation. We take the limit $s \to j \omega$, so that we end up with the Fourier transformed quantities, where $j$ is the imaginary unit and $\omega = 2\pi f$ is the radial frequency, while $f$ denotes the frequency of operation. In our numerical work, we use the fast Fourier transform (FFT) to compute the pertaining Fourier transforms.

In the conductive domain $D$ of the workpiece, the components of the electromagnetic diffusive field are obtained as

$$\begin{cases}
\hat{H}_z = \hat{I}_S \left[ A \frac{I_0(\gamma r)}{I_0(\gamma a)} + B \frac{K_0(\gamma r)}{K_0(\gamma a)} \right], \\
\hat{E}_z = -Z s \hat{I}_S \left[ A \frac{I_1(\gamma r)}{I_0(\gamma a)} - B \frac{K_1(\gamma r)}{K_0(\gamma a)} \right],
\end{cases}$$

(34)

where $I_{0,1}$ and $K_{0,1}$ are the modified Bessel function of the first and second kind, while the quantities

$$\gamma = \sqrt{\sigma \mu_0}, \quad Z = \sqrt{s \mu_0 / \sigma}$$

(35)

are the diffusion coefficient and the impedance of the diffusive electromagnetic field, respectively. In the vacuum domains, we have zero conductivity and the following quasi-static equations hold:

$$\begin{cases}
\partial_t \hat{H}_z = 0, \\
\frac{1}{r} \partial_r (r \hat{E}_z) + s \mu_0 \hat{H}_z = 0.
\end{cases}$$

(36)

The first equation indicates that in a vacuum domain the magnetic field is constant, while the second equation shows that the electric field is a linear combination of $r$ and $1/r$. The constants $A$ and $B$ are determined from the continuity conditions for the
magnetic and electric fields at \( r = a \), the excitation condition that the magnetic field at \( r = b \) is equal to \( \mathbf{H}_S \), and the solutions in the vacuum domains. In effect, in the domain \( r \in [a, b] \), we arrive at

\[
\hat{H}_z = \hat{I}_S \frac{m(\gamma a, \gamma r)}{n(\gamma a, \gamma b)} \quad \hat{E}_\varphi = -Z \hat{I}_S \frac{m(\gamma a, \gamma b)}{n(\gamma a, \gamma b)}
\]  

(37)

where

\[
n(\gamma a, \gamma r) = I_0(\gamma r)[K_1(\gamma a) + \eta a K_0(\gamma a)] \\
+ K_0(\gamma r)[I_1(\gamma a) - \eta a I_0(\gamma a)]
\]  

(38)

\[
m(\gamma a, \gamma r) = I_1(\gamma r)[K_1(\gamma a) + \eta a K_0(\gamma a)] \\
- K_0(\gamma r)[I_1(\gamma a) - \eta a I_0(\gamma a)]
\]  

(39)

with \( \eta = (1/2)\gamma \mu_0 / \mu \).

Within the homogeneous isotropic domain \( D \) with \( r \in [a, b] \), thus in the cylindrical domain representing the workpiece, the application of (24) gives

\[
f^V_\varphi(r, t) = \sigma \mu E_\varphi(r, t) H_z(r, t).
\]  

(40)

In view of the continuity of the magnetic field strength at the boundaries at \( r = a \) and \( r = b \) of the workpiece, the electromagnetic surface force density on the outer boundary \( b \) is obtained from (12) as

\[
f^S_\varphi(b, t) = \frac{1}{2}(\mu - \mu_0)[H_z(b, t)]^2
\]  

(41)

while on the inner boundary \( a \) of the workpiece we have

\[
f^S_\varphi(a, t) = \frac{1}{2}(\mu_0 - \mu)[H_z(a, t)]^2.
\]  

(42)

After an inverse Fourier transform of the field quantities given by (37), all the quantities in the right-hand side of (40)–(42) are known for the whole time interval of investigation and within the whole configuration. The electromagnetic volume force density and the electromagnetic surface force density can then be calculated very easily. For nonmagnetic workpieces, the surface force densities are zero, while for magnetic workpieces, they have nonzero values.

B. Elastodynamic Solution

When we turn to the elastodynamic problem, we observe that the sources, exciting the elastic field, are the electromagnetic force densities. In our configuration at hand, only the radial components of these force densities are nonzero, viz., the electromagnetic force density \( f^{V\varphi}_r \) in the workpiece and the electromagnetic surface force densities \( f^{S\varphi}_r \), at the boundaries of the workpiece. In a cylindrical coordinate system, the components of the stress, strain, and particle displacement are \( \{\tau_{rr}, \tau_{r\varphi}, \tau_{zz}, \tau_{r\varphi}, \tau_{\varphi\varphi}, \tau_{zz}, \tau_{\varphi z}, \tau_z, \tau_{\varphi z} \} \), \( \{e_{rr}, e_{r\varphi}, e_{zz}, e_{r\varphi}, e_{\varphi\varphi}, e_{zz}, e_{\varphi z}, e_z \} \), and \( \{u_r, u_\varphi, u_z \} \), respectively. With these components, the equations of motion are

\[
\partial_t \tau_{rr} + \frac{\partial \tau_{r\varphi}}{r} + \partial_z \tau_{zz} + \frac{\tau_{rr} - \tau_{r\varphi}}{r} = \rho_m \partial_t^2 u_r = -f^V_r,
\]  

(43)

\[
\partial_t \tau_{r\varphi} + \frac{\partial \tau_{zz}}{r} + \partial_z \tau_{r\varphi} + \frac{2 \tau_{r\varphi}}{r} = -\rho_m \partial_t^2 u_\varphi = 0
\]  

(44)

\[
\partial_t \tau_{zz} + \frac{\partial \tau_{\varphi z}}{r} + \partial_z \tau_{zz} + \frac{\tau_{zz}}{r} = -\rho_m \partial_t^2 u_z = 0
\]  

(45)

Since the workpiece has been assumed homogeneous and isotropic, the constitutive relations are given by

\[
\begin{align*}
\tau_{rr} &= (\lambda_L + 2\mu_L) e_r + \lambda_L e_\varphi + \lambda_L e_z \\
\tau_{r\varphi} &= \lambda_L e_\varphi + (\lambda_L + 2\mu_L) e_\varphi + \lambda_L e_z \\
\tau_{zz} &= \lambda_L e_z + \lambda_L e_\varphi + (\lambda_L + 2\mu_L) e_\varphi
\end{align*}
\]  

(46)

In the same coordinate system, the components of the strain are defined as

\[
\begin{align*}
e_{rr} &= \partial_t u_r \\
e_{r\varphi} &= r^{-1} (\partial_r u_\varphi + u_r) \\
e_{zz} &= \partial_z u_z \\
e_{r\varphi} &= (r^{-1} \partial_r u_\varphi + \partial_z u_\varphi - r^{-1} u_\varphi)/2 \\
e_{\varphi z} &= (\partial_z u_\varphi + r^{-1} \partial_r u_\varphi)/2
\end{align*}
\]  

(47)

There exists no radial stress within the air, so the boundary conditions to be applied at \( r = a \) and \( r = b \) are

\[
\begin{align*}
\lim_{r \to a} \tau_{rr}(r, t) &= -f^S_\varphi(a, t) \\
\lim_{r \to b} \tau_{rr}(r, t) &= f^S_\varphi(b, t)
\end{align*}
\]  

(48)

(49)

respectively. In the modeled cylindrical configuration with infinite length, we will assume that the stresses and strains are uniform along the length of the workpiece. This assumption, combined with the rotational symmetry of the configuration gives that all elements of the stress and strain are functions only of the radius \( r \) and time \( t \), and all \( \partial_{r\varphi} \) and \( \partial_z \) operators result in zero. Moreover, \( u_\varphi = 0 \). In our cylindrical configuration with infinite length, two particular cases related to the elastic field may be distinguished: the plane stress case and the plane strain case. The plane stress case models a tube with free ends, thus the longitudinal displacement \( u_z \) is allowed and the longitudinal stress \( \tau_{zz} \) is assumed to be zero. Although there is a longitudinal displacement \( u_z \), the infinite length of the configuration does not allow the calculation of the longitudinal displacement \( u_z \). The plane strain case models a tube with fixed ends, thus the longitudinal displacement \( u_z \) is zero, and further the longitudinal strain \( e_{zz} \) is also zero. The infinite length of our model can deal with this case. Moreover, the case \( u_z = 0 \) characterizes the practical situation in which the finite workpiece is clamped at the ends. Therefore, we will solve (43) for the case of plane strain.

The simplest way to construct solutions of the present elastodynamic equations is again to use the Laplace transformation with respect to time. As for the calculation of the electromagnetic field, we take \( s \to j \omega \), where \( \omega = 2\pi f \) is the radial frequency, while \( f \) denotes the frequency of operation. In the Laplace transform domain, the nonzero components of the strain tensor and of the stress tensor, in the case of plane strain \( (e_{zz} = 0) \), are written as

\[
\hat{\tau}_{rr} = \partial_t \hat{u}_r, \quad \hat{\tau}_{r\varphi} = r^{-1} \hat{u}_r,
\]  

(50)

and

\[
\begin{align*}
\hat{\tau}_{rr} &= (\lambda_L + 2\mu_L) \partial_r \hat{u}_r + \lambda_L r^{-1} \hat{u}_r \\
\hat{\tau}_{r\varphi} &= \lambda_L \partial_\varphi \hat{u}_r + (\lambda_L + 2\mu_L) r^{-1} \hat{u}_r \\
\hat{\tau}_{zz} &= \lambda_L \partial_z \hat{u}_r + \lambda_L r^{-1} \hat{u}_r
\end{align*}
\]  

(51)
After substitution of $\tilde{\tau}_{rr}$ and $\tilde{\tau}_{\varphi z}$ in the Fourier transformed counterpart of (43), it becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \frac{1}{r^2} \frac{\partial}{\partial r} u_r - \frac{1}{r^2} \frac{\partial^2}{\partial r^2} u_r + k^2 u_r = -\frac{1}{\lambda_L + 2\mu_L} \tilde{f}_r^V,$$  \hspace{1cm} (52)

where $k = \omega/c_P$ is the compressional wave number and $c_P = (\lambda_L + 2\mu_L)/\rho_m^{1/2}$ is the compressional or P-wave speed.

When the solution of the above equation is found (see the Appendix), all nonzero components of the strain and stress tensor may be calculated in the frequency domain, in accordance to (50)–(51). The results may then be transformed back to the time domain using an inverse FFT.

VI. METHOD OF EQUIVALENT SURFACE FORCES

In the literature related to electromagnetic forming (see [13]–[15], [17]), the electromagnetic forming problem is dealt with the use of an equivalent surface force (pressure) accounting for the total electromagnetic force exerted on the workpiece during the deformation process. Although in the literature related to electromagnetic forming, the elastic field is not dealt with in detail, it seems to suggest a solution of the present elastodynamic problem with the help of these equivalent surface forces as external sources. Thus, the electromagnetic force density $f_r^V$ in the right-hand side of (43) is assumed to be zero and we have to solve the following equation of motion:

$$\frac{\partial^2}{\partial t^2} u_r + \frac{1}{r} \frac{\partial}{\partial r} u_r - \frac{1}{r^2} \frac{\partial^2}{\partial r^2} u_r = 0,$$ \hspace{1cm} (53)

supplemented with boundary conditions. Due to the existence of the equivalent surface forces that act at the boundaries of the workpiece, the boundary conditions to be applied are different from the ones in (48). The new boundary conditions to be applied result from the fact that the total electromagnetic force acting on the workpiece is replaced by an equivalent electromagnetic surface force, calculated with the Maxwell stress-tensor formula. Therefore, two equivalent surface forces act on the workpiece, and the boundary conditions in (48) and (49) become

$$\lim_{r \to a} \tau_{rr}(r, t) = -\frac{1}{2} \mu_0 [H_z(a, t)]^2,$$ \hspace{1cm} (54)

$$\lim_{r \to b} \tau_{rr}(r, t) = -\frac{1}{2} \mu_0 [H_z(b, t)]^2.$$ \hspace{1cm} (55)

In the next section we will use the method of equivalent surface forces as presented in (53)–(54) to compare the numerical results with the ones obtained with our theory.

VII. NUMERICAL RESULTS

In this section, we present some numerical results for a typical electromagnetic forming system designed for compression of hollow circular cylindrical workpieces. The workpiece subjected to electromagnetic compression has an inner radius $r = a = 20$ mm and an outer radius $r = b = 22$ mm. With this geometry, two types of materials have been chosen: one nonmagnetic ($\mu = \mu_0$) and one hypothetical linear magnetic material ($\mu = 100 \mu_0$). In both cases, the electrical conductivity of the workpiece is $\sigma = 3.6 \times 10^7$ S/m and the cylindrical current sheet accounting for the forming coil is located at $r = c = 24$ mm.

Further, in both cases, the workpiece has the same elastic properties and it has a linear elastic behavior within the whole range of stresses and strains. The workpiece has the Lamé coefficients of elasticity $\lambda_L = 17 \times 10^{10}$ N/m$^2$, $\mu_L = 8 \times 10^{10}$ N/m$^2$, and a mass density $\rho_m = 2.7 \times 10^3$ kg/m$^3$. We first compare the results of the electromagnetic problem and subsequently the results of the elastodynamic problem. In particular, we will present results for the radial displacement $u_r$, the radial strain $\varepsilon_r = \varepsilon_{rr}$, the tangential strain $\varepsilon_\varphi = \varepsilon_{\varphi\varphi}$, the radial stress $\sigma_r = \tau_{rr}$, the tangential stress $\sigma_\varphi = \tau_{\varphi\varphi}$, and the longitudinal stress $\sigma_z = \tau_{zz}$.

A. Electromagnetic Results

In our examples, we have considered that the current per unit length $I_S(t)$ flowing in the sheet antenna is a damped pulse typically used in electromagnetic forming processes. In the left plot of Fig. 5, the current per unit length in the sheet antenna in the time domain $I_S(t)$ is presented, while its frequency domain counterpart $|I_S(f)|$ is presented in the right plot of this figure. As we will observe later, the electromagnetic volume force density in the frequency domain reaches its maximum value at about 100–500 Hz. Therefore, at 500 Hz, we present the spatial distribution of the complex magnetic field inside and outside the workpiece (see Fig. 6). In the magnetic workpiece, we observe a much larger decay of the magnetic field than in the nonmagnetic workpiece.

In Fig. 7, the electromagnetic volume force density at the outer interface $r = b$ of the cylindrical workpiece has been presented in the time domain and in the frequency domain. The results are normalized with $(\mu/\mu_0)^{-3/2}$, since from their expressions it directly follows that the absolute values of the electromagnetic force density in a magnetic material are $(\mu/\mu_0)^{3/2}$ times larger than the ones obtained in a nonmagnetic material. We observe that the time domain results are roughly the same, although in the magnetic workpiece, the electromagnetic volume force density decays to zero more rapidly than in a nonmagnetic one. In the time domain, the electromagnetic volume force density $f_r^V$ decays...
more rapidly than the current per unit length \( I_S(t) \) in the sheet antenna. In fact, it contains higher frequency components, which can also be observed from its frequency spectrum. Although the dominant part of the electromagnetic force density \( f_0^{\text{EM}} \) is negative, there is an extended time interval where the electromagnetic force density is positive. The positive values of the electromagnetic force density are negligible as compared with the maximum value of the electromagnetic force density \( f_0^{\text{EM}} \) of about \( -2 \times 10^{11} \text{ N/m}^3 \).

In Fig. 8, the evolution of the electromagnetic force density in the space-time domain both for a nonmagnetic and for a magnetic workpiece is presented. For the magnetic workpiece, significant (nonzero) results of the electromagnetic force density are expected only for a small part of the workpiece, i.e., only for a cylindrical domain located near the outer boundary \( r = b \) facing the electric current sheet. Therefore, we have presented the space-time evolution of the electromagnetic force density only for a small part \( (21.8 \text{ mm} < r < 22 \text{ mm}) \) of the workpiece. Comparing the results for the magnetic and the nonmagnetic workpieces, we observe indeed the large decay of the electromagnetic force density in the negative radial direction of the magnetic workpiece, together with a different decay in time.

In case of a magnetic workpiece, there exists also an electromagnetic surface force density on both sides of the workpiece. The time-domain results for the electromagnetic surface force density are presented in Fig. 9, for \( r = a \) and \( r = b \). As expected, on the outer boundary \( r = b \) closest to the sheet antenna, the absolute values of the electromagnetic surface force density are larger than the ones of the electromagnetic surface force density on the inner boundary \( r = a \).

Comparing Figs. 7 and 9, we notice that the electromagnetic volume force density (Fig. 7) decays to zero more rapidly than the electromagnetic surface force density (Fig. 9). This is due to the fact the volume force density depends linearly on the magnetic field [see (40)], while the surface force density is related to the square of the magnetic field [see (41) and (42)].

B. Elastodynamic Results

In Fig. 10, the components of the elastic field at \( r = a \) in the nonmagnetic \((\mu = \mu_0)\) and in the magnetic \((\mu = 100 \mu_0)\) workpiece are presented. As the elastic problem is almost quasistatic, the radial displacement is most of the time negative and its shape is very similar to that of the dominant electromagnetic volume force densities. With the chosen elastic properties of the material, the values of the radial displacement \( u_r \) are very small, and these values yield very small radial and tangential strains, \( \varepsilon_r \) and \( \varepsilon_\theta \), respectively. The tangential stress \( \sigma_\theta \) has the largest values from all the stress components, for both nonmagnetic and magnetic workpieces. In the nonmagnetic workpiece, the radial stress \( \sigma_r \) is very small and can be neglected, while for the magnetic workpiece it has nonzero values. The longitudinal stress \( \sigma_z \) has negative values that are about three times smaller than the values of the tangential stress \( \sigma_\theta \).

From our simulations, we have observed that the radial displacement at \( r = b \) is slightly smaller than the radial displacement at \( r = a \). The results confirm the fact that during the electromagnetic compression the workpiece becomes thicker.

C. Comparison With Method of Equivalent Surface Sources

The space-time evolution of the radial displacement \( u_r \) has been calculated using our theory presented in Section V and with the method of equivalent surface forces presented in Section VI.
Fig. 10. Elastic field components at $r = a$ in the time domain for the nonmagnetic ($\mu = \mu_0$) workpiece (dashed lines) and for the magnetic ($\mu = 100 \mu_0$) workpiece (solid lines).

For the nonmagnetic workpiece, the radial displacements computed with both methods are roughly the same (see Fig. 11). A similar behavior is noticed for the pertaining stresses. Therefore, we do not present these results.

In order to have a better picture of the real differences between the results obtained with the two methods, in Fig. 12 we present the temporal evolution of the radial displacement and of the tangential stress at the inner boundary $r = a$ and at the outer boundary $r = b$ of the workpiece. We notice that both methods yield almost similar results.

Subsequently, for the magnetic workpiece, after using the two methods, in Fig. 13 we present the temporal evolution of the radial displacement $u_r$ and of the tangential stress $\sigma_\varphi$ at the inner boundary $r = a$ and at the outer boundary $r = b$ of the workpiece. We notice that the radial displacement $u_r$ has negative values when calculated with both methods, though the values calculated with our method are about four times larger than with the method of equivalent surface forces. For the tangential stress $\sigma_\varphi$ at the inner boundary $r = a$ of the workpiece, the same observation is valid, while at the outer boundary $r = b$ of the workpiece the two methods give results that differ very much from each other.

The method of equivalent surface forces gives at the outer boundary $r = b$ a tangential stress $\sigma_\varphi$ that has the same behavior as the one at the inner boundary. Our method gives a tangential stress $\sigma_\varphi$ that has a different behavior than the similar tangential stress $\sigma_\varphi$ calculated with the method of equivalent surface forces, due to the existence of surface forces that are taking into account the magnetic nature of the workpiece. Because these surface forces are negligible at the boundary $r = a$, their influence on the tangential stress behavior is very small, while at the outer boundary $r = b$ the surface forces are very large and it has a considerable influence on the behavior of the tangential stress $\sigma_\varphi$.

Fig. 11. Radial displacement $u_r$, in space-time domain, calculated with the present theory (a) and with the method of equivalent surface forces (b), for the nonmagnetic ($\mu = \mu_0$) workpiece.

Fig. 12. Radial displacement $u_r$ and tangential stress $\sigma_\varphi$ in time domain at $r = a$ (upper graphs) and $r = b$ (lower graphs), calculated with the present theory (solid lines) and with the method of equivalent surface forces (dashed lines), for the nonmagnetic ($\mu = \mu_0$) workpiece.

Further, from Fig. 13 we observe that the tangential stress at the outer boundary $r = b$, in the case of a magnetic workpiece presents more important sign variations when compared with the results given in Fig. 12 for a nonmagnetic workpiece. The reason is that near $r = b$ the surface force density decays very
Fig. 13. Radial displacement \( u_r \) and tangential stress \( \sigma_\varphi \) in time domain at \( r = a \) (upper graphs) and \( r = b \) (lower graphs), calculated with the present theory (solid lines) and with the method of equivalent surface forces (dashed lines), for the magnetic (\( \mu = 100 \mu_0 \)) workpiece.

Fig. 14. Radial displacement \( u_r \) in space-time domain, calculated with the present theory (a) and with the method of equivalent surface forces (b), for the magnetic (\( \mu = 100 \mu_0 \)) workpiece.

Fig. 15. Tangential stress \( \sigma_\varphi \) in space-time domain, calculated with the present theory (a) and with the method of equivalent surface forces (b), for the magnetic (\( \mu = 100 \mu_0 \)) workpiece.

rapidly, while the volume force density, having an opposite sign, decays much slower (compare also Figs. 7 and 9).

In Figs. 14 and 15, we present also the full space-time evolution of the radial displacement and of the tangential stress \( \sigma_\varphi \), when calculated with the presented theory and with the method of equivalent surface forces. We observe that the maximum values of the radial displacement \( u_r \), when calculated with our method, are about four times larger than the corresponding values calculated with the method of equivalent surface forces, and their space-time evolution is very different. The same observation is valid for the tangential stress \( \sigma_\varphi \).

VIII. CONCLUSION

In this paper, the electromechanical force densities associated with pulsed electromagnetic fields in piecewise homogeneous, isotropic, linear media with conductive losses have been discussed, in the context of their application in a process of shaping metal objects. It has been shown that the conductivity and the gradients in permittivity and in permeability lead to volume force densities, while jump discontinuities in permittivity and permeability lead to surface force densities. These electromagnetic force densities are assumed to act as volume (body) source densities in the elastodynamic equations and as surface source densities in the corresponding boundary conditions that govern the elastic and anelastic motion of deformable matter.
The practical configuration wherein the developed theory has been applied consisted of a hollow cylindrical domain with a high electrical conductivity (representing the workpiece) placed inside a cylindrical sheet antenna (representing the forming coil) carrying a given electric current per unit length. The configuration has been assumed to have infinite length and to be axially symmetric. It has also been assumed that the displacement current may be neglected and the diffusive field equations may be applied in the conducting cylindrical domain. Further, in the domains containing air with zero electrical conductivity $\sigma$, the quasi-static field equations have been applied. Our theory can be applied for nonaxially symmetric geometries as well. However, for these more general geometries the pertinent differential equations, both the electromagnetic and elastic ones, have to be solved by numerical methods, e.g., a finite-element method developed by Lee et al. [19], [20] and Besbes et al. [21].

The electromagnetic force, assumed to be the source of the elastic field, has been computed in the cylindrical domain with high electrical conductivity for two types of materials: one nonmagnetic ($\mu = \mu_0$) and one linear magnetic ($\mu = 100\mu_0$). The values of the electromagnetic volume force density are much larger in a magnetic material than in a nonmagnetic one. The numerical results showed also that, in both cases, the electromagnetic force density decays rapidly in time and space. For the magnetic material, the electromagnetic surface force density has also been calculated. Its values are much smaller that the integrated value of the electromagnetic volume force density over the thickness of the highly conducting domain.

In the literature related to electromagnetic forming, an equivalent pressure calculated with the use of the Maxwell stress tensor is used for the calculation of radial displacements. This formula uses only the values of the magnetic field at the inner and outer boundaries of the cylindrical domain, as they have been calculated with our model. Obviously, the theory of equivalent surface forces does not take correctly into account the magnetic nature of the object (workpiece). As a result, the theory of equivalent surface sources yields elastic deformations that are also assumed to be incorrect. We finally remark that our comparison has been made within the linear approximation. For nonlinear media, we anticipate that the method of the equivalent surface sources will not be useful.

**Appendix**

**Solution of the Elastodynamic Equations**

In order to arrive at the solution of (52), we write it as

$$\hat{u}_r(r) = \hat{u}_{r\text{part}}^p(r) + \hat{u}_{r\text{gen}}^p(r)$$

(56)

where $\hat{u}_{r\text{part}}^p(r)$ denotes the particular solution and $\hat{u}_{r\text{gen}}^p(r)$ denotes the general solution of the homogeneous form of (52).

**A. Particular Solution $\hat{u}_{r\text{part}}^p(r)$**

We may construct the particular solution $\hat{u}_{r\text{part}}^p(r)$ using the Green’s function $\hat{G}(r, r')$ as follows:

$$\hat{u}_{r\text{part}}^p(r) = \frac{1}{\lambda_L + 2\mu_L} \int_{r'=a}^b \hat{G}(r, r') \hat{f}^p_r(r')r'dr'$$

(57)

where $\hat{G}(r, r')$ is the solution of equation

$$\frac{1}{r} \partial_r(r \partial_r \hat{G}) + \left( k^2 - \frac{1}{r^2} \right) \hat{G} = -\frac{1}{r^2} \delta(r - r').$$

(58)

In an unbounded domain with a cylindrical source placed at $r = r'$, the solution of (58) is

$$\hat{G}(r, r') = \begin{cases} -\frac{\pi}{2} I_1(kr)Y_1(kr'), & \text{for } r \leq r' \\ -\frac{\pi}{2} Y_1(kr)J_1(kr'), & \text{for } r \geq r'. \end{cases}$$

(59)

In our further analysis, we need also the radial derivative of the particular solution. This is obtained as

$$\partial_r \hat{u}_{r\text{part}}^p(r) = \frac{1}{\lambda_L + 2\mu_L} \int_{r'=a}^b \partial_r \hat{G}(r, r') \hat{f}^p_r(r')r'dr'$$

(60)

where

$$\partial_r \hat{G}(r, r') = \begin{cases} -\frac{\pi}{2} \left[ k^2 I_0(kr) - r^{-1} J_1(kr) \right] Y_1(kr'), & r < r' \\ -\frac{\pi}{2} \left[ k^2 Y_0(kr) J_1(kr) + J_0(kr) Y_1(kr) \right] + \frac{\pi}{2} Y_1(kr) J_1(kr'), & r = r' \\ -\frac{\pi}{2} \left[ k^2 Y_0(kr) - r^{-1} J_1(kr) \right] J_1(kr'), & r > r'. \end{cases}$$

(61)

A special case for the calculation of the elastodynamic field is $k \to 0 (f \to 0)$. Then, (59) and (61) become

$$\lim_{k \to 0} \hat{G}(r, r') = \begin{cases} r/(2r'), & r \leq r' \\ r'/2r, & r \geq r' \end{cases}$$

(62)

and

$$\lim_{k \to 0} \partial_r \hat{G}(r, r') = \begin{cases} 1/(2r'), & r < r' \\ 0, & r = r' \\ -r'/(2r^2), & r > r' \end{cases}$$

(63)

respectively. Until here, since all quantities in (57) and (60) are known, we have calculated the particular solution $\hat{u}_{r\text{part}}^p(r)$ and its radial derivative $\partial_r \hat{u}_{r\text{part}}^p(r)$ as functions of the radius $r$. Thus, we also have the values of the particular solution and its radial derivative at the boundaries $r = a$ and $r = b$ of the cylindrical domain representing the workpiece.

**B. General Solution $\hat{u}_{r\text{gen}}^p(r)$**

The general solution $\hat{u}_{r\text{part}}^p(r)$ of the homogeneous form of (52) is given by

$$\hat{u}_{r\text{gen}}^p(r) = C J_1(kr) + D Y_1(kr)$$

(64)

where the coefficients $C$ and $D$ are obtained from the boundary conditions

$$\lim_{r \to a} \hat{u}_{r\text{gen}}^p = -\hat{\beta}_r^p(a)$$

(65)

$$\lim_{r \to b} \hat{u}_{r\text{gen}}^p = \hat{\beta}_r^p(b).$$

(66)
Similar to (56), we write \( \hat{\tau}_{rr} \) as

\[
\hat{\tau}_{rr}(r) = \hat{\tau}_{rr}^{\text{part}}(r) + \hat{\tau}_{rr}^{\text{gen}}(r)
\]  

(67)

where

\[
\hat{\tau}_{rr}^{\text{part}}(r) = (\lambda_L + 2\mu_L)\hat{\varepsilon}_r + \lambda_L r^{-1}\hat{\varepsilon}_r
\]

and

\[
\hat{\tau}_{rr}^{\text{gen}}(r) = CM(kr) + DN(kr)
\]

(68)

(69)

with

\[
M(kr) = k(\lambda_L + 2\mu_L)J_0(kr) - \frac{2\mu_L}{r}J_1(kr)
\]

(70)

\[
N(kr) = k(\lambda_L + 2\mu_L)Y_0(kr) - \frac{2\mu_L}{r}Y_1(kr)
\]

(71)

From (65)–(66), we obtain the following system of equations in matrix form:

\[
\begin{pmatrix}
M(ka) & N(ka) \\
M(kb) & N(kb)
\end{pmatrix}
\begin{pmatrix}
C \\
D
\end{pmatrix}
= \begin{pmatrix}
h(a) \\
h(b)
\end{pmatrix}
\]

(72)

where the elements of the known vector are given by

\[
h(a) = \hat{\varepsilon}_r^S(a) - \hat{\tau}_{rr}^{\text{part}}(a)
\]

(73)

\[
h(b) = \hat{\varepsilon}_r^S(b) - \hat{\tau}_{rr}^{\text{part}}(b)
\]

(74)

Thus, the coefficients \( C \) and \( D \) may be calculated as

\[
C = \frac{N(kb)h(a) - N(ka)h(b)}{M(ka)N(kb) - N(ka)M(kb)}
\]

(75)

\[
D = \frac{M(kb)h(a) + M(ka)h(b)}{M(ka)N(kb) - N(ka)M(kb)}.
\]

(76)

When \( k \to 0 \), the general solution in (64) may be written as

\[
\hat{\varepsilon}_r^{\text{gen}}(r) = Cr + D\frac{1}{r}
\]

(77)

Similar with the procedure presented above, we get

\[
\hat{\tau}_{rr}^{\text{gen}}(r) = (\lambda_L + \mu_L)C - 2\mu_L\frac{1}{r^2}D.
\]

(78)

The coefficients \( C \) and \( D \) are obtained from the application of the boundary conditions in (65)–(66). This procedure yields the following system of equations:

\[
2(\lambda_L + \mu_L)C - 2\mu_L\frac{1}{\alpha^2}D = h(a)
\]

(79)

\[
2(\lambda_L + \mu_L)C - 2\mu_L\frac{1}{\beta^2}D = h(b)
\]

(80)

where the functions \( h(a) \) and \( h(b) \) have been defined in (73)–(74). The coefficients \( C \) and \( D \) are

\[
C = \frac{\alpha^2h(a) - \beta^2h(b)}{2(\lambda_L + \mu_L)(\alpha^2 - \beta^2)}
\]

(81)

\[
D = \frac{\alpha^2\beta^2[h(a) - h(b)]}{2\mu_L(\alpha^2 - \beta^2)}.
\]

(82)

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