A NUMERICAL STUDY OF AN INITIAL VALUE PROBLEM FOR
A SET OF DIFFUSIVE WAVE EQUATIONS

A.P.M. Baayens and M.F.H. Schuurmans

Department of Physics, Technological University of Eindhoven, The Netherlands

1. INTRODUCTION

In this report we study the set of partial differential equations

\[ r_t + r_x = \mu (r_{xx} - s_{xx}), \]  \hspace{1cm} (1)

\[ s_t - s_x = \mu (s_{xx} - r_{xx}), \]  \hspace{1cm} (2)

subject to the initial conditions

\[ r(x,0) = f(x), \]

\[ s(x,0) = 0, \]

where \( x \) runs through \( (-\infty, \infty) \), \( 0 \leq t \leq T < \infty \), \( r \) and \( s \) are real functions of \( x \) and \( t \), \( \mu \) is a (small) positive constant, the subscript \( x \) (or \( t \)) denotes partial differentiation with respect to \( x \) (or \( t \)) and \( f(x) \) is a sufficiently smooth function. The reasons for our interest are, very briefly stated, the following:

If \( \mu \ll 1 \) it seems likely that for some small interval of time the behaviour of the \( r \)-mode can be described with sufficient accuracy in some sense by the solution \( \tilde{r} \) of the so-called Burgers equation

\[ \tilde{r}_t + \tilde{r}_x - \mu \tilde{r}_{xx} = 0, \]

subject to the initial condition

\[ \tilde{r}(x,0) = f(x). \]
However one might wonder whether this would be true for all $t \geq 0$.

An incomplete answer to this problem was given by L.J.F. Broer and the second author [1]. It turned out that, for an interesting class of initial functions $f(x)$ given by

$$f(x) = \begin{cases} x^n \cos (k_0 x) \exp (\delta x), & x \leq 0, \\ 0, & x > 0 \end{cases}$$

($n = 1, 2, \ldots$, $k_0$ and $\delta$ are real positive numbers), the above-mentioned solution $\tilde{r}$, as $t \to \infty$, may be used as a quite satisfactory approximation in the sense that

$$\int_0^\infty \frac{(r - \tilde{r})^2}{dx} \leq K t^{-1} \int_0^\infty |r|^2 \, dx, \quad t \to \infty,$$

where $K$ is a real positive constant.

However, it was not clear at all whether one might speak of an accurate approximation (in some sense) for all times $t \geq 0$. As we only knew the behaviour of $r$ and $s$ for small and large times this was quite a difficult problem. To get some more insight we decided to investigate the behaviour of the solutions $r$ and $s$ by means of a computer. This has been done for the initial-value function

$$f(x) = \begin{cases} x^{6} \exp (6x), & x \leq 0, \\ 0, & x > 0 \end{cases}$$

2. THE MIXED INITIAL AND BOUNDARY VALUE PROBLEM

At first sight it seemed useful to start from the integral representation of $r$ and found in [1]. However, in this attempt a number of problems were met. Both integrands are strongly oscillating functions with a "period" depending not only on $x$ and $t$ but also on the variable of integration, called $z$.

The amplitude of this oscillation varies rapidly with $z$, $x$ and $t$. However, as from a basic point of view the use of a difference method for the set of partial differential equations seemed to be a more interesting one, we did use the latter method.
For the construction of an unconditionally convergent scheme (a precise definition will follow later on) an implicit difference scheme should be used. The latter in fact implies that the pure initial-value problem should be translated into a mixed initial-boundary-value problem, which must be chosen such that it represents in some (as yet undefined) norm the original problem sufficiently well.

Let $D$ be the rectangular region $|x| < a < \infty, 0 < t < T < \infty$ and $C$ its boundary. Then consider the following initial and boundary data

\[
\begin{align*}
    g(x), & \quad -a \leq x \leq -a + \varepsilon, \\
    r(x,0) = \begin{cases} 
        x^6 \exp (6x), & -a + \varepsilon \leq x \leq 0, \\
        0, & 0 < x \leq a,
    \end{cases} \\
    s(x,0) = 0, & \quad |x| \leq a, \\
    r(-a,t) = r(a,t) = s(-a,t) = s(a,t) = 0, & \quad t \geq 0
\end{align*}
\]

(1) in connection with (1.1) and (1.2) and look for the solutions $r$ and $s$ in $D$. As the problem should be well posed (c.f. Richtmeyer [3]), the function $g(x)$ will be chosen such that $r(x,0)$ is a twice continuously differentiable function that goes to zero together with these derivatives as $|x| \rightarrow a$. These conditions are sufficient but certainly not necessary for the well-posedness of the problem. As $\varepsilon$ may be chosen arbitrary small and we are going to use a numerical procedure in which only a few significant digits of a result are of interest, the precise choice of $g(x)$ is not of interest at all.

Finally the question remains to what extent the solution of this problem agrees with that of our original one. The answer is given quite easily. In the final computations we have chosen $T = 20$. It then turns out that $a$ may be chosen equal to 40 because a further increase of $a$ is of no influence on the significant digits of the numerical solution. This has been verified experimentally.
3. **THE METHOD OF SOLUTION**

Cover the domain \( D + C \) by a lattice of discrete points with coordinates \( (x_m, t_n) \) given by

\[
x_m = -a + m h, \quad m = 0, 1, \ldots, M+1,
\]

\[
t_n = n k, \quad n = 0, 1, \ldots, N+1,
\]

where \( h = \frac{2a}{M+1}, \quad k = \frac{T}{N+1} \) are the net spacings.

We shall introduce the notation

\[
u(x_m, t_n) = u_{m,n},
\]

and use the following difference approximations:

\[
u_t(m,n+\frac{1}{2}) = \frac{u_{m,n+1} - u_{m,n}}{k} + O(k^2),
\]

\[
u_x(m,n+\frac{1}{2}) = \frac{u_{m+1,n} - u_{m-1,n} + u_{m+1,n+1} - u_{m-1,n+1}}{4h} + O(h^2),
\]

\[
u_{xx}(m,n+\frac{1}{2}) = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n} + u_{m+1,n+1} - 2u_{m,n+1} + u_{m-1,n+1}}{2h^2} + O(h^2).
\]

Using this so-called Cranck-Nicholson scheme of approximations we find for \( m = 1, 2, \ldots, M; \quad n = 0, 1, \ldots, N+1: \)

\[
a r_{m-1,n+1} + d r_{m,n+1} + c r_{m+1,n+1} + b(s_{m-1,n+1} - 2s_{m,n+1} + s_{m+1,n+1}) =
\]

\[
= - \{a r_{m-1,n} + c r_{m,n} + c r_{m+1,n} + b(s_{m-1,n} - 2s_{m,n} + s_{m+1,n})\}, \quad (1)
\]

\[
b(r_{m-1,n+1} - 2r_{m,n+1} + r_{m+1,n+1}) + c s_{m-1,n+1} + d s_{m,n+1} + a s_{m+1,n+1} =
\]

\[
= - \{b(r_{m-1,n} - 2r_{m,n} + r_{m+1,n}) + c s_{m-1,n} + e s_{m,n} + a s_{m+1,n}\}, \quad (2)
\]

where \( a, b, c, d, e \) are defined in appendix 1.
The initial and boundary data are specified in the obvious way:

\[ r_{m,0} = r(x_m,0), \quad s_{m,0} = s(x_m,0) = 0, \quad m = 0, 1, \ldots, M+1, \]

\[ r_{0,n} = s_{0,n} = r_{M+1,n} = s_{M+1,n} = 0, \quad n = 0, 1, \ldots, N+1. \] (3)

For each \( t = t_n \), equations (1) and (2) form a system of \( 2M \) linear equations in \( 2M+4 \) unknowns \( r_{n,m}, s_{n,m} \). The required additional equations are supplied by (3). So we find

\[ A_{\bar{r}}(n+1) + bB_{\bar{s}}(n+1) = - \left[ C_{\bar{r}}(n) + bB_{\bar{s}}(n) \right], \] (4)

\[ bB_{\bar{r}}(n+1) + A_{\bar{s}}(n+1) = - \left[ bB_{\bar{r}}(n) + C_{\bar{s}}(n) \right], \] (5)

where

\[ r(n) = \text{col.} \left( r_1, n, \ldots, r_M, n \right), \]

\[ s(n) = \text{col.} \left( s_1, n, \ldots, s_M, n \right), \]

and the matrices \( A, B, \) and \( C \) are defined in appendix 1. \( A^T \) is the transposed of \( A \).

Now, at first sight it seems impossible to avoid using the methods of Crout or Jacobi and Seidel (Isaacson and Keller [2], page 51) for solving (4) and (5). However, a more direct method, requiring a smaller number of operations, has been found.

Multiplication of (4) from the left by \( A^T \) and of (5) by \( bB \) and subtraction of the resulting equations gives

\[ C_{\bar{r}}(n+1) = 2bD_{\bar{s}}(n) - P(n+1), \]

\[ P(n+1) = b(a-c) \text{col.} \left( s_1, n+1, 0, \ldots, 0, s_M, n+1 \right), \]

where the matrices \( G, D, \) and \( E \) are defined in appendix 1.

As \( s_1, n+1, s_M, n+1 \) are very small (\( a \) has been chosen such that increasing \( a \) has practically no influence which implies that \( r \) and \( s \) go to zero very smoothly as \( x \to \pm \infty \)), we introduce only a very small error by choosing for some fixed \( n \):
\[ s_{1,n+1} = s_{1,n} \quad , \quad s_{M,n+1} = s_{M,n} \quad . \]

Besides it will turn out that in the neighbourhood of \(|x| = a\) the numerical approximation is not very accurate anyhow (see section 5 too). Using (6) and (7) we find

\[ G_r(n+1) = 2bD_s(n) - E_r(n) - p(n). \quad (8) \]

Multiplication of (4) to the left by \(bb\) and of (5) by \(A\), followed by a subtraction of the resulting equations and an approximation similar to (6) and (7) gives

\[ H_s(n+1) = 2bD_F(n) - F_s(n) + q(n), \quad (9) \]

\[ q(n) = b(a-c) \left[ r_{1,n}, 0, \ldots, 0, r_{M,n} \right], \]

where \(H, D, F\) can be found in appendix 1. By using a triangular decomposition of \(G\) and \(H\), (8) and (9) can be solved easily. The latter requires only \(12M\) operations consisting of multiplication and division, while the operational count for the Crout method requires \(O(M^3)\) (c.f. [2], page 52).

4. CONSISTENCY, CONVERGENCE AND STABILITY

In this section we shall denote the solution of the difference problem by a capital letter and that of the exact problem by a lower case. Let us represent the partial differential equations (1.1) and (1.2) and the boundary and initial data (2.1), (2.2) and (2.3) symbolically by

\[ L u = 0 \quad (x,t) \in D, \quad (1) \]

\[ B u = g(x,t) \quad (x,t) \in C, \quad (2) \]

where

\[ u = \text{col. } (r,s). \]
In a similar way the difference problem may be represented by

\[
L_{\Delta} U = 0 \quad (x,t) \in D, \\
B_{\Delta} U = g(x,t) \quad (x,t) \in C,
\]

where \( B_{\Delta} = B \) and

\[
k \sum_{j=-1,0,1} L_{\Delta} U = \sum_{j=-1,0,1} B_j U(x+jh, t+k) - C_j U(x+jh, t).
\]

The matrices \( B_j \) and \( C_j \) are defined in appendix 1.

For numerical work equations (3) and (4) are used only at the lattice points, but they will be taken to apply equally well to other points of the interval \( |x| \leq a \) such that if \( U(x,t) \) is specified for all \( |x| \leq a \), \( U(x,t+k) \) is determined for \( |x| \leq a \) by equations (3) and (4). Starting from this point of view we are able to use the Hilbert-space \( L_2([-a,a]) \). It contains all square-(Lebesgue) integrable two-component vector-valued functions on \([-a,a]\], with inner product \((,)\) and norm \( || \cdot || \) defined by

\[
(u,v) = \frac{1}{2a} \int_{-a}^{a} u^+(x)v(x)dx; \quad ||u|| = (u,u)^{\frac{1}{2}},
\]

where

\[
u = \text{col.} \ (u_1(x), u_2(x)),
\]

and \( u^+ \) is the hermitian transpose of \( u \).

**Def. 1**

Let \( \phi(t,x) \) be any function with sufficiently many continuous partial derivatives in \( D+C \). For each such function and every point \( (x,t) \in D+C \), define the truncation error by

\[
\tau(\phi(t,x)) = L(\phi(t,x)) - L_{\Delta}(\phi(x,t))
\]

and for every point \( (x,t) \in C \) let the truncation error be

\[
\beta(\phi(t,x)) = B(\phi(t,x)) - B_{\Delta}(\phi(t,x)).
\]

Then the difference problem (3), (4) is unconditionally consistent with problem (1), (2) iff
\[ \tau(\delta) \to 0, \quad \beta(\delta) \to 0, \]
when \( h \to 0, \ k \to 0 \) in any manner.

From a Taylor expansion we deduce that \( \tau = \Theta(h^2 + k^2) \). Furthermore \( \beta = 0 \) and so unconditional consistency is clearly satisfied.

**Def. 2**

The difference scheme (3), (4) is stable iff there exists a constant \( K \), independent of the net spacing, such that

\[ ||U(t=nk)|| \leq K ||U(0)||, \quad n = 0, 1, \ldots, N+1, \]

for any \( U(x,0) \in L_2([-a,a]) \).

To prove stability we shall proceed in the following way. As for all \( 0 \leq t \leq T \) the solution \( U(t,x) \) is zero at the boundaries \( x = a \) and \( x = -a \), we may formally expand \( U \) in a Fourier series:

\[ U(t,x) = \sum_{j=-\infty}^{\infty} V(t,j) \exp \left( \frac{\pi j x}{a} \right), \quad 0 \leq t \leq T. \]

Substituting this in (3) and (4) we find

\[ AV(t+k,j) = BV(t,j), \quad (5) \]

\[ V(j,0) = \frac{1}{2a} \int_{-a}^{a} U(x,0) \exp \left( -\frac{\pi j x}{a} \right) dx, \]

where

\[ A = \begin{bmatrix} \beta + \alpha & -\beta \\ -\beta & \beta + \alpha \end{bmatrix}, \quad B = \begin{bmatrix} \bar{\alpha} - \beta & \beta \\ \beta & \alpha - \beta \end{bmatrix}, \]

\[ \beta = -8 \frac{uk}{h^2} \sin^2 \frac{\pi j h}{2a}, \]

\[ \alpha = -4 - 2i \frac{k}{h} \sin \frac{\pi j h}{a}, \]

and \( \bar{\alpha} \) is the complex conjugate of \( \alpha \).

As

\[ \text{det}(A) = 16 + 64 \mu \lambda \sin^2 \frac{\pi j h}{2a} + 4 \lambda^2 h^2 \sin^2 \frac{\pi j h}{a} > 0, \]

we may conclude from (5) that
\[ Y(t = nk, j) = G^n Y(0, j), \]

where

\[ G = A^{-1} B = (|\alpha|^2 - 8\beta)^{-1} \begin{pmatrix} \pi^2 & \beta(\alpha + \beta) \\ \beta(\alpha + \beta) & \alpha^2 \end{pmatrix}. \]

Usually, \( G \) is called the amplification matrix.

Now using Parseval's theorem we see that

\[ ||U(t)|| \leq \max_j ||G(j, h, k)|| \cdot ||U(0)||, \]

where the norm of the matrix \( G \) is defined by

\[ ||G(j, h, k)|| = \sup_{v \neq 0} \frac{v^+ G v}{v^+ v} \]

and \( G^+ \) is the hermitian transpose of \( G \).

The two eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( G^+ G \) are given by

\[ 0 \leq \lambda_1 = \frac{(|\alpha|^2 + 8\beta)^2}{(|\alpha|^2 - 8\beta)^2} \leq 1, \]

\[ \lambda_2 = 1, \]

and so

\[ ||G|| \leq 1, \]

from which the stability immediately follows.

**Def. 3**

The difference solution is unconditionally convergent to the exact solution iff for any \( u(x, 0) \in L_2([-a, a]) \)

\[ ||U(t, x) - u(t, x)|| \to 0 \]

as \( h \to 0, k \to 0 \) in any manner.
According to Richtmeyer ([3], page 56), our consistency definition implies consistency in the sense of Lax and Richtmeyer. The definitions 2 and 3 are entirely equivalent to those of Lax and Richtmeyer and, as our continuous problem is well posed, we may conclude from Lax's equivalence theorem to the convergence of the solution of the difference scheme to that of the exact problem.

5. SOME EXPERIMENTAL DATA

The computations were done on the EL-X8 computer of the Technological University of Eindhoven, using an Algol-60 program. Stability and convergence of the numerical solution in the sense of the definitions given in the preceding section were confirmed experimentally. To eliminate the influence of the truncation error we used the Romberg-Stiefel extrapolation method. However, as we were limited by the totally "available" computer-time, we could not make both mesh widths as small as we wanted. Some trial runs indicated that the influence of $h$ seems more important than that of $k$. So we decided to use the Romberg-Stiefel method only with respect to $h$ and to hold $k$ fixed, in fact equal to 0.1. The solutions $r$ and $s$ consisted of some wave crests separated and surrounded by valleys of very small amplitude. Comparison of the amplitudes in the wave crests for various values of $h$ showed that the relative error made in choosing $h = 0.05$ varied from about 0.1 to a few per cent (the latter of course depending on where one wants to cut off the wave crest(s)). In fact, down from the top of a wave crest of one of the functions the absolute error only slowly decreases while the function itself mostly decreases quite rapidly. So at the top the relative error is much smaller than far below the top. Use of the Romberg-Stiefel procedure in these areas gave a still better result.

In the valleys, however, the relative error could be considerably larger, up to (if the amplitude was very small) 100 per cent. The cause of this large relative error probably must be found in loss of significant digits. Of course the Romberg-Stiefel procedure was of no use in these areas. Fortunately the solution there is of no interest at all.

In drawing the graphs we have not used the Romberg-Stiefel values but the values of $r$ and $s$ obtained with $h = 0.05$. This was done because the difference between the two values was hardly discernable in the graphs. In drawing the graphs we confined ourselves to the relevant part of the wave crests.
6. ON THE GRAPHS

The graphs themselves (which can be found in appendix 2) hardly need any comment. The development of left- and right moving r and s-waves is clearly demonstrated. We have not been able to carry out the computations beyond \( t = 20 \), because of practical reasons (e.g. available computer-time). Fortunately this is not necessary. The development of the solution when \( t > 20 \) can be accounted for by the asymptotic analysis as given in [1].

First we shall pay attention to the s-mode. From our asymptotic information ([1]) we infer that as \( t \to \infty \) there are only two dominant wave crests having sharp peaks around and extrema along \( x = t \) and \( x = -t \). The first extremum is a maximum, the second one a minimum.

Looking at the numerical solution at \( t = 20 \) we see that two wave crests are situated around \( x = t \) and \( x = -t \). They have the expected signs. But there are two additional crests. By comparing the absolute values of the extrema of the latter with those of the first ones, we found that the wave crests situated nearest \( x = t \) or \( x = -t \) decrease more slowly than the other ones. So we may expect the solution \( s \) to go to the asymptotic solution (in this respect) indeed.

The same situation arises for the other mode \( r \). It is easily seen that in the wave running to the left the minimum becomes dominant over the maximum. Therefore we may expect the numerical solution to go to the asymptotic solution again.

For clarity the situation for \( t \to \infty \) is sketched in the figures below.

\[
\text{fig. 1: the r-mode} \quad \text{fig. 2: the s-mode}
\]
Looking at the graphs in appendix 2 another interesting feature can be noted. For the waves travelling to the right the relation \( s = \frac{\mu}{2} r_x \) seems to be satisfied approximately. For the backward running waves the analogous relation would be \( r = \frac{\mu}{2} s_x \).

Substituting \( s = \frac{\mu}{2} r_x \) in (1.1) and (1.2) gives

\[
\frac{r}{x_{xxx}} = 0
\]

and so

\[
r(x,t) = A(t)x^3 + B(t)x^2 + C(t)x + D(t), \tag{1}
\]

\[
s(x,t) = \frac{\mu}{2} [3 A(t)x^2 + 2 B(t)x + C(t)]. \tag{2}
\]

Using \( r = \frac{\mu}{2} s_x \) we find (1) and (2) but with \( r \) and \( s \) interchanged in the two formulae. Thus in the regions where \( r \) (or \( s \)) can be described (to a certain accuracy) by a polynomial of degree three the relation \( s = \frac{\mu}{2} r_x \) \((r = \frac{\mu}{2} s_x)\) holds with the same degree of accuracy. Looking more precisely we see that these relations are only valid for small intervals of the \( x \)-axis and are not of much practical use. The approximation \( s = \frac{\mu}{2} r_x \) has been used for the first time (so far as we know) by Lighthill [4] in his theory of real gases.

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APPENDIX 1.

A number of "constants" and matrices will be given.

\[ \lambda = \frac{k}{h^2} \]

\[ a = \lambda(h+2\mu) \]

\[ b = -2\mu \lambda \]

\[ c = \lambda(2\mu - h) \]

\[ d = -4(1 + \mu \lambda) \]

\[ e = 4(1 - \mu \lambda) \]

\[
A = \begin{pmatrix}
d & c & 0 \\
ad & c & 0 \\
0 & ad & c \\
0 & 0 & ad
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2 \\
1 & -2 & 1
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
e & c & 0 \\
ea & c & 0 \\
0 & a & c \\
0 & 0 & a & c
\end{pmatrix}
\]

\[
D_1 = \begin{pmatrix}
-8-4\lambda h & 4 & 4 & 0 \\
4 & -8 & 4 & 0 \\
4 & -8 & 4 & 0 \\
0 & 4 & -8-4\lambda h & 0
\end{pmatrix}
\]

\[
D_2 = \begin{pmatrix}
-8-4\lambda h & 4 & 0 \\
-8 & 4 & 0 \\
4 & -8 & 0 \\
4 & -8 & 0 \\
0 & 4 & -8 & 0
\end{pmatrix}
\]
Putting

\[ u = ac - b^2 \]
\[ v = ce + ad + 4b^2 \]
\[ w = c^2 + a^2 + de - 6b^2 \]
\[ x = cd + ae + 4b^2 \]
\[ y = w + b^2 - c^2 \]
\[ z = w + b^2 - a^2 \]
\[ \alpha = cd + ad + 4b^2 \]
\[ \beta = a^2 + d^2 + c^2 - 6b^2 \]
\[ \gamma = \beta + b^2 - c^2 \]
\[ \delta = \beta + b^2 - a^2 \]

and

\[ E = A^T C - b^2 B^2 \]
\[ F = C A^T - b^2 B^2 \]
\[ G = A^T A - b^2 B^2 \]
\[ H = A A^T - b^2 B^2 \]

we find

\[ E = \begin{bmatrix}
    y & x & u & 0 \\
    v & w & x & u \\
    u & v & w & x u \\
    0 & 0 & 0 & 0 \\
\end{bmatrix} \]
\[ F = \begin{bmatrix}
    z & x & u & 0 \\
    v & w & x & u \\
    u & v & w & x u \\
    0 & 0 & 0 & 0 \\
\end{bmatrix} \]
\[ G = \begin{bmatrix}
    \gamma & \alpha & u & 0 \\
    \alpha & \beta & a & u \\
    a & \beta & a & u \\
    0 & 0 & 0 & 0 \\
\end{bmatrix} \]
\[ H = \begin{bmatrix}
    \delta & \alpha & u & 0 \\
    \alpha & \beta & a & u \\
    a & \beta & a & u \\
    0 & 0 & 0 & 0 \\
\end{bmatrix} \]
All mentioned matrices are of dimension MxM.

Finally we define

\[
\begin{align*}
B_{-1} &= -C_{-1} = \\
&= \begin{bmatrix} a & b \\ b & c \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
B_1 &= -C_1 = \\
&= \begin{bmatrix} c & b \\ b & a \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
B_0 &= \\
&= \begin{bmatrix} d & -2b \\ -2b & d \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
C_0 &= \\
&= \begin{bmatrix} e & -2b \\ -2b & e \end{bmatrix}.
\end{align*}
\]

APPENDIX II.

The pictures below still deserve some comment. Along the vertical axis the value of the amplitude of the s-mode (S), the forward-running part - (Rr) or the backward-running part (Rl) of the r-mode has been plotted. At \( t = 0.6 \) these parts can hardly be separated. Therefore we simply wrote R. This has been done for \( t = 0 \) too. Along the horizontal axis we have plotted at \( t = 0 \) the value of \( x \), at all times \( t > 0 \) the value of \( x/t \).
REFERENCES


