SOUND PROPAGATION IN SLOWLY VARYING LINED FLOW DUCTS OF ARBITRARY CROSS SECTION

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Abstract

Sound transmission through ducts of constant cross section with a uniform inviscid mean flow and a constant acoustic lining (impedance wall) is classically described by a modal expansion, where the modes are eigenfunctions of the corresponding Laplace eigenvalue problem along a duct cross section. A natural extension for ducts with cross section and wall impedance that are varying slowly (compared to a typical acoustic wave length and a typical duct radius) in axial direction is a multiple-scales solution. This was previously done for the simpler problem of circular ducts with homentropic irrotational flow. In the present paper, this solution is generalised to the problem of ducts of arbitrary cross section. It is shown that the multiple-scales problem allows an exact solution, given the cross sectional Laplace eigensolutions. The formulation includes both hollow and annular type of geometries. In addition, the turning point analysis is given for a single hard wall cut-on, cut-off transition. This appears to yield the same reflection and transmission coefficients as in the circular duct problem.

1 Introduction

The sound field in a duct of constant cross section with linear boundary conditions that are independent of the axial coordinate may be described by an infinite sum of modes, consisting of the eigenfunctions of the Laplace operator corresponding to a duct cross section. Consider the two-dimensional area $\mathcal{A}$ with a smooth boundary $\partial \mathcal{A}$ and an externally directed unit normal $\mathbf{n}$. For physical relevance $\mathcal{A}$ should be simply connected, otherwise we would have several independent ducts. When we consider, for definiteness, this area be part of the $y, z$-plane, it describes the duct $\mathcal{D}$ given by

$$\mathcal{D} = \{x = (x, y, z) | (0, y, z) \in \mathcal{A}\}$$

with axial cross sections being copies of $\mathcal{A}$ and the normal vectors $\mathbf{n}$ are the same for all $x$. Assume in the duct a field $\phi$ satisfying the reduced wave equation with boundary conditions

$$\nabla^2 \phi + \omega^2 \phi = 0 \quad \text{for } x \in \mathcal{D}, \quad \mathcal{B}(\phi) = 0 \quad \text{for } x \in \partial \mathcal{D},$$

where $\mathcal{B}$ is typically of the form

$$\mathcal{B}(\phi) = a(y, z) \mathbf{n} \cdot \nabla \phi + b(y, z) \phi + c(y, z) \frac{\partial^2 \phi}{\partial x^2}$$

although more derivatives with respect to $x$ would not essentially alter the result.

The solution of this problem may be given by

$$\phi(x, y, z) = \sum_{n=0}^{\infty} C_n \psi_n(y, z) e^{-ik_n x}$$
where \( \psi_n \) are the eigenfunctions of the Laplace operator, i.e., solutions of

\[
(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\psi = -\alpha^2 \psi \quad \text{for} \quad (y, z) \in \mathcal{A}, \quad \text{with} \quad \hat{B}(\psi; \alpha) = 0 \quad \text{for} \quad (y, z) \in \partial \mathcal{A},
\]

and \(-\alpha^2\) is the corresponding eigenvalue. The axial wave number \( k \) is defined through the dispersion relation \( k^2 = \omega^2 - \alpha^2 \) and the reduced boundary condition operator \( \hat{B} \) is given by

\[
\hat{B}(\psi; \alpha) = a(y, z)n \cdot \nabla \psi + b(y, z)\psi - ik(\alpha)c(y, z)\psi.
\]

If the duct cross section is circular or rectangular while the boundary condition is uniform everywhere, the solutions of the eigenvalue problem are relatively simple: combinations of exponentials and Bessel functions in the circular case and combinations of trigonometric functions in the rectangular case. Some other geometries, like ellipses, do also allow explicit solutions. For other geometries one has to fall back on numerical methods for the eigenvalue problem.

Each term in the series expansion, i.e., \( \psi_n(y, z) e^{-i k n \cdot x} \), is called a duct mode. Mathematically, these modes are interesting because (in general) they form a complete basis by which any other solution can be represented. Physically, they are interesting because the usually complicated behaviour of the total field is easier understood via the simple properties of the elements.

If the duct contains a uniform mean flow, the above solution is only little different. A very important application of this kind is the sound propagation in the inlet or exhaust duct of a turbofan aircraft engine. Near the fan, the duct cross section is necessarily circular, but it varies slowly in diameter. By using this slow variation the above analysis may be extended [1], in the case of an isentropic irrotational mean flow, by way of an application of the method of multiple scales or the WKB method. For hard walled ducts, this analysis was extended by Peake and Cooper [5] to varying cross sections of elliptic shape, which is very relevant to engine inlet ducts. In the following we will show how the analysis can be generalised to ducts of arbitrary cross section and arbitrary boundary conditions of impedance type.

The analysis for slowly varying modes in hard-walled circular ducts with mean flow with swirl was done by Cooper and Peake in [6]. It is far more complicated than the one for irrotational flow, as the occurring equations are not self-adjoint. We will not attempt to include that type of mean flow in the present analysis.

## 2 The problem

### 2.1 The equations

In the acoustic realm of a perfect gas that we will consider, we have for pressure \( \bar{p} \), velocity \( \bar{v} \), density \( \bar{\rho} \), entropy \( \bar{s} \), and soundspeed \( \bar{c} \)

\[
\begin{align*}
\frac{\partial}{\partial t} \bar{p} &= -\bar{\rho} \overline{\nabla \cdot \bar{v}}, & \bar{p} \frac{\partial}{\partial t} \bar{v} &= -\nabla \bar{p}, & \frac{\partial}{\partial t} \bar{s} &= 0, \\
\bar{s} &= C_V \log \bar{p} - C_P \log \bar{\rho}, & \bar{c}^2 &= \frac{\gamma \bar{p}}{\bar{\rho}}, & \gamma &= \frac{C_P}{C_V},
\end{align*}
\]

where \( \gamma, C_P \) and \( C_V \) are gas constants. \( C_V \) is the heat capacity at constant volume, \( C_P \) is the heat capacity at constant pressure, and \( \gamma = C_P / C_V \). When we have a stationary mean flow with unsteady time-harmonic perturbations of frequency \( \omega \), given, in the usual complex notation, by

\[
\bar{v} = V + \text{Re}(\nu e^{i \omega t}), \quad \bar{p} = P + \text{Re}(\rho e^{i \omega t}), \quad \bar{\rho} = D + \text{Re}(\rho e^{i \omega t}), \quad \bar{s} = S + \text{Re}(s e^{i \omega t}),
\]

\((\omega > 0)\) and linearize for small amplitude, we obtain for the mean flow

\[
\nabla \cdot (DV) = 0, \quad D(V \cdot \nabla)V = -\nabla P, \quad (V \cdot \nabla)S = 0, \quad S = C_V \log P - C_P \log D, \quad C^2 = \frac{\gamma P}{D}
\]

\[3\]

\[\]
and the perturbations

\[ i\omega \rho + \nabla \cdot (V \rho + vD) = 0 \quad (4a) \]
\[ D(i\omega + V \cdot \nabla)v + D(v \cdot \nabla)V + \rho(V \cdot \nabla)V = -\nabla p \quad (4b) \]
\[ (i\omega + V \cdot \nabla)s + v \cdot \nabla S = 0 \quad (4c) \]

while

\[ s = \frac{C_v}{P} p - \frac{C_p}{D\rho} = \frac{C_v}{P}(p - C^2 \rho) \quad (4d) \]

Assuming that the flow field \( \tilde{v} \) is irrotational and isentropic everywhere (homentropic), we can introduce a potential for the velocity, where \( \tilde{v} = \nabla \tilde{\phi} \) and \( \tilde{\phi} = H + \text{Re}(\phi e^{i\omega t}) \), and express \( \tilde{p} \) as a function of \( \tilde{\phi} \) only, such that we can integrate the momentum equation (Bernoulli’s law, with constant \( E \)), to obtain for the mean flow

\[ \frac{1}{2}V^2 + \frac{C^2}{y - 1} = E, \quad \nabla \cdot (DV) = 0, \quad \frac{P}{Dy} = \text{constant} \quad (5) \]

and for the acoustic perturbations

\[ (i\omega + V \cdot \nabla)\rho + \rho \nabla \cdot V + \nabla \cdot (DV\phi) = 0, \quad D(i\omega + V \cdot \nabla)\phi + p = 0, \quad p = C^2 \rho. \quad (6) \]

These last equations are further simplified (eliminate \( p \) and \( \rho \) and use the fact that \( \nabla \cdot (DV) = 0 \)) to the rather general convected wave equation

\[ D^{-1}\nabla \cdot (DV\phi) - (i\omega + V \cdot \nabla)\left[C^{-2}(i\omega + V \cdot \nabla)\phi\right] = 0. \quad (7) \]

### 2.2 Nondimensionalisation

Without further change of notation, we will assume throughout this paper that the problem is made dimensionless: lengths on a typical duct radius, time on typical sound speed \( l \) typical duct radius, etc.

### 2.3 The geometry

The domain of interest consists of a duct \( V \) of arbitrary cross section, slowly varying in axial direction (see figure 1).

For definiteness, it is given by the function \( S \) in cylindrical coordinates as follows

\[ S(X, r, \theta) = r - R(X, \theta) \leq 0 \quad (8) \]

where \( X = \varepsilon x \geq 0 \) is a so-called slow variable while \( \varepsilon \) is small. A cross section \( A(X) \) at \( X \) has surface area \( A(X) \).

At the duct surface \( S = 0 \) the gradient \( \nabla S \)

\[ \nabla S = -\varepsilon e_x R_X + e_r - e_\theta \frac{1}{R} R_\theta, \quad \text{with} \quad \nabla = e_x \frac{\partial}{\partial x} + e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta}. \quad (9) \]
(where an index denotes a partial derivative) is a vector normal to the surface, so

\[ n = \frac{\nabla S}{|\nabla S|}, \]  

(10)

while the transverse gradient \( \nabla_\perp S \)

\[ \nabla_\perp S = e_r - e_\theta \frac{1}{R} R_\theta, \quad \text{with} \quad \nabla_\perp = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta}, \]  

(11)

is directed in the plane of a cross section \( \mathcal{A}(X) \), and normal to the perimeter \( \partial \mathcal{A} \). So if \( n_\perp \) denotes the component of the surface normal vector \( n \) in the plane of a cross section, we have

\[ n_\perp = \frac{\nabla_\perp S}{|\nabla_\perp S|}. \]  

(12)

Note that

\[ n = n_\perp - \varepsilon \frac{RR_X}{\sqrt{R^2 + R_\theta^2}} e_x + O(\varepsilon^2). \]  

(13)

2.4 Boundary conditions

The duct wall is impermeable to the mean flow, so we have the mean flow boundary condition

\[ V \cdot n = 0 \quad \text{at} \quad S = 0. \]  

(14)

If we denote the mean flow by \( V = U e_x + V_\perp \), with the axial component \( U \) and the cross-wise component \( V_\perp \), the mean flow mass flux, given by

\[ \iint_A DU \, d\sigma = \mathcal{F}, \]  

(15)

is independent of \( x \). The mean flow is assumed to be determined by the slowly varying geometry only. The acoustic boundary condition of an impedance wall along a curved wall with mean flow is, according to Myers [7], given by

\[ i\omega (v \cdot n) = \left[ i\omega + V \cdot \nabla - n \cdot (n \cdot \nabla V) \right] \left( \frac{\rho}{Z} \right) \quad \text{at} \quad S = 0 \]  

(16)

The impedance \( Z \) may vary with position, as long as it varies slowly in \( x \)-direction, so \( Z = Z(X, \theta) \).

3 Mean flow

Since we assumed the mean flow to be determined by the slowly varying geometry only, we write all flow variables as a function of \((X, r, \theta; \varepsilon)\), and expand each in a regular Poincaré expansion in powers of \( \varepsilon^2 \) (the small parameter that appears in the equations). From elementary order of magnitude considerations it follows that \( U = \mathcal{O}(1), V_{\perp,0} = \mathcal{O}(\varepsilon), H = \mathcal{O}(\varepsilon^{-1}), D = \mathcal{O}(1), C = \mathcal{O}(1), \) and \( P = \mathcal{O}(1) \). So we have

\[ H = \varepsilon^{-1} H_0 + \varepsilon H_1 + \mathcal{O}(\varepsilon^3), \quad U = U_0 + \mathcal{O}(\varepsilon^2), \quad V_\perp = \varepsilon V_{\perp,0} + \mathcal{O}(\varepsilon^3), \]  

\[ D = D_0 + \mathcal{O}(\varepsilon^2), \quad P = P_0 + \mathcal{O}(\varepsilon^2), \quad C = C_0 + \mathcal{O}(\varepsilon^2), \]  

(17)

where each term in the expansion is independent of \( \varepsilon \). We substitute these expansions in the equation of mass conservation and in the boundary conditions at \( S = 0 \) and collect the terms of like powers of \( \varepsilon \). Then we get to leading order

\[ \nabla_\perp \cdot (D_0 \nabla_\perp H_0) = 0 \quad \text{with} \quad \nabla_\perp H_0 \cdot n_\perp = 0 \quad \text{at} \quad r = R. \]  

(18)
A solution for $H_0$ is evidently $\nabla \cdot H_0 \equiv 0$, in other words $H_0 = H_0(X)$. Moreover, this solution is indeed unique, as may be seen from the following integral along a cross section $A$

$$
\int_A H_0 \nabla \cdot (D_0 \nabla H_0) \, d\sigma = \int_A \nabla \cdot [(H_0 D_0 \nabla H_0) - D_0 \nabla \cdot H_0]^2 \, d\sigma = 0
$$

So for any $D_0 > 0$, $|\nabla \cdot H_0|^2 \equiv 0$.

From the leading order of Bernoulli’s equation and the relations between $P$, $D$ and $C$

$$\frac{1}{2} U^2 + \frac{C_0^2}{\gamma - 1} = E, \quad C_0^2 = \frac{\nu P_0}{D_0}, \quad \frac{P_0}{D_0^2} = \text{constant} \tag{20}$$

we find that $C_0$ and thus $D_0$ and $P_0$ are also a function of $X$ only. $U_0$ is found from the given mass flux $F$ through a cross section $A$ with surface $A(X)$

$$U_0(X) = \frac{F}{D_0(X) A(X)}, \tag{21}$$

and $D_0$ (and hence $C_0$ and $P_0$) is found as the root of the algebraic equation that results from Bernoulli’s equation

$$\frac{F^2}{2 D_0^2 A^2} + \frac{D_0^{\gamma - 1}}{\gamma - 1} = E. \tag{22}$$

Altogether we have a nearly uniform mean flow

$$V(X, r, \theta; \epsilon) = U_0(X) e_x + \epsilon V_{\perp 0}(X, r, \theta; \epsilon) + \mathcal{O}(\epsilon^2),$$

$$D(X, r, \theta; \epsilon) = D_0(X) + \mathcal{O}(\epsilon^2), \quad C(X, r, \theta; \epsilon) = C_0(X) + \mathcal{O}(\epsilon^2). \tag{23}$$

The cross-wise component of the mean flow is defined by

$$\frac{\partial}{\partial X} (D_0 U_0) + \nabla \cdot (D_0 V_{\perp 0}) = 0 \quad \text{with} \quad V_{\perp 0} \cdot n_\perp = \frac{RRX}{\sqrt{R^2 + R_0^2}} U_0 \quad \text{at} \quad S = 0. \tag{24}$$

However, as it does not appear in the final result, so we will not try to determine it.

We finally note that the operator that occurs in the acoustic boundary condition becomes

$$i \omega + V \cdot \nabla - n \cdot (n \cdot \nabla V) = i \omega + U_0 \nabla X + \epsilon \left( V_{\perp 0} \cdot \nabla - n_\perp \cdot (n_\perp \cdot \nabla V_{\perp 0}) \right) + \mathcal{O}(\epsilon^2). \tag{25}$$

### 4 Acoustic field

The equation for the acoustic field $\phi$ becomes under the above approximation

$$\phi_{xx} + \nabla^2 \phi - C_0^{-2} \left[ -\omega^2 \phi + 2 i \omega U_0 \phi_x + U_0^2 \phi_{xx} \right] + \epsilon \left[ D_0^{-1} D_{0,x} \phi_x - i \omega U(C_0^{-2})_x \phi - U_0(U_0 C_0^{-2})_x \phi_x \right. \left. - 2 i \omega C_0^{-2} (V_{\perp 0} \cdot \nabla \phi) - 2 U_0 C_0^{-2} (V_{\perp 0} \cdot \nabla \phi_x) \right] + \mathcal{O}(\epsilon^2) = 0 \tag{26}$$
The assumption of a multiple scales solution is here equivalent to the WKB-Ansatz:

\[ \phi = \Phi(X, r, \theta; \varepsilon) e^{-i f^\mu_{\varepsilon}(x; \varepsilon) \frac{d \varepsilon}{\varepsilon}} \] (27a)

\[ \phi_x = (-i \mu \Phi + \varepsilon \Phi X) e^{-i f^\mu_{\varepsilon}(x; \varepsilon) \frac{d \varepsilon}{\varepsilon}} \] (27b)

\[ \phi_{xx} = (-\mu^2 \Phi - i \varepsilon \mu X \Phi - 2i \varepsilon \mu \Phi X + \varepsilon^2 \Phi_{XX}) e^{-i f^\mu_{\varepsilon}(x; \varepsilon) \frac{d \varepsilon}{\varepsilon}} \] (27c)

\[ p = -D_0(i \Omega \Phi + \varepsilon U_0 \Phi X + \varepsilon V_{\perp0} \cdot \nabla_{\perp} \Phi) e^{-i f^\mu_{\varepsilon}(x; \varepsilon) \frac{d \varepsilon}{\varepsilon}} \] (27d)

Introduce

\[ \Omega = \omega - \mu U_0, \] (28)

and substitute to obtain after some simplifications

\[ \nabla^2_{\perp} \Phi + \left( \frac{\Omega^2}{C_0^2} - \mu^2 \right) \Phi = \frac{i \varepsilon}{D_0} \Phi \left[ \left( \frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \Phi^2 \right]_X + \nabla_{\perp} \cdot \left( \frac{\Omega D_0}{C_0^2} \Phi^2 \nabla_{\perp0} \right) + \mathcal{O}(\varepsilon^2) \] (29)

where use is made of \(-D_0 U_0 X = \nabla_{\perp} \cdot (D_0 V_{\perp0})\). We obtain for the boundary condition

\[ i \omega \left( n_{\perp} \cdot \nabla_{\perp} \Phi \right) - \frac{\Omega^2 D_0}{Z} \Phi = \varepsilon \omega \mu \frac{RRX}{\sqrt{R^2 + R_0^2}} \Phi \]

\[ - i \varepsilon \left[ U_0 \left( \frac{D_0 \Omega \Phi}{Z} \right)_X + U_0 \frac{D_0 \Omega}{Z} \Phi_X + D_0 \Omega V_{\perp0} \cdot \nabla_{\perp} \left( \frac{\Phi}{Z} \right) + \frac{D_0 \Omega}{Z} V_{\perp0} \cdot \nabla_{\perp} \Phi \right] \]

\[ + i \varepsilon n_{\perp} \cdot (n_{\perp} \cdot \nabla_{\perp} V_{\perp0}) \frac{D_0 \Omega \Phi}{Z} \] (30)

We expand in the usual way

\[ \Phi(X, r, \theta; \varepsilon) = \Phi_0(X, r, \theta) + \varepsilon \Phi_1(X, r, \theta) + \mathcal{O}(\varepsilon^2), \quad \mu(X; \varepsilon) = \mu(X) + \mathcal{O}(\varepsilon^2). \] (31)

where we ignored for notational convenience the subscript of \( \mu \).

To leading order we obtain

\[ \nabla^2_{\perp} \Phi_0 + \left( \frac{\Omega^2}{C_0^2} - \mu^2 \right) \Phi_0 = 0, \quad \text{with} \quad i \omega \left( n_{\perp} \cdot \nabla_{\perp} \Phi_0 \right) - \frac{\Omega^2 D_0}{Z} \Phi_0 = 0 \quad \text{at} \quad r = R. \] (32)

This is formally solved by the solution of the following eigenvalue problem in a cross-sectional plane \( A \), with \( X \) acting as a parameter,

\[ \nabla^2_{\perp} \psi = -\alpha_n^2 \psi, \quad \text{with} \quad (n_{\perp} \cdot \nabla_{\perp} \psi) = \frac{\Omega^2 D_0}{i \omega Z} \psi \quad \text{at} \quad r = R \] (33)

and \( \Omega = \Omega(\alpha) \) because it satisfies the dispersion relation

\[ \frac{\Omega^2}{C_0^2} - \left( \frac{\omega - \Omega}{U_0^2} \right)^2 = \alpha_n^2. \]

We consider the \( n \)-th eigenvalue \(-\alpha_n^2 \) with eigensolution \( \psi_n \). We assume that \( \iint_A \psi_n^2 \, d\sigma \neq 0 \) so that \( \psi_n \) can be normalized as

\[ \iint_A \psi_n^2 \, d\sigma = 1. \] (34)

Then we have

\[ \Phi_0 = N(X) \psi_n(X, r, \theta), \quad \text{while} \quad \mu_n = \frac{\omega - \Omega(\alpha_n)}{U_0}. \] (35)
The amplitude $N$ is still unknown. This will be determined from a solvability condition for the next order $\Phi_1$. We have
\[
v^2_{\perp} \Phi_1 + \alpha_n^2 \Phi_1 = \frac{i}{D_0 \Phi_0} \left[ \left( \frac{\Omega U_0}{C_0} + \mu \right) D_0 \Phi_0^2 \right]_X + \nabla_{\perp} \cdot \left( \frac{\Omega D_0}{C_0} \Phi_0^2 V_{\perp,0} \right).\tag{36}\]
with
\[
io (n_{\perp} \cdot \nabla_{\perp} \Phi_1) - \frac{\Omega^2 D_0}{Z} \Phi_1 = \frac{\alpha_n}{2\sqrt{R^2 + R^2_0}} \Phi_0 \cdot \frac{\nabla_{\perp} \Phi_0}{R} \Phi_0 - \frac{1}{\alpha_n} \left[ U_0 \left( \frac{D_0 \Omega \Phi_0}{Z} \right)_X + U_0 \frac{D_0 \Omega}{Z} \Phi_{0,X} + D_0 \Omega V_{\perp,0} \cdot \nabla_{\perp} \left( \frac{\Phi_0}{Z} \right) + \frac{D_0 \Omega}{Z} \nabla_{\perp,0} \cdot \nabla_{\perp} \Phi_0 \right] + \frac{in_{\perp} \cdot (n_{\perp} \cdot \nabla_{\perp} \Phi_1)}{D_0 \Phi_0}.\tag{37}\]

Multiply equation (36) by $D_0 \Phi_0$ and equation (32) by $D_0 \Phi_1$. Integrate their difference over a cross section $A$ to get
\[
D_0 \int_A \Phi_0 \nabla_{\perp}^2 \Phi_1 - \Phi_1 \nabla_{\perp}^2 \Phi_0 \, d\sigma = i \int_A \left( \frac{\Omega U_0}{C_0} + \mu \right) D_0 \Phi_0^2 \, d\sigma + i \int_A \nabla_{\perp} \cdot \left( \frac{\Omega D_0}{C_0} \Phi_0^2 V_{\perp,0} \right) \, d\sigma.\tag{38}\]
The first integral on the right-hand side may be recast into
\[
\int_A \left( \frac{\Omega U_0}{C_0} + \mu \right) D_0 \Phi_0^2 \, d\sigma = \int_0^{2\pi} \int_0^R \frac{\partial}{\partial X} \left( \frac{\Omega U_0}{C_0} + \mu \right) D_0 \Phi_0^2 \, r \, dr \, d\theta =\]
\[
\frac{\partial}{\partial X} \left( \frac{\Omega U_0}{C_0} + \mu \right) D_0 \int_A \Phi_0^2 \, d\sigma = \frac{\Omega U_0}{C_0} D_0 \int_0^{2\pi} \Phi_0^2 \bigg|_{r=R} \, R X \, d\theta.\tag{39}\]
The second one becomes with equation (24)
\[
\int_A \nabla_{\perp} \cdot \left( \frac{\Omega D_0}{C_0} \Phi_0^2 V_{\perp,0} \right) \, d\sigma = \frac{\Omega D_0}{C_0} \int_{A_Y} \Phi_0^2 (V_{\perp,0} \cdot n_{\perp}) \, d\ell = \frac{\Omega D_0}{C_0} \int_0^{2\pi} \Phi_0^2 \bigg|_{r=R} (V_{\perp,0} \cdot n_{\perp}) \sqrt{R^2 + R^2_0} \, d\theta = \frac{\Omega D_0}{C_0} U_0 \int_0^{2\pi} \Phi_0^2 \bigg|_{r=R} \, R X \, d\theta.\tag{40}\]

Together this yields
\[
D_0 \int_A \Phi_0 \nabla_{\perp}^2 \Phi_1 - \Phi_1 \nabla_{\perp}^2 \Phi_0 \, d\sigma = i \frac{\partial}{\partial X} \left( \frac{\Omega U_0}{C_0} + \mu \right) D_0 \int_A \Phi_0^2 \, d\sigma - i \mu D_0 \int_0^{2\pi} \Phi_0^2 \bigg|_{r=R} \, R X \, d\theta.\tag{41}\]

On the other hand, by using the boundary conditions for $\Phi_0$ and $\Phi_1$ (equations (32) and (37)), this is also equal to:
\[
= D_0 \int_{A_Y} \Phi_0 (n_{\perp} \cdot \nabla \Phi_1) - \Phi_1 (n_{\perp} \cdot \nabla \Phi_0) \, d\ell
\]
\[
= -\frac{D_0}{\omega} \int_{A_Y} U_0 \Phi_0 \left( \frac{D_0 \Omega}{Z} \Phi_0 \right)_X + U_0 \frac{D_0 \Omega}{Z} \Phi_{0,X} + D_0 \Omega \Phi_0 V_{\perp,0} \cdot \nabla \left( \frac{\Phi_0}{Z} \right) + \frac{D_0 \Omega}{Z} \nabla_{\perp,0} \Phi_0 - n_{\perp} \cdot (n_{\perp} \cdot \nabla_{\perp,0}) \frac{\Phi_0^2 D_0 \Omega}{Z} \bigg|_{r=R} \, d\ell - i \mu D_0 \int_0^{2\pi} \Phi_0^2 \bigg|_{r=R} \, R X \, d\theta.\tag{42}\]

Using equation (24) and the fact that $V \cdot n = 0$, we can combine equations (41) and (42) to get
\[
-\io \frac{d}{dX} \left( \frac{\Omega U_0}{C_0} + \mu \right) D_0 N^2 = \int_{A_Y} e^{-1} \mathcal{M} \left( \frac{\Omega D_0}{Z} \Phi_0^2 V \right) \, d\ell + O(\varepsilon^2)\tag{43}\]
where 
\[ \mathcal{M}(F) = \nabla \cdot F - n \cdot (n \cdot \nabla F). \]

Before we continue we need an auxiliary result (see also [4]).

**Theorem 1**
For any sufficiently smooth vectorfield \( f \) with \( f \cdot n = 0 \) at \( r = R \), we have
\[
\int_{\partial A} \nabla \cdot f - n \cdot (n \cdot \nabla f) \, d\ell = \frac{d}{dx} \int_{\partial A} (n \times f) \cdot d\ell.
\]

**Proof.** See Appendix A. \( \Box \)

Since \((n \times V) \cdot d\ell = U_0 \, d\ell + O(\varepsilon^2)\),

we have the result
\[
\int_{\partial A} \varepsilon^{-1} \mathcal{M} \left( \frac{\Omega D_0^2 \Phi_0^2}{Z} V \right) \, d\ell = \frac{d}{dx} \int_{\partial A} \frac{\Omega D_0^2 U_0 \Phi_0^2}{Z} \, d\ell + O(\varepsilon^2) = \frac{d}{dx} \left[ \Omega D_0^2 U_0 N^2 \int_{\partial A} \frac{1}{Z} \psi_n^2 \, d\ell \right]
\]
and so we get altogether the adiabatic invariant
\[
\frac{d}{dx} \left[ \imath \omega \left( \frac{\Omega U_0}{C_0^2} + \mu \right) D_0 N^2 + D_0^2 \Omega U_0 N^2 \int_{\partial A} \frac{1}{Z} \psi_n^2 \, d\ell \right] = 0 \quad (44)
\]

It is convenient to introduce the reduced axial wave number
\[
\sigma = \sqrt{1 - \frac{\alpha^2}{\omega^2}}
\]
so that
\[
\mu = \frac{\omega C_0}{C_0^2 - U_0^2}, \quad D_0 \frac{U_0 \Omega}{C_0^2} + \mu = \frac{\omega C_0}{C_0^2 - U_0^2}, \quad \Omega = \frac{\omega C_0}{C_0^2 - U_0^2}.
\]

This yields finally for the amplitude \( N \)
\[
\frac{Q^2}{N^2} = \frac{\omega \sigma D_0}{C_0} + \frac{D_0^2 \Omega U_0}{\imath \omega} \int_{\partial A} \frac{1}{Z} \psi_n^2 \, d\ell \quad (45)
\]

where \( Q^2 \) is an integration constant. It represents the conserved quantity, and is to be fixed at some position \( X = X_0 \). In the case of an annular duct the analysis is only little different, and we get (with outer perimeter denoted by \( \partial A_2 \) and inner perimeter by \( \partial A_1 \))
\[
\frac{Q^2}{N^2} = \frac{\omega \sigma D_0}{C_0} + \frac{D_0^2 \Omega U_0}{\imath \omega} \left( \int_{\partial A_2} \frac{1}{Z} \psi_n^2 \, d\ell - \int_{\partial A_1} \frac{1}{Z} \psi_n^2 \, d\ell \right). \quad (46)
\]

This is the main result of this paper.

### 5 Applications

#### 5.1 Hard walls
If \( Z \to \infty \), we obtain: \( \frac{\omega \sigma D_0}{C_0} N^2 = Q^2 = \text{constant} \).
5.2 No mean flow

If $U = 0$, $D$ and $C$ are constant and we obtain: $\mu N^2 = \text{constant}$.

5.3 Axi-symmetric duct with constant impedance

If $R = R(X)$, the eigenfunctions are given by $\psi = K(X) J_m(\alpha X r) \left\{ \cos(m \theta) \right\}$ for $m \in \mathbb{N}$. If $m \neq 0$, $K$ is determined by the relation

$$K^2 \int_0^{2\pi} \left\{ \cos^2(m \theta) \right\} d\theta \int_0^R J_m(\alpha r)^2 r dr = K^2 \left( \frac{m^2}{\alpha^2} \right) J_m(\alpha R)^2 + R^2 J_m'(\alpha R)^2 = 1,$$  \hspace{0.5cm} (47a)

and if $m = 0$ by

$$K^2 R^2 [J_0(\alpha R)^2 + J_0'(\alpha R)^2] = 1.$$ \hspace{0.5cm} (47b)

If the pair $e^{im\theta}$ and $e^{-im\theta}$ were taken, instead of $\cos(m \theta)$ and $\sin(m \theta)$, it would not have been possible to normalise $\psi$ in the way assumed here. Finally, the line integral along $\partial A$ is given by

$$\int_{\partial A} \psi^2 d\ell = K^2 J_m(\alpha R)^2 R \pi \left( \ldots 2\pi \text{ if } m = 0 \right).$$ \hspace{0.5cm} (48)

The resulting expression for $N$ is indeed equivalent to what was found in [1].

5.4 Elliptic hard-walled duct

The analysis of [1], restricted to hard walled ducts, was extended by Cooper and Peake [5] to ducts of elliptic cross section. The present solution includes their results, as may seen by comparing their equation (38) (or (36))

$$M_{\sigma^2}^2(X) = \frac{Q_0^2 C_0^2(X)}{(C_0^2 \mu + \Omega D_0) I},$$

with our equation (45) with $Z = \infty$, and noting that we normalized the eigenfunctions such that their integral $I$ becomes equal to unity.

6 Turning point analysis

In the case of hard walls, the above analysis fails when $\sigma \to 0$. So when the medium and diameter vary in such a way that at some point $X = X_t$ wave number $\sigma$ vanishes, the present solution breaks down. In a small interval around $X_t$ the mode does not vary slowly and locally a different approximation is necessary. In the terminology of Matched Asymptotic Expansions [12], this is a boundary layer in variable $X$. The analysis follows closely the circular duct case presented in [3], and we use a similar notation.

![Figure 1: Turning point $X_t$, where a mode changes from cut-on to cut-off.](image)

When $\sigma^2$ changes sign, and $\sigma$ changes from real into imaginary, the mode changes from cut-on to cut-off. If $X_t$ is isolated, such that there are no interfering neighbouring points of vanishing $\sigma$, no power is transmitted beyond $X_t$, and the wave has to reflect at $X_t$. The incident propagating mode is split up into a
cut-on reflected mode and a cut-of transmitted mode. Therefore, a point where wave number \( \sigma \) vanishes is called a “turning point”.

Assume at \( X = X_t \), a transition from cut-on to cut-off, so

\[
\sigma_t = 0, \quad \frac{d}{dx} \sigma_t^2 < 0, \quad \mu_t = 1, \quad \mu'_t > 0, \quad \frac{C_{0t}C_{1t} - U_{0t}U'_{0t}}{C_{0t}^2 - U_{0t}^2} + \frac{\alpha'_t}{\alpha_t} > 0,
\]

where subscript “\( t \)” indicates evaluation at \( X = X_t \).

Consider an incident, reflected and transmitted wave of the type found above. So in \( X < X_t \), where \( \sigma \) is real positive, we have the incident and reflected waves

\[
\phi = \frac{n(X)}{\sqrt{\sigma(X)}} \psi(r, \theta; X) e^{\frac{1}{2} \int_{X_0}^{X} \omega_0 U_0(x') \, dx'} - \frac{1}{2} \int_{X_0}^{X} \omega_0 \sigma_0 \, dx' + \mathcal{R} e^{\frac{1}{2} \int_{X_0}^{X} \omega_0 \sigma_0 \, dx'}
\]

with reflection coefficient \( \mathcal{R} \) to be determined and

\[
n(X) = Q \left( \frac{C_0}{\omega D_0} \right)^{1/2}.
\]

In \( X > X_t \), where \( \sigma \) is imaginary negative, we have the transmitted wave

\[
\phi = T \frac{n(X)}{\sqrt{\sigma(X)}} \psi(r, \theta; X) e^{\frac{1}{2} \int_{X_0}^{X} \omega_0 U_0(x') \, dx'} - \frac{1}{2} \int_{X_0}^{X} \omega_0 \sigma_0 \, dx',
\]

with transmission coefficient \( T \) to be determined, while \( \sqrt{\sigma} = e^{-\frac{1}{2} \pi i} \sqrt{\sigma} \) will be taken.

This set of approximate solutions of equation (7), valid outside the turning point region, constitute the outer solution. Inside the turning point region this approximation breaks down. The approximation is invalid here, because neglected terms of equation (7) are now dominant, and another approximate equation is to be used. This will give us the inner or boundary layer solution. To determine the unknown constants (here: \( \mathcal{R} \) and \( T \)), inner and outer solution are asymptotically matched.

For the matching it is necessary to determine the asymptotic behaviour of the outer solution in the limit \( X \to X_t \), and the boundary layer thickness (i.e., the appropriate local coordinate).

From the limiting behaviour of the outer solution in the turning point region (see below), we can estimate the order of magnitude of the solution. From a balance of terms in the differential equation (7) it transpires that the turning point boundary layer is of thickness \( X - X_t = \mathcal{O}(\varepsilon^{2/3}) \), leading to a boundary layer variable \( \xi \) given by

\[
X = X_t + \varepsilon^{2/3} \lambda^{-1} \xi
\]

where \( \lambda \) is introduced for notational convenience later, and is given by

\[
\lambda^3 = \frac{2\alpha^2 C_{0t}^2}{(C_{0t}^2 - U_{0t}^2)^2} \left( \frac{C_{0t}C_{1t} - U_{0t}U'_{0t}}{C_{0t}^2 - U_{0t}^2} + \frac{\alpha'_t}{\alpha_t} \right)
\]

By assumption is \( \lambda = \mathcal{O}(1) \). Since for \( \varepsilon \to 0 \)

\[
\sigma^2(X) = \sigma^2(X_t + \varepsilon^{2/3} \lambda^{-1} \xi) = -2\varepsilon^{2/3} \left( \frac{C_{0t}C_{1t} - U_{0t}U'_{0t}}{C_{0t}^2 - U_{0t}^2} + \frac{\alpha'_t}{\alpha_t} \right) \lambda^{-1} \xi + \mathcal{O}(\varepsilon^{4/3} \xi^2),
\]

we have

\[
\frac{1}{\varepsilon} \int_{X_t}^{X} \omega_0 \sigma_0 \, dx' = \begin{cases} \frac{2}{3} (-\xi)^{3/2} = -\xi, & \text{if } \xi < 0 \\ -i\frac{4}{9} \xi^{3/2} = -i\xi, & \text{if } \xi > 0 \end{cases}
\]

where we introduced \( \xi = \frac{2}{3} |\xi|^{3/2} \). The limiting behaviour for \( X \to X_t \) is now given by

\[
\phi \simeq \frac{n_t}{\varepsilon^{1/3} (-\xi)^{1/4}} \left( \frac{\omega_0 C_{0t}}{\lambda (C_{0t}^2 - U_{0t}^2)} \right)^{1/2} \psi(r, \theta; X_t) \left( e^{i\xi} + \mathcal{R} e^{-i\xi} \right). \tag{51}
\]
while it is for $X \downarrow X_t$ given by
\[
\phi \simeq T \frac{n_t}{\epsilon^{1/6} \xi^{1/4}} \left( \frac{\omega C_{0t}}{\lambda(C_0^2 - U_{0t}^2)} \right)^{1/2} e^{\frac{1}{2} \pi i} \psi(r, \theta; X_t) e^{-\xi}.
\] (52)

Since the boundary layer is relatively thin, also compared to the radial coordinate, the behaviour of the incident mode remains rather unaffected in radial direction, and we can assume in the turning point region
\[
\phi(x, r, \theta) = \chi(\xi) \psi(r, \theta; X_t) e^{\frac{1}{2} \int_{X_t}^{X} \frac{\omega C_{0t}}{C_0^2} dX'} e^{\frac{1}{2} \pi i}\]
where $X = X_t + \epsilon^{2/3} \lambda^{-1} \xi$ and $\xi = \mathcal{O}(1)$. Substitution in equation (7), and using the defining equation (33) of $\psi$, we arrive at
\[
\epsilon^{2/3} \left( 1 - \frac{U_{0t}^2}{C_0^2} \right) \lambda^2 \psi(r, \theta; X_t) \left( \chi'' - \xi \chi \right) = \mathcal{O}(\epsilon).
\]
So to leading order we have Airy's equation
\[
\frac{d^2 \chi}{d\xi^2} - \xi \chi = 0.
\]
This has the general solution (figure 2)
\[
\chi(\xi) = a \text{Ai}(\xi) + b \text{Bi}(\xi),
\]
where $a$ and $b$, parallel with $R$ and $T$, are to be determined from matching. Using the asymptotic expressions (55a,55b) for Airy functions, we find that for $\xi$ large with $1 \ll \xi \ll \epsilon^{-2/3}$, equation (52) matches the inner solution if
\[
\frac{a}{2 \sqrt{\pi} \xi^{1/4}} e^{-\xi} + \frac{b}{\sqrt{\pi} \xi^{1/4}} e^\xi \sim T \frac{n_t}{\epsilon^{1/6} \xi^{1/4}} e^{\frac{1}{2} \pi i} \left( \frac{\omega C_{0t}}{\lambda(C_0^2 - U_{0t}^2)} \right)^{1/2} e^{-\xi}.
\]
Since $e^\xi \to \infty$, we can only have $b = 0$, and thus
\[
a = \frac{2n_t \sqrt{\pi}}{\epsilon^{1/6}} \left( \frac{\omega C_{0t}}{\lambda(C_0^2 - U_{0t}^2)} \right)^{1/2} e^{\frac{1}{2} \pi i} T
\]
If $-\xi$ is large with $1 \ll -\xi \ll \epsilon^{-2/3}$ we use the asymptotic expression (55a), and find that equation (51) matches the inner solution if
\[
a \frac{\cos(\xi - \frac{1}{4} \pi)}{\sqrt{\pi} (-\xi)^{1/4}} \sim \frac{n_t}{\epsilon^{1/6} (-\xi)^{1/4}} \left( \frac{\omega C_{0t}}{\lambda(C_0^2 - U_{0t}^2)} \right)^{1/2} (e^{i\xi} + \mathcal{R} e^{-i\xi}).
\]
which is equivalent to the following identity in variable $\zeta$

$$
T \ e^{i\zeta} + T \ i \ e^{-i\zeta} \equiv e^{i\zeta} + R \ e^{-i\zeta}.
$$

This is true for any $\zeta$ if

$$
T = 1, \quad R = i.
$$

The amplitudes of these reflection and transmission coefficients could of course be guessed by conservation of energy arguments. This is not the case with the phase. It appears that the wave reflects with a phase change of $\frac{1}{2} \pi$, while the transmission is without phase change.

7 Conclusions

The problem of sound propagation in slowly varying lined ducts of arbitrary cross section with isentropic irrotational mean flow is solved in principle. No attempt has been done yet to illustrate the results with numerical examples, because the corresponding eigenvalue problem in a cross section is not elementary. Still, we made a lot progress because we reduced a hard three-dimensional problem to a much simpler two-dimensional problem. Further work is underway to implement the present results numerically.

The present generalisation gives much insight in previous results for circular and elliptic ducts (which are special cases of the present results), because the form of the solution is seen to become very simple by the used normalisation of the eigenfunctions.

An interesting phenomenon of modes propagating in hard-walled ducts of varying cross section is their change from propagating (cut-on) to exponentially decaying (cut-off) at a so-called turning point. The present multiple scales solution allows the analysis of this turning point behaviour. The results are rather similar to those for the circular duct case. It seems possible to extend our analysis also to the quasi-turning point behaviour in ducts with lined walls reported by Ovenden [14].

A Proof of Theorem 1

For any sufficiently smooth vectorfield $f$ with $f \cdot n = 0$ at $r = R$, we have

$$
\int_{\partial A} \nabla \cdot f - n \cdot (n \cdot \nabla f) \, d\ell = \frac{d}{dx} \int_{\partial A} (n \times f) \cdot d\ell.
$$

**Proof.** Extend the field $n$ to a small environment of the surface $r = R$, such that $n \cdot n = 1$ and $f \cdot n = 0$ while it is continuous differentiable everywhere. We obtain from the vector identity

$$
a \cdot \nabla (b \cdot c) = b \cdot (a \cdot \nabla c) + c \cdot (a \cdot \nabla b)
$$

that

$$
\frac{1}{2} f \cdot \nabla (n \cdot n) = n \cdot (f \cdot \nabla n) = 0.
$$

From the vector identity

$$
\nabla(a \cdot b) = a \cdot \nabla b + b \cdot \nabla a + a \times (\nabla \times b) + b \times (\nabla \times a)
$$

it follows that

$$
\nabla(f \cdot n) = f \cdot \nabla n + n \cdot \nabla f + f \times (\nabla \times n) + n \times (\nabla \times f) = 0.
$$

Take the inner product with $n$, and use the identity

$$
a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)
$$

to obtain

$$
n \cdot (f \cdot \nabla n) + n \cdot (n \cdot \nabla f) + n \cdot (f \times (\nabla \times n)) + n \cdot (n \times (\nabla \times f)) =
n \cdot (n \cdot \nabla f) + (\nabla \times n) \cdot (n \times f) + (\nabla \times f) \cdot (n \times n) = n \cdot (n \cdot \nabla f) - (\nabla \times n) \cdot (f \times n) = 0.
$$
From the vector identity
\[ a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \]
it follows that
\[ n \times (f \times n) = f(n \cdot n) - n(n \cdot f) = f. \]
With
\[ \nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b) \]
it follows from the above results that
\[ \nabla \cdot f = \nabla \cdot (n \times (f \times n)) = (f \times n) \cdot (\nabla \times n) - n \cdot (\nabla \times (f \times n)) = n \cdot (n \cdot \nabla f) + n \cdot (\nabla \times (n \times f)), \]
which yields
\[ \nabla \cdot f = n \cdot (n \cdot \nabla f) = n \cdot (\nabla \times (n \times f)), \]
a version of Möhring’s result found in [9].

Now, following an idea of Eversman [10], we integrate along a strip \( \delta \), wrapped around the duct surface between \( x \) and \( x + \Delta x \), i.e., between perimeters \( \partial A(x) = \partial A_0 \) and \( \partial A(x + \Delta x) = \partial A_\Delta \). Using Stokes’s theorem we get
\[
\iint_{\delta} \nabla \cdot f - n \cdot (n \cdot \nabla f) \, dS = \iint_{\delta} \nabla \times (n \times f) \cdot dS = \int_{\partial A_\Delta} (n \times f) \cdot d\ell - \int_{\partial A_0} (n \times f) \cdot d\ell.
\]
Divide by \( \Delta x \), and the result follows by taking the limit for \( \Delta x \to 0 \) and using continuity of the integrand in \( x \). □

**B Airy functions**

Related to Bessel functions of order \( \frac{1}{4} \) are the Airy functions \( \text{Ai} \) and \( \text{Bi} \), solution of
\[ y'' - xy = 0, \quad (54) \]
with the following asymptotic behaviour (introduce \( \zeta = \frac{2}{3} \sqrt[3]{|x|} \))
\[
\text{Ai}(x) \simeq \begin{cases} 
\frac{\cos(\zeta - \frac{1}{4}\pi)}{\sqrt{\pi} \sqrt[3]{|x|}^{1/4}} & (x \to -\infty), \\
\frac{1}{2} e^{-\zeta} & (x \to \infty),
\end{cases} \quad (55a)
\]
\[
\text{Bi}(x) \simeq \begin{cases} 
\frac{\cos(\zeta + \frac{1}{4}\pi)}{\sqrt{\pi} \sqrt[3]{|x|}^{1/4}} & (x \to -\infty), \\
\frac{1}{\sqrt{\pi} \sqrt[3]{x}^{1/4}} & (x \to \infty).
\end{cases} \quad (55b)
\]

**References**


