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$L^p$-INVERSION OF THE
DIFFUSION EQUATION
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Abstract

The ranges of the propagation operator for the diffusion equation are characterized. Thus, the formal inversion formula for the diffusion equation is made precise in $L^p$ space setting.

Consider the diffusion equation in the space of tempered distributions $S'$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Here the differentiation with respect to $x$ is in the sense of that of tempered distributions, while the differentiation with respect to $t$ is in the topology of $S'$. Since the Fourier transform $F$ and its inverse $F^{-1}$ are continuous on $S'$, equation (1) is equivalently transformed into

$$\frac{\partial (Fu)}{\partial t} = -x^2 F u. \tag{2}$$

From this we get immediately the propagator of the initial value problem of (1):

$$T(t) u = e^{i \frac{x^2}{4t}} u = F^{-1} \left[ e^{-xt^2} F u \right] = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t} \text{ } *_{(x)} u} \tag{3}$$

where $*_{(x)}$ denotes convolution with respect to $x$. One may directly check that the family

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\{T(t) \mid t \geq 0\} of operators defined in (3) is indeed a \(C_0\) semigroup on each of the Banach spaces \(L^p(\mathbb{R})\) (\(1 \leq p < \infty\)). Since \(S\), the Schwartz test space of rapidly decreasing functions, is dense in \(L^p(\mathbb{R})\) and \(T(t)S \subset S\) for all \(t \geq 0\), it serves as a core for the generator \(A_p\) of \{\(T(t) \mid t \geq 0\}\} in \(L^p(\mathbb{R})\). Of course, one can also show straightforwardly that the operator \(\frac{\partial^2}{\partial x^2}\) as defined on \(S\) is an essentially dissipative operator in \(L^p(\mathbb{R})\) and so generates a \(C_0\) semigroup on \(L^p(\mathbb{R})\).

The purpose of this note is to answer the twin questions: Which are the possible states of the system at time \(t\) if the system starts evolution according to equation (1) from initial states in \(L^p(\mathbb{R})\)? What is the initial state in \(L^p(\mathbb{R})\) if the state at time \(t\) is known?

**Proposition 1.** If \(u \in L^p(\mathbb{R})\), then \(v = T(t)u\) extends to an entire function \(v(z) (z = x + iy)\) such that

\[
\|v(x+iy)\|_{p,x} \leq e^{\gamma^2/4t} \|u\|_p. \tag{4}
\]

**Proof.** Define

\[ v(z) = v(x + iy) = \frac{1}{2^{\frac{1}{2}}} e^{-\frac{(x+iy)^2}{4t}} u(x) \]

\[ = \frac{1}{2^{\frac{1}{2}}} e^{\gamma^2/4t} \int_{\mathbb{R}} e^{-\frac{(x-\zeta)^2+2i(x-\zeta)y}{4t}} u(\zeta) d\zeta. \tag{5} \]

Since the integral in (5) converges uniformly on each compact set of \(z\), \(v(z)\) is a well defined entire function. In view of (3) it is an extension of \(v(x) = T(t)u\). From equality (5) by Young' inequality we have immediately (4). Indeed

\[
\frac{1}{2^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{(x-\zeta)^2+2i(x-\zeta)y}{4t}} d\zeta = \frac{1}{2^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\gamma^2/4t} dt = 1. \]

**Definition 2.** Given \(1 \leq p \leq \infty\) and \(s > 0\). Let \(A^{p,s}\) denote the normed space of entire functions \(v(\zeta)\) such that

\[
\|v\|_{p,s} = \sup_{y \in \mathbb{R}} e^{-\gamma^2} \int_{\mathbb{R}} |v(x+iy)|^p dx^{1/p} < \infty. \tag{6}
\]

In the case \(p = \infty\) the above equality should be understood as follows:
Lemma 3. $A^{p,s} \subset A^{\infty,s'}$ for any $1 \leq p < \infty$ and $0 < s < s' < \infty$. Namely $A^{p,s} \subset A^{\infty,s'}$ and there exists a constant $\alpha$ depending only on $p, s$ and $s'$ such that

$$
\sup_{x+iyeC} |v(x+iy)| \leq \alpha \|u\|_{p,s} e^{s'y'^2}.
$$

Proof. Let $R > 0$ be fixed. By the mean value theorem we have

$$
v(x+iy) = \frac{1}{\pi R^2} \int_{|\zeta+i\eta| < R} v[(x+\zeta)+i(y+\eta)] d\zeta d\eta.
$$

So

$$
|v(x+iy)| \leq \frac{1}{\pi R^2} (\pi R^2)^{-p} \int_{|\zeta+i\eta| < R} |v[(x+\zeta)+i(y+\eta)]|^p d\zeta d\eta)^{1/p}
$$

$$
\leq (\pi R^2)^{-1/p} (2R) \sup_{|\eta-y| < R} \left( \int_{|\zeta+i\eta| < R} |v(\zeta+i\eta)|^p d\zeta \right)^{1/p}
$$

$$
\leq (2/\pi R)^{1/p} \|u\|_{p,s} e^{s'(y+R)^2}.
$$

This implies that $A^{p,s} \subset A^{\infty,s'}$ and inequality (8) holds, for $2 |y| R \leq \epsilon y^2 + \epsilon^{-1} R^2$ ($\epsilon > 0$ arbitrary). That is, $A^{p,s} \subset A^{\infty,s'}$. 

Corollary 4. For $1 \leq p \leq \infty$ and $s > 0$ the normed space $A^{p,s}$ is complete so it is a Banach space.

Proof. Let us first show that $A^{\infty,s}$ is complete. Let $\{v_n\}$ be a Cauchy sequence in $A^{\infty,s}$. Then for any $\epsilon > 0$ there exists $N$ such that

$$
e^{-s'y^2} |v_n(x+iy) - v_m(x+iy)| \leq \epsilon \text{ for all } n, m \geq N.
$$

Hence the sequence $\{v_n(x+iy)\}$ of functions converges uniformly on each compact subset of $\mathcal{C}$, so to an entire function $v(x+iy)$. Letting $m \to \infty$ in the last inequality we have

$$
e^{s'y^2} |v_n(x+iy) - v(x+iy)| \leq \epsilon \text{ for } n \geq N.
$$

Therefore $v_n - v$ belongs to $A^{\infty,s}$, so does $v$. Moreover $\{v_n\} \to v$ in $A^{\infty,s}$; $A^{\infty,s}$ is complete.
Next let \( 1 \leq p < \infty \). Let \( \{v_n\} \) be a Cauchy sequence in \( A^{p,s} \). So, for any \( \varepsilon > 0 \) there exists \( N \) such that

\[
e^{-sy^2} \left( \int_{\mathbb{R}} |v_n(x+iy) - v_m(x+iy)|^p \, dx \right)^{1/p} \leq \varepsilon \text{ for all } n, m > N \text{ and } y \in \mathbb{R}.
\]  

(11)

Lemma 3 shows that \( \{v_n\} \) is a Cauchy sequence in \( A^{\infty,s'} \) for \( s' > s \). By the completeness of \( A^{\infty,s'} \) proved above \( \{v_n\} \) converges to an entire function \( v \). Letting \( m \to \infty \) in (11) by Lebesque's dominance convergence theorem we then obtain

\[
e^{-sy^2} \left( \int_{\mathbb{R}} |v_n(x+iy) - v(x+iy)|^p \, dx \right)^{1/p} \leq \varepsilon \text{ for } n > N.
\]

Therefore \( v_n - v \) belongs to \( A^{p,s} \), so does \( v \) and \( \{v_n\} \to v \) in \( A^{p,s} \). We have thus proved that \( A^{p,s} \) is complete.

Proposition 5. Assume that \( v \in A^{p,s} \) \( (1 \leq p < \infty, s > 0) \) and \( t < 1/4s \). Then

(i) The function

\[
u(x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{4t}y^2} v(i\eta) \, d\eta, \quad x \in \mathbb{R}
\]  

(12)

is well defined and is equivalently given by

\[
u(x) = \frac{1}{2\sqrt{\pi t}} \int_{c-i\infty}^{c+i\infty} e\left(\frac{1}{4t}(x-\zeta)^2\right) v(\zeta) \, d\zeta, \quad x \in \mathbb{R}
\]  

(13)

\( (c \in \mathbb{R} \text{ arbitrary}) \), in particular

\[
u(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{4t}\eta^2} v(x+i\eta) \, d\eta.
\]  

(14)

(ii) \( \nu(x) \in L^p(\mathbb{R}) \), and for any \( a \in (s, 1/4t) \) there holds the estimate \( \left( q = p^* = \frac{p}{p-1} \right) \):

\[
\|\nu\|_p \leq \frac{|v|_{p,s}}{2\sqrt{\pi t}} \|e^{-\frac{1}{4t}\eta^2}\|_q \|e^{-(a-\eta)^2}\|_p.
\]  

(15)

(iii) \( T(t)u = e^{\frac{t}{2\tau^s}}u = v \).

Proof:

(i) For any \( s' \in (s, 1/4t) \) Lemma 3 ensures the existence of some constant \( \alpha \) such that inequality
(8) holds. Therefore a function $u$ is well defined by equality (12). Furthermore, by Cauchy’s contour integral theorem we have

$$u(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-\xi)^2} v(\xi) d\xi$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-\xi)^2} v(\xi) d\xi.$$ 

In particular $u$ is given by (14) if $c$ assumes $x$.

(ii) By Hölder’s inequality and Fubini’s theorem we have

$$(2\sqrt{\pi t})^p \int_{-\infty}^{\infty} |u(x)|^p \, dx$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x+i\eta)^2} v(x+i\eta) \, d\eta \right)^p \, dx \quad \text{(by (14))}$$

$$\leq \int_{-\infty}^{\infty} dx \left( \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x+i\eta)^2} d\eta \right)^{p/q} \left( \int_{-\infty}^{\infty} e^{p\eta^2} v(x+i\eta) \, d\eta \right)^q$$

$$= \|e^{-\frac{1}{4t}a\eta^2}\|_q^p \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{p\eta^2} v(x+i\eta) \, d\eta \right)^q \, dx$$

$$\leq \|e^{-\frac{1}{4t}a\eta^2}\|_q^p \|v\|_{L_p}^p \int_{-\infty}^{\infty} e^{-p\eta^2} \, d\eta$$

$$= \|e^{-\frac{1}{4t}a\eta^2}\|_q^p \|v\|_{L_p}^p.$$

Thus $u \in L^p(\mathbb{R})$ and there holds the estimate (15).

(iii) Put $w(x) = e^{-\frac{1}{2t}u(x)}$. Then, by (5) and (14) we have

$$w(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{1}{4t}u(x)} u(x)$$

$$= \frac{1}{4\pi t} \int \int e^{-\frac{(x-\xi)^2}{4t}} \frac{1}{4t} e^{-\frac{1}{4t} \eta^2} v(\xi+i\eta) \, d\xi \, d\eta.$$
\[
= \frac{1}{4\pi t} \int_0^\infty e^{-\frac{r^2}{4t}} r \int_0^{2\pi} v(x+re^{i\theta}) d\theta dr
\]
\[
= \frac{1}{4\pi t} \int_0^\infty 2\pi v(x) r e^{-\frac{r^2}{4t}} dr
\]
\[
= v(x) \quad (x \in \mathbb{R}). \quad (17)
\]

We remark that the form of the above inversion formula (12) was suggested by Widder in [2]. See also [1], Section 5.4.

Definition 6. For \( s \in (0,\infty) \) let \( A^{p,s+} = \bigcup_{\sigma < s} A^{p,\sigma} \) be the inductive limit of the family of Banach spaces \( \{A^{p,\sigma} \mid \sigma < s\} \). For \( s \in [0,\infty) \) let \( A^{p,s-} = \bigcap_{\sigma > s} A^{p,\sigma} \) be the projective limit of the family of Banach spaces \( \{A^{p,\sigma} \mid \sigma > s\} \).

Let \((L^p)_{\exp}(\frac{d^2}{dx^2})\) be the range at time \( t \) of the propagator of the diffusion equation (1) in \( L^p(\mathbb{R}) \), i.e., \( R(T(t)) = R(e^{-\frac{t}{2\sqrt{s}}}) \). With the graph norm it is a Banach space. Moreover, \((L^p)_{\exp}(\frac{d^2}{dx^2}) \to (L^p)_{\exp}(\frac{d^2}{dx^2})\) if \( t > s \). Let \((L^p)_{\exp}(\frac{d^2}{dx^2}) = \bigcup_{\sigma < t} (L^p)_{\exp}(\frac{d^2}{dx^2}) \) be the inductive limit, and \((L^p)_{\exp}(\frac{d^2}{dx^2}) = \bigcap_{\sigma > t} (L^p)_{\exp}(\frac{d^2}{dx^2}) \) be the projective limit.

Summarizing Propositions 1 and 5 we obtain the characterization:

Theorem 7. For \( t \in [0,\infty) \), \((L^p)_{\exp}(\frac{d^2}{dx^2}) = A^{p,(1/4t)+}\) topologically. For \( t \in (0,\infty) \), \((L^p)_{\exp}(\frac{d^2}{dx^2}) = A^{p,(1/4t)-}\) topologically.

Therefore, in an obvious sense we have \((L^p)_{\exp}(\frac{d^2}{dx^2}) \cong A^{p,1/4t}\). In the special case \( p = 2 \), however, we can characterize each of the Hilbert spaces \((L^2)_{\exp}(\frac{d^2}{dx^2})\) exactly.

Theorem 8. \((L^2)_{\exp}(\frac{d^2}{dx^2})\) is isometrically equivalent to the Hilbert space of entire functions \( v \) such that
\[ \| v \|^2 = \frac{1}{\sqrt{2\pi t}} \int_{\mathcal{E}} |v(x + iy)|^2 e^{-iy^2t} \, dx \, dy < \infty. \]

**Proof.** For \( u \in L^2 \) and \( v = e^{i \frac{\partial^2}{\partial x^2}} u \), using Plancherel's theorem we have

\[
\frac{1}{\sqrt{2\pi t}} \int_{\mathcal{E}} |v(x + iy)|^2 e^{-iy^2t} \, dx \, dy \]

\[
= \frac{1}{\sqrt{2\pi t}} \int_{\mathcal{E}} e^{-iy^2t} \, dy \int_{\mathcal{E}} dx \left( \int_{\mathcal{E}} e^{i\lambda k} e^{-\lambda k^2} (F(u)) \, dk \right)^2 \]

\[
= \frac{1}{\sqrt{2\pi t}} \int_{\mathcal{E}} e^{-2\lambda k^2} |F(u)|^2 \left( \int_{\mathcal{E}} e^{-y^2t-2ky} \, dy \right) \, dk \]

\[
= \int_{\mathcal{E}} |F(u)|^2 \, dk \]

\[
= \int_{\mathcal{E}} |u|^2 \, dx. \tag{18} \]

This together with Theorem 3 in [Z-S] completes the proof. \( \square \)
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References


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