Some remarks on the gap metric

Citation for published version (APA):
Eindhoven: Technische Universiteit Eindhoven.

Document status and date:
Published: 01/01/1988

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
SOME REMARKS ON THE GAP METRIC

S.Q. Zhu and M.L.J. Hautus

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Eindhoven, December 1988
The Netherlands
SOME REMARKS ON THE GAP METRIC

S.Q. Zhu  M.L.J. Hautus
Faculty of mathematics and computing science
Eindhoven university of technology
the Netherlands

Abstract

This paper presents the following results on the gap metric defined on the space of closed linear operators: 1) the topology introduced by the gap metric is a diagonal product topology; 2) if the gap of two closed operators is smaller than one, then the two directed gaps are the same; 3) a normalized right coprime factorization is constructed for given densely defined closed linear operators, and using this result an equivalent form of the gap metric is presented.

KEY WORDS: Closed linear operator, the gap metric, normalized right coprime factorization.

INTRODUCTION

The gap metric was introduced by [Kre.1947] in order to measure perturbations of closed linear operators. It was shown [Cor.1963, Kat.1966, Kra.1972, Zam.1980, Zhu.1987*] that the gap metric is a valuable tool for the perturbation analysis of closed linear operators and related problems. Many properties and some of their applications were collected in [Kat.1966]. In 1980, [Zam.1980] used the gap metric in system and control theory for the analysis of robust stabilization. [Zhu.1987] proved that the topology introduced by the gap metric, restricted to the set of linear time invariant (LTI) systems, which is a subspace of the space of the closed linear operators, has the diagonal product property. It was shown in [Zhu1988] that, if the gap of two LTI systems is smaller than one, then the two directed gaps are the same (for
simplicity, we refer to this property as coincidence property). The diagonal product property of the
gap metric on the set of systems can be extended easily to the space of closed linear operators.
However, we have to give an new procedure to prove that the gap metric defined on the space of the
closed linear operators has the coincidence property. A normalized right coprime factorization is
constructed for a given densely defined closed linear operator. Using this result, we present an
equivalent form of the gap metric. In section 1, the definition of the gap metric is introduced. In
section 2, we present the diagonal product property of the gap metric on the space of closed linear
operators. The coincidence property is proved in section 3. Finally, in section 4, we construct a
normalized right coprime factorization for a given densely defined closed linear operator and we give
an equivalent form of the gap metric.

1 PRELIMINARY We first define the gap metric on the space consisting of all closed subspaces in
a Hilbert space \( H \). Suppose that \( \varphi \) and \( \psi \) are closed subspaces in \( H \) and let \( \Pi(\varphi) \) denotes the
orthogonal projection onto \( \varphi \). Then the gap between \( \varphi \) and \( \psi \) is defined as

\[
\delta(\varphi, \psi) = \max \{ -\delta^{-}\!(\varphi, \psi), -\delta^{-}\!(\psi, \varphi) \}
\]

where \( -\delta^{-}\!(\varphi, \psi) \) is called the directed gap from \( \varphi \) to \( \psi \) and defined as

\[
-\delta^{-}\!(\varphi, \psi) = \sup_{x \in \varphi} \inf_{y \in \psi} \| x - y \| \quad \| x \| = 1
\]

It is easy to see that the directed gap has the following equivalent form

\[
-\delta^{-}\!(\varphi, \psi) = \| (I - \Pi(\psi)) \Pi(\varphi) \|
\]

One can easily prove that \( \delta(\cdot, \cdot) \) is a metric on the space consisting of all closed subspaces in
\( H \).

Let \( X \) and \( Y \) be two Hilbert spaces and denote by \( C(X, Y) \) the space of all closed linear
operators mapping \( X \) to \( Y \). We define the gap metric on the space \( C(X, Y) \). For each element \( P \in
C(X, Y) \), the graph of \( P \) denoted by \( G(P) \) is a closed subspace of \( X \times Y \). For any two elements \( P_1 \) and
\( P_2 \) in \( C(X, Y) \), the gap

\[
\delta(P_1, P_2) \text{ between } P_1 \text{ and } P_2 \text{ is defined as the gap } \delta(G(P_1), G(P_2)) \text{ between their graphs i.e.}
\]
and the directed gap \( \delta^{\rightarrow}(P_1, P_2) \) from \( P_1 \) to \( P_2 \) defined as the directed gap \( \delta^{\rightarrow}(G(P_1), G(P_2)) \) from the graph \( G(P_1) \) of \( P_1 \) to the graph \( G(P_2) \) of \( P_2 \). In the sequel, we denote the orthogonal projection mapping \( X \times Y \) onto \( G(P) \) by \( \Pi(P) \).

2 THE DIAGONAL PRODUCT PROPERTY

In this section, we first introduce two inequalities related to the diagonal form of the closed linear operators. Their proofs can be found in [Zhu.1987]. The diagonal product property, which is our main result in this section, follows directly from the two inequalities.

Let \( X^i \) and \( Y^i \) (\( i=1,2 \)) be Hilbert spaces and define the Hilbert spaces \( X \) and \( Y \) as

\[
X := X^1 \times X^2 \quad \quad Y := Y^1 \times Y^2
\]

Now suppose that \( P_k \in C(X, Y) \) (\( k=1,2 \)) have the following diagonal form

\[
P_k = \begin{bmatrix} P_k^1 & 0 \\ 0 & P_k^2 \end{bmatrix} \quad (k=1,2)
\]

where \( P_k^i \in C(X^i, Y^i) \) (\( i=1,2 \)). Then we have

\[
\delta(P_1, P_2) + \delta(P_2, P_1) \geq \delta(P_1 P_2) \geq \max \{ \delta(P_1^1 P_2^1), \delta(P_1^2 P_2^2) \}
\]

Lemma 2.1 can be proved completely in the same way as the methods presented in [Zhu.1987] for proving the corresponding results for LTI systems.

We introduce a parameter \( \lambda \) which is in a Hausdorff-topological space \( \Gamma \). Assume that, for each \( \lambda \in \Gamma, P_\lambda \in C(X, Y) \) has the following diagonal form

\[
P_\lambda = \begin{bmatrix} P_\lambda^1 & 0 \\ 0 & P_\lambda^2 \end{bmatrix}
\]
where $P_{\lambda}^i \in C(X^i, Y^i)$ (i=1,2). The following result follows from (2.1).

**THEOREM 2.2**

\[
\delta(P_{\lambda}^i, P_{\lambda_0}^i) \rightarrow 0 \quad \lambda \rightarrow \lambda_0
\]

if and only if

\[
\delta(P_{\lambda}^i, P_{\lambda_0}^i) \rightarrow 0 \quad \lambda \rightarrow \lambda_0
\]

holds for both $i=1$ and $i=2$.

It is easy to show that lemma 2.1, and theorem 2.2 are still valid for the case when $P = \text{diag}(P^1, P^2, \ldots, P^l)$. This is the diagonal product property of the gap topology on the space of the linear closed operators. An application of this property can be seen in [Zhu.1987]. Note that the gap topology is not a product topology in the sense that if $P_{\lambda}$ has the following form

\[
P_{\lambda} = \begin{bmatrix}
P_{\lambda}^1 & P_{\lambda}^2 \\
P_{\lambda}^3 & P_{\lambda}^4
\end{bmatrix}
\]

then $\delta(P_{\lambda}, P_{\lambda_0}) \rightarrow 0$ neither implies nor is implied by $\delta(P_{\lambda}^i, P_{\lambda_0}^i) \rightarrow 0$ simultaneously for $i = 1, 2, 3, \text{ and } 4$. This can be seen as follows: In [Zhu. 1987] it was proved that for finite dimensional LTI systems the gap topology is equivalent to the graph topology, and in [Vid. 1985, pp 247] it was shown that the graph topology defined for finite dimensional LTI systems is not a product topology.

### 3 THE COINCIDENCE PROPERTY

In this section, we directly prove that, if the gap between two closed subspaces is smaller than 1, then the two directed gaps are the same.

**THEOREM 3.1** Suppose that $\varphi$ and $\psi$ are closed subspaces in a Hilbert space $H$. If $\delta(\varphi, \psi) < 1$, then $\overrightarrow{\delta}(\varphi, \psi) = \overrightarrow{\delta}(\psi, \varphi)$.

To prove this theorem, we need

**LEMMA 3.2** [Kra.1972, pp206] $\delta(\varphi, \psi) < 1$ if and only if $\Pi(\varphi)$ maps $\psi$ bijectively onto
PROOF OF THEOREM 3.1 Suppose $\delta(\varphi, \psi) < 1$. Define $[\Pi(\psi)]_r$ as the restriction of $\Pi(\psi)$ to $\varphi$. By lemma 3.2, $[\Pi(\psi)]_r$ has a bounded inverse. Accordingly, we have

$$
\delta^{-}(\varphi, \psi)^2 = \| (I - \Pi(\psi)) \Pi(\varphi) \|^2
$$

$$
= \sup_{x \in \varphi} \| (I - \Pi(\psi)) \Pi(\varphi) x \|^2
$$

$$
= \sup_{x \in \varphi} \| x - [\Pi(\psi)]_r x \|^2
$$

$$
= 1 - \inf_{x \in \varphi} \| [\Pi(\psi)]_r x \|^2
$$

$$
= 1 - \| [\Pi(\psi)]_r^{-1} \|^2
$$

Similarly

$$
\delta^{-}(\psi, \varphi)^2 = 1 - \| [\Pi(\varphi)]_r^{-1} \|^2
$$

Since

$$
< [\Pi(\varphi)]_r x - x, y > = < x, [\Pi(\psi)]_r y - y > = 0 \quad \forall x \in \psi \quad \forall y \in \varphi
$$

we have

$$
< [\Pi(\psi)]_r^* x, y > = < x, y > = < x, [\Pi(\psi)]_r y > \quad \forall x \in \psi \quad \forall y \in \varphi
$$

Hence $[\Pi(\psi)]_r^* = [\Pi(\psi)]_r$. Consequently, $([\Pi(\varphi)]_r^{-1})^* = ([\Pi(\psi)]_r^{-1})$ and $\|([\Pi(\varphi)]_r^{-1})\| = \|([\Pi(\psi)]_r^{-1})\|$. This implies that $\delta^{-}(\varphi, \psi) = \delta^{-}(\psi, \varphi)$. Q.E.D.

Since the gap of two closed operators is defined as the gap of their graphs, it is a trivial consequence that, if the gap of two closed operators is smaller than 1, then the two directed gaps are the same.
4 NORMALIZED RIGHT COPRIME FACTORIZATION OF THE CLOSED LINEAR OPERATOR

For a linear operator $P$ mapping Hilbert space $X$ to Hilbert space $Y$, we say $P$ has a right coprime factorization over the bounded linear operators if there exist bounded linear operators $D \in \mathcal{B}(X)$ and $N \in \mathcal{B}(X, Y)$ such that

1) $D$ is invertible;

2) there exist bounded linear operators $F \in \mathcal{B}(X)$ and $H \in \mathcal{B}(Y, X)$ such that

$$FD + HN = I$$

3) $P = ND^{-1}$.

A right coprime factorization $(D, N)$ of $P$ is said to be normalized if

$$D^{*}D + N^{*}N = I.$$ 

In this section we construct a normalized right coprime factorization for an operator $P \in \mathcal{C}(X, Y)$ which has a dense domain. For convenience, let $\mathcal{D}(X, Y)$ denote the subset consisting of all densely defined elements in $\mathcal{C}(X, Y)$.

**THEOREM 4.1** Suppose $P \in \mathcal{D}(X, Y)$, then there exists a bounded linear operator $A$ mapping $X$ onto $\text{G}(P)$ which satisfies

(4.2) \hspace{1cm} A^{*}A = I

(4.3) \hspace{1cm} \Pi(P) = AA^{*}.

If we partition $A$ as

\begin{equation}
(4.4) \hspace{1cm} A = \begin{bmatrix} D \\ N \end{bmatrix}
\end{equation}

such that $D \in \mathcal{B}(X)$ and $N \in \mathcal{B}(X, Y)$, then $(D, N)$ is a normalized right coprime factorization of $P$.

The proof consists of the following lemmas.

**LEMMA 4.2** [Rie.1953, pp. 307–] If $P \in \mathcal{D}(X, Y)$, then $(I + P^{*}P)^{-1}$ (which will be denoted by $R_{P}$) exists as a bounded self-adjoint positive operator mapping $X$ to $\text{Dom}(P)$ bijectively. Moreover, $PR_{P}$ is also bounded and
If $P \in \text{DC}(X, Y)$, then the bounded self-adjoint positive operator $R_P := (I + P^*P)^{-1}$ has a unique bounded self-adjoint positive square root, which we denote by $S_P$, i.e.

$$R_P = S_P S_P^*$$

Moreover, $S_P$ maps $X$ to $\text{Dom}(P)$ bijectively and $PS_P$ is bounded and

$$\|S_P\| \leq 1 \quad \text{and} \quad \|PS_P\| \leq 1.$$

**PROOF OF THEOREM 4.1**

Suppose $P \in C(X, Y)$ and define

$$A := \begin{bmatrix} I & S_P \\ P \end{bmatrix}$$

(4.5)

One can easily check that $A$ satisfies the required conditions: (4.2) follows directly from (4.5) and (4.3) follows from the facts that $(AA^*)^2 = AA^*$, $AA^*$ is self-adjoint, and $\text{Range } AA^* = \text{G}(P)$.

Finally, according to (4.5) the $(D, N)$ in partition (4.4) is $(S_P, PS_P)$, and one can easily show that $(S_P, PS_P)$ is a normalized right coprime factorization of $P$.

**QED**

In the rest of this section, we present an equivalent form of the gap metric using the operator $A$.

**THEOREM 4.4**

Suppose $P_i \in \text{DC}(X, Y)$ and $A_i$ is defined by (4.5) with respect to $P_i$ for $i = 1, 2$. Then $\delta(P_1, P_2) < 1$ if and only if $M := A_1^*A_2$ maps $X$ to $X$ bijectively. Further, if $\delta(P_1, P_2) < 1$, then

$$\delta(P_1, P_2)^2 = 1 - \|M^{-1}\|^{-2}$$

To prove this theorem we need

**LEMMA 4.5**

Let $P \in \text{DC}(X, Y)$ and $B$ be a linear bounded operator mapping $X$ to $\mathbb{R}^X$. Then $B$ maps $X$ bijectively to $\text{G}(P)$ if and only if there exists a unique linear bounded operator $U$ mapping $X$ to $X$ such that $B = A U$, where $A$ is defined by (4.5).

**PROOF**

This is obvious, because $A$ maps $X$ bijectively to $\text{G}(P)$.
Since both $B$ and $A$ map $X$ to $G(P)$ bijectively, for each $x \in X$ there is a unique $y \in X$ such that

\[ (*) \quad B x = A y \]

and vice versa.

By $(* )$ we can define a linear mapping $U$

\[ Ux = y \]

It is obvious that $U$ maps $X$ to $X$ bijectively (hence bounded) and $B = AU$. The uniqueness comes from the bijection of $B$ and $A$. This completes the proof.

**LEMMA 4.6** Assume $P_i (i=1,2) \in DC(X, Y)$ and define the operator $B_1$ as

\[ (4.6) \quad B_1 := \Pi(P_1) A_2 \]

where $A_2$ is defined by (4.5) corresponding to $P_2$. Then $\delta(P_1, P_2) < 1$ if and only if $B_1$ maps $X$ bijectively to $G(P_1)$.

**PROOF** This result follows from lemma 3.2 and the fact, that $\Pi(P_1)$ maps $G(P_2)$ bijectively to $G(P_1)$ if and only if $\Pi(P_1) A_2$ maps $X$ bijectively to $G(P_2)$. Q.E.D.

**PROOF OF THEOREM 4.4** According to (4.3) and (4.6), $B_1 = A_1 A_1^* A_2$. By lemma 4.6, $\delta(P_1, P_2) < 1$ if and only if $B_1$ maps $X$ bijectively to $G(P_1)$. By lemma 4.5, $B_1$ maps $X$ to $G(P_1)$ bijectively if and only if $A_1^* A_2$ maps $X$ to $X$ bijectively. This proves the first part. Suppose that $\delta(P_1, P_2) < 1$, then $M$ is bijective, hence $M^{-1}$ exists as a bounded operator. By definition

\[ \overline{\delta}^2(P_2, P_1) = \| (I - \Pi(P_1)) \Pi(P_2) \|^2 \]

\[ = \sup_{x \in X \times Y \atop \| x \| = 1} \| (I - \Pi(P_1)) x \|^2 \]

\[ = \sup_{x \in \mathcal{G}(P_2) \atop \| x \| = 1} \| (I - \Pi(P_1)) x \|^2 \]

\[ = \sup_{x \in A_2 X \atop \| x \| = 1} \| (I - \Pi(P_1)) x \|^2 \]

\[ (G(P_2) = A_2 X) \]
\[ \sup_{x \in \mathcal{X}, \|x\| = 1} \| (I - \Pi(P_1)) A_2 x \|^2 \quad (\|A_2 x\| = \|x\|) \]

\[ = \sup_{x \in \mathcal{X}, \|x\| = 1} (\| A_2 x \|^2 - \| \Pi(P_1) A_2 x \|^2) \quad (\Pi(P_1) \text{ is orthogonal}) \]

\[ = 1 - \inf_{x \in \mathcal{X}, \|x\| = 1} \| \Pi(P_1) A_2 x \|^2 \]

\[ = 1 - \inf_{x \in \mathcal{X}, \|x\| = 1} \| A_1 A_1^* A_2 x \|^2 \quad (\|A_1 x\| = \|x\|) \]

\[ = 1 - \| (A_1^* A_2)^{-1} \|^{-2} \]

\[ = 1 - \| M^{-1} \|^{-2} \]

According to section 3, the two directed gaps are the same. Consequently, our claim is true. QED.

ACKNOWLEDGEMENT:

The authors would like to thank Drs. A.A. Stoorvogel for his careful reading of the original manuscript.
EFERENCES

[Cor.1963] H.O. Cordes and J.P. Labrousse, The invariance of the index in the metric space of


[Kre.1947] M.G. Krein and M.A. Krasnosel'skii, Fundamental theorems on the extension of
Hermitian operators and some of their applications to the theory of orthogonal

York, 1953.

[Vid.1985] M. Vidyasagar, Control systems synthesis: A factorization approach, M.I.T. Press,

[Zam.1980] G. Zames, A. EL–Sakkary, Unstable systems and feedback: The gap metric, in


stabilization: given by the gap metric, Eindhoven 1987, System and
