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ON CERTAIN MULTIPLE INTEGRALS OCCURRING IN A WAVEGUIDE SCATTERING PROBLEM*

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Abstract. Closed-form results are presented for some $n$-fold integrals where the integrand contains the exponential of a specific quadratic form in $n$ variables. These integrals arise in the ray-optical analysis of reflection and diffraction problems for an open-ended parallel-plane waveguide. The results are obtained by three methods: the first method is elementary, the second method uses an integral equation which is solved by the Wiener-Hopf technique, and the third method is based on a probabilistic interpretation of the integrals.

1. Introduction. This paper deals with the evaluation of the $n$-fold integrals

$$I_{n,q}(\alpha) = \pi^{-n/2} \int_0^\infty \cdots \int_0^\infty x_1^q \exp \left[ -\alpha x_1^2 - 2 \sum_{m=2}^{n-1} x_m^2 \right] dx_1 \cdots dx_n,$$

$$J_{n,q}(\alpha) = \pi^{-n/2} \int_0^\infty \cdots \int_0^\infty x_1^q \exp \left[ -\alpha x_1^2 - 2 \sum_{m=2}^{n-1} x_m^2 + 2 \sum_{m=1}^{n-2} x_m x_{m+1} \right]$$

$$-2x_{n-1}x_n - x_n^2 \right] dx_1 \cdots dx_n,$$

$$n = 2, 3, 4, \ldots, \quad q = 0, 1, 2, \ldots,$$

where the integration extends over the orthant $x_m \geq 0, m = 1, 2, \ldots, n$. By repeated application of the estimate

$$\frac{1}{\pi} \frac{1}{p^{1/2}} \frac{1}{p^{1/2}} \exp \left( s^2/p \right) \leq \int_0^\infty \exp \left[ -px^2 + 2sx \right] dx \leq \pi^{1/2} \frac{1}{p^{1/2}} \exp \left( s^2/p \right),$$

valid for $p > 0, s \geq 0$, it is found that the integrals (1.1) and (1.2) converge if $\alpha > (n-1)/n$ and $\alpha > (n-2)/(n-1)$, respectively.

These integrals were encountered in the ray-optical analysis of (i) the reflection problem for a TM or a TE mode traveling toward the open end of a semi-infinite parallel-plane waveguide [6], [7], (ii) the diffraction problem for a plane wave normally incident on two nonstaggered parallel half-planes [14]. In the course of that analysis explicit results were needed for $I_{n,q}(\alpha), J_{n,q}(\alpha)$ with $q = 0, 1$ and $\alpha = 1$ or $\alpha = 2$. It is the purpose of this paper to provide such results, namely

$$I_{n,0}(2) = \frac{1}{(n+1)^{3/2}}, \quad I_{n,1}(2) = \frac{1}{4n^{1/2}} \sum_{m=1}^{n} \frac{1}{m^{3/2}(n+1-m)^{3/2}},$$

$$J_{n,0}(2) = \frac{(1/2)^n}{n!}, \quad J_{n,1}(2) = \frac{1}{2n^{1/2}} \sum_{m=0}^{n-1} \frac{(1/2)_m}{m!(n-m)^{1/2}},$$

$$J_{n,0}(2) = \frac{-(-1/2)^n}{n!}, \quad J_{n,1}(2) = \frac{1}{2} \sum_{m=0}^{n-1} \frac{(-1/2)_m}{m!(n-m)^{3/2}},$$

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(1.7) \[ J_{n,0}(1) = \frac{1}{2\pi(n-1)^{1/2}}, \quad J_{n,1}(1) = \frac{1}{8\pi^{1/2}} + \frac{1}{8\pi^{3/2}} \sum_{m=1}^{n-2} \frac{1}{m^{1/2}(n-1-m)^{1/2}}, \]

where \((a)_n\) denotes Pochhammer’s symbol defined by

(1.8) \[ (a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1), \quad n = 1, 2, 3, \ldots. \]

The integrals \(I_{n,0}(2)\) and \(I_{n,1}(2)\) were already evaluated [6, Appendix D] thus leading to (1.4). Only recently the author became aware of a previous evaluation of \(I_{n,0}(2)\) by Anis and Lloyd [2] using essentially the same method as in [6, Appendix D]. In §2 of this paper the remaining results (1.5)-(1.7) are derived by elementary methods that involve integration by parts and generating function techniques. For \(q \geq 2\) recurrence relations are presented for \(I_{n,q}, J_{n,q}\), expressed in terms of the same functions with second subscripts \(q-1\) and \(q-2\).

In §§3.1, 3.2 the results (1.4)-(1.7) are rederived by a second and different approach. It is shown that the evaluation of \(I_{n,q}\) and \(J_{n,q}\) can be reduced to the solution of the integral equation

(1.9) \[ \varphi(t) = f(t) + \frac{\lambda}{\pi^{1/2}} \int_0^\infty \exp\left[-(t-s)^2\right] \varphi(s) \, ds, \]

where \(f(t) = e^{-t^2}\) and \(|\lambda| < 1\). The latter equation is solved by Fourier transformation and Wiener–Hopf technique. In §3.3 we consider the related \(n\)-fold integral

(1.10) \[ \mathcal{T}_n(t) = \pi^{-n/2} \int_0^\infty \cdots \int_0^\infty \exp \left[ 2 \frac{e^{-m/4} t x_1 - x_1^2 - 2 \sum_{m=2}^n x_m^2}{m} \right] \right. \]

\[ \left. + 2 \sum_{m=1}^{n-1} x_m x_{m+1} \right] \, dx_1 \cdots dx_n, \quad n = 1, 2, 3, \ldots, \]

while \(\mathcal{T}_0(t) = 1\) by definition. As a side result of the previous analysis it is found that

(1.11) \[ \sum_{n=0}^\infty \lambda^n \mathcal{T}_n(t) = \begin{cases} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\log (1-\lambda e^{-x^2})}{x-e^{m/4} t} \, dx \right], & t < 0, \\ (1-\lambda e^{-it} )^{-1} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\log (1-\lambda e^{-x^2})}{x-e^{m/4} t} \, dx \right], & t > 0, \end{cases} \]

where \(|\lambda| < 1\). Then the right-hand side of (1.11) is expanded in a power series in powers of \(\lambda\) and it turns out that \(\mathcal{T}_n(t)\) can be expressed in terms of Fresnel integrals. The result (1.11) is to be used in the ray-optical solution of the radiation problem for an incident mode traveling toward the open end of a semi-infinite parallel-plane waveguide [8].

The Wiener–Hopf solution of the integral equation (1.9) has been treated in the literature to some extent. Stewartson [18] solved both (1.9) and the associated homogeneous equation with \(f(t) = 0\), in the case when \(\lambda = 1\). As Stewartson points out, these integral equations arise in the evolution theory of comets and in some problems from fluid mechanics. Ghizzetti and Ossicini [11] studied the eigensolutions of the homogeneous equation when \(\lambda > 0\). A related integral equation with a shifted kernel \(\exp \left[-\frac{1}{2}(t+A-s)^2\right]\) was recently discussed by Atkinson [5] in connection with some inference and queuing problems.

In §4 the integrals \(I_{n,0}(\alpha)\) and \(J_{n,0}(\alpha)\) with \(\alpha = 1\) or \(\alpha = 2\) are evaluated by a third, probabilistic method. It is shown that \(I_{n,0}\) and \(J_{n,0}\) can be interpreted in terms of the probability distribution of a sum of random variables which are independent and have
the same normal distribution function. Then the explicit values (1.4)–(1.7) are recovered by means of theorems due to Sparre Andersen [1] and Spitzer [17]. In fact, the same probabilistic approach underlies a previous evaluation of $I_{n,0}(1)$ due to Anis and Lloyd [2], [3].

Integrals similar to (1.1), (1.2), but containing the exponential of a general quadratic form, occur in probability theory and statistics. For example, the so-called orthant probability for the multivariate normal distribution with zero means and variance-covariance matrix $V$ is given by

\begin{equation}
\Phi_n(V) = (2\pi)^{-n/2}|V|^{-1/2} \int_0^\infty \cdots \int_0^\infty \exp \left[ -\frac{1}{2}x'V^{-1}x \right] dx_1 \cdots dx_n,
\end{equation}

where $x' = (x_1, x_2, \ldots, x_n)$; see Ruben [16], Johnson and Kotz [12, Chap. 35]. According to [16], $\Phi_n(V)$ can be expressed in terms of the area of a certain simplex on the unit sphere in $n$-dimensional space. Such an expression is obtained by a suitable linear transformation of (1.12) which reduces $x'V^{-1}x$ to a sum of squares. Then the domain of integration is transformed into a polyhedral cone in $n$-dimensional space, bounded by $n$ hyperplanes through the origin, and the said simplex is the intersection of the cone and the unit sphere. Closed-form results for $\Phi_n(V)$ are readily obtained now in the cases $n = 1, 2, 3$. For $n > 3$, $\Phi_n(V)$ can no longer be expressed in terms of elementary functions; cf. [16, p. 171]. Various other methods for the evaluation of multinormal probabilities are reviewed in [12, Chap. 35]. It is remarked that none of the methods of this paper is applicable to the general integral (1.12). As for the integrals $I_{n,q}(\alpha)$, $J_{n,q}(\alpha)$, closed-form results valid for any $\alpha$ may be derived when $n \leq 3$ by the geometrical approach as described above.

2. Evaluation by elementary means.

2.1. $I_{n,q}(2)$. The integrals $I_{n,0}(2)$ and $I_{n,1}(2)$ were already evaluated, see [6, Appendix D], [2]. Consider now $I_{n,q}(2)$ with $q \geq 2$, as defined by (1.1), and replace the factor $x_1$ in the integrand by

\begin{equation}
x_1^q = \frac{x_1^{q-1}}{2(n+1)} \left[ n(4x_1^2 - 2x_2) + \sum_{m=2}^n (n + 1 - m)(-2x_{m-1} + 4x_m - 2x_{m+1}) \right],
\end{equation}

where $x_{n+1} = 0$ by definition. Then $I_{n,q}(2)$ can be expressed as a sum of integrals which permit integration by parts with respect to $x_1$ and explicit integration with respect to $x_m$, $m = 2, 3, \ldots, n$, respectively. The result comprises an $n$-fold integral which is recognized as $I_{n,q-2}(2)$, and a sum of $(n-1)$-fold integrals which can be expressed as products $I_{m-1,q-1}(2)I_{n-m,0}(2)$, $m = 2, 3, \ldots, n$. On substitution of the actual value of $I_{n-m,0}(2)$, we obtain the recurrence relation

\begin{equation}
I_{n,q}(2) = \frac{n(q-1)}{2(n+1)} I_{n,q-2}(2) + \frac{\pi^{-1/2}}{2(n+1)} \sum_{m=1}^{n-1} \frac{I_{m,q-1}(2)}{(n-m)^{1/2}},
\end{equation}

valid for $q \geq 2$. The same method may be used for the reduction of the integral $I_{n,1}(2)$, yielding

\begin{equation}
I_{n,1}(2) = \frac{\pi^{-1/2}}{2(n+1)} \sum_{m=1}^n \frac{I_{m-1,0}(2)}{(n+1-m)^{1/2}} = \frac{1}{4\pi^{1/2}} \sum_{m=1}^n \frac{1}{m^{3/2}(n+1-m)^{3/2}},
\end{equation}

in accordance with (1.4). The relations (2.2) and (2.3) can be combined to the single
recurrence relation

\[ I_{n,q}(2) = \frac{n(q - 1)}{2(n + 1)} I_{n,q-2}(2) + \frac{\pi^{-1/2}}{2(n + 1)} \sum_{m=0}^{n-1} I_{m,q-1}(2), \]

valid for \( q \geq 1 \), where it is understood that \( (q - 1)I_{n,q-2}(2) = 0 \) for \( q = 1 \), and \( I_{0,q}(2) = \delta_{q0} \) with \( \delta_{00} = 1, \delta_{q0} = 0 \) for \( q \neq 0 \) (Kronecker’s symbol).

2.2. \( I_{n,q}(1), J_{n,q}(2) \). Consider first the integrals \( I_{n,0}(1) \) and \( J_{n,0}(2) \), as defined by (1.1) and (1.2). In the integral \( J_{n,0}(2) \) we set

\[ I_o \exp \left[ -2x_{n-1}x_n - x_n^2 \right] dx_n = \pi^{1/2} \exp \left( x_{n-1}^2 \right) - \int_{0}^{\infty} \exp \left[ 2x_{n-1}x_n - x_{n-1}^2 \right] dx_n; \]

then it is obvious that

\[ J_{n,0}(2) = I_{n-1,0}(1) - I_{n,0}(1), \quad n \geq 2. \]

A second relation between \( I_{n,0}(1) \) and \( J_{n,0}(2) \) is obtained by starting from the identity

\[ \pi^{-n/2} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \left\{ (2x_1 - 2x_2) + \sum_{m=2}^{n-1} (-2x_{m-1} + 4x_m - 2x_{m+1}) \right\} 
+ (-2x_{n-1} + 4x_n + 2x_{n+1}) - (2x_n + 2x_{n+1}) \right\} 
\cdot \exp \left[ -x_1^2 - 2 \sum_{m=2}^{n} x_m^2 + 2 \sum_{m=1}^{n-1} x_m x_{m+1} - 2x_n x_{n+1} \right. 
- x_{n+1}^2 \left. \right] dx_1 \cdots dx_{n+1} = 0, \]

where \( n \geq 2 \). Notice that the successive linear factors are just the derivatives of the exponent with respect to \( x_m, m = 1, 2, \ldots, n + 1 \). Hence, the \((n + 1)\)-fold integral (2.6) can be rewritten as a sum of integrals which permit explicit integration with respect to \( x_m \). Each of the resulting \( n\)-fold integrals is the product of an \((m-1)\)-fold integral equal to \( I_{m-1,0}(1) \), and an \((n + 1 - m)\)-fold integral equal to \( J_{n+1-m,0}(2) \). Thus we find

\[ \sum_{m=0}^{n} I_{m,0}(1)J_{n-m,0}(2) = 0, \quad n \geq 2, \]

where \( I_{0,0}(1) = 1, J_{0,0}(2) = -1, J_{1,0}(2) = \frac{1}{2} \) by definition. By a direct calculation from (1.1) it is found that \( I_{1,0}(1) = \frac{1}{2} \), hence, (2.7) and (2.5) also hold for \( n = 1 \).

In order to determine \( I_{n,0}(1) \) and \( J_{n,0}(2) \), we introduce the generating functions

\[ A_0(\lambda) = \sum_{n=0}^{\infty} I_{n,0}(1) \lambda^n, \quad B_0(\lambda) = \sum_{n=0}^{\infty} J_{n,0}(2) \lambda^n. \]

From (2.5) and (2.7) we then infer

\[ B_0(\lambda) = -(1 - \lambda)A_0(\lambda), \quad A_0(\lambda)B_0(\lambda) = -1, \]

and consequently

\[ A_0(\lambda) = (1 - \lambda)^{-1/2}, \quad B_0(\lambda) = -(1 - \lambda)^{1/2}. \]

By expansion of \((1 - \lambda)^{\pm 1/2}\) in binomial series we readily find \( I_{n,0}(1) \) and \( J_{n,0}(2) \), as stated in (1.5) and (1.6).
Consider next the integral $I_{n,q}(1)$ with $q \geq 1$, as defined by (1.1), and replace the factor $x_1^q$ in the integrand by

$$x_1^q = \frac{1}{2} x_1^{q-1} \left[ n (2x_1 - 2x_2) + \sum_{m=2}^{n} (n + 1 - m) (-2x_{m-1} + 4x_m - 2x_{m+1}) \right],$$

where $x_{n+1} = 0$ by definition. Proceeding as in § 2.1, we are led to the recurrence relation

$$J_{n,q}(1) = \frac{1}{2} n (q - 1) I_{n,q-2}(1) + \frac{1}{2 \pi^{1/2}} \sum_{m=0}^{n-1} (n - m)^{1/2},$$

valid for $q \geq 1$, where $(q - 1) I_{n,q-2}(1) = 0$ for $q = 1$, and $I_{0,q}(1) = \delta_{q0}$ by definition. The present relation was also established by Anis [4] in the same manner. A similar recurrence relation for $J_{n,q}(2)$ is obtained by setting

$$x_2^q = \frac{1}{2} x_2^{q-1} \left[ (4x_1 - 2x_2) + \sum_{m=2}^{n-2} (-2x_{m-2} + 4x_{m-1} - 2x_m) - (2x_{n-1} + 2x_n) \right]$$

in the defining integral (1.2). Thus we find

$$J_{n,q}(2) = \frac{1}{2} (q - 1) J_{n,q-2}(2) - \frac{1}{2 \pi^{1/2}} \sum_{m=0}^{n-1} \frac{(-1/2)_m}{m!} J_{n-1-m,q-1}(2),$$

valid for $q \geq 1$, where $(q - 1) J_{n,q-2}(2) = 0$ for $q = 1$, and $I_{0,q}(2) = \delta_{q0}$ by definition. For $q = 1$ the recurrence relations (2.12) and (2.14) provide the explicit values of $I_{n,1}(1)$ and $J_{n,1}(2)$, as stated in (1.5) and (1.6).

For later use we introduce the generating function

$$A_q(\alpha) = \sum_{n=0}^{\infty} I_{n,q}(1) \alpha^n, \quad q = 0, 1, 2, \cdots.$$  

Then (2.12) can be reduced to a recurrence relation for $A_q(\alpha)$, viz.,

$$A_q(\lambda) = \frac{1}{2} (q - 1) \lambda A^1_{q-2}(\lambda) + \frac{1}{2 \pi^{1/2}} A_{q-1}(\lambda) L(\lambda), \quad q \geq 1,$$

where a prime denotes differentiation with respect to $\lambda$ and

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{1/2}}.$$  

Starting from $A_0(\lambda) = (1 - \lambda)^{-1/2}$, we have

$$A_1(\lambda) = \frac{1}{2 \pi^{1/2}} (1 - \lambda)^{-1/2} L(\lambda),$$

$$A_2(\lambda) = \frac{1}{4} \lambda (1 - \lambda)^{-3/2} + \frac{1}{4 \pi} (1 - \lambda)^{-1/2} L^2(\lambda),$$

and so on; in principle the functions $A_q(\alpha)$ are completely determined.
2.3. \( J_{n,q}(1) \). Starting from (1.2) with \( \alpha = 1 \), we perform an integration by parts with respect to \( x_1 \), yielding

\[
J_{n,q}(1) = \pi^{-n/2} q + 1 \int_{0}^{\infty} \cdots \int_{0}^{\infty} x_1^{q+1}(2x_1 - 2x_2) \exp \left[ -x_1^2 - 2 \sum_{m=2}^{n-1} x_m^2 \right] + 2 \sum_{m=1}^{n-2} x_m x_{m+1} - 2x_{n-1}x_n - x_n^2 \ dx_1 \cdots dx_n,
\]

where it is supposed that \( n \geq 3 \). In the latter integral the factor \( 2x_1 - 2x_2 \) is replaced by

\[
(2.21) \quad 2x_1 - 2x_2 = -\sum_{m=2}^{n-2} (-2x_{m-1} + 4x_m - 2x_{m+1})
\]

Then \( J_{n,q}(1) \) becomes a sum of integrals which permit explicit integration with respect to \( x_m \), \( m = 2, 3, \ldots, n \). Proceeding as before, we find

\[
(2.22) \quad J_{n,q}(1) = \pi^{-1/2} q + 1 \sum_{m=0}^{n-2} \frac{(-1/2)_m}{m!} I_{n-1-m,q+1}(1),
\]

valid for \( n \geq 3 \). In a similar manner it can be verified that (2.22) holds true also for \( n = 2 \). Thus \( J_{n,q}(1) \) has been expressed in terms of the integrals \( I_{m,q+1}(1) \) which are known from § 2.2.

In order to explicitly evaluate \( J_{n,0}(1) \) and \( J_{n,1}(1) \), we introduce the generating function

\[
(2.23) \quad C_q(\lambda) = \sum_{n=2}^{\infty} J_{n,q}(1) \lambda^n.
\]

Then it follows from (2.22) that

\[
(2.24) \quad C_q(\lambda) = \pi^{-1/2} q + 1 \frac{\lambda (1 - \lambda)^{1/2}}{\lambda^{1/2}} A_{q+1}(\lambda),
\]

where \( A_{q+1}(\lambda) \) is defined by (2.15). Referring to (2.18), (2.19), we thus find

\[
(2.25) \quad C_0(\lambda) = (1/2\pi)\lambda L(\lambda),
\]

\[
(2.26) \quad C_1(\lambda) = \frac{1}{8\pi^{1/2}}\lambda^2 (1 - \lambda)^{-1} + \frac{1}{8\pi^{3/2}}\lambda L^2(\lambda).
\]

By expansion of these functions the results (1.7) for \( J_{n,0}(1) \) and \( J_{n,1}(1) \) are readily established.


3.1. \( I_{n,q}(2), I_{n,q}(1) \). Let the functions \( \varphi_n(t) \), \( t \) real, \( n = 0, 1, 2, \ldots \), be defined by

\[
(3.1) \quad \varphi_n(t) = \pi^{-n/2} e^{-t^2} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \exp \left[ 2tx_1 - 2 \sum_{m=1}^{n} x_m^2 + 2 \sum_{m=1}^{n-1} x_m x_{m+1} \right] dx_1 \cdots dx_n,
\]

Then it is easily seen from (1.1) that

\[
(3.2) \quad I_{n,q}(2) = 2^{-q} \left( \frac{d}{dt} \right)^q \left[ e^{t^2} \varphi_n(t) \right]_{t=0}, \quad I_{n,q}(1) = \pi^{-1/2} \int_{0}^{\infty} t^q \varphi_{n-1}(t) \ dt.
\]
By repeated application of (1.3), one is led to the estimate
\[ 0 \leq \varphi_n(t) \leq \pi^{-1/2} n^{-1/2} e^{-t^2} \int_0^\infty \exp \left[ \frac{2tx_1}{n} x_1^2 \right] dx_1. \]
(3.3)

The functions \( \varphi_n(t) \) are connected through the recurrence relation
\[ \varphi_n(t) = \pi^{-1/2} \int_0^\infty \exp \left[ -(t-s)^2 \right] \varphi_{n-1}(s) \, ds, \quad n \geq 1. \]
(3.4)

We now introduce the generating function
\[ \varphi(t) = \sum_{n=0}^\infty \lambda^n \varphi_n(t), \quad |\lambda| < 1; \]
then, in view of (3.3), (3.4), the latter series converges and is precisely the Neumann series associated with the integral equation
\[ \varphi(t) = e^{-t^2} + \frac{\lambda}{\pi^{1/2}} \int_0^\infty \exp \left[ -(t-s)^2 \right] \varphi(s) \, ds. \]
(3.5)

Furthermore, it follows from (3.3) that
\[ \varphi(t) = O(1), \quad t \geq 0; \quad \varphi(t) = O(e^{-t^2}), \quad t \leq 0. \]
(3.6)

(By a more careful analysis the first result can even be improved to \( \varphi(t) = O(e^{-\beta t}) \) as \( t \to \infty \), where \( \beta \) will be specified below; however, we do not need this sharper estimate.)

The integral equation (3.6) is solved by Fourier transformation and Wiener–Hopf technique (cf. Noble [15]). We introduce the Fourier transforms
\[ \Phi_+(w) = \int_0^\infty \varphi(t) e^{iwt} \, dt, \quad \Phi_-(w) = \int_{-\infty}^0 \varphi(t) e^{iwt} \, dt, \]
(3.7)

where \( w \) is a complex variable. Then the estimates (3.7) imply that \( \Phi_+(w) \) is regular in the upper half-plane \( \text{Im} \, w > 0 \), while \( \Phi_-(w) \) is an integral function. Under Fourier transformation the integral equation (3.6) reduces to
\[ \Phi_+(w) + \Phi_-(w) = \pi^{1/2} e^{-w^2/4} + \lambda e^{-w^2/4} \Phi_+(w), \quad \text{Im} \, w > 0, \]
or equivalently
\[ (1-\lambda e^{-w^2/4}) \left[ \frac{\pi^{1/2}}{\lambda} \right] \Phi_+(w) + \left[ \Phi_-(w) - \frac{\pi^{1/2}}{\lambda} \right] = 0, \quad \text{Im} \, w > 0. \]
(3.8)

Before going on we observe that the factor \( 1-\lambda e^{-w^2/4} \) vanishes when \( w = 2 (\log \lambda)^{1/2} \). The zeros closest to the real axis have imaginary parts \( \pm \beta \) where \( \beta = 2|\text{Im} \, (\log \lambda)^{1/2} | \) with the principal value of \( \log \lambda \) to be taken. Thus, \( 1-\lambda e^{-w^2/4} \neq 0 \) in the strip \( -\beta < \text{Im} \, w < \beta \). Then by means of (3.9), extended to \( \text{Im} \, w > -\beta \), \( \Phi_+(w) \) may be analytically continued into the upper half-plane \( \text{Im} \, w > -\beta \). The functional equation (3.9) is now solved by the standard Wiener–Hopf procedure. The key step in this procedure is the factorization of \( 1-\lambda e^{-w^2/4} \) into
\[ 1-\lambda e^{-w^2/4} = K_+(w)/K_-(w), \quad -\beta < \text{Im} \, w < \beta, \]
(3.9)
such that $K_+(w)$ is regular and nonzero in $\text{Im} \, w > -\beta$, and $K_-(w)$ is regular and nonzero in $\text{Im} \, w < \beta$. This factorization can be accomplished by means of Noble [15, § 1.3, Thm. C], yielding

$$
(3.11) \quad K_+(w) = \exp \left[ \frac{1}{2\pi i} \int_{-\infty+ia}^{\infty+ia} \log \left(1 - \frac{e^{-z^2/4}}{z - w} \right) \, dz \right],
$$

$$
(3.12) \quad K_-(w) = \exp \left[ \frac{1}{2\pi i} \int_{-\infty+ib}^{\infty+ib} \log \left(1 - \frac{e^{-z^2/4}}{z - w} \right) \, dz \right],
$$

where $a, b$ are any numbers subject to $-\beta < a < b < \beta$, and the logarithm stands for its principal value. Using (3.10), we rearrange (3.9) as

$$
(3.13) \quad K_+(w) \Phi_+(w) + \frac{\pi^{1/2}}{\lambda} = -K_-(w) \Phi_-(w) - \frac{\pi^{1/2}}{\lambda}, \quad -\beta < \text{Im} \, w < \beta.
$$

Then the functions on the left-hand side of (3.13) are regular in the upper half-plane $\text{Im} \, w > -\beta$, and the functions on the right-hand side are regular in the lower half-plane $\text{Im} \, w < \beta$. Hence, by analytic continuation both sides of (3.13) must equal an integral function $P(w)$, say. From (3.11), (3.12) it is obvious that $K_+(w) \to 1$ as $|w| \to \infty$, $\text{Im} \, w \not\in \pm \beta$; likewise, $\Phi_+(w) \to 0$ as $|w| \to \infty$, $\text{Im} \, w \not\in \pm \beta$, according to the Riemann-Lebesgue lemma. Thus $P(w) \to \pi^{1/2}/\lambda$ as $|w| \to \infty$, and consequently $P(w) = \pi^{1/2}/\lambda$ by Liouville’s theorem. Then the solution for $\Phi_+(w)$ is easily obtained from the left-hand side of (3.13), viz.,

$$
(3.14) \quad \Phi_+(w) = \frac{\pi^{1/2}}{\lambda} \left\{ \frac{1}{K_+(w)} - 1 \right\}, \quad \text{Im} \, w > 0,
$$

where the path of integration has been chosen along the real axis. Finally, the original function $\varphi(t)$ is found by inverse Fourier transformation of $\Phi_+(w)$, viz.,

$$
(3.15) \quad \varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_+(w) e^{-iw \, dt}, \quad t > 0.
$$

The solution for $\varphi(t)$ thus determined is of a rather complicated form. However, it follows from (3.2), (3.5), (3.8) that

$$
(3.16) \quad \sum_{n=0}^{\infty} I_{n, q}(2) \lambda^n = 2^{-q} \left( \frac{d}{dt} \right)^q \left[ e^{t^2} \varphi(t) \right] \bigg|_{t=0},
$$

$$
(3.17) \quad \sum_{n=1}^{\infty} I_{n, q}(1) \lambda^n = \frac{\lambda}{\pi^{1/2}} \int_0^{\infty} t^q \varphi(t) \, dt = \frac{\lambda^q}{\pi^{1/2} \Phi_+(0)},
$$

so the required integrals $I_{n, q}(2)$ and $I_{n, q}(1)$ are completely determined by the derivatives $\varphi^{(m)}(0)$, $m = 0, 1, \ldots, q$, and $\Phi_+(0)$ only. The derivatives $\varphi^{(m)}(0)$ are readily obtained from the asymptotic expansion of $\Phi_+(w)$ as $|w| \to \infty$, $\text{Im} \, w > 0$. In fact, starting from the Taylor series

$$
(3.18) \quad \varphi(t) = \sum_{m=0}^{\infty} \frac{\varphi^{(m)}(0)}{m!} t^m,
$$
one has by Watson’s lemma (see e.g. Erdélyi [9, § 2.2])

$$\Phi_+(w) \sim \sum_{m=0}^{\infty} i^{m+1} \varphi^{(m)}(0) w^{-m-1}, \quad |w| \to \infty, \quad \delta \leq \arg w \leq \pi - \delta,$$

for any positive \(\delta\). Thus the integrals \(I_{n,q}(2)\) and \(I_{n,q}(1)\) can be determined from \(\Phi_+(w)\) only. We shall now evaluate the integrals in the two cases \(q = 0\) and \(q = 1\).

The asymptotic expansion of \(\Phi_+(w)\) is easily obtained from (3.14), viz.,

$$\Phi_+(w) = \frac{\pi^{1/2}}{\lambda} \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log (1 - \lambda e^{-z^2/4}) \, dz \, w^{-1} \right\}$$

$$= -\frac{1}{8\pi^3} \left\{ \int_{-\infty}^{\infty} \log (1 - \lambda e^{-z^2/4}) \, dz \right\} w^{-2} + O(w^{-3})$$

as \(|w| \to \infty, \text{Im} \, w > 0\). Compare the latter expansion to (3.19); then it is found that

$$\varphi(0) = \frac{\pi^{1/2}}{2\pi \lambda} \int_{-\infty}^{\infty} \log (1 - \lambda e^{-z^2/4}) \, dz$$

$$\varphi'(0) = \frac{\lambda}{4\pi^{1/2}} \left[ \varphi(0) \right]^2 = \frac{\lambda}{4\pi^{1/2}} \left[ \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+1)^{3/2}} \right]^2.$$

In view of (3.16), the present results immediately yield \(I_{n,0}(2)\) and \(I_{n,1}(2)\), their values being given by (1.4).

Next we determine \(\Phi_+(0)\) from (3.14) by taking the limit when \(w \to 0\) from the upper side \(\text{Im} \, w > 0\). By Plemelj’s formulae we have

$$\frac{\lambda}{\pi^{1/2}} \Phi_+(0) = \exp \left[ -\frac{1}{2} \log (1 - \lambda) \right] - 1 = (1 - \lambda)^{-1/2} - 1 = \sum_{n=1}^{\infty} \frac{(1/2)_n}{n!} \lambda^n,$$

which should be compared to (3.17). Then the result (1.5) for \(I_{n,0}(1)\) is obvious. From (3.14) the derivative \(\Phi_+'(w)\) is found to be

$$\Phi_+'(w) = \frac{\lambda^{1/2}}{\pi} \left[ \Phi_+(w) + \frac{\pi^{1/2}}{\lambda} \right] \left( -\frac{\lambda}{4\pi} \right) \int_{-\infty}^{\infty} \frac{z e^{-z^2/4}}{1 - \lambda e^{-z^2/4}} \, dz \, w^{-1}, \quad \text{Im} \, w > 0.$$

Then again by Plemelj’s formulae we have

$$\frac{\lambda^{1/2}}{\pi} \Phi_+'(0) = \frac{\lambda}{\pi^{1/2}} \left( \Phi_+(0) + 1 \right) \frac{\lambda}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-z^2/4}}{1 - \lambda e^{-z^2/4}} \, dz,$$

$$= \frac{\lambda}{2\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(1/2)_n}{n!} \lambda^n \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+1)^{1/2}},$$

from which the result (1.5) for \(I_{n,1}(1)\) is easily recovered.

In principle, the integrals \(I_{n,q}(2)\) and \(I_{n,q}(1)\) with \(q \geq 2\) can be evaluated in the same manner. In addition, one may establish recurrence relations for the integrals. However, these recurrence relations turn out to be more complicated than the ones derived in §§ 2.1, 2.2. Therefore we shall not pursue this matter.
3.2. $J_{n,q}(2)$, $J_{n,q}(1)$. The approach is highly similar to that of § 3.1. Let the functions $\psi_n(t)$, $t$ real, $n = 1, 2, 3, \ldots$, be defined by

$$\psi_1(t) = \frac{1}{2} \text{erfc} t = \pi^{-1/2} \int \limits_{-t}^{\infty} e^{-x^2} dx,$$

(3.26)

$$\psi_n(t) = \pi^{-n/2} e^{-t^2} \int \limits_{0}^{\infty} \cdots \int \limits_{0}^{\infty} \exp \left\{ \frac{2tx_1 - 2 \sum \limits_{m=1}^{n-1} x_m^2 + 2 \sum \limits_{m=1}^{n-2} x_m x_{m+1}}{2} - 2x_{n-1}x_n - x_n^2 \right\} dx_1 \cdots dx_n, \quad n = 2, 3, 4, \ldots.$$ 

Then it is easily seen from (1.2) that

$$J_{n,q}(2) = 2^{-q} \left[ \frac{d}{dt} \int \limits_{t=0}^{\infty} e^{t^2} \psi_n(t) \right] t=0, \quad J_{n,q}(1) = \pi^{-1/2} \int \limits_{0}^{\infty} t^q \psi_{n-1}(t) \ dt.$$

The inner $x_n$ integral in (3.26) can be estimated in an obvious manner, thus leading to

$$0 \leq \psi_n(t) \leq \frac{1}{2} \varphi_{n-1}(t) \leq \begin{cases} \frac{1}{4} n^{-1/2} e^{-t^2/n}, & t \geq 0, \\ \frac{1}{4} n^{-1/2} e^{-t^2}, & t \leq 0, \end{cases} \quad n \geq 2,$$

(3.28)
on account of (3.3). The functions $\psi_n(t)$ are connected through the recurrence relation

$$\psi_n(t) = \pi^{-1/2} \int \limits_{0}^{\infty} \exp \left\{ -(t-s)^2 \right\} \psi_{n-1}(s) \ ds, \quad n \geq 2.$$

As we did in (3.5), we introduce the generating function

$$\psi(t) = \sum \limits_{n=1}^{\infty} \lambda^n \psi_n(t), \quad |\lambda| < 1;$$

(3.30)then, in view of (3.29), the latter series is the Neumann series associated with the integral equation

$$\psi(t) = \frac{1}{2} \lambda \text{erfc} t + \frac{\lambda}{\pi^{1/2}} \int \limits_{0}^{\infty} \exp \left\{ -(t-s)^2 \right\} \psi(s) \ ds.$$

(3.31)Furthermore, by use of (3.28) it can be shown that $\psi(t) \to 0$ as $t \to \infty$.

The integral equation (3.31) can again be solved by Fourier transformation and Wiener–Hopf technique. However, a simpler way out is to differentiate (3.31) with respect to $t$ followed by an integration by parts in the integral term, thus yielding

$$\psi'(t) = \frac{\lambda}{\pi^{1/2}} [\psi(0) - 1] e^{-t^2} + \frac{\lambda}{\pi^{1/2}} \int \limits_{0}^{\infty} \exp \left\{ -(t-s)^2 \right\} \psi'(s) \ ds.$$

(3.32)The latter integral equation is of the same form as (3.6), hence, its solution is given by

$$\psi'(t) = \frac{\lambda}{\pi^{1/2}} [\psi(0) - 1] \varphi(t).$$

(3.33)By integration of (3.33) over $[0, \infty)$, we have

$$-\psi(0) = \frac{\lambda}{\pi^{1/2}} [\psi(0) - 1] \int \limits_{0}^{\infty} \varphi(t) \ dt$$

(3.34)

$$= \frac{\lambda}{\pi^{1/2}} [\psi(0) - 1] \Phi_+(0) = [\psi(0) - 1] [(1-\lambda)^{-1/2} - 1],$$
where \( \psi_+(0) \) was taken from (3.23). Thus we find

\[
(3.35) \quad \psi(0) = 1 - (1 - \lambda)^{1/2},
\]
\[
(3.36) \quad \psi'(t) = -\pi^{-1/2} \lambda (1 - \lambda)^{1/2} \varphi(t),
\]
and \( \psi(t) \) is completely determined in this manner.

We now turn to the evaluation of the integrals \( J_{n,q}(2) \) and \( J_{n,q}(1) \). It follows from (3.27), (3.30) that

\[
(3.37) \quad \sum_{n=1}^{\infty} J_{n,q}(2) \lambda^n = 2^{-q} \left( \frac{d}{dt} \right)^q \left[ e^{i2\psi(t)} \right] \bigg|_{t=0},
\]
\[
(3.38) \quad \sum_{n=2}^{\infty} J_{n,q}(1) \lambda^n = \frac{\lambda}{\pi^{1/2}} \int_0^\infty t^q \psi(t) \, dt.
\]

In the cases \( q = 0, q = 1 \), the right-hand side of (3.37) reduces to

\[
(3.39) \quad \psi(0) = 1 - (1 - \lambda)^{1/2} = -\sum_{n=1}^{\infty} \frac{(-1/2)_n}{n!} \lambda^n,
\]
\[
(3.40) \quad \frac{1}{2} \psi'(0) = -\frac{1}{2} \pi^{-1/2} (1 - \lambda)^{1/2} \varphi(0) = -\frac{\lambda}{2\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!} \lambda^n \sum_{n=1}^{\infty} \frac{\lambda^n}{(n + 1)^{3/2}},
\]

where \( \varphi(0) \) was quoted from (3.21). Then it is easily recognized that \( J_{n,0}(2) \) and \( J_{n,1}(2) \) are given by (1.6). In the same manner one may evaluate \( J_{n,q}(2) \) when \( q \geq 2 \). Consider next (3.38) where the right-hand side is reduced through an integration by parts. By replacing \( \psi'(t) \) by (3.36), we obtain

\[
(3.41) \quad \sum_{n=2}^{\infty} J_{n,q}(1) \lambda^n = \frac{\pi^{-1/2}}{q + 1} \lambda^{2(1 - \lambda)^{1/2}} \int_0^\infty t^{q+1} \psi(t) \, dt
\]

on account of (3.17). By equating the coefficients of corresponding powers of \( \lambda \) in (3.41), we re-obtain the recurrence relation (2.22). As shown at the end of § 2.3, the latter relation readily yields the explicit values of \( J_{n,0}(1) \) and \( J_{n,1}(1) \).

### 3.3. \( \mathcal{F}_n(t) \)

We consider the \( n \)-fold integrals \( \mathcal{F}_n(t) \), \( t \) real, \( n = 0, 1, 2, \cdots \), defined by

\[
(3.42) \quad \mathcal{F}_0(t) = 1,
\]
\[
(3.43) \quad \mathcal{F}_n(t) = \pi^{-n/2} \int_0^\infty \cdots \int_0^\infty \exp \left[ 2 e^{-\pi t/4} t x_1 - x_1^2 - 2 \sum_{m=2}^n x_m^2 \right]
+ 2 \sum_{m=1}^{n-1} x_m x_{m+1} \right] \, dx_1 \cdots \, dx_n, \quad n = 1, 2, 3, \cdots
\]

These integrals can be expressed in terms of the functions \( \varphi_n \) as given by (3.1), viz.,

\[
(3.44) \quad \mathcal{F}_n(t) = \pi^{-1/2} \int_0^\infty \exp [2 e^{-\pi t/4} t \varphi_{n-1}(x)] \, dx, \quad n \geq 1.
\]
Then by means of (3.3) one has the estimate

\[
|\mathcal{T}_n(t)| \leq (\pi n)^{-1/2} \int_0^\infty \exp \left[ 2^{1/2} tx - \frac{x^2}{2n} \right] dx \leq \left\{ \begin{array}{ll}
\frac{1}{2}, & t \geq 0, \\
1, & t \leq 0.
\end{array} \right.
\]

We now introduce the generating function

\[
G(\lambda, t) = \sum_{n=0}^\infty \lambda^n \mathcal{T}_n(t),
\]

where \( \lambda \) is a complex variable. In view of (3.44), the latter series certainly converges for \( |\lambda| < \exp (-t^2/2) \) when \( t \geq 0 \), and for \( |\lambda| < 1 \) when \( t \leq 0 \). Replace \( \mathcal{T}_n(t) \) by (3.43), then \( G(\lambda, t) \) reduces to

\[
G(\lambda, t) = 1 + \frac{\lambda}{\pi^{1/2}} \int_0^\infty \exp \left[ 2 e^{-\pi/4} tx \right] \varphi(x) dx = 1 + \frac{\lambda}{\pi^{1/2}} \Phi_+(-2 e^{-\pi/4} t),
\]

on account of (3.5), (3.8). The Fourier transform \( \Phi_+(w) \) was determined in § 3.1—see (3.14); it was also found that \( \Phi_+(w) \) is regular in the upper half-plane \( \text{Im } w > -\beta \), where \( \beta = 2 |\text{Im } (\log \lambda)^{1/2}| \). For the present purpose we rewrite the solution (3.14) with the path of integration shifted to \( \text{Im } w = a \) where \( a \) is any number such that \( -\beta < a \leq 0 \). Then we have, under an obvious change of variable,

\[
1 + \frac{\lambda}{\pi^{1/2}} \Phi_+(-2 e^{-\pi/4} t) = \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log (1 - \lambda e^{-x^2})}{x - e^{-\pi/4} t} \right], \quad t < -2^{-1/2} a,
\]

or equivalently,

\[
1 + \frac{\lambda}{\pi^{1/2}} \Phi_+(-2 e^{-\pi/4} t) = \left\{ \begin{array}{ll}
\exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log (1 - \lambda e^{-x^2})}{x - e^{-\pi/4} t} \right], & t < 0, \\
(1 - \lambda e^{-it^2})^{-1} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log (1 - \lambda e^{-x^2})}{x - e^{-\pi/4} t} \right], & 0 < t < 2^{-1/2} \beta.
\end{array} \right.
\]

Notice that the inequality \( 0 < t < 2^{-1/2} \beta \) is certainly satisfied when \( t > 0, \ |\lambda| < \exp (-t^2/2) \). For fixed \( t \) the right-hand side of (3.47) is a regular function of \( \lambda \) in the region \( |\lambda| < 1 \). Hence, its Taylor series, that is the series (3.45), will be convergent when \( |\lambda| < 1 \). Thus we obtain the final result (1.11) for the generating function of the integrals \( \mathcal{T}_n(t) \).

Starting from (1.11), we shall express \( \mathcal{T}_n(t) \) in terms of Fresnel integrals \( F \), generally defined by

\[
F(t) = \pi^{-1/2} e^{-\pi/4} e^{-it^2} \int_{-\infty}^{t} e^{is^2} ds.
\]

To that purpose, the exponent in the right-hand side of (1.11) is expanded in a power series in powers of \( \lambda \). Then the coefficient of \( \lambda^n, n = 1, 2, 3, \cdots \), can be reduced to

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-nx^2}}{x - e^{-\pi/4} t} dx = \left\{ \begin{array}{ll}
-F(n^{1/2} t), & t < 0, \\
e^{-int^2} - F(n^{1/2} t), & t > 0,
\end{array} \right.
\]

according to a well-known integral representation for the Fresnel integral. On sub-
stitution of the latter result, we find

$$\sum_{n=0}^{\infty} \lambda^n \mathcal{F}_n(t) = \exp \left[ \sum_{n=1}^{\infty} \frac{\lambda^n}{n} F(n^{1/2}t) \right],$$

valid for $|\lambda| < 1$ and all $t$. Then, by equating the coefficients of corresponding powers of $\lambda$ in (3.50), we have

$$\mathcal{F}_0(t) = 1, \quad \mathcal{F}_1(t) = F(t),$$

(3.51)

$$\mathcal{F}_2(t) = \frac{1}{2} F(2^{1/2}t) + \frac{1}{2} F^2(t),$$

$$\mathcal{F}_3(t) = \frac{1}{2} F(3^{1/2}t) + \frac{1}{2} F(2^{1/2}t) F(t) + \frac{1}{2} F^3(t),$$

and so on. In addition to these explicit results, differentiation of (3.50) with respect to $\lambda$ yields

$$\sum_{n=1}^{\infty} n \lambda^{n-1} \mathcal{F}_n(t) = \sum_{n=0}^{\infty} \lambda^n \mathcal{F}_n(t) \sum_{n=1}^{\infty} \lambda^{n-1} F(n^{1/2}t),$$

(3.52)

from which we derive the simple recurrence relation

$$\mathcal{F}_n(t) = \frac{1}{n} \sum_{m=0}^{n-1} \mathcal{F}_m(t) F((n-m)^{1/2}t), \quad n \geq 1.$$  

(3.53)

It is clear that the integrals $\mathcal{F}_n(t)$ are completely determined by (3.53) and the initial value $\mathcal{F}_0(t) = 1$.

4. Evaluation by probabilistic means.

4.1. $I_{n,0}(1), I_{n,0}(2)$. Let the functions $F_n(t)$, $t$ real, $n = 1, 2, 3, \ldots$, be defined by

$$F_n(t) = \pi^{-n/2} \int_0^{\infty} \cdots \int_0^{\infty} \int_t^{\infty} \exp \left[ -2 \sum_{m=1}^{n-1} x_m^2 + \sum_{m=1}^{n-1} x_m x_{m+1} - x_n^2 \right] dx_1 \cdots dx_n.$$  

(4.1)

Here it is understood that the lower limit $t$ pertains to the integration with respect to $x_n$, all other integrations having lower limits 0. It is easily seen from (1.1) that

$$F_n(0) = I_{n,0}(1), \quad -F'_n(0) = \pi^{-1/2} I_{n-1,0}(2).$$

Consider the exponent in (4.1) which is rewritten as

$$2 \sum_{m=1}^{n-1} x_m^2 - 2 \sum_{m=1}^{n-1} x_m x_{m+1} + x_n^2 = x_1^2 + \sum_{m=2}^{n} (x_m - x_{m-1})^2.$$  

(4.3)

We introduce the new variables

$$y_1 = x_1; \quad y_m = x_m - x_{m-1}, \quad m = 2, 3, \ldots, n,$$

(4.4)

and conversely,

$$x_m = \sum_{j=1}^{m} y_j, \quad m = 1, 2, \ldots, n.$$  

(4.5)

Then (4.1) transforms into

$$F_n(t) = \pi^{-n/2} \int_{D_n} \cdots \int_{D_n} \exp \left[ - \sum_{m=1}^{n} y_m^2 \right] dy_1 \cdots dy_n.$$  

(4.6)
where $D_n$ is an $n$-dimensional domain given by

$$D_n: \sum_{j=1}^{m} y_j \geq 0, \quad m = 1, 2, \ldots, n-1; \quad \sum_{j=1}^{n} y_j \geq t.$$  

The integral (4.6) admits of a simple probabilistic interpretation. Let $y_1, y_2, \cdots, y_n$ be independent random variables with a common normal density function $\pi^{-1/2} \exp (-y_i^2)$, $m = 1, 2, \cdots, n$, and let their partial sums be denoted by $S_m = \sum_{j=1}^{m} y_j$, $m = 1, 2, \cdots, n$. Then $F_n(t)$, as given by (4.6), is equal to the probability $P\{S_1 \geq 0, \cdots, S_{n-1} \geq 0, S_n \geq t\}$.

In particular we now have, from (4.2),

$$I_{n,0}(1) = P\{S_1 \geq 0, S_2 \geq 0, \cdots, S_n \geq 0\},$$  

$$I_{n-1,0}(2) = \pi^{1/2} P\{S_1 \geq 0, \cdots, S_{n-1} \geq 0, S_n = 0\},$$

where $\pi$ denotes the probability density.

The probability (4.8) can be determined by means of the generating function relation

$$1 + \sum_{n=1}^{\infty} P\{S_1 \geq 0, S_2 \geq 0, \cdots, S_n \geq 0\} \lambda^n = \exp \left[ \sum_{n=1}^{\infty} \frac{\lambda^n}{n} P\{S_n \geq 0\} \right], \quad |\lambda| < 1,$$

first proved by Sparre Andersen [1, Thm. 1]; for later, different proofs, see Spitzer [17, p. 330], Feller [10, XII.7]. In the present case one has $P\{S_n \geq 0\} = \frac{1}{2}$ for all $n$, thus leading to

$$1 + \sum_{n=1}^{\infty} I_{n,0}(1) \lambda^n = \exp \left[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \right] = \exp \left[ -\frac{1}{2} \log (1-\lambda) \right] = (1-\lambda)^{-1/2},$$

from which the result (1.5) for $I_{n,0}(1)$ is easily recovered. The same result was also derived by Anis and Lloyd [2], [3]. In fact, these authors were prior to Sparre Andersen [1] in proving (4.10) for the special case when $P\{S_n \geq 0\} = \frac{1}{2}$.

The probability density (4.9) remains the same when all inequalities $\geq$ are replaced by strict inequalities $>$. We now employ a combinatorial result due to Spitzer [17, Thm. 2.1], rephrased as follows for the present purpose: “Let $y = (y_1, y_2, \cdots, y_n)$ be a vector such that $y_1 + y_2 + \cdots + y_n = 0$, but no other partial sum of distinct components vanishes. Let $y_{m+n} = y_m$, and $y(m) = (y_m, y_{m+1}, \cdots, y_{m+n})$, $m = 1, 2, \cdots, n$. Then exactly one of the $n$ cyclic permutations $y(m)$ of $y$ has the property that its successive partial sums are all positive except the last one which vanishes.” Then it is easily seen that

$$I_{n-1,0}(2) = (\pi^{1/2}/n) p\{S_n = 0\} = 1/n^{3/2},$$

since $p\{S_n = t\} = (\pi n)^{-1/2} \exp [-t^2/n]$.

4.2. $J_{n,0}(2), J_{n,0}(1)$. Consider first the integral $J_{n,0}(2)$, as defined by (1.2). Proceeding as in § 4.1, we now introduce the new variables

$$y_1 = x_1; \quad y_m = x_m - x_{m-1}, \quad m = 2, 3, \cdots, n-1; \quad y_n = -x_n - x_{n-1}.$$  

Then the integral $J_{n,0}(2)$ reduces to a form which is readily interpreted as a probability, namely,

$$J_{n,0}(2) = P\{S_1 \geq 0, \cdots, S_{n-1} \geq 0, S_n \leq 0\}$$
with $S_m$ as defined in § 4.1. Compare (4.14) with (4.8), then it is obvious that

\[(4.15) \quad J_{n,0}(2) = I_{n-1,0}(1) - I_{n,0}(1) = \frac{(-1/2)n}{n!},\]

in accordance with (1.6).

Secondly, the integral $J_{n+1,0}(1)$, as defined by (1.2), may be expressed in the form

\[(4.16) \quad J_{n+1,0}(1) = \pi^{-(n+1)/2} \int_{E^{n+1}} \exp \left[ -x_1^2 - 2 \sum_{m=2}^{n} x_m^2 + 2 \sum_{m=1}^{n-1} x_m x_m+1 - 2x_n x_{n+1} - x_{n+1}^2 \right] \prod_{m=1}^{n+1} H(x_m) \, dx_1 \cdots dx_{n+1},\]

where $E^{n+1}$ is the $(n + 1)$-dimensional Euclidean space, and $H(x)$ stands for the unit step function, i.e., $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. Consider the exponent in (4.16) which is rewritten as

\[(4.17) \quad \sum_{m=2}^{n} x_m^2 - \sum_{m=1}^{n-1} x_m x_m+1 + 2x_n x_{n+1} + x_{n+1}^2 = \sum_{m=1}^{n-1} (x_m - x_{m+1})^2 + (x_n + x_{n+1})^2.\]

We now introduce the new variables

\[(4.18) \quad y_1 = -x_n - x_{n+1}; \quad y_m = x_{n+2-m} - x_{n+1-m}, \quad m = 2, 3, \ldots, n; \quad y_{n+1} = -x_{n+1}.

Then, conversely,

\[(4.19) \quad x_{n+1-m} = y_{n+1} - \sum_{j=1}^{m} y_j, \quad m = 1, 2, \ldots, n; \quad x_{n+1} = -y_{n+1},\]

and (4.16) transforms into

\[(4.20) \quad J_{n+1,0}(1) = \pi^{-(n+1)/2} \int_{E^n} \exp \left[ -\sum_{m=1}^{n} y_m^2 \right] \prod_{m=1}^{n} H(y_{n+1} - \sum_{j=1}^{m} y_j) \cdot H(-y_{n+1}) \, dy_1 \cdots dy_{n+1}.

Here, the integration with respect to $y_{n+1}$ can be carried out, thus leading to

\[(4.21) \quad J_{n+1,0}(1) = \pi^{-(n+1)/2} \int_{E^n} \exp \left[ -\sum_{m=1}^{n} y_m^2 \right] g(y) \, dy_1 \cdots dy_n,

where

\[(4.22) \quad g(y) = -\min \left[ \sum_{1 \leq m \leq n} \sum_{j=1}^{m} y_j \right].\]

For a probabilistic interpretation of (4.21), let $y_1, y_2, \ldots, y_n$ be independent random variables with a common normal density function $\pi^{-1/2} \exp (-y_m^2)$, $m = 1, 2, \ldots, n$. Then $J_{n+1,0}(1)$, as given by (4.21), is equal to the expectation $\pi^{-1/2} E(g(y))$. By means of the notations

\[(4.23) \quad S_m = \sum_{j=1}^{m} y_j, \quad T_m = \sum_{j=2}^{m+1} y_j, \quad a^+ = \max [0, a],\]
we reduce (4.22) to
\[ g(y) = -\min \left[ 0, \max_{1 \leq m \leq n} S_m \right] = \max \left[ 0, \max_{1 \leq m \leq n} S_m \right] - \max_{1 \leq m \leq n} S_m \]
(4.24)
\[ = \max_{1 \leq m \leq n} S_m^+ - y_1 - \max_{1 \leq m \leq n-1} T_m^+. \]

Inserting (4.24) into (4.21), we may set
\[ J_{n+1,0}(1) = \pi^{-1/2} E \left( \max_{1 \leq m \leq n} S_m^+ \right) - \pi^{-1/2} E \left( \max_{1 \leq m \leq n-1} S_m^+ \right), \]
(4.25)

since the random variables \( S_m \) and \( T_m \) have the same distribution. The present result can be further reduced by means of the relation
\[ E \left( \max_{1 \leq m \leq n} S_m^+ \right) = \sum_{m=1}^{n} \frac{1}{m} E(S_m^+), \]
(4.26)
quoted from Spitzer [17, p. 330], and originally due to Kac [13, Thm. 4.1]. Thus we obtain as our final result
\[ J_{n+1,0}(1) = \pi^{-1/2} E(S_n^+) = \frac{1}{n \pi n^{3/2}} \int_{0}^{\infty} t e^{-t^2/n} dt = \frac{1}{2 \pi n^{1/2}}. \]

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REFERENCES

