Strong normalization in a typed lambda calculus with lambda structured types

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STRONG NORMALIZATION IN
A TYPED LAMBDA CALCULUS WITH
LAMBDA STRUCTURED TYPES

R.P. Nederpelt

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LAMBDA-STRUCTURED TYPES

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CONVENTIONS AS REGARDS REFERENCES

References to literature are denoted by the name(s) of the author(s) and a number in square brackets which is sometimes followed by a further indication (e.g. Church [2, p. 36]). These references are listed at the end of this thesis.

This thesis contains three chapters, which are denoted by capital Roman numerals. Each chapter consists of a number of sections. In referring, for example, to § 3 of Chapter II, we write Section II.3. In each section we number definitions, theorems and notation rules consecutively, indicating the section number in front of the serial number. Thus, in Section II.3 we may propose Theorem 3.5. A reference to this theorem in the pertaining chapter is written as Th. 3.5, in other chapters as Th. II.3.5.

CHAPTER I. INTRODUCTION AND SUMMARY

§ 1. LAMBDA-CALCULUS

The lambda-notation was originally introduced as a useful notation by Church in two papers developing a system of formal logic [2]. He extended this notation in his calculus of lambda-conversion (*lambda-calculus*). This calculus was meant to describe a general class of functions which have the feature that they can be applied to functions of this same class.

For historical comment see Curry and Feys [3, Ch. 0, § D and Ch. 3, § S1] and Barendregt [1, Ch. 1, § 1.1]. In the latter reference the importance of lambda-calculus for the development of recursive functions is mentioned. The calculus has also been brought into relation with the theory of ordinal numbers, predicate calculus and other theories. From the very beginning, lambda-calculus was strongly linked to the theory of combinatory logic.

We shall later mention some major results achieved concerning lambda-calculus. Right here we stress the contribution of lambda-calculus to ordinary mathematics at a purely notational level. The mathematical custom to use the notation $f(x)$, both for the function itself and for the value of this function at an undetermined argument x , obscures the mathematical notion "function". According to Curry and Feys "this defect is especially striking in theories which employ functional operations (functions which admit other functions as arguments)". For an example showing that the usual mathematical function notation is defective not only for understanding, but also in use, see Curry and Feys [3, Ch. 3, § A2].

We shall give an example of the lambda-notation. Consider the function which assigns to x the value $x+2$. This function is denoted in lambda-notation as $\lambda x.x+2$. We can apply the function to an argument, say 3. The application of this function to the argument 3 is denoted as $(\lambda x.x+2)3$. The result of this application must clearly be $3+2$.

This suggests that there exists an order between the terms $(\lambda x.x+2)3$ and $3+2$ (the latter term is "closer to the outcome"). The transitive and reflexive relation corresponding to such an order is called a *reduction*. In the above case it is called a β -reduction, often denoted by $\geq_\beta$. Thus we have the relation $(\lambda x.x+2)3 \geq_\beta 3+2$.

The reduction relation is also *monotonous*, i.e.: if term S reduces to term T, then $\lambda x.S$ reduces to $\lambda x.T$, $(U)S$ to $(U)T$ and $(S)U$ to $(T)U$. So from the relation $(\lambda x.x+2)3 \geq_\beta 3+2$ follows, for example, that $\lambda y.((\lambda x.x+2)3) \geq_\beta \lambda y.(3+2)$.

The relation compares two *terms* (viz. $(\lambda x.x+2)3$ and $3+2$); the fact that these terms have the common value 5 in the usual interpretation, plays no rôle here.

If we do not take 3, but x as argument for the above function, then we obtain $(\lambda x.x+2)x \geq_\beta x+2$. So lambda-calculus makes a clear distinction between the function: $\lambda x.x+2$ and the value of this function for an undetermined argument: $x+2$.

We are used to the fact that the terms $\lambda x.x+2$ and $\lambda y.y+2$ denote the same function. The two terms are called α -*equivalent*, and the passage of the one into the other is called α -reduction, often denoted by $\geq_\alpha$. In this way we also have the relation $\lambda x.x+2 \geq_\alpha \lambda y.y+2$. It is quite a nuisance that this α -reduction, which is simply a renaming of variables, plays a rôle in the lambda-notation. One can avoid this by considering α -equivalence classes instead of separate terms. Another nice and practical way out is given by De Bruijn [8], who completely suppresses the use of names of variables by means of a notational system referring to the positions of a variable in a term.

We wish to state that the desire to eliminate variables is one of the things giving rise to combinatory logic. The method used in combinatory logic to obtain this elimination is, however, different from De Bruijn's.

A third reduction, which is commonly used and strongly related to extensionality (see Barendregt [1, Th. 1.1.17 and Th. 1.1.18]),

is called η -reduction. This relation, commonly denoted by $\geq_\eta$, is based on the following rule: If x is not free in the term M , then $\lambda x \cdot (M)x \geq_\eta M$. An intuitive justification is that, for any argument X , the sides of the relation have comparable values: this value is $(\lambda x \cdot (M)x)X$ for the left-hand side and $(M)X$ for the right-hand side, and $(\lambda x \cdot (M)x)X \geq_\beta (M)X$.

A sequence of reductions obtained by successive application of reductions is called a *reduction sequence*.

For each of the reduction relations explained above, the corresponding symmetric and transitive closure is called a *conversion* relation. One of the first important results in lambda-calculus concerns the dependence between conversion and reduction. This is called the *Church-Rosser theorem*, which states: If X converts to Y , then there is a Z such that X reduces to Z and Y reduces to Z (see Curry and Feys [3, Ch. 4]). For interesting historical comments see Barendregt [1, Th. 1.2.9 and the remarks in 1.2.18 plus footnote]. In Appendix II of the latter reference the latest and nicest proof of the Church-Rosser theorem is given (1971 by W.W. Tait and P. Martin-Löf). For a precise description see Schulte Mönting [24].

In this thesis we shall use the name "*Church-Rosser property*" for the following statement: If A reduces to B and to C , then there is a D such that B and C reduce to D . This property is equivalent to the Church-Rosser theorem.

§ 2. NORMALIZATION AND STRONG NORMALIZATION

An important issue in lambda-calculus is the question of the *normalization* of terms. This is a termination problem. For example, a β -reduction such as $(\lambda x \cdot x)y \geq_\beta y$ cannot be continued in a non-trivial manner: there is no reduction for y , except those trivial on account of the reflexivity of α -, β - and η -reduction. In this case $(\lambda x \cdot x)y$ is said to *normalize* into a normal form y .

In lambda-calculus, which allows all functions as arguments of functions, such a termination of the reduction is not guaranteed. See Church's nice example: $\omega_2 = (\lambda x.xx)(\lambda x.xx)$. There is a non-trivial β -reduction, by applying the rule $(\lambda x.xx)A \geq_{\beta} (A)A$ with $A = \lambda x.xx$. This produces $\omega_2 \geq_{\beta} \omega_2$. It is clear that the reduction of ω_2 by repeated use of the above non-trivial β -reduction will never come to an end.

There are more and stranger examples of such terms, the reduction of which never terminates. For example: put $\omega_3 = \lambda x.xxx$. Then $\omega_3\omega_3 \geq_{\beta} \omega_3\omega_3\omega_3 \geq_{\beta} \dots$. Barendregt even constructed a *universal generator* with the property that it has a reduction sequence in which all terms of lambda-calculus occur as subterms.

A term in lambda-calculus is called *normalizable* if there is some reduction sequence which terminates. A term is *strongly normalizable* if *each* of its reduction sequences terminates. The last term of a terminating sequence is called a *normal form*.

It is obvious that strong normalization implies normalization. The reverse implication does not hold. For example: put again $\omega_2 = \lambda x.xx$, then $(\lambda x.(\lambda y.y))(\omega_2\omega_2)$ reduces to $\lambda y.y$ if the function $\lambda x.(\lambda y.y)$ is applied to the argument $\omega_2\omega_2$, but it reduces to itself if the function ω_2 is applied to the argument ω_2 . Since $\lambda y.y$ is in normal form, $(\lambda x.(\lambda y.y))(\omega_2\omega_2)$ is normalizable, but not strongly normalizable.

In this example we see a term that normalizes if one application of a function to an argument is assigned priority over another. There is a general theorem in lambda-calculus (the *standardization* theorem, cf. Curry and Feys [3, Ch. 4E1]), which states that any normalizable term can be normalized by assigning priority to the "leftmost" application in the term.

The fact that some term in lambda-calculus have non-terminating reduction sequences is related to the feature that one can use all functions as arguments for functions. (Even the function itself can be used as an argument, see the above-mentioned example by Church. This is called *self-application*.)

The same things can happen in programming languages and in the theory of partial recursive functions, where normalization- (or termination-) problems arise too.

In lambda-calculus the question of the normalizability of terms has been shown to be undecidable.

There are systems in which normalization implies strong normalization. For example, in a restricted lambda-calculus (λI -calculus) this implication holds (the so-called second Church-Rosser theorem, see Curry and Feys [3, Ch. 4, § E]), but the proof is not trivial.

Prawitz [17] proved normalization for derivations in natural deduction. He also proved strong normalization for these derivations in [18]. Note that in the latter proof he does not use his results from [17], but quite a different proof technique developed by Tait [21].

An interesting problem concerning normalization is the question of uniqueness of normal forms. If a term A has the property that every terminating reduction sequence leads to the same normal form (but for α -reduction), then A is said to have a *unique* normal form. We note that the Church-Rosser theorem implies the uniqueness of the normal form if this exists.

In this thesis we shall show that, if in a system all terms are normalizable into a unique normal form, then each term is strongly normalizable. This will be proved for a certain lambda-calculus called Λ , the method can, however, be applied to more systems, and we suggest this as a field of further investigation.

§ 3. NORMALIZATION IN SYSTEMS OF TYPED LAMBDA-CALCULUS

In ordinary mathematics one, sometimes tacitly, assumes that each object has a certain type (in our example of a term in lambda-notation: $\lambda x.x+2$, we assumed that x has a type (e.g. that of the natural numbers) in which addition is possible). In systems of typed lambda-calculus one attaches a type to each term. In so doing

and in restricting the formation of terms in accordance with the types (see the "applicability condition" explained in Section I.4) one brings lambda-calculus nearer to usual mathematical systems.

We note here that there is a strong correspondence between derivations in systems of natural deduction and terms in systems of typed lambda-calculus, as well as between formulae in the one and types in the other: a derivation D proving a formula F corresponds to a term D' with type F' . This is called the "formulae-as-type notion".

The latter notion has recently been investigated by various authors in developing a theory of construction and in studying functional interpretations. The first indication in this direction was given in Curry and Feys [3, p. 312-315]. We further mention Läuchli [11], De Bruijn, who developed and applied this notion with a large variety of types in his mathematical language Automath ([4]), Howard [10], Prawitz [18] and Girard [9].

Normalization problems also arise in systems of typed lambda-calculus. Sanchis [19] investigated a lambda-calculus with types (essentially Gödel's theory of functionals of finite type) and found all terms in this calculus to be strongly normalizable.

Martin-Löf [12] admitted more general types and obtained normalization for his terms. His system is close to the requirements of common mathematics in the sense that usual mathematical notions such as the logical connectives and the recursion operator are incorporated.

In this thesis we shall regard a typed lambda-calculus, in which the types themselves have lambda-structure. Our typed lambda-calculus, which we call Λ , has a large overlap with the mathematical language Automath [4]. (See the following section for the relation between Automath and our system Λ .)

In particular, a single-line version of Automath (AUT-SL, see [7]) introduced by De Bruijn has led us to the investigations in this thesis. Preliminary work in the direction of AUT-SL can be found in our notes on Lambda-Automath ([13] and [14]), in which

some syntactical notions of Automath were unified. In AUT-SL this unification was extended considerably.

De Bruijn defined AUT-SL by means of a recursive programme. Our definition of system Λ (given in Chapter III) follows more orthodox recursive lines. Nevertheless, the resulting systems are the same.

In these systems there is no syntactical distinction between terms and types. We therefore use the word expression rather than term or type. There is one basic constant in the system, called τ . To each expression which does not end in τ we shall assign a type in a natural manner.

We say that expressions ending in τ have *degree* 1. Each other expression has some degree $n > 1$, while the degree of such an expression A is defined to be one more than the degree of the type of A . In this manner we have expressions of any finite degree at our disposal.

In Automath and in Martin-Löf's system there is a restriction to the degrees permitted. Both systems have only terms and types of degree 1, 2 or 3.

Our system has in common with Automath that logical connectives, the recursion operator and a basic set of numbers (e.g. natural numbers) are not incorporated. The proofs of normalization results concerning these systems can be formalized in first order arithmetic.

Yet it is possible to interpret into these systems mathematical theories containing, for instance, logical connectives and the recursion operator by introducing new primitive equality relations which extend the existing equality relations which correspond to conversion.

We shall prove normalization and strong normalization for our system in Chapter III. As mentioned above, we shall introduce a method for deriving strong normalization from normalization together with the uniqueness of normal forms (see Section I.6).

§ 4. THE RELATION TO THE MATHEMATICAL LANGUAGE AUTOMATH

Automath (see [4] and [5]) was designed by De Bruijn as a language for mathematics. It has the property that the interpretation of a text written in Automath is correct mathematics if the text is syntactically correct.

Many such systems have been developed for logic. For mathematics, Russell and Whitehead's Principia Mathematica was the first successful attempt in the direction of formalization. There have since been many other attempts.

However, in the majority of these systems important parts of the mathematical argumentation were not incorporated in the formal system, but were dealt with at a meta-level. For example, in systems based on axioms and inference rules a theorem is true if it can be inferred by successive application of a number of axioms and rules. But one hardly ever says exactly (in terms of the formal language) *which* axioms and rules were used, and in which order. Moreover, the use of an axiom scheme was usually not substantiated by a formalized indication of the substitution instance employed.

Admittedly, there is a gap in the completeness of the formalization in Automath, too. The gap is that, in the case of "definitionally equal" expressions, there is no indication of how this equality can be established on the basis of the language definition. It is left to algorithms to justify these definitional equalities.

The existence of terminating algorithms for this purpose can be proved by means of normalization properties. The question of practical efficiency of such algorithms is, of course, a different one, and is not considered in this thesis.

Two expressions in Automath are called *definitionally equal* if one expression can be transferred into the other by (1) conversions and (2) the elimination of abbreviations.

A major problem for automatic checking in Automath is whether definitional equality of two expressions is decidable. The latter is clearly the case if each expression is effectively normalizable into a unique normal form. In this respect, see Kreisel [23].

The main aim of this thesis is to prove the existence and uniqueness of normal forms for Λ . Since Λ does not use an abbreviation system as a syntactical element like Automath does, we may restrict ourselves to conversions. We note that the omission of abbreviations is no severe restriction, since abbreviations are relatively simple operations usually, considered to be only notational devices without mathematical content.

The mere typing of lambda-calculus expressions does not guarantee the property of normalization. We need more.

Automath permits only a restricted class of expressions. In this class only those expressions E are included which obey the so-called *applicability condition*: for each part of E which has the form of a function F applied to an argument A it is required that (1) F has a domain D , and (2) the type of A is definitionally equal to D .

These requirements are natural for a system which is so closely linked to ordinary mathematics. The following examples in lambda-notation will make this clear. In the first place, it would be unnatural to supply an expression which is not a function with an argument: one can attach an argument to $\lambda x.x+2$, but it looks strange to provide the number 7 with an argument.

Secondly, let us assume that x in $\lambda x.x+2$ is required to have the natural numbers as type. This defines the domain of the function. Then one may write the application $(\lambda x.x+2)3$, since 3 has the same type as x . But it would be quite unnatural to write the application $(\lambda x.x+2)\underline{a}$, where $\underline{a}$ represents a vector in R_3 .

In AUT-SL and in Λ , expressions have to obey the applicability condition, like in Automath. This condition is sufficiently strong to guarantee normalizability (even a weaker condition suffices, see Section I.6).

We note that Automath has the property that assignment of a type to an expression of degree 3 is different to that for expressions of degree 2. Expressions of degree 3 have lambda-structured types, whereas expressions of degree 2 all have the same type, viz.

the expression denoted by the underlined symbol type. (This symbol type is the Automath version of the symbol τ used in our system Λ .)

As an illustration we give an example in lambda-notation. Suppose that the term $\lambda x.x+2$ has Nat as type for x , and type as type for Nat. Then $\lambda x.x+2$ has degree 3. In the manner of Automath it has $\lambda x.\text{Nat}$ as type. The latter expression, having degree 2, has as type the expression type.

In AUT-SL and in Λ , however, the assignment of types to expressions of any degree ≥ 2 is treated in a uniform manner, comparable to the assignment of types to expressions of degree 3 in Automath.

If the term in the above example ($\lambda x.x+2$) were treated in the Λ -way, its type would again be $\lambda x.\text{Nat}$, but the type of $\lambda x.\text{Nat}$ would be $\lambda x.\tau$.

We note that an extension of Automath, called AUT-QE ("Automath with quasi-expressions", see [6]), has more expressions of degree 1 than only type; it admits as expressions of degree 1 some of those admitted in AUT-SL and Λ . However, AUT-QE allows a choice to be made for some expressions of degree 2, between essentially different types.

Again using the above example as an illustration: in the manner of AUT-QE one may choose either $\lambda x.\text{type}$ or type as type of $\lambda x.\text{Nat}$.

It is to be noted that the above-mentioned difference between Automath (or AUT-QE) and AUT-SL (or Λ) has the important consequence that neither Automath nor AUT-QE is a subsystem of AUT-SL (or Λ). The results for Λ obtained in this thesis are therefore not immediately transferable either to Automath or to AUT-QE.

Normalization for a simpler form of AUT-QE, which does form a subsystem of AUT-SL, was proved by Van Benthem Jutting [22], using the norm introduced in this thesis (we shall call this norm ρ ; cf. Section I.6). The normalization theorem of this thesis is a generalization of that of [22].

Strong normalization for a system resembling Automath was recently studied by R.C. de Vrijer on the basis of Tait's ideas exposed in [21], and for Automath and AUT-QE by D.T. van Daalen (private communications).

The uniqueness of normal form has only been proved with respect to β -reduction. Uniqueness of normal form with respect to β - η -reduction is as yet an open question (see also Section 1.6).

§ 5. CHANGE OF NOTATIONAL CONVENTIONS

In the lambda-notation as usually employed the quantifiers (such as λx) are written to the left of the expressions they operate upon, whereas applications are written to the right. This corresponds to mathematical notational traditions to write quantifiers (such as $\forall x$, $\sum_{n=1}^{\infty}$, ...) to the left, and to write the argument of a function f to the right (as in $f(a)$).

In ordinary mathematics these two kinds of operations have nothing in common, but in lambda-calculus they are closely related by β - and η -reduction. In a sense, quantification (also called *abstraction*) and application are inverse operations. Sequences of such operations can be applied in various orders, and it is most convenient to write them all on the same side of an expression, thus showing clearly in which order the expression has been formed from its constituents.

In Automath applications and abstractions are all written to the left. Instead of writing abstractions in the form λx , Automath writes $[x, A]$, in which A stands for the type of the variable x . Applications are indicated by writing the expression in braces; instead of the usual mathematical notation $f(a)$ we write $\{a\}f$.

For example: the term given in lambda-notation as $(\lambda x.x+2)3$ reads in Automath as: $\{3\}[x, \text{Nat}] \text{plus}(x, 2)$. (Here we assume that x has as type the natural numbers, abbreviated Nat ; a minor difference is that Automath uses only prefix notation for operators.)

Note that the pair β indicates the possibility of β -reduction. Sometimes, but not always, the pair η indicates the possibility of η -reduction.

This notation for abstraction and application renders the use of parentheses $()$ entirely superfluous, since there can be no doubt as to the order in which abstractions and applications appear. The separation dot as used in $\lambda x.x+2$ disappears as well. Automath uses the parentheses $()$, but for a different purpose.

In AUT-QE, AUT-SL and in the system Λ which we shall develop in this thesis, these slightly different notational conventions are also adopted.

§ 6. SUMMARY OF THE CONTENTS OF THIS THESIS

This thesis contains a chapter on the formal system Δ (Chapter II) and a chapter on the formal system Λ (Chapter III). In the latter chapter we develop the main results of the thesis.

System Λ forms part of system Δ , containing those expressions of Δ which obey the applicability condition (explained in Section I.4).

We shall now discuss the contents of Chapter II. There we define expressions inductively by: x and τ are expressions; $[x,A]B$ and $\{A\}B$ are expressions if A and B are so (x is a variable).

In system Δ we only include those expressions which are "distinctly bound", i.e. (1) which do not contain free variables and (2) which have distinct binding variables.

Our preference for bound (also called closed) expressions (expressions without free variables) is noticeable throughout this thesis. We give the following justification for this preference. We believe that in a typed lambda-calculus the feature of typing can only be meaningful if every typable expression has an effectively computable type. Since free variables have no traceable type in our system, this implies that only bound expressions are admissible. If in this thesis we deviate from this agreement by considering ex-

pressions with free variables, this will be in cases in which it is clear from the context which types belong to these free variables.

The consequence of the above agreement is that many expressions under discussion begin with an abstractor chain Q . (An abstractor chain is a string of abstractors; an abstractor has the form $[x, A]$, A being an expression.)

The fact that we require all binding variables in an expression in Δ to be distinct has only practical reasons (cf. Section II.5).

We stress that system Δ is not a typed lambda-calculus in the usual sense, since the types have no influence whatsoever on the formation of expressions. The types, which themselves have a lambda-structure, will only be treated as formal expressions. It is not until Chapter III, dealing with the restricted system Λ , that the types will play the usual rôle in the formation of expressions. This is due to the applicability condition imposed upon expressions in Λ .

We shall formulate the relations α -, β - and η -reduction inside Δ and we shall prove a number of properties of these reductions in the system Δ (in Sections II.4, II.5 and II.7, respectively).

In Section II.6 we shall consider some reductions related to β -reduction. Our proof of strong normalization in Λ (Section III.3) is based on these reductions. The more important one of these reductions will be called β_1 -reduction.

We shall explain its characteristic property by reducing the term which we previously used as an example: $(\lambda x.x+2)3$, or, in Automath-notation: $\{3\}[x, \text{Nat}] \text{plus}(x, 2)$ (cf. the previous section).

As for β -reduction, we have the relation $\{3\}[x, \text{Nat}] \text{plus}(x, 2) \geq_{\beta} \text{plus}(3, 2)$. But with β_1 -reduction, which we denote by $\geq_{\beta_1}$, we have: $\{3\}[x, \text{Nat}] \text{plus}(x, 2) \geq_{\beta_1} \{3\}[x, \text{Nat}] \text{plus}(3, 2)$. Here the part $\{3\}[x, \text{Nat}]$ is left intact on the right-hand side. (Actually β_1 -reduction is more complicated; see Section II.6.)

The following feature of β_1 -reduction is worth noting: Application of β -reduction sometimes enables one to eliminate a non-

normalizable subterm (in this respect we recall the example $(\lambda x.(\lambda y.y))(\omega_2\omega_2) \geq_{\beta} \lambda y.y$ of Section I.2), but with β_1 -reduction this is impossible.

In Section II.6 we prove the Church-Rosser property for β_1 -reductions, using a proof technique of Tait and Martin-Löf. This property implies the uniqueness of normal form for β_1 -reductions. From the Church-Rosser property for β_1 -reductions follows the Church-Rosser property for β -reductions (although the latter could also be proved directly).

Unfortunately the Church-Rosser property for β - η -reductions does *not* hold in our system Δ . The trouble here arises from the typed character of our lambda-calculus. We explain this in greater detail in Section II.7. (However, we conjecture the Church-Rosser property for β - η -reductions in Λ ; see the end of the present section.)

In Section II.7 we also prove a theorem concerning the "postponement of η -reductions" in a sequence of β - and η -reductions, by means of a method suggested by Barendregt.

At the end of Section II.7 we define lambda-equivalence for Δ : A and B are lambda-equivalent if there is a C such that A and B reduce to C. This lambda-equivalence is not necessarily transitive since the Church-Rosser property for β - η -reductions does not hold in Δ .

In Section II.8 we define a formal type-operator called Typ , which assigns a type to an expression not ending in τ . The action of this type-operator is syntactically simple and is in agreement with what we mentioned about the assignment of types in Section I.3.

In Section II.8 we also define the degree-function Deg , which is in agreement with our description of degree as given in Section I.3. In our system Δ we can apply the type-operator Typ a finite number of times.

For each expression A in Δ there is an $n \geq 0$ such that $\text{Typ}^n A$ ends in τ , which implies that $\text{Typ}^n A$ has no type. (Here $\text{Typ}^n A$ is obtained by n applications of the type-operator.) This n is the

degree of A minus one. We define $\text{Typ}^* A$ to be $\text{Typ}^n A$ for that particular n .

We begin Chapter III with the definition of the formal system Λ (Section III.1). Among the theorems in Section III.1 there is one which states that the type of an expression in Λ again belongs to Λ .

In Section III.2 we prove the normalization theorem for Λ . We use a norm ρ , which is a partial function on Δ . The norm $\rho(A)$ for a certain A in Δ is itself an expression in Δ . The norm of A is defined if A obeys a weak form of the applicability condition, which amounts to the following: for each part of an expression E which has the form of a function F applied to an argument A : (1) F has a domain D , and (2) the norms of A and D are defined and equal (apart from α -reduction).

Applied to expressions for which the norm is defined (so-called ρ -normable expressions), the norm ρ has two powerful properties: (1) If A reduces to B , the norms of A and B are (essentially) equal, and (2) the norm of an expression is (essentially) the same as the norm of its type.

The norm $\rho(A)$ of a ρ -normable expression A can be obtained by (1) replacing non-binding variables by their types, repeating this process until no non-binding variable remains, and (2) cancelling adjacent pairs $\{C\}[x, D]$.

We show in Section III.2 that all expressions in Λ are ρ -normable. We subsequently show that each ρ -normable expression has a normal form for β -reductions. It follows in particular that Λ is normalizable for β -reductions. It now easily follows that Λ is also normalizable for β - η -reductions. Our proofs show that the normal form of A in Λ is *effectively* (viz. primitively recursively) computable.

In Section III.3 we prove strong normalization for Λ . We use the β_1 -reduction introduced in Section II.6. We show that expressions in Λ are normalizable for β_1 -reductions, using the same methods as in the corresponding proof for β -reductions in Section III.2. By using the Church-Rosser property for β_1 -reductions as proved in

Section II.6 we obtain the uniqueness of normal form for β_1 -reductions.

The special features of β_1 -reduction enable us to conclude *strong* normalization in Λ for β_1 -reductions from the normalization and the uniqueness of normal form. Strong normalization in Λ for β -reductions is a consequence, as well as strong normalization for β - η -reductions.

The uniqueness of normal form in Λ is proved for β -reductions but not for β - η -reductions. Nevertheless, the latter would be a consequence if we could prove the following conjecture:

Conjecture I. In Λ the Church-Rosser property holds for β - η -reductions.

(The difficulties in proving this arise in the same place where the corresponding statement for Δ turns out to be false; see Section II.7.)

As to Λ , there is an important conjecture on closure:

Conjecture II. If A is an expression in Λ and if A reduces to B , then B is an expression in Λ .

In [15] we stated this as a theorem, but the proof turned out to be incorrect. The latter conjecture has no influence upon the results in this thesis; it is, however, of importance for the construction of an efficient checking-algorithm for expressions in Λ .

CHAPTER II. THE FORMAL SYSTEM Δ

§ 1. ALPHABET AND SYNTACTICAL VARIABLES

We use the following symbols as our *alphabet*:

(i) an infinite set of (individual) *variables*:

$\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, \dots$;

(ii) a single constant, called the *base*: τ ;

(iii) the *improper symbols*: $[,] , \{ , \} , . .$

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As syntactical variables denoting certain well-structured symbol strings (possibly empty) we use small Latin letters $a, b, c, \dots$ and Latin capitals $A, B, C, \dots$ (primed or subscripted if required). In special definitions, called Notation Rules, we restrict the use of some syntactical variables (and its primed or subscripted variants).

For example, we agree upon:

Notation Rule 1.1. As a syntactical variable for arbitrary strings of symbols from the alphabet we use the Latin capital S . $\square$

Such a string can be empty. The empty string itself is denoted by ϕ .

Notation Rule 1.2. As syntactical variables for individual variables we use the small Latin letters x, y and z . $\square$

(Instead of "individual variable" we often say "variable".)

Hence from now on each use of a syntactical variable S (or S_1, S' , etc.) denotes a string of symbols from the alphabet, and each use of a syntactical variable x (or y, x_1 , etc.) denotes an individual variable.

It is usual to build strings of symbols from the alphabet and syntactical variables, concatenated. For example, $[x, \alpha][y, \alpha]x$ is such a string. We shall call this kind of string a *mixed string*.

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Equality of mixed strings will be expressed in the discussion language by the symbol $\equiv$. For example, if we wish to express that the strings S and $[x, \alpha]\beta$ are the same, we write $S \equiv [x, \alpha]\beta$. The symbol $\neq$ is the negation of $\equiv$.

There are said to be two "occurrences" of α in the string $[\alpha, \beta]\alpha$. We shall formalize this notion *occurrence*. We define that S "occurs in S after S' " if there is an S'' such that $S \equiv S'S''S''$. Hence, in the above example: α occurs in $[\alpha, \beta]\alpha$ after $[$, and α occurs in $[\alpha, \beta]\alpha$ after $[\alpha, \beta]$. In this manner we can distinguish between occurrences.

$$S \equiv S'S_1S'' \equiv S'S''S_2S''S''$$

The following statement is clear: If S_1 occurs in S after S' and S_2 occurs in S_1 after S'' , then S_2 occurs in S after $S'S''$.

Consider the mixed string $[x, y]x$, in which there are two occurrences of x . If x denotes α , then $[x, y]x$ denotes $[\alpha, y]\alpha$: both occurrences of x are replaced by α . If, moreover, y denotes β , then $[x, y]x$ denotes $[\alpha, \beta]\alpha$. It is, however, also possible that both x and y denote α . Then $[x, y]x$ denotes $[\alpha, \alpha]\alpha$ (see also Schoenfield [20, p. 7]).

Syntactical variables are used in two hardly distinguishable rôles: as abbreviations ("We abbreviate $[x, \alpha]\beta$ as S ") and as variables ("Let S be a string of the form ..."). It is also good usage to state something in the nature of: "Let $A \equiv [x, B]C$ ", meaning: "Assume that A has the form $[x, B]C$ for certain x, B and C " (in this manner one economizes in the use of the existential quantifier).

We shall define many specific sets and relations in an *inductive* manner (see Schoenfield [20, p. 4]). The proof technique linked with this kind of definition, which amounts to induction on the construction, is often called (somewhat confusingly) *induction on the length of proof* (or induction on theorems, see Schoenfield [20, p. 5]).

We shall call an application of one rule of the inductive definition a *derivation-step*. If a relation is defined inductively by a number of rules, then the relation is also said to be *generated* by these rules. When speaking of a transitive (or reflexive, etc.) relation generated by a number of rules, one wishes to express that

the rule of transitivity (or reflexivity, etc.) is to be added to that number of rules.

If S denotes a certain symbol string, then the *length* of S is the number of symbols in that string. We denote the length of S by $|S|$. For example, if $S \equiv [\alpha, \beta]\alpha$, then $|S| = 6$.

§ 2. EXPRESSIONS

The expressions of our systems are inductively defined as follows (we use the word expression rather than the words term or type):

Definition 2.1.

- (1) A variable is an expression.
- (2) τ is an expression.
- (3) If x is a variable and if A and B are expressions, then $[x, A]B$ is an expression.
- (4) If A and B are expressions, then $\{A\}B$ is an expression. □

Note that this definition gives a unique construction of an expression.

Notation Rule 2.2. As syntactical variables for expressions we use the Latin capitals $A, B, C, \dots, N$. □

Definition 2.3. A symbol string of the form $[x, C]$ is called an *abstractor*, a symbol string of the form $\{D\}$ an *applicator*. A *lambda-phrase* is either an abstractor or an applicator. A (possibly empty) string of abstractors (applicators, λ -phrases) is called an *abstractor chain* (an *applicator chain*, a *lambda-phrase chain*). □

Notation Rule 2.4. As a syntactical variable for abstractor chains we use the Latin capital Q , for applicator chains the Latin capital R and for λ -phrase chains the Latin capital P . □

The number of entries in a string forming an abstractor chain Q (an applicator chain P , a lambda-phrase chain R) is denoted by $\|Q\|$ ($\|P\|$, $\|R\|$ respectively). Hence $\|Q\| = 0$ if $Q \equiv \phi$, and $\|Q[x, C]\| = \|Q\| + 1$.

An expression B can be a *subexpression* of an expression A , denoted $B \subset A$. This relation is inductively defined as follows:

Definition 2.5.

- (1) $A \subset A$.
- (2) If $C \subset A$ or $C \subset B$, then $C \subset [x, A]B$ and $C \subset \{A\}B$. □

Note: if $B \subset A$, then $A \equiv S_1 B S_2$, i.e.: a subexpression of an expression A is an expression which forms a connected part of A .

Instead of $B \subset A$ we sometimes say: A contains B . If $B \subset A$ and $B \neq A$, we call B a *proper* subexpression of A .

Theorem 2.6. If $F \subset E$ and $E \subset D$, then $F \subset D$.

Proof. Induction on $|D|$. □

If $B \subset A$, then B occurs in A , but there may evidently be more occurrences of B in A . In the following we wish to be able to distinguish between such occurrences of B in A . We shall indicate the occurrence meant by saying " $B \subset A$ after S " if $B \subset A$ and B occurs in A after S .

Definition 2.7. Let B occur in A after S_1 and let C occur in A after S_2 . We call these occurrences *disjoint* if either $S_2 \equiv S_1 B S'$ or $S_1 \equiv S_2 C S''$. □

implied in Theorem 2.8. Let B occur in A after S_1 , let C occur in A after S_2 , let $B \subset A$ and $C \subset A$. Then (1) B and C occur disjointly in A or (2) $B \subset C$ or (3) $C \subset B$.

Proof. Induction on the length of proof of $B \subset A$. □

Let $B \subset A$ after S . We shall inductively define the *factor* of A with respect to S and B (denoted $A|<S;B>$) in definition 2.9.

In this definition the occurrence of B meant is precisely described.

However, it will often be clear from the context which occurrence of B is meant in case $B \subset A$. In that case the precise indication of this occurrence is superfluous, and instead of $A| \langle S; B \rangle$ we shall write $A|B$.

Informally we can concentrate the inductive definition of $A|B$, under the condition that in each of the following rules the occurrences of B under discussion are "in corresponding places":

- (1) If $A \equiv B$, then $A|B \equiv A$.
- (2) If $A \equiv [x, C]D$, then $A|B \equiv C|B$ in case $B \subset C$ and $A|B \equiv [x, C](D|B)$ in case $B \subset D$.
- (3) If $A \equiv \{C\}D$, then $A|B \equiv C|B$ in case $B \subset C$ and $A|B \equiv D|B$ in case $B \subset D$.

For a description of a characteristic property of $A|B$, which justifies its introduction, see the following section (after Th. 3.6).

The formal inductive definition of $A| \langle S; B \rangle$ is the following:

Definition 2.9. Let $B \subset A$ after S.

- (1) If $A \equiv B$, then $A| \langle S; B \rangle \equiv A$.
- (2) Let $A \equiv [x, C]D$. If $B \subset C$ after S_1 and $[x, S_1] \equiv S$, then $A| \langle S; B \rangle \equiv C| \langle S_1; B \rangle$. If $B \subset D$ after S_2 and $[x, C]S_2 \equiv S$, then $A| \langle S; B \rangle \equiv [x, C](D| \langle S_2; B \rangle)$.
- (3) Let $A \equiv \{C\}D$. If $B \subset C$ after S_1 and $\{S_1\} \equiv S$, then $A| \langle S; B \rangle \equiv C| \langle S_1; B \rangle$. If $B \subset D$ after S_2 and $\{C\}S_2 \equiv S$, then $A| \langle S; B \rangle \equiv D| \langle S_2; B \rangle$. □

Note: the parentheses () in $[x, C](D| \langle S_2; B \rangle)$ belong to the discussion language and are meant to fix the scope of |.

Let $B \subset A$ after S. It will be clear that $B \subset A|B$, or, a fortiori: $A|B$ ends in B (here, of course, $A|B$ is meant to be $A| \langle S; B \rangle$). It is also evident that $A|B \equiv QB$ and $(QA)|B \equiv Q(A|B)$.

We state the following theorems:

Theorem 2.10. If $C \subset B \subset A$, then $(A|B)|C \equiv A|C$.

Proof. Induction on $|A|$. □

Theorem 2.11. If $B \subset A$ and $A|B \equiv Q_1[x, C]Q_2B$, then $C \subset A$.

Proof. Induction on $|A|$, using Th. 2.6. □

Theorem 2.12. If $E \equiv Q_1[x, C]D$ and $B \subset C$ after S , then $E|B \equiv Q_1(C|B)$ (here $E|B$ is $E| < Q_1[x, S; B] >$ and $C|B$ is $C| < S; B >$).

Proof. Induction on $\|Q_1\|$. □

Theorem 2.13.

- (1) If $[x, C]D \subset A$ after S_1 and $B \subset D$ after S_2 , then $A|B \equiv Q_1[x, C]Q_2B$ (here $A|B$ is $A| < S_1[x, C]S_2; B >$).
- (2) If $B \subset A$ after S and $A|B \equiv Q_1[x, C]Q_2B$, then there is a D such that $[x, C]D \subset A$ after S_1 , $B \subset D$ after S_2 and $S \equiv S_1[x, C]S_2$ (here $A|B$ is $A| < S; B >$).

Proof. In both parts of the theorem: induction on $|A|$. □

We conclude with an inductive definition of the function Tail, which maps expressions to expressions:

Definition 2.14.

- (1) $\text{Tail}(x) \equiv x$.
- (2) $\text{Tail}(\tau) \equiv \tau$.
- (3) $\text{Tail}([x, A]B) \equiv \text{Tail}(B)$.
- (4) $\text{Tail}(\{A\}B) \equiv \text{Tail}(B)$. □

Note that $\text{Tail}(A)$ can only be a variable or τ . An expression A can always be written (uniquely) as $A \equiv P \text{Tail}(A)$ (in which P denotes a λ -phrase chain, see Notation Rule 2.4).

§ 3. BOUND EXPRESSIONS

An occurrence of a variable in an expression can be a free, a bound or a binding occurrence. We shall introduce these well-known notions in our system too. An occurrence of a variable in an expression is *binding* if and only if that occurrence immediately follows an opening bracket [.

If D contains an occurrence of x (i.e.: $x \in D$), then that occurrence of x is either bound (and there is a unique binding occurrence of x which binds that bound occurrence) or free. A formal description is given in the following inductive definition. In this definition we often encounter "corresponding" occurrences of x . For easy understanding we shall not use our formalism concerning occurrences (see Section II.1), but we shall introduce "a certain x " and refer to it as "that x ".

Definition 3.1.

- (1) x is free in x .
- (2) Let a certain x be free in A or B . Then that x is free in $\{A\}B$.
- (3) Let a certain x be free in A . Then that x is free in $[y, A]B$ (both if $y \equiv x$ and if $y \neq x$).
- (4) Let a certain x be free in B . Then that x is free in $[y, A]B$ if $y \neq x$, but that x is bound in $[x, A]B$ (by the binding x occurring in $[x, A]B$ after [).
- (5) Let a certain x in A be bound by a binding x in A , or let a certain x in B be bound by a binding x in B . Then that x is bound by the corresponding binding x in both $\{A\}B$ and $[y, A]B$ (also if $y \equiv x$). □

The binding x occurring in $[x, A]B$ after [binds precisely the free x 's in B (if any).

We shall mainly be interested in expressions in which no variable is free, called *bound expressions* (in the literature also called closed expressions). In bound expressions the same binding variable can occur in different instances. This cannot, however,

give rise to confusion as to the connection between a bound variable and the binding variable by which it is bound.

Yet, for practical reasons, we wish to avoid such expressions. We call a bound expression in which all binding variables are different, a *distinctly bound* expression, and we restrict ourselves to the set of all distinctly bound expressions, which we call Δ .

This is no essential restriction. Every interesting theory concerning bound expressions can be restricted to distinctly bound expressions.

Let $x \in D$ after S and let this x be bound in D . It follows from Def. 3.1 that we have $D \equiv S_1[x, E]FS_2$ such that $[x, E]F \in D$, $x \in F$ after S_3 , $S_1[x, E]S_3 \equiv S$ and the x occurring in D after S_1 binds the x occurring in D after S . We shall call $[x, E]$ the *binding abstractor* of the bound x .

From this and Th. 2.13 (1) it follows:

Theorem 3.2. If $x \in D \in \Delta$, then $D|x \equiv Q_1[x, E]Q_2x$ and $[x, E]$ is the binding abstractor of the bound x in D . $\square$

It follows from Th. 2.13 (2):

Theorem 3.3. If for each $x \in D$ there are Q_1 , A and Q_2 such that $D|x \equiv Q_1[x, A]Q_2x$, then D is a bound expression. $\square$

The following theorem expresses in an intricate manner the obvious observation that, in case $x \in K \in D \in \Delta$, the x is bound by a binding abstractor either outside or inside K .

Theorem 3.4. If $x \in K \in D \in \Delta$, then either

- (i) $D|K \equiv Q_1[x, A]Q_2K$, $D|x \equiv Q_1[x, A]Q_2Q'x$ and $K|x \equiv Q'x$, or
- (ii) $K|x \equiv Q_1[x, A]Q_2x$ and $D|x \equiv QQ_1[x, A]Q_2x$.

In both cases $[x, A]$ is the binding abstractor of x in D .

Proof. Let $D|K \equiv QK$, then $D|x \equiv$ (by Th. 2.10) $(D|K)|x \equiv (QK)|x \equiv Q(K|x) \equiv QQ'x$. Hence, by Th. 3.2, either $Q \equiv Q_1[x, A]Q_2$, or $Q' \equiv Q_1[x, A]Q_2$. $\square$

Theorem 3.5. If $Q\{A\}B \in \Delta$, then QA and $QB \in \Delta$; if $Q[x,A]B \in \Delta$, then $QA \in \Delta$.

Proof. Apply Th. 3.3. □

Theorem 3.6. If $A \in \Delta$ and $B \subset A$ after S , then $A|B \in \Delta$.

Proof. First assume that $x \subset B$. Then by Th. 2.10: $(A|B)|x \equiv A|x \equiv Q_3[x,C]Q_4x$.

Next assume that $A|B \equiv QB$ and $x \subset Q$. Then $Q \equiv Q_1[y,D]Q_2$ and $x \subset D$.

By Th. 2.11: $D \subset A$. So $(QB)|x \equiv Q_1(D|x) \equiv (Q_1D)|x \equiv (A|D)|x \equiv A|x \equiv Q_3[x,C]Q_4x$. Apply Th. 3.3, and note that the binding variables in $A|B$ must be distinct. □

The above theorem states an essential feature of the factor $A|B$. If $A \in \Delta$ and $B \subset A$, then not necessarily $B \in \Delta$. But $A|B$, which is QB for a certain abstractor chain Q , "closes" B in Δ by placing in front of B those abstractors which necessarily bind all free variables of B . This, together with our previously uttered wish to restrict our expressions as much as possible to Δ , justifies our introduction in Section II.2 of the factor $A|B$.

We continue with three theorems, related to one another.

2.10.11
Th 3.3

Theorem 3.7. If $Q[x,A]B \in \Delta$ and $x \not\subset B$, then $QB \in \Delta$.

Proof. Observe the various places where a variable y ($\neq x$) can occur in QB . □

Theorem 3.8. If $QB \in \Delta$ and $Q[x,A]B \in \Delta$, then $x \not\subset B$.

Proof. The assumption $x \subset B$ leads to a contradiction. □

Theorem 3.9. If QA and $QB \in \Delta$, if the binding variables in A and B are distinct and if x does not occur as a binding variable in QA or QB , then $Q[x,A]B \in \Delta$.

Proof. Again observe the various places where a variable y ($\neq x$) can occur in $Q[x,A]B$. □

$D \equiv [y, A][x, B]x$
mit $\in \Delta$ volgens 3.10

$D|x \equiv D$? $D \in \Delta$?

In general: If QB and $QPB \in \Delta$ and x occurs as a binding variable in P , then not $x \in B$. We say: P has no *binding influence* on B .

The description of Δ , which we gave so far, began with (general) expressions and selected the distinctly bound expressions among these. This method is not so practical for theoretical investigations. In the following two theorems we shall indicate how we can compose expressions in Δ from expressions in Δ . Or, rather, we shall show how expressions in Δ can be decomposed into smaller expressions, which also belong to Δ .

Theorem 3.10.

- (1) $\tau \in \Delta$.
- (2) If $QA \in \Delta$ and if x does not occur in QA , then $Q[x, A]x \in \Delta$ and $Q[x, A]\tau \in \Delta$.
- (3) If QA and $Qy \in \Delta$, if x does not occur in QA and if $x \neq y$, then $Q[x, A]y \in \Delta$.
- (4) If QA and $QB \in \Delta$ and if the binding variables in A and B are distinct, then $Q\{A\}B \in \Delta$.

Proof. It is trivial that $Q[x, A]x$, $Q[x, A]\tau$, $Q[x, A]y$ and $Q\{A\}B$ respectively are again expressions. These are also clearly bound expressions, moreover distinctly bound by the conditions given in the theorem. □

We may consider the four parts of the previous theorem as derivation rules.

Definition 3.11. We call K Δ -constructible if we can establish that $K \in \Delta$ by a (finite) number of applications of the rules in Th. 3.10. □

The proof of the following theorem is technical. Yet it is interesting to see how we can establish Δ -constructibility. For better understanding, we shall express the main lines at the end of the present section.

Theorem 3.12. If $K \in \Delta$, then K is Δ -constructible.

Proof. Induction on $|K|$. If $|K| = 1$, then $K \equiv \tau$ and K is Δ -constructible by rule (1).

Assume that $|K| > 1$, and let all distinctly bound expressions K' with $|K'| < |K|$ be Δ -constructible (first induction hypothesis). Then $K \equiv P_1 P_2 \dots P_n \text{Tail}(K)$ for some $n \geq 1$, where each of the P_i is a lambda-phrase (i.e. either an abstractor or an applicator).

We can now prove the lemma: "For all i for which $1 \leq i \leq n+1$ it holds that: $K|(P_1 \dots P_n \text{Tail}(K))$ is Δ -constructible", by induction on $n+1-i$.

(1) Let $i = n+1$ (i.e. $P_1 \dots P_n \equiv \phi$). If $|K|\text{Tail}(K)| < |K|$, then the first induction hypothesis leaves nothing to prove. If $|K|\text{Tail}(K)| = |K|$, then $K|\text{Tail}(K) \equiv K \equiv [z_1, E_1] \dots [z_n, E_n]\text{Tail}(K)$ for $n \geq 1$.

(1a) Assume that $\text{Tail}(K) \equiv x$. Then for exactly one $s : z_s \equiv x$. Abbreviate $[z_1, E_1] \dots [z_t, E_t]$ as Q_t for $0 \leq t \leq n$. We distinguish the cases $z_n \equiv x$ and $z_n \not\equiv x$. If $z_n \equiv x$, then K is Δ -constructible by rule (2) since $Q_{n-1}E_n \in \Delta$ by Th. 3.5. If $z_n \not\equiv x$, then K is Δ -constructible by rule (3), since $Q_{n-1}E_n \in \Delta$ and $Q_{n-1}z_n \in \Delta$ (the latter by Th. 3.7).

(1b) Assume that $\text{Tail}(K) \equiv \tau$. Then $Q_{n-1}E_n \in \Delta$ by Th. 3.5 and Δ -constructible by induction. By rule (2) we then find that K is Δ -constructible.

(2) Let $1 \leq i \leq n$ and assume that $K|(P_j \dots P_n \text{Tail}(K))$ is Δ -constructible if $i < j \leq n+1$ (second induction hypothesis). If

$|K|(P_i \dots P_n \text{Tail}(K))| < |K|$, then again the first induction hypothesis leaves nothing to prove. So let $|K|(P_i \dots P_n \text{Tail}(K))| = |K|$. Then $K \equiv K|(P_i \dots P_n \text{Tail}(K)) \equiv [z_1, E_1] \dots [z_t, E_t]P_i \dots P_n \text{Tail}(K)$.

(2a) Let $P_i \equiv [z_{t+1}, E_{t+1}]$. Then $K|(P_i \dots P_n \text{Tail}(K)) \equiv K|(P_{i+1} \dots P_n \text{Tail}(K))$, which is Δ -constructible by the second induction hypothesis.

(2b) Let $P_i \equiv \{F\}$. Then $Q_t F \in \Delta$ by Th. 3.5 and Δ -constructible by the second induction hypothesis, and the same holds for $Q_t P_{i+1} \dots P_n \text{Tail}(K)$. Hence, by rule (4):

$Q_t\{F\}P_{i+1} \dots P_n \text{Tail}(K) \equiv K$ is Δ -constructible.

From this lemma it follows that $K|(P_1 \dots P_n \text{Tail}(K)) \equiv K|K \equiv K$ is Δ -constructible.

(In this proof we did not check the conditions concerning variables in rules (1) to (4). It is easy to see that these are fulfilled in the appropriate places.) $\square$

From Th. 3.10 and Th. 3.12 follows that rules (1) to (4) generate the relation $K \in \Delta$. Hence, we can consider these four rules as a *second definition of Δ* . The advantage of the latter is that we have an inductive definition of Δ , whereas the original definition was restrictive with respect to the set of all expressions.

In a proof by induction the four rules of Th. 3.10 are much easier to use. The notion "induction on the length of proof" usually refers to a proof (in fact a construction) produced by an inductive definition, as is the second definition of Δ . From now on we shall refer to this latter definition when giving a proof by "induction on the length of proof of $K \in \Delta$ ".

Note that, given a $K \in \Delta$, only one of the derivation steps in Th. 3.10 can give $K \in \Delta$ as a conclusion (if $K \equiv \tau$, then this must be rule (1). If $K \equiv Q\{A\}B$, then this must be rule (4). If $K \equiv Q\tau$ and $Q \neq \phi$, then this must be part 2 of rule (2). If $K \equiv Qx$ and $Q \equiv Q'[x,A]$, then this must be part 1 of rule (2). If $K \equiv Qx$ and $Q \equiv Q'[y,A]$ for some $y \neq x$, then this must be rule (3)).

The proof of the previous theorem suggests how we can establish that $K \in \Delta$ by using the four rules of Th. 3.10 (i.e. that K is Δ -constructible). We can express this in words as follows:

Let K be a distinctly bound expression.

(1) we first establish that $K|\text{Tail}(K)$ is Δ -constructible:

(1a) if $\text{Tail}(K) \equiv x$, then find the binding abstractor $[x,A]$ in K which binds x , establish that $K|A \equiv QA$ is Δ -constructible, and apply rule (2) to obtain $Q[x,A]x \in \Delta$. Let $Q'_1, Q'_2, \dots, Q'_k$ be the abstractors occurring in $K|\text{Tail}(K)$ "between" $[x,A]$ and x . Insert these abstractors, starting with Q'_1 (from left to right), by inserting Q'_i in $Q[x,A]Q'_1 \dots Q'_{i-1}x$ between Q'_{i-1} and x (by rule (3)).

In this manner we establish that $K|Tail(K)$ is Δ -constructible.

(1b) If $Tail(K) \equiv \tau$, then $K|Tail(K) \equiv \tau$ and we may use rule (1) immediately, or $K|Tail(K) \equiv Q[x, A]\tau$.

In the latter case: establish that QA is Δ -constructible, and apply rule (2) to establish that $K|Tail(K)$ is also Δ -constructible.

(2) We established that $K|Tail(K) \equiv [x_1, A_1] \dots [x_n, A_n]Tail(K)$ is Δ -constructible. In K we find applicators $\{B_1\}, \dots, \{B_\ell\}$ "between" the abstractors $[x_i, A_i]$. Insert these B_i , starting with B_ℓ and ending with B_1 (from right to left) in the appropriate places, using rule (4). In this manner we establish that K is Δ -constructible.

(Note the following. If we establish that K is Δ -constructible, then we use the Δ -constructibility of $K|E$ for all $E \in K$. We can prove this by induction on the length of proof of $K \in \Delta$.)

§ 4. REPLACEMENT, RENOVATION AND α -REDUCTION

If we replace a certain variable x in all its occurrences (free, bound or binding) in an expression A by a variable y , then we denote the result of this replacement by $((x:=y))A$. An inductive definition of *simple replacement* is the following (induction here is on the length of the expression):

Definition 4.1. For each pair x and y , $((x:=y))$ is a function from expressions to expressions.

(1) $((x:=y))x \equiv y$; $((x:=y))z \equiv z$ if $z \neq x$; $((x:=y))\tau \equiv \tau$.

(2) $((x:=y))[z, A]B \equiv [((x:=y))z, ((x:=y))A]((x:=y))B$.

(3) $((x:=y))\{A\}B \equiv \{((x:=y))A\}((x:=y))B$. □

The simple replacement of certain variables by others will be used for making the binding variables in an expression distinct. This we shall call the *renovation* of the expression.

Renovation is in fact nothing but a repeated renaming of variables. Renaming does not affect relevant properties of expressions

(under a reasonable interpretation of "variable"; see also what we said concerning α -equivalent terms in Section I.1).

We have maintained names for variables for reasons of tradition and legibility. This is at the expense of the renovation selector (to be introduced in this section) and the so-called α -reduction (our wishes concerning bound expressions in the previous section had nothing to do with names for variables, but with our dislike of the occurrence of free variables; the additional wish to have *distinctly* bound expressions, however, does concern names for variables, and leads us to introduce renovation).

We shall discuss the process of renovation. The mathematical meaning of renovation is not profound. If one dislikes a formalization of an intuitively clear concept, one can continue reading at Def. 4.4 (concerning α -reduction).

There is, of course, one precaution which one must take in the renovation process: the relation between bound and binding variables should remain unaffected in a natural way. For example, in $[x,y][x,x]x$ the binding x occurring after $[$ binds the bound x occurring after $[x,y][x,$; the binding x occurring after $[x,y][$ binds the bound x occurring at the end of the expression. In changing this expression into an expression with distinct binding variables, we might obtain: $[x,y][z,x]z$ for a certain $z \neq x$.

Such a variable z as introduced in the latter example by the renovation process plays a special rôle. It has to be chosen with care. At any rate it should be different to all binding variables in the expression under discussion, or, as we shall say: it has to be *fresh* with respect to that expression.

We introduce the renovation selector Fr_V , operating on expressions. In using the renovation selector Fr_V with an expression A we have it preceded by a lambda phrase chain P , giving $PFr_V A$. The subscript V denotes a finite set of variables, which can be empty. We shall not specify the variables belonging to V until the following section, where we use Fr_V in the formal definition of substitution.

The expression $P\text{Fr}_V A$ can informally be described as being PA' , where (1) A' is obtained from A by renovation of A and (2) the fresh variables chosen during this renovation do not occur in P or V and are mutually distinct.

The following inductive definition gives a formalization of this concept:

Definition 4.2. Let V be a finite set of variables.

- (1) $P\text{Fr}_V x \equiv Px$; $P\text{Fr}_V \tau \equiv P\tau$.
- (2) $P\text{Fr}_V ([y, B]C) \equiv P[z, B']\text{Fr}_V (((y:=z))C)$, with $PB' \equiv P\text{Fr}_V B$, while z does not occur in PB' and $z \notin V$.
- (3) $P\text{Fr}_V (\{B\}C) \equiv P\{B'\}\text{Fr}_V C$, with $PB' \equiv P\text{Fr}_V B$. □

From the above definition we see that the renovation of an expression takes place from left to right. For instance, the renovation resulting in $P\text{Fr}_V (\{B\}C)$ first requires the renovation resulting in $P\text{Fr}_V B$, and subsequently the one resulting in $P\{B'\}\text{Fr}_V C$. This implies that the fresh variables chosen in the renovation process are mutually distinct.

Of course uncertainties remain as to the choice of fresh variables. An order in the set of all variables could turn the selector Fr_V into an operator. We shall, however, not push the formalization this far.

In the following section we shall describe substitution by the aid of the renovation selector, and we shall in turn use substitution in describing the β -reduction. Our use of the renovation selector is meant to keep an expression distinctly bound after β -reduction. We shall use the renovation selector in typing an expression, as is described in Section II.8, with the same purpose.

We usually begin renovation with $V = \phi$ (here, of course, ϕ denotes the empty set and not the empty string).

Definition 4.3. $P\text{Fr}A \equiv P\text{Fr}_{\phi} A$. □

In fact $P\text{Fr}A$ is the concatenation of P and $\text{Fr}_W A$, where W is the set of variables occurring in P .

Let $P\text{Fr}_V A \equiv PA'$ be the result of a renovation. If we again write $\text{Fr}_V A$, in the same context, we do not require a new renovation, but mean A' . If we wish another renovation in the same context, then we supply Fr_V with primes: $P\text{Fr}'_V A$ can be such a new renovation.

We shall now define the α -reduction relation. For an informal discussion of α -reduction see Section I.1. We restrict α -reduction to expressions in Δ :

Definition 4.4. α -reduction, denoted by $\geq_\alpha$, is the transitive relation generated by:

(1) If $A \in \Delta$ and if y does not occur in A , then $A \geq_\alpha ((x:=y))A$. $\square$

The α -reduction is clearly an equivalence relation (reflexivity: take x to be a variable which does not occur in A ; symmetry: note that x does no longer occur in $((x:=y))A$).

If two expressions are related by α -reduction ($A \geq_\alpha B$), we speak of "the α -reduction $A \geq_\alpha B$ ". This is clearly abuse of language, although it cannot give rise to confusion.

A renaming of a single variable in a distinctly bound expression is called a *single-step* α -reduction, denoted as $A \geq'_\alpha B$ (so $A \geq'_\alpha B$ if and only if $A \in \Delta$ and $B \equiv ((x:=y))A$ where x occurs as a binding variable in A and y does not occur in A).

The following theorems are trivial:

Theorem 4.5. If $PA \in \Delta$, then $P\text{Fr}_V A \in \Delta$; if $A \in \Delta$ and $A \geq_\alpha B$, then $B \in \Delta$ and $|A| = |B|$. $\square$

Theorem 4.6. Let PA , PB , $PP'A$ and $PP'B \in \Delta$. Then $PA \geq_\alpha PB$ if and only if $PP'A \geq_\alpha PP'B$. $\square$

§ 5. SUBSTITUTION AND β -REDUCTION

Substitution is an operation acting on expressions. We denote "the result of the substitution of A for x in B" by $(x:=A)B$. One can use several definitions of substitution which are equivalent.

We shall use the definitions given in Def. 5.1 and Def. 5.2. These definitions of substitution can informally be described as follows: $P(x:=A)_V B$ is the expression which we obtain from PB by replacing all free x's in B with renovations of A, in which the fresh variables are chosen in the following manner: they have to be mutually distinct for all renovations of A replacing the free x's, and they have to be distinct from the variables occurring in B, P or V (we do not replace the binding variables in B by fresh ones). Here V denotes a finite set of variables, which can be empty.

This careful dealing with fresh variables is necessary to guarantee that an expression with distinct binding variables has again distinct binding variables after β -reduction (to be defined in this section); substitution is an essential part of β -reduction.

The following part of this section as far as Def. 5.5., will formalize the above notion of substitution. As with renovation, our formalization of substitution may be cumbersome to the reader. One may continue with Def. 5.5, without impairing understanding.

An inductive definition of $P(x:=A)_V B$ is the following (induction is here on the length of B):

Definition 5.1.

- (1) $P(x:=A)_V x \equiv P \text{Fr}_V A$; $P(x:=A)_V y \equiv Py$ if $y \neq x$; $P(x:=A)_V \tau \equiv P\tau$.
- (2) If $y \neq x$, then $P(x:=A)_V [y, B]C \equiv P[y, B'](x:=A)_V C$, where $PB' \equiv P(x:=A)_W B$, W being the union of V and the set of all variables occurring in C.
If $y \equiv x$, then $P(x:=A)_V [y, B]C \equiv P[y, B']C$, where B' is obtained as above.
- (3) $P(x:=A)_V \{B\}C \equiv P\{B'\}(x:=A)_V C$, where B' is obtained as above. $\square$

From the above definition we see that substitution (as renovation) takes place from left to right in an expression: for instance, the substitution resulting in $P(x:=A)_V\{B\}C$ first requires the substitution resulting in $P(x:=A)_W B$, and subsequently the substitution resulting in $P\{B'\}(x:=A)_V C$.

In part (2) of the definition we see the importance of the subscript used in $(x:=A)$: in executing the substitution resulting in $P(x:=A)_W B$ we must have at our disposal the set of variables occurring in C in order to be able to choose fresh variables different from the variables in C . The set W contains the latter variables.

The set V can be the empty set. When we begin a substitution, V is usually empty.

Definition 5.2. $P(x:=A)B \equiv P(x:=A)_\phi B$. □

In the above definitions there are two more or less unusual parts. Usually $(x:=A)x$ is defined as A ; however, with a view to our wish to keep distinctly bound expressions distinctly bound after some substitution, we deviate from this. Next, one sometimes defines $(x:=A)[y, B]C$ as $[z, (x:=A)B](x:=A)(y:=z)C$, in either of the cases that $x \equiv y$ or $x \neq y$, z being a fresh variable. The latter definition prevents so-called "confusion of variables" (cf. Curry and Feys [3, Ch. 3D2]; see also the example below), but gives rise to an additional amount of simple substitutions of variables of the form $(y:=z)$, which we find cumbersome.

In using Def. 5.1. and Def. 5.2. confusion of variables may occur if the use of the substitution operator is not restricted. For example: $[y, A](x:=y)[y, \tau]x \equiv [y, A][y, \tau]y$, where the final y in the latter expression is influenced by $[y, \tau]$, and not by $[y, A]$ as it should be.

In general: confusion of variables may arise as a consequence of the substitution resulting in $P(x:=A)_V B$ if a free variable y of A (with $y \neq x$) occurs as a binding variable in B , and if there is a free x in B within the "scope" of that binding variable y .

A sufficient condition for avoiding this is, that the free variables of A do not occur as binding variables in B.

We use substitution only in the relation β -reduction, defined later in this section. The above condition is there fulfilled. Hence confusion of variables cannot arise in our system.

Note that $(x:=A)$ operates on free x's, $((x:=A))$ on all x's, and Fr on all binding variables in an expression.

We also define the substitution operator for lambda-phrase chains:

Definition 5.3. If $(x:=A)Pr \equiv P'r$, then $(x:=A)P \equiv P'$. □

One may interchange the substitution operator and the renovation selector under certain conditions:

Theorem 5.4. Let $PA \in \Delta$ and $P[x,B]D \in \Delta$. Then

$P\text{Fr}(x:=A)D \geq_{\alpha} P(x:=A)\text{Fr}D$ if no binding variable of D occurs in A.

Proof. Induction on $|D|$, using Th. 4.5 and the lemma:

" $((y:=z))(x:=A)C \equiv (x:=A)((y:=z))C$ if $y \notin A$ (but for renaming)". □

Substitution is used in the more interesting reduction in lambda-calculus called β -reduction, which we denote by $\geq_{\beta}$. The interpretation linked with β -reduction is the application of a function to an argument (see also the informal description in Section I.1).

We shall restrict β -reduction to expressions belonging to Δ . This is unusual. One usually conceives of a reduction as a formal relation between expressions in which free variables may occur. It is only our preference for distinctly bound expressions which makes us choose the definitions given below.

Note that our β -reduction is not essentially different from the usual concept. The use of the Q in Def. 5.5 is a little obscuring in this respect.

We first define *single-step β -reduction*, denoted by $\geq'_{\beta}$:

Definition 5.5. Single-step β -reduction is the relation generated by:

- (1) If $Q\{A\}[x, B]C \in \Delta$, then $Q\{A\}[x, B]C \geq'_\beta Q(x:=A)C$.
- (2) Let $Q\{A\}C$ and $Q\{A\}D \in \Delta$. If $QC \geq'_\beta QD$, then $Q\{A\}C \geq'_\beta Q\{A\}D$.
- (3) Let $Q[x, A]C$ and $Q[x, B]C \in \Delta$. If $QA \geq'_\beta QB$, then $Q[x, A]C \geq'_\beta Q[x, B]C$.
- (4) Let $Q\{A\}C$ and $Q\{B\}C \in \Delta$. If $QA \geq'_\beta QB$, then $Q\{A\}C \geq'_\beta Q\{B\}C$. $\square$

Note: one rule appears to be missing (viz.: let $Q[x, A]C$ and $Q[x, A]D \in \Delta$. If $QC \geq'_\beta QD$, then $Q[x, A]C \geq'_\beta Q[x, A]D$). But this is a derived rule, see Cor. 5.14.

Rules (2), (3) and (4) in the above definition are called the *monotony rules* of single-step β -reduction; we call rule (1) the rule of *elementary β -reduction*.

If K and L are related by a single-step β -reduction, we obtain L from K by replacing a certain subexpression $\{A\}[x, B]C$ in K by $(x:=A)C$. This is, in terms of interpretation, a 'single functional application'.

If $K \geq'_\beta L$, then we have a construction (a "proof") which establishes that relation according to Def. 5.5. Such a construction consists of one derivation step which is an elementary β -reduction (rule (1) of Def. 5.5) followed by a number of derivation steps which are monotony steps (rules (2) to (4)). Note that a *single-step β -reduction* is achieved from a *number* of derivation steps.

We note that, since $Q\{A\}[x, B]C \in \Delta$, no free variable of A can occur as binding variable in C . This is sufficient, as previously stated, to prevent "confusion of variables".

Definition 5.6. β -reduction is the reflexive and transitive closure of single-step β -reduction. $\square$

- (This means that (1) if $K \geq'_\beta L$, then $K \geq_\beta L$; (2) $K \geq_\beta K$;
(3) if $K \geq_\beta L$ and $L \geq_\beta M$, then $K \geq_\beta M$.)

If A and B are related by a single-step β -reduction, we speak of "the single-step β -reduction $A \geq'_\beta B$ ". As with α -reduction, this

is abuse of language. Analogously we speak of "the β -reduction $A \geq_{\beta} B$ ".

If $A_0 \geq'_{\beta} A_1, A_1 \geq'_{\beta} A_2, \dots, A_{n-1} \geq'_{\beta} A_n$, we also write $A_0 \geq'_{\beta} A_1 \geq'_{\beta} \dots \geq'_{\beta} A_n$ or $A_0 \geq'_{\beta} A_n$. We call this a *composite* single-step β -reduction, or an *n-step β -reduction* (a zero-step β -reduction has, of course, the form $A \geq'_{\beta} A$). If $A_0 \geq_{\beta} A_1, \dots, A_{n-1} \geq_{\beta} A_n$, we also write $A_0 \geq_{\beta} A_1 \geq_{\beta} \dots \geq_{\beta} A_n$.

It follows from the definition of β -reduction that each β -reduction $K \geq_{\beta} L$ can be presented as an *n-step β -reduction* $K \equiv A_0 \geq'_{\beta} \dots \geq'_{\beta} A_n \equiv L$. This splitting is called a *decomposition* of a β -reduction.

Each of the monotony rules has the form: "If reduction (i) holds, then reduction (ii) holds". It is usual to call reduction (ii) the *direct consequence* of reduction (i). For example: $Q\{A\}C \geq_{\beta} Q\{A\}D$ is a direct consequence of $QC \geq_{\beta} QD$. We recall that "the length of proof of $K \geq'_{\beta} L$ " is the total number of derivation steps in the proof of $K \geq'_{\beta} L$.

A proof of $K \geq'_{\beta} L$ begins with an elementary β -reduction $Q\{A\}[x, B]C \geq_{\beta} Q(x=A)C$. In this case we say that $Q\{A\}[x, B]C$ *generates* the single-step β -reduction $K \geq'_{\beta} L$. The following theorem holds:

Theorem 5.7. Let $K \in \Delta$. Then $Q\{A\}[x, B]C$ generates a single-step β -reduction of the form $K \geq'_{\beta} L$ if and only if $\{A\}[x, B]C \subset K$ and $Q\{A\}[x, B]C \equiv K \setminus \{A\}[x, B]C$.

Proof. $\Rightarrow$: Induction on the length of proof of $K \geq'_{\beta} L$.

$\Leftarrow$: We state the following *lemma*: "Let $K \equiv Q'M \in \Delta$, and let

$E \equiv \{A\}[x, B]C \subset M$. Then $K|E$ generates a single-step β -reduction

$K \equiv Q'M \geq'_{\beta} Q'N$ ". We prove this lemma by induction on $|M|$:

(1) Let $M \equiv E$. Then $K|E \equiv K \equiv Q'\{A\}[x, B]C \geq'_{\beta} Q'(x=A)C$.

(2a) Let $M \equiv [x, F]G$ and $E \subset F$. Call $K' \equiv Q'F \in \Delta$; then $E \subset F$, hence, by induction: $K'|E \equiv Q'(F|E)$ generates a single-step β -reduction $K' \equiv Q'F \geq'_{\beta} Q'F'$. By applying monotony rule (3) it follows that $K'|E$ also generates $Q'M \equiv Q'[x, F]G \geq'_{\beta} Q'[x, F']G$. Moreover, $K|E \equiv K'|E$.

(2b) Let $M \equiv [x, F]G$ and $E \subset G$. Take $K' \equiv K \equiv Q'[x, F]G \equiv Q''G$; then $E \subset G$, hence, by induction: $K' \mid E$ generates a single-step β -reduction $Q''G \geq'_\beta Q''G'$, which can be rewritten as $Q'M \geq'_\beta Q'[x, F]G'$.

(3a) Let $M \equiv \{F\}G$ and $E \subset F$. The proof is analogous to the one in case (2a), using monotony rule (4) instead of (3).

(3b) Let $M \equiv \{F\}G$ and $E \subset G$. Again the proof is analogous to the one in case (2a), now using monotony rule (2) instead of (3).

This proves the lemma. The "if-part" of the theorem follows immediately from the lemma. $\square$

We state the "closure theorem for Δ with respect to single-step β -reduction":

Theorem 5.8. If $K \in \Delta$ and $K \geq'_\beta L$, then $L \in \Delta$.

Proof. Induction on the length of proof of $K \geq'_\beta L$. Note that our definition of substitution for a variable with the aid of the ren-ovation selector is essential. In the proof we can use Th. 3.4. $\square$

Corollary 5.9. If $K \in \Delta$ and $K \geq_\beta L$, then $L \in \Delta$. $\square$

One may conceive of the β -reduction relation, not as a relation between expressions, but as a relation between α -equivalence classes (in the α -equivalence class of K we include all K' such that $K \geq_\alpha K'$).

The following theorem gives a justification for this conception of β -reduction:

Theorem 5.10. Let $A \in \Delta$, $A \geq'_\beta B$ and $A \geq_\alpha C$. Then there is a reduction $C \geq'_\beta D$ such that $B \geq_\alpha D$.

Proof. It is sufficient to assume that $A \geq'_\alpha C$. Apply induction on the length of proof of $A \geq'_\beta B$. $\square$

In the sequel we shall sometimes refer to the above conception of β -reduction, by inserting the words "but for α -reduction" in a statement (for example: " $A \geq_\beta B$ but for α -reduction" means that

there are A' and B' such that $A \geq_{\alpha} A' \geq_{\beta} B' \geq_{\alpha} B$). However, we often omit the words "but for α -reduction".

We shall proceed with a number of theorems, in which especially the rôle of the abstractor chain Q in a β -reduction is considered, Q occurring in the beginning of an expression. (The definition of $\|P\|$ was given after notation rule 2.4.)

Theorem 5.11. If $QE \in \Delta$, $QE \geq'_{\beta} Q'F$ and $\|Q\| = \|Q'\|$, then $Q \equiv Q'$ or $E \equiv F$. In the latter case $Q \equiv Q_1[x, K]Q_2$, $Q' \equiv Q_1[x, L]Q_2$ and $Q_1K \geq'_{\beta} Q_1L$.

Proof. Induction on the length of proof of $QE \geq'_{\beta} Q'F$. There are four possible cases for the last derivation step in the proof of $QE \geq'_{\beta} Q'F$. In three of these cases the conclusion is: $Q \equiv Q'$. The fourth case is that the last derivation step in the proof of $QE \geq'_{\beta} Q'F$ is: " $Q_1K \geq'_{\beta} Q_1L$, so $QE \equiv Q_1[x, K]M \geq'_{\beta} Q_1[x, L]M \equiv Q'F$ ". Now if $\|Q\| \leq \|Q_1\|$, then $Q \equiv Q'$, and if $\|Q\| > \|Q_1\|$, then $E \equiv F$, $Q \equiv Q_1[x, K]Q_2$ and $Q' \equiv Q_1[x, L]Q_2$. □

Theorem 5.12. If $QE \in \Delta$ and $QE \geq'_{\beta} K$, then $K \equiv Q'F'$ for certain Q' and F' with $\|Q'\| = \|Q\|$.

Proof. The reduction $QE \geq'_{\beta} K$ must be the conclusion of one of the rules of Def. 5.5. 5 It is easy to see that the statement of the theorem holds ~~good~~ ⁱⁿ in all these cases. □

If a reduction $QC \geq_{\beta} Q'D$ is given, in which $\|Q\| = \|Q'\|$, one can conceive of an accompanying "reduction" of C to D . The following theorem shows this.

Theorem 5.13. If QC , Q_0C and $Q_0D \in \Delta$, $QC \geq_{\beta} Q'D$ and $\|Q\| = \|Q'\|$, then $Q_0C \geq_{\beta} Q_0D$.

Proof. First assume that $QC \geq'_{\beta} Q'D$. Then by Th. 5.11 $Q \equiv Q'$ or $C \equiv D$. In the latter case it is trivial that $Q_0C \geq_{\beta} Q_0D$. So assume $Q \equiv Q'$. Then induction on the length of proof of $QC \geq'_{\beta} Q'D$ and Th. 5.12 yield $Q_0C \geq'_{\beta} Q_0D$. The general theorem follows. □

The apparently missing monotony rule, announced immediately after Def. 5.5, is a consequence:

Corollary 5.14. Let QC , QD , $Q[x,A]C$ and $Q[x,A]D \in \Delta$. If $QC \geq'_\beta QD$, then $Q[x,A]C \geq'_\beta Q[x,A]D$. $\square$

We defined β -reduction as being transitive and reflexive. We shall now show that β -reduction is also monotonous:

Theorem 5.15. The monotony rules hold for β -reduction, i.e.:

- (2) If $Q\{A\}C$ and $Q\{A\}D \in \Delta$, and $QC \geq_\beta QD$, then $Q\{A\}C \geq_\beta Q\{A\}D$.
- (3) If $Q[x,A]C$ and $Q[x,B]C \in \Delta$, and $QA \geq_\beta QB$, then $Q[x,A]C \geq_\beta Q[x,B]C$.
- (4) If $Q\{A\}C$ and $Q\{B\}C \in \Delta$, and $QA \geq_\beta QB$, then $Q\{A\}C \geq_\beta Q\{B\}C$.

Proof. We shall prove rule (2).

Since $QC \geq_\beta QD$, we know that $QC \geq'_\beta E_1 \geq'_\beta E_2 \dots \geq'_\beta E_n \geq'_\beta QD$. From Th. 5.12, Th. 5.13 and induction on n it follows that there is also a reduction $QC \geq'_\beta QF_1 \geq'_\beta QF_2 \dots \geq'_\beta QF_n \geq'_\beta QD$. Repeated application of the corresponding monotony rule for single-step β -reduction gives: $Q\{A\}C \geq'_\beta Q\{A\}F_1 \geq'_\beta \dots \geq'_\beta Q\{A\}D$, hence $Q\{A\}C \geq_\beta Q\{A\}D$.

The other two monotony rules for β -reduction can be proved analogously. $\square$

One may extend Cor. 5.14 to β -reduction:

Theorem 5.16. Let QC , QD , $Q[x,A]C$ and $Q[x,A]D \in \Delta$. If $QC \geq_\beta QD$, then $Q[x,A]C \geq_\beta Q[x,A]D$. $\square$

Theorem 5.17. If QC , PC and $PD \in \Delta$, $QC \geq_\beta Q'D$ and $\|Q\| = \|Q'\|$, then $PC \geq_\beta PD$.

Proof. Let $PC|C \equiv Q_0C$, then $Q_0C \geq_\beta Q_0D$ follows from Th. 5.13. The theorem is proved by repeated application of monotony rule (2) for β -reduction (see Th. 5.15). $\square$

Note: the converse of this theorem does not hold ("If PC, QC and $QD \in \Delta$ and $PC \geq_{\beta} PD$, then $QC \geq_{\beta} QD$ ").

Example ($Q \equiv [x, \tau]$, $P \equiv [x, \tau]\{x\}$):

$[x, \tau]\{x\}[y, \tau]\{x\}[z, \tau]y \geq_{\beta} [x, \tau]\{x\}[z, \tau]x$, but not:

$[x, \tau][y, \tau]\{x\}[z, \tau]y \geq_{\beta} [x, \tau][z, \tau]x$.

Theorem 5.18. If $P\{A\}[x, B]C \in \Delta$, then $P\{A\}[x, B]C \geq_{\beta}^1 P(x:=A)C$.

Proof. This is a consequence of the following:

$(P\{A\}[x, B]C) \mid \{A\}[x, B]C \equiv Q\{A\}[x, B]C \geq_{\beta}^1 Q(x:=A)C$. Apply Th.5.17. $\square$

Theorem 5.19. If QK, QM and $Q'M \in \Delta$, $QK \geq_{\beta} Q'L$ and $\|Q\| = \|Q'\|$, then $QM \geq_{\beta} Q'M$.

Proof. Along the same lines as the proof of Th. 5.16. $\square$

The following theorems are trivial consequences of the preceding.

Theorem 5.20. If $Q[y, K]L \in \Delta$ and $Q[y, K]L \geq_{\beta} Q[y, K']L'$, then $QK \geq_{\beta} QK'$. $\square$

Theorem 5.21. If $QK \in \Delta$, $QK \geq_{\beta} Q'K'$ and $\|Q\| = \|Q'\|$, then $QK \geq_{\beta} QK' \geq_{\beta} Q'K'$ and $QK \geq_{\beta} Q'K \geq_{\beta} Q'K'$. $\square$

We define the beta-equivalence relation as follows:

Definition 5.22. Let A and $B \in \Delta$. We call A *beta-equivalent* to B (denoted: $A \sim_{\beta} B$) if there is an expression C such that $A \geq_{\beta} C$ and $B \geq_{\beta} C$. $\square$

It is clear that beta-equivalence is reflexive and symmetric. The transitivity of beta-equivalence will be proved in Th. 7.35, using Th. 6.43 (in the literature the transitive closure of beta-equivalence is called beta-conversion).

Theorem 5.23. Let QK and $QL \in \Delta$. If $QK \sim_{\beta} QL$, there is an N such that $QK \geq_{\beta} QN$ and $QL \geq_{\beta} QN$.

Proof. Since $QK \sim_{\beta} QL$: $QK \geq_{\beta} A$ and $QL \geq_{\alpha} A$. From Th. 5.12 it follows that $A \equiv Q'N$ with $\|Q\| = \|Q'\|$. Then from Th. 5.21: $QK \geq_{\beta} QN$ and $QL \geq_{\beta} QN$. □

From this theorem, together with Th. 5.15, it easily follows that the monotony rules hold for beta-equivalence (parts (a), (c) and (d) of the following theorem); part (b) follows from Th. 5.23 and Th. 5.16:

Theorem 5.24.

- (a) If $QC, QD, Q\{A\}C, Q\{A\}D \in \Delta$ and $QC \sim_{\beta} QD$, then $Q\{A\}C \sim_{\beta} Q\{A\}D$.
- (b) If $QC, QD, Q[x,A]C, Q[x,A]D \in \Delta$ and $QC \sim_{\beta} QD$, then $Q[x,A]C \sim_{\beta} Q[x,A]D$.
- (c) If $QA, QB, Q\{A\}C, Q\{B\}C \in \Delta$ and $QA \sim_{\beta} QB$, then $Q\{A\}C \sim_{\beta} Q\{B\}C$.
- (d) If $QA, QB, Q[x,A]C, Q[x,B]C \in \Delta$ and $QA \sim_{\beta} QB$, then $Q[x,A]C \sim_{\beta} Q[x,B]C$. □

§ 6. OTHER β -REDUCTIONS

In a β -reduction we eliminate a pair of the form $\{A\}[x,B]$ occurring in an expression, obtaining "copies" of A (to be precise: expressions A' , A'' , etc., which are renovations of A) instead of the non-binding x 's in that expression. We sometimes wish to retain information concerning "past" β -reductions, as a kind of "scar" in an expression.

The easiest way to do this is to maintain the pair $\{A\}[x,B]$ in an expression after β -reduction. We shall formalize this kind of β -reduction, calling it β_1 -reduction. Another β -reduction, called β_2 -reduction, will be introduced especially to eliminate the "scars" $\{A\}[x,B]$ as soon as they are no longer required. We shall show that a β -reduction can be decomposed into a β_1 -reduction and a β_2 -reduction.

We describe in this section β_1 - and β_2 -reduction as a preparation for Section III.3.

The fact that we wish to keep the pair $\{A\}[x, B]$ in an expression after β_1 -reduction complicates matters since we wish each sequence of β -reductions to have a corresponding sequence of β_1 -reductions. For example, a sequence of β -reductions $Q\{A\}\{B\}[x, \tau][y, x]y \geq'_\beta Q\{A\}[y, FrB]y \geq'_\beta QFrA$ should have its counterpart in β_1 -reductions: $Q\{A\}\{B\}[x, \tau][y, x]y \geq'_{\beta_1} Q\{A\}\{B\}[x, \tau][y, FrB]FrA$. Note that in the latter single-step β_1 -reduction we have to ignore the scar $\{B\}[x, \tau]$ located between $\{A\}$ and $[y, FrB]$ (a property of such a scar $\{B\}[x, \tau]$ is that the x does not occur in the expression following it).

However, β_1 -reduction permits more. One may ignore the pair $\{B\}[x, \tau]$ in $Q\{A\}\{B\}[x, \tau][y, x]y$, in spite of the fact that x does occur in $[y, x]y$. This gives the single-step β_1 -reduction: $Q\{A\}\{B\}[x, \tau][y, x]y \geq'_{\beta_1} Q\{A\}\{B\}[x, \tau][y, x]FrA$, by applying $[y, x]y$ to A . This is a real extension of the usual β -reduction concept. We shall call a string like $\{B\}[x, \tau]$ in the above example, which may be located between "function" and "argument" of a β_1 -reduceable expression, a β -chain.

Moreover, we shall agree that the relation $Q\{A\}P[y, C]D \geq'_{\beta_1} Q\{A\}P[y, C](y:=A)D$ (in which P is a β -chain) does only hold if y occurs in D . If we did not require this, one could continue the β_1 -reduction with the latter expression, thus producing a non-terminating β_1 -reduction sequence.

The β_2 -reduction relation, on the other hand, eliminates an applicator and an abstractor as in $Q\{A\}P[y, C]E \geq'_{\beta_2} QPE$ (in which P is again a β -chain); the latter relation only holds, however, if y does *not* occur in E .

We give an inductive definition of β -chain:

Definition 6.1.

- (1) If $P \equiv \phi$, then P is a β -chain.
- (2) If P is a β -chain, then $\{B\}P[x, C]$ is a β -chain.
- (3) If P_1 and P_2 are β -chains, then P_1P_2 is a β -chain. □

Example: $\{A\}\{B\}[x, C]\{D\}[y, E][z, F]$ is a β -chain.

Notation Rule 6.2. We write $\hat{P}$ to indicate that P is a β -chain.

We write $[x, B]C$ to indicate in $[x, B]C$ that $x \neq C$. $\square$

A β -chain $\hat{P}$ has the property that the number of applicators in P is equal to the number of abstractors in P . Moreover, if $\hat{P} \equiv P_1 P_2$, the number of applicators in P_1 is at least equal to the number of abstractors in P_1 .

We can also express this by means of a valuation v of lambda-phrase chains, defined inductively by (1) $v(\phi) = 0$,

(2) $v(\{A\}P) = v(P) + 1$, (3) $v([x, A]P) = v(P) - 1$.

Then for a β -chain $\hat{P}$ it holds that (i) $v(\hat{P}) = 0$, and (ii) if $\hat{P} \equiv P_1 P_2$, then $v(P_1) \geq 0$. These conditions are also sufficient.

The following theorem concerning β -chains can be proved by the aid of the above-mentioned valuation properties:

Theorem 6.3.

(1) If $\hat{P} \neq \phi$, then $\hat{P} \equiv P'\{B\}[x, C]P''$.

(2) $P \equiv P_1 P_3$ is a β -chain if and only if $P' \equiv P_1 \hat{P}_2 P_3$ is a β -chain. $\square$

Note that for each $\hat{P} \neq \phi$ there is a unique decomposition $\hat{P} \equiv \hat{P}_1 \hat{P}_2 \dots \hat{P}_n$ with $\hat{P}_i \neq \phi$ and $\hat{P}_i \neq \hat{P}_i' \hat{P}_i''$ for some $P_i' \neq \phi$ and $P_i'' \neq \phi$.

The following theorem shows that a β -chain has a compact structure:

Theorem 6.4. If $\{A\}\hat{P}[x, B]C \subset \hat{P}_1 F$, then either $\{A\}\hat{P}[x, B]$ is a part of $\hat{P}_1$ (i.e. $\hat{P}_1 \equiv S_1\{A\}\hat{P}[x, B]S_2$), or $\{A\}\hat{P}[x, B]C \subset F$.

Proof. The essential part of the theorem is that the following cannot occur: $P \equiv P_2 P_3$, $P_1 \equiv P_4\{A\}P_2$ and $F \equiv P_3[x, B]C$ (" $\{A\}\hat{P}[x, B]C$ occurs partially in P_1 , partially in F "). This can be proved by the aid of the valuation properties for β -chains. $\square$

We continue with the definitions of single-step β_1 - and β_2 -reduction:

Definition 6.5. Single-step β_1 -reduction is the relation generated by

- (1) If $Q\{A\}\hat{P}[x, B]C \in \Delta$ and $x < C$, then
 $Q\{A\}\hat{P}[x, B]C \geq'_{\beta_1} Q\{A\}\hat{P}[x, B](x:=A)C$, and
- (2) the monotony rules (see Def. 5.5 (2), (3) and (4), reading $\geq'_{\beta_1}$ instead of $\geq'_\beta$). □

Definition 6.6. β_1 -reduction is the reflexive and transitive closure of single-step β_1 -reduction. □

Definition 6.7. Single-step β_2 -reduction is the relation generated by

- (1) If $Q\{A\}\hat{P}[x, B]C \in \Delta$, then $Q\{A\}\hat{P}[x, B]C \geq'_{\beta_2} Q\hat{P}C$, and
- (2) the monotony rules. □

Definition 6.8. β_2 -reduction is the reflexive and transitive closure of single-step β_2 -reduction. □

Note that in Def. 6.7 x may not occur in C .

As in the case of β -reduction (see the previous section), we speak of *elementary* β_1 - or β_2 -reduction, *n-step* β_1 - or β_2 -reduction, and a single-step β_1 - or β_2 -reduction *generated by* $Q\{A\}\hat{P}[x, B]C$ or $Q\{A\}\hat{P}[x, B]C$ respectively. The "length of proof of $K \geq'_{\beta_1} L$ " or " $\dots K \geq'_{\beta_2} L$ " is defined as for single-step β -reduction.

Finally we define β_1 -equivalence ($K \sim_{\beta_1} L$) and β_2 -equivalence ($K \sim_{\beta_2} L$) analogously to β -equivalence (see the previous section).

Theorem 6.9. If $K \in \Delta$ and $K \geq'_{\beta_1} L$ or $K \geq'_{\beta_2} L$, then $L \in \Delta$.

Proof. Induction on the length of proof of $K \geq'_{\beta_1} L$ or $K \geq'_{\beta_2} L$ respectively. Cf. the proof of Th. 5.8. □

Definition 6.10. We shall write $P \geq_{\beta_1} P'$ etc. if $P\tau \geq_{\beta_1} P'\tau$ etc. □

Theorem 6.11. If $QA \in \Delta$ and $QA \geq_{\beta_1}' K$ (or $QA \geq_{\beta_2}' K$), then $K \equiv Q'A'$ with $\|Q\| = \|Q'\|$, and either

- (1) $Q \geq_{\beta_1}' Q'$ (or $Q \geq_{\beta_2}' Q'$ respectively) and $A \equiv A'$, or
- (2) $Q \equiv Q'$ and $QA \geq_{\beta_1}' QA'$ (or $QA \geq_{\beta_2}' QA'$ respectively).

Proof. Cf. the proofs of Th. 5.11 and Th. 5.12. □

Theorem 6.12. The monotony rules hold for β_1 -reduction and for β_2 -reduction.

Proof. Cf. the proof of Th. 5.15. □

Theorem 6.13. If QC, QPC and $QPD \in \Delta$, and $QC \geq_{\beta_1} QD$ (or $QC \geq_{\beta_2} QD$), then $QPC \geq_{\beta_1} QPD$ (or $QPC \geq_{\beta_2} QPD$ respectively).

Proof. Cf. the proof of Th. 5.17. □

The following two theorems deal with the relation between β -reduction on the one hand and β_1 - and β_2 -reduction on the other.

Theorem 6.14. If $K \geq_{\beta_1}' L$, then $K \sim_{\beta} L$.

Proof. Induction on the length of proof of $K \geq_{\beta_1}' L$.

I. $K \geq_{\beta_1}' L$ is $Q\{A\}\hat{P}[x, B]C \geq_{\beta_1}' Q\{A\}\hat{P}[\overset{\circ}{x}, B](x:=A)C$.

Note that, if $\hat{P} \neq \phi$, then $\hat{P} \equiv P'\{B\}[x, C]P''$, and $P_1 \equiv P'P''$ is again a β -chain by Th. 6.3. It follows that there is a β -reduction for K and for L . Continuation of this β -reduction process gives:

$K \geq_{\beta} Q\{A\}[x, B']C' \geq_{\beta} Q(x:=A)C'$, and $L \geq_{\beta} Q\{A\}[\overset{\circ}{x}, B](x:=A)C' \geq_{\beta} Q(x:=A)C'$. In the latter reduction one should note that the substitutions $(y:=D)$ introduced in the reduction $L \geq_{\beta} Q\{A\}[\overset{\circ}{x}, B](x:=A)C'$ do not influence A , since $y \neq A$. Together with the statement that in this case $(y:=D)(x:=A)E \equiv (x:=A)(y:=D)E$ (but for renaming) this results in us obtaining the same C' (but for renaming) in reducing L as we obtained in reducing K . It follows that $K \geq_{\beta} L$.

II. $K \geq'_{\beta_1} L$ is $Q\{A\}C \geq'_{\beta_1} Q\{A\}D$ as a direct consequence of $QC \geq'_{\beta_1} QD$.

Then, by induction: $QC \sim_{\beta} QD$. Hence, by Th. 5.24: $Q\{A\}C \sim_{\beta} Q\{A\}D$.

III, IV. The two other monotony cases are proved similarly to II. $\square$

Theorem 6.15. If $K \in \Delta$ and $K \geq'_{\beta} L$, then $K \geq'_{\beta_1} M \geq'_{\beta_2} L$ or $K \geq'_{\beta_2} L$.

Proof. Let $K \geq'_{\beta} L$ be generated by $Q\{A\}[x, B]C \geq'_{\beta} Q(x:=A)C$.

If $x \in C$, then $Q\{A\}[x, B]C \geq'_{\beta_1} Q\{A\}[\overset{\circ}{x}, B](x:=A)C \geq'_{\beta_2} Q(x:=A)C$.

If $x \notin C$, then $K \geq'_{\beta} L$ is a β_2 -reduction. The remainder follows from the fact that the monotony rules for β -, β_1 - and β_2 -reduction are similar. $\square$

We shall now prove a number of theorems leading to a theorem on the possibility of postponement of β_2 -reductions after β_1 -reductions (Th. 6.19).

Theorem 6.16. If $K \in \Delta$, $K \geq'_{\beta_2} L \geq'_{\beta_1} M$, then $K \geq'_{\beta_1} L' \geq'_{\beta_2} M$ for a certain $n \geq 1$.

Proof. Let $L \geq'_{\beta_1} M$ be generated by $Q\{A\}\hat{P}[x, B]C \geq'_{\beta_1} Q\{A\}\hat{P}[x, B](x:=A)C$, and $K \geq'_{\beta_2} L$ by $Q'\{D\}\hat{P}_1[\overset{\circ}{y}, E]F \geq'_{\beta_2} Q'\hat{P}_1F$. Then $\{A\}\hat{P}[x, B]C \in L$ and $\hat{P}_1F \in L$.² Now we can distinguish three cases: (1) $\{A\}\hat{P}[x, B]C$ and $\hat{P}_1F$ occur in L in disjoint places, (2) $\{A\}\hat{P}[x, B]C \in \hat{P}_1F$, (3) $\hat{P}_1F \in \{A\}\hat{P}[x, B]C$ and $\hat{P}_1F \neq \{A\}\hat{P}[x, B]C$.

(1) In this case it is clear that the theorem holds for $n = 1$.

(2) We may distinguish (see also Th. 6.4):

(a) $\hat{P}_1 \equiv P_2\{G\}P_3$ or $\hat{P}_1 \equiv P_2[z, G]P_3$ and $\{A\}\hat{P}[x, B]C \in G$.

The theorem holds for $n = 1$.

(b) $\hat{P}_1 \equiv P_2\{A\}\hat{P}[x, B]P_3$. Idem.

(c) $\{A\}\hat{P}[x, B]C \in F$. Idem.

These cases (a)-(c) are exhaustive if $\{A\}\hat{P}[x, B]C \in \hat{P}_1F$.

(3) (a) Let $\hat{P}_1F \in A$. Now the theorem holds for n is the number of occurrences of x in C plus one.

- (b) Let $\hat{P} \equiv P_2 \hat{P}_1 P_3$. The theorem holds for $n = 1$.
 (c) Let $\hat{P} \equiv P_2 \{G\} P_3$ or $\hat{P} \equiv P_2 [z, G] P_3$ and $\hat{P}_1 F \subset G$. Idem.
 (d) Let $\hat{P}_1 F \subset [x, B] C$. Idem.

These cases (a)-(d) are exhaustive if $\hat{P}_1 F \subset \{A\} \hat{P} [x, B] C$ and $\hat{P}_1 F \neq \{A\} \hat{P} [x, B] C$. (In this proof we several times use the lemma: "If $\hat{P} \equiv P_1 \hat{P}_2 P_3$, then $P_1 \{G\} \hat{P}_2 [z, H] P_3$ is also a β -chain", which is a consequence of Th. 6.3.) $\square$

Theorem 6.17. If $K \in \Delta$, $K \geq_{\beta_2} L \geq_{\beta_1}^p M$, then $K \geq_{\beta_1}^p L' \geq_{\beta_2} M$.

Proof. Induction on the number of steps of $K \geq_{\beta_2} L$, using Th. 6.16. $\square$

Theorem 6.18. If $K \in \Delta$, $K \geq_{\beta_2} L \geq_{\beta_1}^p M$, then $K \geq_{\beta_1}^p L' \geq_{\beta_2} M$.

Proof. Induction on p , using Th. 6.17. $\square$

Theorem 6.19. If $K_1 \geq' K_2 \geq' \dots \geq' K_n$ by single-step β_1 - and β_2 -reductions, the total number of β_1 -reductions being p , there is a reduction $K_1 \equiv L \geq_{\beta_1}^p M \geq_{\beta_2} N \equiv K_n$.

Proof. Combine the successive single-step β_1 -reductions in $K_1 \geq' K_2 \geq' \dots \geq' K_n$, and do the same with the successive single-step β_2 -reductions: we obtain $K_1 \equiv L_1 \geq_{\beta_2} M_1 \geq_{\beta_1} L_2 \geq_{\beta_2} M_2 \geq_{\beta_1} \dots \geq_{\beta_1} L_\ell \geq_{\beta_2} M_\ell \geq_{\beta_1} K_n$. Induction on ℓ yields the proof. $\square$

We shall now prove what we call the *Church-Rosser property* (CR) for β_1 -reduction, which we formulate as follows: If $K \geq_{\beta_1} L$ and $K \geq_{\beta_1} M$, there is an N such that $L \geq_{\beta_1} N$ and $M \geq_{\beta_1} N$. We can express this Church-Rosser property in a diagram, as follows:

Now Tait and Martin-Löf defined a new "single"-step reduction (which we shall call single-step *nested* reduction to avoid confusion). The latter reduction has the property that with each pair of single-step nested reductions " $K \geq^* L$ " and " $K \geq^* M$ " there can be found an N such that " $L \geq^* N$ " and " $M \geq^* N$ ", both last-mentioned reductions being single-step nested reductions as well. Moreover, each multiple-step reduction can be decomposed into single-step nested reductions and each single-step nested reduction is a composition of (ordinary) single-step reductions.

If one now begins with multiple-step reductions " $K \geq L$ " and " $K \geq M$ ", one can decompose these reductions into single-step nested reductions and apply the above. Then one obtains, for example, a situation as is expressed in the following diagram:

In each of these restangles the four sides represent single-step nested reductions.

Moreover, the nested reductions " $L \geq^* L' \geq^* N$ " and " $M \geq^* M' \geq^* N$ " can be decomposed into (ordinary) single-step reductions, which combine into " $L \geq N$ " and " $M \geq N$ ". Thus we obtain CR.

In the following we shall define a single-step nested β_1 -reduction, which we shall call single-step γ -reduction. Our γ -reduction is a little more complex than the nested reduction of Tait and Martin-Löf, but it yields essentially the same results.

The "nested" character of γ -reduction can be explained as follows. Let a β_1 -reduction be generated by $Q\{A\}\hat{P}[x,B]C$, let

$QA \geq_Y QA'$ and $Q\hat{P}[x, B]C \geq_Y Q\hat{P}'[x, B']C'$. Then also $Q\{A\}\hat{P}[x, B]C \geq_Y Q\{A'\}\hat{P}'[x, B'](x:=A')C'$ (if the suggested β_1 -reduction is preceded by single-step nested reductions "inside" A, P, B and C, the composite reduction is a single-step nested reduction). The reductions take place in a "nested" order.

With the aid of γ -reduction we shall prove CR for β_1 -reduction. We shall subsequently prove CR for β -reduction, using CR for β_1 -reduction (we could also prove CR for β -reductions directly, along the same lines pointed out by Tait and Martin-Löf).

Definition 6.20. Single-step γ -reduction, denoted by $\geq_Y'$, is the reflexive relation generated by

- (1) If $Q\{A\}\hat{P}[x, B]C \in \Delta$, $x \in C$, $QA \geq_Y QA'$ and $Q\hat{P}[x, B]C \geq_Y Q\hat{P}'[x, B']C'$, then $Q\{A\}\hat{P}[x, B]C \geq_Y Q\{A'\}\hat{P}'[x, B'](x:=A')C'$.
- (2) If $Q\{A\}C$ and $Q\{A'\}C' \in \Delta$, $QA \geq_Y QA'$ and $QC \geq_Y QC'$, then $Q\{A\}C \geq_Y Q\{A'\}C'$.
- (3) If $Q[x, A]C$ and $Q[x, A']C' \in \Delta$, $QA \geq_Y QA'$ and $Q[x, A]C \geq_Y Q[x, A']C'$, then $Q[x, A]C \geq_Y Q[x, A']C'$.
- (4) If $A \in \Delta$, $A \geq_Y B$ and $B \geq_\alpha B'$, then $A \geq_Y B'$. □

We call (2) and (3) the *monotony rules* for single-step γ -reduction. We also define γ -equivalence ($K \sim_Y L$) analogously to β -equivalence.

Definition 6.21. γ -reduction, denoted by $\geq_Y$, is the transitive closure of single-step γ -reduction. □

We continue with some theorems concerning γ -reduction (it will be clear that $Q \geq_Y Q'$ if and only if $Q\tau \geq_Y Q'\tau$).

Theorem 6.22. If $QA \in \Delta$ and $QA \geq_Y K$, then $K \equiv Q'A'$, where $\|Q\| = \|Q'\|$, $Q \geq_Y Q'$ and $QA \geq_Y QA'$.

Proof. Induction on the length of proof of $QA \geq_Y K$.

- (1) If $QA \geq_Y K$ by reflexivity, the theorem is trivial.
- (2) If $QA \geq_Y K$ is $Q_0\{B\}\hat{P}[x, C]D \geq_Y Q_0\{B'\}\hat{P}'[x, C'](x:=B')D'$ as a

- direct consequence of $Q_0B \geq_Y' Q_0B'$ and $Q_0\hat{P}[x,C]D \geq_Y' Q_0\hat{P}'[x,C']D'$, then $\|Q\| \leq \|Q_0\|$, so $Q_0 \equiv QQ''$, and the theorem follows.
- (3) If $QA \geq_Y' K$ is $Q_0\{B\}C \geq_Y' Q_0\{B'\}C'$ as a direct consequence of $Q_0B \geq_Y' Q_0B'$ and $Q_0C \geq_Y' Q_0C'$, then again $\|Q\| \leq \|Q_0\|$ and the theorem follows.
- (4) Let $QA \geq_Y' K$ be $Q_0[x,B]C \geq_Y' Q_0[x,B']C'$ as a direct consequence of $Q_0B \geq_Y' Q_0B'$ and $Q_0[x,B]C \geq_Y' Q_0[x,B']C'$.
- (i) If $\|Q\| \leq \|Q_0\|$ or $Q \equiv Q_0[x,B]$ then the theorem follows.
- (ii) If $C \equiv [y_1, B_1] \dots [y_n, B_n]C_0$ and $Q \equiv Q_0[x,B][y_1, B_1] \dots [y_n, B_n]$, then $QA \equiv Q_0[x,B]C \equiv Q_0[x,B][y_1, B_1] \dots [y_n, B_n]A \geq_Y' Q_0[x,B]C' \equiv Q_0[x,B][z_1, B'_1] \dots [z_n, B'_n]A'$ (by induction) with $QA \geq_Y' QA'$ and $Q \geq_Y' Q_0[x,B][z_1, B'_1] \dots [z_n, B'_n]$. It follows that $Q \geq_Y' Q_0[x,B'][z_1, B'_1] \dots [z_n, B'_n]$. Also $C' \equiv [z_1, B'_1] \dots [z_n, B'_n]A'$, so $Q_0[x,B']C' \equiv Q_0[x,B'][z_1, B'_1] \dots [z_n, B'_n]A'$. Consequently the theorem holds if we take $Q' \equiv Q_0[x,B'][z_1, B'_1] \dots [z_n, B'_n]$.
- (5) Let $QA \geq_Y' K$ be a direct consequence of $QA \geq_Y' K'$ and $K' \geq_\alpha K$. Then by induction the theorem holds for $QA \geq_Y' K'$, and trivially also for $QA \geq_Y' K$. $\square$

Theorem 6.23. The monotony rules hold for γ -reduction.

Proof. Cf. the proof of Th. 5.15. $\square$

Theorem 6.24. If QC, PC and $PD \in \Delta$, and $QC \geq_Y' QD$, then $PC \geq_Y' PD$.

Proof. Analogous to the proof of Th. 5.17. $\square$

The following two theorems deal with the relation between β_1 - and γ -reduction.

Theorem 6.25. If $K \in \Delta$ and $K \geq_{\beta_1}' L$, then $K \geq_Y' L$.

Proof. Induction on the length of proof of $K \geq_{\beta_1}' L$. The rule of elementary β_1 -reduction is covered by Def. 6a.11(1) (take $QA \geq_Y' QA'$ to be $QA \geq_Y' QA$ by reflexivity, etc.), the monotony rules for β_1 -

reduction are covered by the monotony rules for γ -reduction (again using the reflexivity of γ -reduction in appropriate places). $\square$

Theorem 6.26. If $K \in \Delta$ and $K \geq_Y' L$, then $K \geq_{\beta_1} L$ but for α -reduction.

Proof. Induction on the length of proof of $K \geq_Y' L$. For example: let $K \geq_Y' L$ be $Q\{A\}\hat{P}[x,B]C \geq_Y' Q\{A'\}\hat{P}'[x,B'](x:=A')C'$ as a direct consequence of $x < C$, $QA \geq_Y' QA'$ and $QP[x,B]C \geq_Y' QP'[x,B']C'$.

By induction the last two reductions can also be obtained by β_1 -reductions and α -reductions, and $Q\{A\}\hat{P}[x,B]C \geq_{\beta_1} Q\{A\}\hat{P}'[x,B']C' \geq_{\beta_1} Q\{A'\}\hat{P}'[x,B']C' \geq_{\beta_1} Q\{A'\}\hat{P}'[x,B'](x:=A')C'$ (but for α -reduction).

In the last β_1 -reduction we use the lemma: "If $x < C$, if x occurs as a binding variable in Q_1 and if $Q_1C \geq_{\beta_1} Q_1C'$, then $x < C'$ ". $\square$

Theorem 6.27. If $K \in \Delta$ and $K \geq_Y L$, then $L \in \Delta$.

Proof. Follows from Th. 6.26 and Th. 6.9. $\square$

We inductively define *similarity* of two lambda-phrase chains (not necessarily β -chains):

Definition 6.28.

- (1) If $P_1 \equiv \phi$, then P_1 and P_2 are similar.
- (2) If P_1 and P_2 are similar, then $\{A\}P_1$ and $\{B\}P_2$ are similar, and $[x,A]P_1$ and $[x,B]P_2$ are similar. $\square$

The following theorems are a preparation for Th. 6.37, which expresses CR for γ -reduction. In order to prove Cor. 6.31 and Th. 6.34 it is convenient to extend the notion of β -chain, as in Def. 6.29: a number of β -chains, connected by abstractors, will be called a β -chain complex. A β -chain complex may be empty.

Definition 6.29. Let $P_1, P_2, \dots, P_i$ be (possibly empty) β -chains. Then a lambda chain $P_1[x_1, A_1]P_2[x_2, A_2] \dots P_{i-1}[x_{i-1}, A_{i-1}]P_i$ is called a β -chain complex. $\square$

We denote a β -chain complex P by $\bar{P}$.

The following statement can be proved by the aid of valuations:
If P is a β -chain complex and $P \equiv P_1 P_2$, then P_2 is a β -chain complex.

Theorem 6.30. If $Q\bar{P}A \in \Delta$ and $Q\bar{P}A \geq_Y' K$, then $K \equiv Q'\bar{P}'A'$, where $\|Q\| = \|Q'\|$, $Q \geq_Y' Q'$, $Q\bar{P}A \geq_Y' Q\bar{P}'A'$ and P and P' are similar.

Proof. If $P \equiv Q_1$ (including $P \equiv \phi$), then Th. 6.22 gives the proof. Let $P \neq Q_1$. We proceed with induction on the length of proof of $Q\bar{P}A \geq_Y' K$. Note that there must be at least one applicator in the lambda chain P , on account of our assumption $P \neq Q_1$.

(1) Assume that $Q\bar{P}A \geq_Y' K$ by reflexivity. The proof is now trivial.

(2) Assume that $Q\bar{P}A \geq_Y' K$ is $QQ_1\{C\}\hat{P}[y,D]E \geq_Y' QQ_1\{C'\}\hat{P}'[y,D'](y:=C')E'$ as a direct consequence of $QQ_1C \geq_Y' QQ_1C'$ and $QQ_1\hat{P}[y,D]E \geq_Y' QQ_1\hat{P}'[y,D']E'$. Now it must hold that $E \equiv \bar{P}_2A$, while $Q_1\hat{P}[y,D]P_2$ is a β -chain complex. By induction: $E' \equiv \bar{P}_2'A'$ and P_2 and P_2' are similar. The remainder is easy.

(3) Assume that $Q\bar{P}A \geq_Y' K$ is $QQ_1\{C\}D \geq_Y' QQ_1\{C'\}D'$ as a direct consequence of $QQ_1C \geq_Y' QQ_1C'$ and $QQ_1D \geq_Y' QQ_1D'$. Now $\bar{P} \equiv Q_1\{C\}\bar{P}_2$ and $D \equiv \bar{P}_2A$. Here $\bar{P}_2$ is a β -chain complex, hence by induction $D' \equiv \bar{P}_2'A'$, in which P_2 and P_2' are similar. The remainder follows easily.

(4a) Assume that $Q\bar{P}A \geq_Y' K$ is $QQ_1[y,C]D \geq_Y' QQ_1[y,C']D'$ as a direct consequence of $QQ_1C \geq_Y' QQ_1C'$ and $QQ_1[y,C]D \geq_Y' QQ_1[y,C']D'$. Then $\bar{P} \equiv Q_1[y,C]\bar{P}_2$ and $D \equiv \bar{P}_2B$. The completion of the proof is similar to that in the last part of the previous case.

(4b) Assume that $Q\bar{P}A \geq_Y' K$ is $Q_0[y,C]Q_1\bar{P}A \geq_Y' Q_0[y,C']Q_1D'$ as a direct consequence of $Q_0C \geq_Y' Q_0C'$ and $Q_0[y,C]Q_1\bar{P}A \geq_Y' Q_0[y,C']Q_1D'$. Then, by induction, $D' \equiv Q_1'\bar{P}'A'$ with $\|Q_1\| = \|Q_1'\|$, while $\bar{P}$ and $\bar{P}'$ are similar. The remainder follows.

(5) Assume that $Q\bar{P}A \geq_Y' K$ is a direct consequence of $Q\bar{P}A \geq_Y' K'$ and $K' \geq_\alpha K$. The proof is again by induction. $\square$

Corollary 6.31. If $Q\hat{P}A \in \Delta$ and $Q\hat{P}A \geq_Y' K$, then $K \equiv Q'\hat{P}'A'$, where $\|Q\| = \|Q'\|$, $Q \geq_Y' Q'$, $Q\hat{P}A \geq_Y' Q\hat{P}'A'$, while P and P' are similar. $\square$

The following four theorems are lemmas for Th. 6.36. Th. 6.34 might have been called "the substitution lemma for γ -reduction".

Theorem 6.32.

(1) Let $Q\hat{P}[x, B]C \in \Delta$ and $Q\hat{P}[x, B]C \geq_Y' K$. Then $K \equiv Q'\hat{P}'[x, B']C'$ such that $Q \geq_Y' Q'$, $Q\hat{P}[x, B]C \geq_Y' Q\hat{P}'[x, B']C'$, $\|Q\| = \|Q'\|$, while P and P' are similar.

(2) Let $Q\{A\}\hat{P}[x, B]C \in \Delta$ and $Q\{A\}\hat{P}[x, B]C \geq_Y' K$. Then either (i) $K \equiv Q'\{A'\}\hat{P}'[x, B']C'$, or (ii) $K \equiv Q'\{A'\}\hat{P}'[x, B'](x:=A')C'$, where in both cases $Q \geq_Y' Q'$, $QA \geq_Y' QA'$, $Q\hat{P}[x, B]C \geq_Y' Q\hat{P}'[x, B']C'$ and $\|Q\| = \|Q'\|$, while P and P' are similar.

(3) Let $Q\{A\}B \in \Delta$ and $Q\{A\}B \geq_Y' K$. Then either (i) $K \equiv Q'\{A'\}B'$ where $Q \geq_Y' Q'$, $QA \geq_Y' QA'$, $QB \geq_Y' QB'$ and $\|Q\| = \|Q'\|$, or (ii) $Q\{A\}B \equiv Q\{A\}\hat{P}[x, C]D$, $x \in C$, $K \equiv Q'\{A'\}\hat{P}'[x, C']D'$, $Q \geq_Y' Q'$, $\|Q\| = \|Q'\|$, $QA \geq_Y' QA'$ and $Q\hat{P}[x, C]D \geq_Y' Q\hat{P}'[x, C']D'$.

Proof. See Cor. 6.31 and the possibilities for single-step γ -reduction. $\square$

Theorem 6.33. If $Q\hat{P}[x, B]C \in \Delta$, $Q\hat{P}[x, B]C \geq_Y' Q'\hat{P}'[x, B']C'$ and $x \in C$, then $x \in C'$.

Proof. In a subexpression we can only eliminate free variables by substitution, and substitution can only originate from a β_1 -reduction. Note that a β_1 -reduction yielding a substitution $(x:=A)$ can not occur in the above. $\square$

Theorem 6.34. Let QA and $Q\bar{P}B \in \Delta$, let no binding variable of $\bar{P}B$ occur in QA and let x not occur in $\bar{P}$. Let $Q\bar{P}B \geq_Y' Q\bar{P}'B'$ (where $\|P\| = \|P'\|$) and $QA \geq_Y' QA'$. Then $Q\bar{P}(x:=A)B \geq_Y' Q\bar{P}'(x:=A')B'$.

Proof. First consider the case that $P \equiv Q_1$, so $P' \equiv Q'_1$.

We prove the theorem by induction on $|B|$. If $B \equiv y \neq x$ or $B \equiv \tau$ then the theorem is trivial. If $B \equiv x$, note that $QQ_1A \geq_Y' QQ'_1A'$, so also $QQ_1A \geq_Y' QQ'_1A'$ by induction on the length of proof of $QQ_1x \geq_Y' QQ'_1x$ and monotony rule (3) for single-step γ -reduction.

(1) Let $B \equiv [y, E]F$. Then $B' \equiv [y, E']F'$ by Th. 6.32(1),
 $QQ_1 E \geq_Y' QQ_1 E'$ and $QQ_1 [y, E]F \geq_Y' QQ_1 [y, E']F'$. So also (by induction)
 $QQ_1 (x:=A)E \geq_Y' QQ_1 (x:=A')E'$ and $QQ_1 [y, E](x:=A)F \geq_Y' QQ_1 [y, E](x:=A')F'$.
 It easily follows from the latter reduction that

$QQ_1 [y, (x:=A)E](x:=A)F \geq_Y' QQ_1 [y, (x:=A)E](x:=A')F'$. It follows that
 $QQ_1 (x:=A)B \geq_Y' QQ_1 (x:=A')B'$, hence also $QQ_1 (x:=A)B \geq_Y' QQ_1 (x:=A')B'$.

(2) Let $B \equiv \{E\}F$. Now by Th. 6.32(3) either $B' \equiv \{E'\}F'$ with
 $QQ_1 E \geq_Y' QQ_1 E'$ and $QQ_1 F \geq_Y' QQ_1 F'$, or $B \equiv \{E\}\hat{P}_1[y, G]H$, and
 $QQ_1 B \equiv QQ_1 \{E\}\hat{P}_1[y, G]H \geq_Y' QQ_1 \{E'\}\hat{P}_1[y, G'](y:=E')H' \equiv QQ_1 B'$.
 In the first case the proof is similar to the proof in case (1).
 In the second case we can follow analogous lines, using the fact
 that $(y:=(x:=A')E')(x:=A')H' \equiv (x:=A')(y:=E')H'$ but for re-
 naming.

Now consider the case that $P \neq Q_1$, whence at least one appli-
 cator must occur in chain P . We proceed with induction on the
 length of proof of $Q\bar{P}B \geq_Y' Q\bar{P}'B'$.

(1) Assume that $Q\bar{P}B \geq_Y' Q\bar{P}'B'$ by reflexivity. This case can be
 proved similarly to the case that $\bar{P} \equiv Q_1$.

(2) Assume that $Q\bar{P}B \geq_Y' Q\bar{P}'B'$ is $QQ_1 \{C\}\hat{P}_1[y, D]E \geq_Y'$
 $QQ_1 \{C'\}\hat{P}_1[y, D'](y:=C')E'$ as a direct consequence of $QQ_1 C \geq_Y' QQ_1 C'$
 and $QQ_1 \hat{P}_1[y, D]E \geq_Y' QQ_1 \hat{P}_1[y, D']E'$. Now it must hold that $E \equiv \bar{P}_2 B$
 and $E' \equiv \bar{P}_2' B'$, where $B' \equiv (y:=C')B''$. Since $Q_1 \hat{P}_1[y, D]\bar{P}_2$ is also a
 chain complex, it follows by induction:

$QQ_1 \hat{P}_1[y, D]\bar{P}_2(x:=A)B \geq_Y' QQ_1 \hat{P}_1[y, D']\bar{P}_2'(x:=A')B''$, so $Q\bar{P}(x:=A)B \equiv$
 $QQ_1 \{C\}\hat{P}_1[y, D]\bar{P}_2(x:=A)B \geq_Y' QQ_1 \{C'\}\hat{P}_1[y, D'](y:=C')(\bar{P}_2'(x:=A')B'') \equiv$
 $QQ_1 \{C'\}\hat{P}_1[y, D']((y:=C')\bar{P}_2)(x:=A')(y:=C')B'' \equiv Q\bar{P}'(x:=A')B'$ (here we
 changed $(y:=C')(x:=A')B''$ into $(x:=A')(y:=C')B''$, which is allowed
 by Th. 6.33 and by the conditions imposed upon the variables).

(3) Assume that $Q\bar{P}B \geq_Y' Q\bar{P}'B'$ is $QQ_1 \{C\}D \geq_Y' QQ_1 \{C'\}D'$ as a direct
 consequence of $QQ_1 C \geq_Y' QQ_1 C'$ and $QQ_1 D \geq_Y' QQ_1 D'$. Then $\bar{P} \equiv Q_1 \{C\}\bar{P}_2$
 and $\bar{P}' \equiv Q_1 \{C'\}\bar{P}_2'$; $D \equiv \bar{P}_2 B$ and $D' \equiv \bar{P}_2' B'$. By induction:

$QQ_1 \bar{P}_2(x:=A)B \geq_Y' QQ_1 \bar{P}_2'(x:=A')B'$, so also $Q\bar{P}(x:=A)B \geq_Y' Q\bar{P}'(x:=A')B'$.

(4) Assume that $Q\bar{P}B \geq_Y' Q\bar{P}'B'$ is $QQ_1 [y, C]D \geq_Y' QQ_1 [y, C']D'$ as a
 direct consequence of $QQ_1 C \geq_Y' QQ_1 C'$ and $QQ_1 [y, C]D \geq_Y' QQ_1 [y, C']D'$.

Then $\bar{P} \equiv Q_1[y, C] \bar{P}_2$ and $\bar{P}' \equiv Q_1[y, C'] \bar{P}_2'$; $D \equiv \bar{P}_2 B$ and $D' \equiv \bar{P}_2' B'$.

The remainder of the proof is analogous to that in case (3).

(5) Assume that $Q\bar{P}B \geq_Y' Q\bar{P}'B'$ is a direct consequence of

$Q\bar{P}B \geq_Y' Q''\bar{P}''B''$ and $Q''\bar{P}''B'' \geq_\alpha Q\bar{P}'B'$. Then $Q\bar{P}(x:=A)B \geq_Y' Q''\bar{P}''(x:=A')B''$ by induction; the remainder is easy. $\square$

Theorem 6.35. If QA and $Q'A' \in \Delta$, $Q \geq_Y' Q'$ and $QA \geq_Y' QA'$, then $QA \geq_Y' Q'A'$.

Proof. Induction on $\|Q\|$, using monotony rule (3) of single-step γ -reduction. $\square$

Theorem 6.36. If $K \in \Delta$, $K \geq_Y' L$ and $K \geq_Y' M$, there is an N such that $L \geq_Y' N$ and $M \geq_Y' N$.

Proof. Induction on the length of proof of $K \geq_Y' L$.

We shall use Th. 6.24 and Th. 6.35 several times without saying so.

(1) Let $K \geq_Y' L$ by reflexivity. Take $N \equiv M$.

(2) Let $K \geq_Y' L$ be $Q\{\hat{A}\}\hat{P}[x, B]C \geq_Y' Q\{\hat{A}'\}\hat{P}'[x, B']C'(x:=A')C'$ as a direct consequence of $QA \geq_Y' QA'$ and $Q\hat{P}[x, B]C \geq_Y' Q\hat{P}'[x, B']C'$. Now by Th.

6.32(2) M can have the form (i) $M \equiv Q''\{\hat{A}''\}\hat{P}''[x, B'']C''$ or (ii)

$M \equiv Q''\{\hat{A}''\}\hat{P}''[x, B''](x:=A'')C''$, where in both cases $Q \geq_Y' Q''$,

$QA \geq_Y' QA''$, $Q\hat{P}[x, B]C \geq_Y' Q\hat{P}''[x, B'']C''$, $\|Q\| = \|Q''\|$ and $\|P\| = \|P''\|$.

By induction and by Th. 6.22 there is an A''' such that $QA' \geq_Y' QA'''$

and $QA'' \geq_Y' QA'''$. Again by induction and by Th. 6.32(1) there are

$\hat{P}'''$, B''' and C''' such that $Q\hat{P}'[x, B']C' \geq_Y' Q\hat{P}'''[x, B''']C'''$ and

$Q\hat{P}''[x, B'']C'' \geq_Y' Q\hat{P}'''[x, B''']C'''$. By Th. 6.34:

$Q\hat{P}'[x, B'](x:=A')C' \geq_Y' Q\hat{P}'''[x, B'''](x:=A''')C'''$, hence by monotony

and Th. 6.35: $L \equiv Q\{\hat{A}'\}\hat{P}'[x, B'](x:=A')C' \geq_Y' Q''\{\hat{A}'''\}\hat{P}'''[x, B''']$

$(x:=A''')C'''$. Call the latter expression N .

It also holds that $Q''\hat{P}''[x, B'']C'' \geq_Y' Q''\hat{P}'''[x, B''']C'''$, and $x < C'''$ by

Th. 6.33. So $Q''\{\hat{A}''\}\hat{P}''[x, B'']C'' \geq_Y' N \equiv Q''\{\hat{A}'''\}\hat{P}'''[x, B'''](x:=A''')C'''$

by an elementary γ -reduction. This completes this part of the

proof in case (i). In case (ii) we first establish that

$Q\hat{P}''[x, B''](x:=A'')C'' \geq_Y' Q\hat{P}'''[x, B'''](x:=A''')C'''$ by Th. 6.34, yielding

by monotony that $M \geq_Y' N$.

(3) Let $K \geq_Y' L$ be $Q\{A\}C \geq_Y' Q\{A'\}C'$ as a direct consequence of $QA \geq_Y' QA'$ and $QC \geq_Y' QC'$. Now by Th. 6.32(3) M can have the form (i) $M \equiv Q''\{A''\}C''$, with $Q \geq Q''$, $\|Q\| = \|Q''\|$, $QA \geq_Y' QA''$ and $QC \geq_Y' QC''$, or (ii) $M \equiv Q''\{A''\}\hat{P}''[x,D''](\dot{x}:=A'')E''$, where $K \equiv Q\{A\}\hat{P}[x,D]E$, $x \in E$, $QA \geq_Y' QA''$, $Q \geq_Y' Q''$ and $Q\hat{P}[x,D]E \geq_Y' Q\hat{P}''[x,D'']E''$.

In case (i) we can find by induction A''' and C''' such that $QA' \geq_Y' Q''A'''$, $Q''A'' \geq_Y' Q''A'''$, $QC' \geq_Y' Q''C'''$ and $Q''C'' \geq_Y' Q''C'''$ and we can take $N \equiv Q''\{A'''\}C'''$. In case (ii) we are in a position similar to case (i) of (2), with L and M permuted.

(4) Let $K \geq_Y' L$ be $Q[x,A]C \geq_Y' Q[x,A']C'$ as a direct consequence of $QA \geq_Y' QA'$ and $Q[x,A]C \geq_Y' Q[x,A']C'$. Then $M \geq_\alpha Q''[x,A'']C''$ by Th. 6.32(1), where $Q \geq_Y' Q''$, $QA \geq_Y' QA''$ and $Q[x,A]C \geq_Y' Q[x,A']C'$. Take $N \equiv Q''[x,A''']C'''$, where A''' and C''' are obtained as in (3), case (i).

(5) Let $K \geq_Y' L$ as a direct consequence of $K \geq_Y' L'$ and $L' \geq_\alpha L$. Then by induction we can find an N such that $L' \geq_Y' N$ and $M \geq_Y' N$, and also $L \geq_Y' N$. □

Theorem 6.37 (CR for γ -reduction). If $K \in \Delta$, $K \geq_Y L$ and $K \geq_Y M$, then $L \approx_Y M$.

Proof. This is a consequence of the previous theorem. □

Theorem 6.38 (CR for β_1 -reduction). If $K \in \Delta$, $K \geq_{\beta_1} L$ and $K \geq_{\beta_1} M$, then $L \sim_{\beta_1} M$ but for α -reduction.

Proof. Decompose $K \geq_{\beta_1} L$ and $K \geq_{\beta_1} M$, apply Th. 6.25, Th. 6.37 and Th. 6.26: we obtain an N' such that $L \geq_{\beta_1} N \geq_\alpha N'$ and $M \geq_{\beta_1} N'' \geq_\alpha N'$. □

Theorem 6.39. If $K \in \Delta$, $K \geq_{\beta_1} L$ and $K \geq_{\beta_2} M$, then there is an n such that $M \geq_{\beta_1} N$ and $L \geq_{\beta_2}^n N$ with $n \geq 1$.

Proof. Let $K \geq_{\beta_1} L$ be generated by $Q\{A\}\hat{P}[x,B]C \geq_{\beta_1} Q\{A\}\hat{P}[\dot{x},B](\dot{x}:=A)C$, and $K \geq_{\beta_2} M$ by $Q'\{D\}\hat{P}_1[\dot{y},E]F \geq_{\beta_2} Q'\hat{P}_1F$. If $\{D\}\hat{P}_1[\dot{y},E]F \in A$, we need n

β_2 -reductions for the n A's in $Q\{A\}\hat{P}[\overset{\circ}{x}, B](x:=A)C$. If not, we need only one. The theorem easily follows. $\square$

Theorem 6.40. If $K \in \Delta$, $K \geq_{\beta_1} L$ and $K \geq_{\beta_2} M$, there is an N such that $M \geq_{\beta_1} N$ and $L \geq_{\beta_2} N$.

Proof. Apply Th. 6.39 repeatedly. This can be illustrated by the following diagram

Here we assume that $K \geq_{\beta_1} L$ can be decomposed into $K \geq_{\beta_1}^* L' \geq_{\beta_1}^* L$, and $K \geq_{\beta_2} M$ into $K \geq_{\beta_2}^* M' \geq_{\beta_2}^* M$. In the diagram all edges (in the sense usual in graph theory) parallel to the edge from K to L' represent single-step β_1 -reductions, those in the direction of $K \geq_{\beta_2}^* M'$ represent single-step β_2 -reductions. $\square$

Theorem 6.41. If $K \in \Delta$, $K \geq_{\beta_2}^* L$ and $K \geq_{\beta_2}^* M$, there is an N such that $L \geq_{\beta_2}^* N$ or $L \equiv N$, and $M \geq_{\beta_2}^* N$ or $M \equiv N$.

Proof. Let $K \geq_{\beta_2}^* L$ be generated by $Q\{A\}\hat{P}[\overset{\circ}{x}, B]C \geq_{\beta_2}^* Q\hat{P}C$, and $K \geq_{\beta_2}^* M$ by $Q'\{D\}\hat{P}_1[\overset{\circ}{y}, E]F \geq_{\beta_2}^* Q'\hat{P}_1F$. If $\{D\}\hat{P}_1[\overset{\circ}{y}, E]F \subset A$ or $\subset B$ then $M \geq_{\beta_2}^* L$; if $\{A\}\hat{P}[\overset{\circ}{x}, B]C \subset D$ or $\subset E$, then $L \geq_{\beta_2}^* M$. If $\{D\}\hat{P}_1[\overset{\circ}{y}, E]F \equiv \{A\}\hat{P}[\overset{\circ}{x}, B]C$ then $L \equiv M$. In all other cases there is clearly an N such that $L \geq_{\beta_2}^* N$ and $M \geq_{\beta_2}^* N$. $\square$

Theorem 6.42 (CR for β_2 -reduction). If $K \in \Delta$, $K \geq_{\beta_2} L$ and $K \geq_{\beta_2} M$, then $L \sim_{\beta_2} M$.

Proof. Apply Th. 6.41 repeatedly. □

Theorem 6.43 (CR for β -reduction). If $K \in \Delta$, $K \geq_{\beta} L$ and $K \geq_{\beta} M$, then $L \sim_{\beta} M$.

Proof. Decompose $K \geq_{\beta} L$ and $K \geq_{\beta} M$, according to Th. 6.19, into $K \geq_{\beta_1} L' \geq_{\beta_2} L$ and $K \geq_{\beta_1} M' \geq_{\beta_2} M$ respectively. The remainder of the proof is illustrated by the following diagram:

We find N' from Th. 6.38, N'' and N''' from Th. 6.40 and finally N from Th. 6.42. □

§ 7. η -REDUCTION, REDUCTION AND LAMBDA-EQUIVALENCE

A third reduction in lambda-calculus (apart from α - and β -reduction) is called η -reduction and denoted by $\geq_{\eta}$. We shall incorporate it in our system.

We first define *single-step η -reduction*, denoted by $\geq'_{\eta}$:

Definition 7.1. Single-step η -reduction is the relation generated by:

(1) If $Q[x, A]\{x\}B \in \Delta$ and $x \notin B$, then $Q[x, A]\{x\}B \geq'_{\eta} QB$.

- (2) Let $Q\{A\}C$ and $Q\{A\}D \in \Delta$. If $QC \geq'_\eta QD$, then $Q\{A\}C \geq'_\eta Q\{A\}D$.
- (3) Let $Q[x,A]C$ and $Q[x,B]C \in \Delta$. If $QA \geq'_\eta QB$, then $Q[x,A]C \geq'_\eta Q[x,B]C$.
- (4) Let $Q\{A\}C$ and $Q\{B\}C \in \Delta$. If $QA \geq'_\eta QB$, then $Q\{A\}C \geq'_\eta Q\{B\}C$. $\square$

Rules (2), (3) and (4) are called the *monotony rules* of single-step η -reduction; they are similar to those of single-step β -reduction. Rule (1) is called the rule of *elementary η -reduction*.

Definition 7.2. η -reduction is the reflexive and transitive closure of single-step η -reduction. $\square$

If A and B are related by a (single-step) η -reduction, we speak of "the (single-step) η -reduction $A \geq'_\eta B$ ". The notions *n-step η -reduction* and *decomposition* of an η -reduction are defined analogously to the corresponding notions for β -reduction. If the first derivation step of a single-step η -reduction has the form $Q[x,A]\{x\}B \geq'_\eta QB$, we say that $Q[x,A]\{x\}B$ *generates* the single-step η -reduction.

Theorem 7.3. Let $K \in \Delta$. Then $Q[x,A]\{x\}B$ generates a single-step η -reduction of the form $K \geq'_\eta L$ if and only if $[x,A]\{x\}B \subset K$ and $Q[x,A]\{x\}B \equiv K| [x,A]\{x\}B$.

Proof. Similar to the proof of Th. 5.7. $\square$

Theorem 7.4. If $K \in \Delta$ and $K \geq'_\eta L$, then $L \in \Delta$.

Proof. Induction on the length of proof of $K \geq'_\eta L$. The proof is similar to the proof of Th. 5.8. $\square$

Theorem 7.5. If $QE \in \Delta$, $QE \geq'_\eta Q'F$ and $\|Q\| = \|Q'\|$, then (i) $Q \equiv Q'$, (ii) $E \equiv F$, or (iii) $Q \equiv Q_0[x,A]$, $E \equiv \{x\}[y,B]E'$, $Q' \equiv Q_0[y,B]$, $x \notin [y,B]E'$ and $F \equiv E'$. In the second case $Q \equiv Q_1[x,K]Q_2$, $Q' \equiv Q_1[x,L]Q_2$ and $Q_1K \geq'_\eta Q_1L$.

Proof. Induction on the length of proof of $QE \geq'_\eta Q'F$. The proof is comparable to the proof of Th. 5.11, except for the case in which

$QE \geq'_\eta Q'F$ is an elementary η -reduction. In this case we have to note the possibility that QE and $Q'F$ are as in (iii). $\square$

Theorem 7.6. If $QE \in \Delta$ and $QE \geq'_\eta K$, then $K \equiv Q'F'$ for certain Q' and F' with $\|Q'\| \geq \|Q\| - 1$.

Proof. Similar to the proof of Th. 5.12. $\square$

Theorem 7.7. If $QE \in \Delta$ and $QE \geq'_\eta K \geq_\eta QG$, then $K \equiv QF$.

Proof. If $Q \equiv Q_1[x, A]$ and $QE \geq'_\eta K$ is $Q_1[x, A]\{x\}B \geq'_\eta Q_1B$, then the binding variable x of K has disappeared, and we cannot regain it by η -reduction. Hence by Th. 7.6: $K \equiv Q'F$ and $\|Q'\| = \|Q\|$, and the case expressed in Th. 7.5 (iii) does not hold. In the derivation steps leading to $K \geq_\eta QG$ the final ones of the first $\|Q\|$ abstractors cannot disappear by an elementary η -reduction for the same reason as above. Assume that $Q \neq Q'$. Then by Th. 7.5 (ii): $Q \equiv Q_1[x, K]Q_2$, $Q' \equiv Q_1[x, L]Q_2$ and $Q_1K \geq'_\eta Q_1L$. It is clear that $|L| < |K|$. Since the length of an expression cannot increase by η -reduction it follows that we cannot regain Q from Q' . Hence $Q \equiv Q'$. $\square$

Theorem 7.8. The monotony rules hold for η -reduction.

Proof. Similar to the proof of Th. 5.15, using Th. 7.5. $\square$

Theorem 7.9. If QE, PE and $PF \in \Delta$, and $QE \geq_\eta QF$, then $PE \geq_\eta PF$.

Proof. Analogous to the proof of Th. 5.17; use Th. 7.7. $\square$

The converse of this theorem holds too.

Given an η -reduction $QK \geq_\eta M$, it need not follow that $M \equiv Q'N'$ with $\|Q\| = \|Q'\|$, since the final abstractors of Q may have been cancelled in η -reductions. For example: Let $Q \equiv Q'[x, A]$ and $K \equiv \{x\}\tau$, then $QK \geq'_\eta Q'\tau$, where $\|Q'\| = \|Q\| - 1$. This kind of η -reductions plays an important rôle in the following. We shall call them η !-reductions. We shall prove a number of theorems concerning η !-reductions. In Th. 7.14 we shall show that we can postpone η !-reductions until after other η -reductions. Cor. 7.17 will result from our discussions of η !-reductions.

Definition 7.10.

- (1) $K \geq_{\eta}^{\prime} L$ is called a *single η !-reduction* (denoted by $K \geq_{\eta}^{(1)} L$) if $K \equiv Q[x, A]\{x\}B$ and K generates $K \geq_{\eta}^{\prime} L$ (i.e. if $K \geq_{\eta}^{\prime} L$ is an elementary η -reduction). This reduction is called *of order p* if $\|Q[x, A]\| = p$.
- (2) $K \geq_{\eta}^{\prime} L$ is called a *k-fold η !-reduction* (denoted by $K \geq_{\eta}^{(k)} L$) if there are $K_i \equiv Q[x_i, A_i] \dots [x_i, A_i]\{x_i\} \dots \{x_i\}B$ such that $K \equiv K_k \geq_{\eta}^{(1)} K_{k-1} \dots \geq_{\eta}^{(1)} K_0 \equiv QB \equiv L$ and K_i generates $K_i \geq_{\eta}^{(1)} K_{i-1}$. This reduction $K \geq_{\eta}^{(k)} L$ is called *of order p* if $K_k \geq_{\eta}^{(1)} K_{k-1}$ is of order p. □

Theorem 7.11. If $K \in \Delta$ and $K \geq_{\eta}^{(1)} L \geq_{\eta}^{\prime} N$, then either $K \geq_{\eta}^{(2)} N$ or there is a reduction $K \geq_{\eta}^{\prime} M \geq_{\eta}^{(1)} N$ where $K \geq_{\eta}^{\prime} M$ is not an η !-reduction.

Proof. $K \equiv Q[x, A]\{x\}B \geq_{\eta}^{(1)} QB \equiv L$. Consider the possibilities for $QB \equiv L \geq_{\eta}^{\prime} N$. □

Theorem 7.12. If $K \in \Delta$ and $K \geq_{\eta}^{(k)} L \geq_{\eta}^{\prime} N$, where $K \geq_{\eta}^{(k)} L$ is of order p, then either $K \geq_{\eta}^{(k+1)} N$ of order p, or there is a reduction $K \geq_{\eta}^{\prime} M \geq_{\eta}^{(k)} N$ where $M \geq_{\eta}^{(k)} N$ is of order p and $K \geq_{\eta}^{\prime} M$ is not an η !-reduction.

Proof. Compare with the previous theorem. □

Theorem 7.13. Let $K \in \Delta$ and $K \geq_{\eta}^{(k)} L \geq_{\eta}^{\prime} N$, where $K \geq_{\eta}^{(k)} L$ is of order p. Then there is a reduction $K \geq_{\eta}^{\prime} L' \geq_{\eta}^{(\ell)} N$, where a decomposition of $K \geq_{\eta}^{\prime} L$ contains no η !-reductions and $L' \geq_{\eta}^{(\ell)} N$ is of order p.

Proof. Decompose $L \geq_{\eta}^{\prime} N$ into $L \equiv E_1 \geq_{\eta}^{\prime} \dots \geq_{\eta}^{\prime} E_r \equiv N$. We proceed with induction on r. If $r = 1$ there is nothing to prove. Let $r > 1$. Consider the reduction $K \geq_{\eta}^{(k)} L \equiv E_1 \geq_{\eta}^{\prime} E_2$. By the previous theorem we have either $K \geq_{\eta}^{(k+1)} E_2$ or a reduction $K \geq_{\eta}^{\prime} L'' \geq_{\eta}^{(k)} E_2$ where $K \geq_{\eta}^{\prime} L''$ is not an η !-reduction. Applying the

induction hypothesis on $K \geq_{n!}^{(k+1)} E_2 \geq_n N$ or $L'' \geq_{n!}^{(k)} E_2 \geq_n N$, we obtain $K \geq_n L' \geq_{n!}^{(\ell)} N$, where a decomposition of $K \geq_n L'$ contains no $n!$ -reductions. $\square$

Theorem 7.14. If $QK \in \Delta$ and $QK \geq_n L$, there is a reduction $QK \geq_n Q'K' \geq_{n!}^{(k)} L$ where $\|Q\| = \|Q'\|$, $Q'K \geq_{n!}^{(k)} L$ is of order $\|Q\|$, and where a decomposition of $QK \geq_n Q'K'$ contains no $n!$ -reductions of order $\|Q\|$.

Proof. Decompose $QK \geq_n L$ into single-step n -reductions $QK \equiv L_1 \geq'_n \dots \geq'_n L_n \equiv L$. Let i be the smallest integer such that $L_i \geq'_n L_{i+1}$ is an $n!$ -reduction of order $\|Q\|$. Apply the previous theorem on $L_i \geq_{n!}^{(1)} L_{i+1} \geq_n L_n \equiv L$. We obtain a reduction $QK \geq_n L' \geq_{n!}^{(k)} L$ as desired. The fact that $L' \equiv Q'K'$ with $\|Q'\| = \|Q\|$ follows from Th. 7.5. $\square$

Theorem 7.15. If $QK \in \Delta$, $QK \geq_n Q'K'$, $\|Q\| = \|Q'\| = p$ and a decomposition of $QK \geq_n Q'K'$ contains no $n!$ -reductions of order p , there is a reduction $QK \geq_n QK' \geq_n Q'K'$ and a reduction $QK \geq_n Q'K \geq_n Q'K'$.

Proof. See Th. 7.5. $\square$

Theorem 7.16. If $Q_1K \in \Delta$, $Q_2L \in \Delta$, $Q_1K \geq_n M$, $Q_2L \geq_n M$, $Q_1 \equiv [x_1, A_1] \dots [x_p, A_p]$ and $Q_2 \equiv [x_1, B_1] \dots [x_p, B_p]$, there is an N such that $Q_1K \geq_n Q_1N$ and $Q_2L \geq_n Q_2N$.

Proof. By the aid of Th. 7.14 and Th. 7.15 we can find reductions $Q_1K \geq_n Q_1K' \geq_n Q_1'K' \geq_{n!}^{(k)} M$ and $Q_2L \geq_n Q_2L' \geq_n Q_2'L' \geq_{n!}^{(\ell)} M$, where $Q_1'K' \geq_{n!}^{(k)} M$ and $Q_2'L' \geq_{n!}^{(\ell)} M$ are of order p . Note that $Q_1' \equiv [x_1, A_1'] \dots [x_p, A_p']$ and $Q_2' \equiv [x_1, B_1'] \dots [x_p, B_p']$. Now both $Q_1'K'$ and $Q_2'L' \in \Delta$, so $k = \ell$: assume $k > \ell$, then $M \equiv [x_1, A_1'] \dots [x_{p-k}, A_{p-k}'] M' \equiv [x_1, B_1'] \dots [x_{p-k}, B_{p-k}'] \dots [x_{p-\ell}, B_{p-\ell}'] M''$; it follows that $[x_{p-\ell}, B_{p-\ell}']$ occurs in M' , hence

also in K' ; this contradicts the fact that $Q_1'K' \in \Delta$, since we found two binding variables $x_{p-\ell}$ in $Q_1'K'$. It follows that $K' \equiv L'$; we can take $N \equiv K' \equiv L'$. $\square$

Corollary 7.17. If $QK, QL \in \Delta$, $QK \geq_\eta M$ and $QL \geq_\eta M$, there is a QN such that $QK \geq_\eta QN$ and $QL \geq_\eta QN$. $\square$

Theorem 7.18. If $QD \in \Delta$, $QD \geq_\eta Q'E$, $Q \equiv [x_1, A_1] \dots [x_p, A_p]$ and $Q' \equiv [x_1, A_1'] \dots [x_p, A_p']$, then $QD \geq_\eta QE$.

Proof. Resulting from Th. 7.14 we can find a reduction $QD \geq_\eta Q''D' \geq_{\eta!}^{(k)} Q'E$ where $Q''D' \geq_{\eta!}^{(k)} Q'E$ is of order $\|Q\|$. Now $Q'' \equiv [x_1, A_1''] \dots [x_p, A_p'']$, hence $k = 0$ (because $Q''D' \in \Delta$; see the proof of Th. 7.16). Then also $QD \geq_\eta QE$ by Th. 7.15. $\square$

We shall now prove a theorem concerning the so-called "postponement of η -reductions" for Δ . What we want to prove is that every reduction $K \geq M$ which takes place by means of single-step β - and η -reductions in arbitrary order, can be replaced by a reduction $K \geq_\beta L \geq_\eta M$, in which all β -reductions precede all η -reductions.

It is easy to show that each reduction $A \geq_\eta' B \geq_\beta' C$ can be replaced either by a reduction $A \geq_\beta' B' \geq_\eta^r C$ (where $r \geq 0$) or by a reduction $A \geq_\beta' B' \geq_\beta' C$. But this does not suffice to prove the theorem. It is not sure that this process of interchanging η 's and β 's terminates for a given reduction $K \geq M$.

In Curry and Feys [3, Ch. 4, § D2] a compound β -reduction is introduced for the purpose of proving the above mentioned theorem. In our opinion there is an error in their proof (viz., the case that R is $M_k N$ and L is some $M_j y_j$, for $j \leq k$ is missing). Nevertheless, their idea can be extended in such a manner that the theorem on the postponement of η -reductions can be proved. We have carried this out by defining a "compound β -reduction" $A \geq_\beta^* B$ with the property that each reduction $A \geq_\eta B \geq_\beta^* C$ can be replaced by a reduction $A \geq_\beta^* B' \geq_\eta C$. However, this compound β -reduction looks rather complicated.

Barendregt suggested to us another way of proving the theorem (private communication). He proposed a "nested" η -reduction (which we call κ -reduction and denote by $\geq'_\kappa$) with the property that a reduction $A \geq'_\kappa B \geq_\beta C$ can be replaced by a reduction $A \geq_\beta B' \geq'_\kappa C$. The nested character of this κ -reduction is comparable to that of γ -reduction discussed in the previous section.

We prefer the latter way of proving because it is easier to understand.

Definition 7.19. Single-step κ -reduction, denoted by $\geq'_\kappa$, is the reflexive relation generated by

- (1) If $Q[x, A]\{x\}B \in \Delta$, $x \notin B$ and $QB \geq'_\kappa QC$, then $Q[x, A]\{x\}B \geq'_\kappa QC$.
- (2) If $Q\{A\}C \in \Delta$, $QA \geq'_\kappa QA'$ and $QC \geq'_\kappa QC'$, then $Q\{A\}C \geq'_\kappa Q\{A'\}C'$.
- (3) If $Q[x, A]C \in \Delta$, $QA \geq'_\kappa QA'$ and $Q[x, A]C \geq'_\kappa Q[x, A']C'$, then $Q[x, A]C \geq'_\kappa Q[x, A']C'$. □

We call rule (1) in this definition the rule of *elementary* single-step κ -reduction, rules (2) and (3) the *monotony rules* for κ -reduction.

The following two theorems deal with the relation between η - and κ -reduction.

Theorem 7.20. If $K \in \Delta$ and $K \geq'_\eta L$, then $K \geq'_\kappa L$.

Proof. Induction on the length of proof of $K \geq'_\eta L$. □

Theorem 7.21. If $K \in \Delta$ and $K \geq'_\kappa L$, then $K \geq_\eta L$.

Proof. Induction on the length of proof of $K \geq'_\kappa L$.

For example, if $K \geq'_\kappa L$ is $Q[x, A]\{x\}B \geq'_\kappa QC$, as a direct consequence of $QB \geq'_\kappa QC$, then by induction $QB \geq_\eta QC$, and $Q[x, A]\{x\}B \geq'_\eta QC$. □

We shall now prove a number of theorems which are lemmas for the theorem on the postponement of η -reductions (Th. 7.28).

Theorem 7.22. If $K \in \Delta$ and $K \geq'_K L$, then $L \in \Delta$.

Proof. Follows from Th. 7.21 and Th. 7.4. □

Theorem 7.23. If $QE \in \Delta$ and $QE \geq'_K Q[y, G]H$, then

$QE \equiv Q[x_1, A_1][x_1][x_2, A_2][x_2] \dots [x_n, A_n][x_n][y, G']H'$, with

$QG' \geq'_K QG$, $Q[y, G']H' \geq'_K Q[y, G]H$ and $x_i \notin [x_{i+1}, A_{i+1}] \dots [x_n][y, G']H'$.

Proof. Induction on the length of proof of $QE \geq'_K Q[y, G]H$.

If the latter reduction results from reflexivity, the proof is completed.

(1) Let $QE \geq'_K Q[y, G]H$ be $Q'[x, A][x]B \geq'_K Q'C$, as a direct consequence of $Q'B \geq'_K Q'C$. If $Q' \equiv QQ''$, induction yields the proof.

If $Q \equiv Q'[x, A]$, then C begins with $[x, A]$. This implies that $[x, A]$ occurs in B (since κ -reduction can only omit abstractors and applicators without influencing the remainder of the expression), which is impossible since $QE \in \Delta$. So this latter case cannot apply.

(2) Let $QE \geq'_K Q[y, G]H$ be $Q'\{A\}C \geq'_K Q'\{A'\}C'$. Then $QE \equiv Q[y, G]F$.

(3) Let $QE \geq'_K Q[y, G]H$ be $Q'[x, A]C \geq'_K Q'[x, A']C'$, as a direct consequence of $Q'A \geq'_K Q'A'$ and $Q'[x, A]C \geq'_K Q'[x, A']C'$. There are the following possibilities: (a) $Q \equiv Q'$, (b) $Q \equiv Q'[x, A]Q_1$ and (c) $Q' \equiv QQ_1$ with $\|Q_1\| > 0$. In all three cases the proof is easy. □

Theorem 7.24. Let QA and $Q[x, B]C \in \Delta$, $QA \geq'_K QA'$ and $Q[x, B]C \geq'_K Q[x, B]C'$. Then $Q(x:=A)C \geq'_K Q(x:=A')C'$.

Proof. Induction on $|C|$. If $C \equiv \tau$, $C \equiv x$ or $C \equiv y \neq x$, the proof is easy.

(1) Let $C \equiv [y, E]F$. There are two possible cases:

(a) $Q[x, B]C \geq'_K Q[x, B]C'$ is $Q[x, B][y, E]F \geq'_K Q[x, B][y, E']F'$, as a direct consequence of $Q[x, B]E \geq'_K Q[x, B]E'$ and $Q[x, B][y, E]F \geq'_K Q[x, B][y, E']F'$. By induction: $Q(x:=A)E \geq'_K Q(x:=A')E'$, and $Q[y, (x:=A)E](x:=A)F \geq'_K Q[y, (x:=A)E](x:=A')F'$ (the latter because $Q[y, (x:=A)E][x, B]F \geq'_K Q[y, (x:=A)E][x, B]F'$). Hence $Q(x:=A)C \geq'_K Q(x:=A')C'$.

(b) $Q[x, B]C \geq'_\kappa Q[x, B]C'$ is $Q[x, B][y, E]\{y\}G \geq'_\kappa Q[x, B]G'$, as a direct consequence of $Q[x, B]G \geq'_\kappa Q[x, B]G'$. By induction $Q(x:=A)G \geq'_\kappa Q(x:=A')G'$, so $Q(x:=A)[y, E]\{y\}G \equiv Q[y, (x:=A)E]\{y\}(x:=A)G \geq'_\kappa Q(x:=A')G'$.

(2) Let $C \equiv \{E\}F$. Then $Q[x, B]C \geq'_\kappa Q[x, B]C'$ is $Q[x, B]\{E\}F \geq'_\kappa Q[x, B]\{E'\}F'$, as a direct consequence of $Q[x, B]E \geq'_\kappa Q[x, B]E'$ and $Q[x, B]F \geq'_\kappa Q[x, B]F'$. The theorem results from the induction. (Note that $Q[x, B]C \geq'_\kappa Q[x, B]C'$ cannot be $Q[x, B]\{x\}G \geq'_\kappa QG'$.) $\square$

Theorem 7.25. Let $A \in \Delta$ and $A \geq'_\kappa B \geq'_\beta C$. Then $A \geq_\beta B' \geq'_\kappa C$.

Proof. Induction on the length of proof of $A \geq'_\kappa B$.

If the last derivation step results from reflexivity, nothing remains to be proved.

(1) Let $A \geq'_\kappa B$ be $Q[x, D]\{x\}E \geq'_\kappa QE'$ as a direct consequence of $QE \geq'_\kappa QE'$, and let $B \geq'_\beta C$ be generated by $Q'\{F\}[y, G]H \geq'_\beta Q'(y:=F)H$. The following cases may apply:

(a) $\{F\}[y, G]H \subset Q$. There is clearly a reduction $A \geq_\beta B' \geq'_\kappa C$.
 (b) $\{F\}[y, G]H \subset E'$. Let $B \geq'_\beta C$ be $QE' \geq'_\beta QE''$, then by induction there is a reduction

$QE \geq_\beta QE'' \geq'_\kappa QE'''$, hence $Q[x, D]\{x\}E \geq_\beta Q[x, D]\{x\}E'' \geq'_\kappa QE'''$.

(2) Let $A \geq'_\kappa B$ be $Q\{D\}E \geq'_\kappa Q\{D'\}E'$ as a direct consequence of $QD \geq'_\kappa QD'$ and $QE \geq'_\kappa QE'$, and let $B \geq'_\beta C$ be generated by $Q'\{F\}[y, G]H \geq'_\beta Q'(y:=F)H$.

The following cases may apply:

(a) $\{F\}[y, G]H \subset Q$. Clearly $A \geq_\beta B' \geq'_\kappa C$.
 (b) $\{F\}[y, G]H \equiv \{D'\}E'$. Then $QE' \equiv Q[y, G]H$, so $QE \equiv Q[x_1, A_1]\{x_1\} \dots [x_n, A_n]\{x_n\}[y, G']H'$, with $QG' \geq'_\kappa QG$, $Q[y, G']H' \geq'_\kappa Q[y, G']H$ and $x_i \notin [x_{i+1}, A_{i+1}] \dots [x_n, A_n]\{x_n\}[y, G']H'$ by Th. 7.23. Then $Q\{D\}E \geq_\beta Q(y:=D)H'$. By Th. 7.24: $Q(y:=D)H' \geq'_\kappa Q(y:=F)H \equiv C$.

(c) $\{F\}[y, G]H \subset D'$. Then $C \equiv Q\{D''\}E'$ with $QD' \geq'_\beta QD''$, and by induction $QD \geq_\beta QD'' \geq'_\kappa QD'''$, hence $Q\{D\}E \geq_\beta Q\{D''\}E \geq'_\kappa Q\{D'''\}E' \equiv C$.

(d) $\{F\}[y, G]H \subset E'$. Then $C \equiv Q\{D'\}E'''$ with $QE' \geq'_\beta QE''$, and by induction $QE \geq_\beta QE'' \geq'_\kappa QE'''$, hence $Q\{D\}E \geq_\beta Q\{D'\}E'' \geq'_\kappa Q\{D'\}E''' \equiv C$.

(3) Let $A \geq_{\kappa}' B$ be $Q[x, D]E \geq_{\kappa}' Q[x, D']E'$ as a direct consequence of $QD \geq_{\kappa}' QD'$ and $Q[x, D]E \geq_{\kappa}' Q[x, D]E'$, and let again $B \geq_{\beta}' C$ be generated by $Q'\{F\}[y, G]H \geq_{\beta}' Q'(y:=F)H$.

The following cases may apply:

(a) $\{F\}[y, G]H \subset Q$. Clearly $A \geq_{\beta} B' \geq_{\kappa}' C$.

(b) $\{F\}[y, G]H \subset D'$. Then $C \equiv Q[x, D''']E'$ with $QD' \geq_{\beta}' QD'''$.

By induction: $QD \geq_{\beta} QD'' \geq_{\kappa}' QD'''$, so $Q[x, D]E \geq_{\beta} Q[x, D'']E \geq_{\kappa}' Q[x, D''']E'$ (where we require the lemma: $Q[x, D]E \geq_{\kappa}' Q[x, D]E'$, then $Q[x, D'']E \geq_{\kappa}' Q[x, D'']E'$).

(c) $\{F\}[y, G]H \subset E'$. Then $C \equiv Q[x, D']E'''$.

Also: $Q[x, D]E' \geq_{\beta}' Q[x, D]E'''$, so by induction $Q[x, D]E \geq_{\beta} Q[x, D]E'' \geq_{\kappa}' Q[x, D]E'''$. Hence $Q[x, D]E \geq_{\beta} Q[x, D]E'' \geq_{\kappa}' Q[x, D']E''' \equiv C$. $\square$

Theorem 7.26. Let $A \in \Delta$ and $A \geq_{\kappa}' B \geq_{\beta}^P C$. Then $A \geq_{\beta} B \geq_{\kappa}' C$.

Proof. Induction on p , using the previous theorem. $\square$

Theorem 7.27. Let $A \in \Delta$ and let $A \geq C$ by means of a number of single-step κ - and β -reductions in arbitrary order. Then there is a reduction $A \geq_{\beta} B \geq_{\kappa}' C$.

Proof. Induction on the number of single-step κ -reductions in $A \geq C$. If this number is zero, the proof is completed. Else, let $A \geq C$ be $A \geq A' \geq_{\kappa}' B \geq_{\beta}^P C$. Apply the previous theorem, obtaining $A \geq A' \geq_{\beta} B' \geq_{\kappa}' C$, and apply the induction on $A \geq A' \geq_{\beta} B'$. $\square$

Theorem 7.28. Let $A \in \Delta$ and let $A \geq C$ by means of a number of single-step η - and β -reductions in arbitrary order. Then there is a reduction $A \geq_{\beta} B \geq_{\eta} C$.

Proof. Since each η -reduction can be considered as a κ -reduction (Th. 7.20), we can apply the previous theorem, obtaining $A \geq_{\beta} B \geq_{\kappa}' C$. But $B \geq_{\kappa}' C$ implies $B \geq_{\eta} C$ (Th. 7.21), so $A \geq_{\beta} B \geq_{\eta} C$. $\square$

The remainder of this section will concern (general) reduction, defined as a sequence of single-step α -, β - and η -reductions.

Definition 7.29. *Single-step reduction* (denoted by $\geq'$) is the relation obeying:

$A \geq' B$ if and only if $A \geq'_\alpha B$, $A \geq'_\beta B$ or $A \geq'_\eta B$. □

Definition 7.30. *Reduction* (or *general reduction*, denoted by $\geq$) is the reflexive and transitive closure of single-step reduction. □

Theorem 7.31. The monotony rules hold for reduction.

Proof. Use Th. 5.15 and Th. 7.8. □

We shall prove a theorem (Th. 7.33) which expresses that the Q is in a certain sense irrelevant in a reduction $QC \geq QE$: it can be replaced by any P such that PC and $PE \in \Delta$. This corresponds with general usage in lambda-calculus to define reduction for expressions which may contain free variables. Our choice to define reductions inside Δ is apparently not in disagreement with that general usage.

Theorem 7.32. If $QC \in \Delta$ and $QC \geq QE$ by means of β - and η -reductions, there is a reduction $QC \geq_\beta QD \geq_\eta QE$.

Proof. There is a reduction $QC \geq_\beta K \geq_\eta QE$ by Th. 7.28.

Now by Th. 5.12: $K \equiv Q'D$ with $\|Q\| = \|Q'\|$.

If $Q \equiv [x_1, A_1] \dots [x_p, A_p]$, then $Q' \equiv [x_1, A'_1] \dots [x_p, A'_p]$,

so by Th. 7.18: $Q'D \geq_\eta Q'E$. From Th. 5.21: $QC \geq_\beta QD$, and from Th.

7.9: $QD \geq_\eta QE$. □

Theorem 7.33. If QC , PC and $PE \in \Delta$, and $QC \geq QE$, then $PC \geq PE$.

Proof. See Th. 7.32, Th. 5.17 and Th. 7.9. □

Reduction is a non-symmetric relation between expression in Δ , which is reflexive and transitive. We shall define lambda-equivalence. The definition of beta-equivalence was given in Def. 5.22. In Th. 7.35 we shall prove that beta-equivalence is the symmetric closure of beta-reduction.

Definition 7.34. Let A and $B \in \Delta$. We call A *lambda-equivalent* to B (denoted: $A \sim B$) if there is an expression C such that $A \geq C$ and $B \geq C$. □

Theorem 7.35. Beta-equivalence is reflexive, symmetric and transitive.

Proof. Reflexivity and symmetry are trivial. Transitivity follows from Th. 6.43 (CR for β -reduction): let $A \sim_{\beta} B$ and $B \sim_{\beta} C$, then there are D and E such that $A \geq_{\beta} D$, $B \geq_{\beta} D$, $B \geq_{\beta} E$ and $C \geq_{\beta} E$. Moreover, there is an F such that $D \geq_{\beta} F$ and $E \geq_{\beta} F$ (Th. 6.43), so $A \geq_{\beta} F$ and $C \geq_{\beta} F$. Hence $A \sim_{\beta} C$. □

Unfortunately, there is no similar theorem for lambda-equivalence. Of course lambda-equivalence is symmetric and reflexive, but not necessarily transitive. The reason for this is that CR does not hold for (general) reduction: for example, let $K \geq L$ and $K \geq M$, let $K \equiv Q[x, A]\{x\}[y, B]C$ where $x \notin [y, B]C$, let $K \geq L$ be $Q[x, A]\{x\}[y, B]C \geq_{\eta} Q[y, B]C$ and let $K \geq M$ be $Q[x, A]\{x\}[y, B]C \geq_{\beta} Q[x, A](y:=x)C (\geq_{\alpha} Q[y, A]C)$. Now we cannot in general find an N such that $L \geq N$ and $M \geq N$, since we know nothing concerning a relation between A and B .

We note the following. We can embed ordinary lambda-calculus into Δ , since there is a one-to-one correspondence between expressions from lambda-calculus and those expressions in Δ in which only abstractors of the form $[x, \tau]$ occur. If we restrict ourselves in Δ to the latter expressions, the example above changes into $K \equiv Q[x, \tau]\{x\}[y, \tau]C$, $L \equiv Q[y, \tau]C$ and $M \geq_{\alpha} Q[y, \tau]C$. Now there is no problem as regards CR. Indeed, in lambda-calculus the Church-Rosser property holds (see Barendregt [1, Appendix II]).

The following theorem expresses that lambda-equivalence of QK and QL implies the existence of an N such that $QK \geq QN$ and $QL \geq QN$ or, otherwise stated: the abstractor chain Q can remain unaffected.

Theorem 7.36. Let QK and $QL \in \Delta$. If $QK \sim QL$, there exists an N such that $QK \geq QN$ and $QL \geq QN$.

Proof. There must be an M : $QK \geq M$ and $QL \geq M$. By postponement of η -reductions we obtain reductions $QK \geq_{\beta} M_1 \geq_{\eta} M$ and $QL \geq_{\beta} M_2 \geq_{\eta} M$. Th. 5.12 implies that $M_1 \equiv Q_1 K'$, $M_2 \equiv Q_2 L'$, $\|Q\| = \|Q_1\| = \|Q_2\|$. Then, according to Th. 5.21, we also have $QK \geq_{\beta} QK' \geq_{\beta} Q_1 K' \geq_{\eta} M$ and $QL \geq_{\beta} QL' \geq_{\beta} Q_2 L' \geq_{\eta} M$. It is easy to show that Q_1 and Q_2 have the form as required in Th. 7.16, hence there is an N such that $Q_1 K' \geq_{\eta} Q_1 N$ and $Q_2 L' \geq_{\eta} Q_2 N$. From Th. 7.9 it follows that $QK' \geq_{\eta} QN$ and $QL' \geq_{\eta} QN$. So $QK \geq QN$ and $QL \geq QN$. $\square$

The monotony rules also hold for lambda-equivalence:

Theorem 7.37.

- (a) If $QC, QD, Q\{A\}C, Q\{A\}D \in \Delta$ and $QC \sim QD$, then $Q\{A\}C \sim Q\{A\}D$.
- (b) If $QC, QD, Q[x,A]C, Q[x,A]D \in \Delta$ and $QC \sim QD$, then $Q[x,A]C \sim Q[x,A]D$.
- (c) If $QA, QB, Q\{A\}C, Q\{B\}C \in \Delta$ and $QA \sim QB$, then $Q\{A\}C \sim Q\{B\}C$.
- (d) If $QA, QB, Q[x,A]C, Q[x,B]C \in \Delta$ and $QA \sim QB$, then $Q[x,A]C \sim Q[x,B]C$.

Proof. See Th. 7.36 and Th. 7.33. $\square$

Theorem 7.38. If QC, QD, PC and $PD \in \Delta$ and $QC \sim QD$, then $PC \sim PD$.

Proof. Th. 7.36 and Th. 7.33. $\square$

§ 8. TYPE AND DEGREE

The notions introduced in the preceding sections are from lambda-calculus (as reduction, lambda-equivalence) or applicable to lambda-calculus (factors, bound expressions), since the types played no essential rôle.

We shall now look into the *typing* of an expression in Δ . With every $A \in \Delta$ for which $\text{Tail } A \neq \tau$ we define a *type*, denoted as $\text{Typ } A$, as follows:

Definition 8.1. Let $A \in \Delta$ and $\text{Tail } A \equiv x$, so $A \equiv P_1[x, B]P_2x$.
Then $\text{Typ } A \equiv P_1[x, B]P_2\text{FrB}$. $\square$

Informally speaking, we may say that B is the type of x in the above expression. Note, however, that we allow Typ to operate only on expressions in Δ .

Theorem 8.2. If $A \in \Delta$ and $\text{Tail } A \neq \tau$, then $\text{Typ } A \in \Delta$.

Proof. Let $A \equiv P_1[x, B]P_2x$ and let $A|x \equiv Q_1[x, B]Q_2x$.

We prove that $\text{Typ } A$ is a bound expression. All non-binding variables in $P_1[x, B]P_2$ are clearly also bound in $\text{Typ } A$. Consider a non-binding variable $z \in \text{FrB} \subset Q_1[x, B]Q_2\text{FrB}$. There is a corresponding $y \in B$, and $A|y \equiv Q_1Q_3y$. So $\text{Typ } A|z \equiv Q_1[x, B]Q_2Q_3'z$ where $Q_3'z$ is a renovation of Q_3y .

Case 1: if y was bound in $A|y$ by a binding variable in Q_3 , z is bound in $\text{Typ } A|z$ by the corresponding binding variable in Q_3' .

Case 2: if y was bound in $A|y$ by a binding variable in Q_1 , $z \equiv y$ is still bound by the same binding variable in Q_1 since all binding variables of $[x, B]Q_2Q_3'$ are different from y .

So $\text{Typ } A$ is bound. Clearly $\text{Typ } A$ is also distinctly bound by the renovation of B . $\square$

We define repeated application of Typ inductively as follows:

Definition 8.3. Let $A \in \Delta$. Then $\text{Typ}^0 A \equiv A$; if $\text{Typ}^{n-1} A$ is defined for $n \geq 1$ and if $\text{Tail } \text{Typ}^{n-1} A \neq \tau$, then $\text{Typ}^n A \equiv \text{Typ}(\text{Typ}^{n-1} A)$. $\square$

If $A \in \Delta$ and $\text{Typ}^n A$ is defined, we call n *permissible* for A ($n = 0$ is always permissible for $A \in \Delta$).

Theorem 8.4. If $A \in \Delta$ and $A \geq_\alpha B$, then $\text{Typ}^n A \geq_\alpha \text{Typ}^n B$ for all n permissible for A and B .

Proof. It is sufficient to prove: if $A \geq_\alpha' B$ and $\text{Tail } A \neq \tau$, then $\text{Typ } A \geq_\alpha \text{Typ } B$. The latter proof is easy. $\square$

With each expression A in Δ we define a degree, denoted $\text{Deg } A$:

Definition 8.5.

- (1) If $A \in \Delta$ and $\text{Tail } A \equiv \tau$, then $\text{Deg}(A) = 1$.
- (2) If $A \in \Delta$, $\text{Tail } A \equiv x$ and $A \equiv P_1[x, B]P_2x$, then
 $\text{Deg}(A) = \text{Deg}(P_1B) + 1$. □

Induction on the length of A shows that $\text{Deg}(A)$ is well-defined by Def. 8.5. Clearly $\text{Deg } A = 1$ if and only if $\text{Tail } A \equiv \tau$.

We shall now prove a number of theorems, leading to the theorem: if $\text{Tail } A \neq \tau$, then $\text{Deg } A = \text{Deg Typ } A + 1$ (Th. 8.12). We could have taken this property as a definition of Deg . In that case, however, the well-definedness of Deg would have been harder to prove.

Theorem 8.6. If $PC \in \Delta$ and $P\{K\}C \in \Delta$, then $\text{Deg } PC = \text{Deg } P\{K\}C$.

Proof. Induction on $|PC|$. □

Corollary 8.7. If $A \in \Delta$ and $A \equiv PC$, then $\text{Deg } A = \text{Deg}(A|C)$. □

Corollary 8.8. If $A \in \Delta$, $\text{Tail } A \equiv x$ and $A|x \equiv Q_1[x, B]Q_2x$, then
 $\text{Deg } A = \text{Deg } Q_1B + 1$. □

Theorem 8.9. If $PC \in \Delta$ and $P[x, K]C \in \Delta$, then $\text{Deg } PC = \text{Deg } P[x, K]C$.

Proof. By Th. 3.8: $x \neq C$. The rest of the proof follows from induction on $|PC|$. □

Theorem 8.10. If $PC \in \Delta$ and $PP'C \in \Delta$, then $\text{Deg } PC \equiv \text{Deg } PP'C$.

Proof. Induction on $\|P'\|$, using Th. 8.6 and Th. 8.9. □

Theorem 8.11. If $A \in \Delta$ and $A \geq_\alpha B$, then $\text{Deg } A = \text{Deg } B$.

Proof. Take $A \geq'_\alpha B$; induction on $|A|$. □

Theorem 8.12. If $A \in \Delta$ and $\text{Tail } A \neq \tau$, then $\text{Deg Typ } A = \text{Deg } A-1$.

Proof. Let $\text{Tail } A \equiv x$ and $A \equiv P_1[x, C]P_2x$, so $\text{Typ } A \equiv P_1[x, C]P_2\text{FrC}$.
Then, $P_1\text{FrC} \in \Delta$ and $\text{Deg } P_1\text{FrC} = \text{Deg } P_1C$ by Th. 4.5 and Th. 8.11.
By Th. 8.10: $\text{Deg } P_1\text{FrC} = \text{Deg Typ } A$.
So $\text{Deg } A = \text{Deg } P_1C + 1 = \text{Deg Typ } A + 1$. □

Corollary 8.13. If $A \in \Delta$, then $\text{Tail}(\text{Typ}^{\text{Deg } A-1} A) \equiv \tau$. □

This optimal exponent of Typ with a certain $A \in \Delta$ is of special importance. We shall introduce an abbreviation:

Definition 8.14. If $A \in \Delta$, then $\text{Typ}^* A \equiv \text{Typ}^{\text{Deg } A-1} A$. □

We stress that the asterisk replaces an exponent n *dependent* upon A . Moreover, note that Typ is a partial function on Δ , but Typ^* is a total function on Δ .

We proceed with a number of theorems on Typ^n , Typ^* and Deg :

Theorem 8.15. If $A \in \Delta$, $\text{Deg } A = 1$ and $A \geq B$, then $\text{Deg } B = 1$. □

Theorem 8.16. If $PC \in \Delta$, then for permissible n $\text{Typ}^n PC \equiv PC'$;
in particular $\text{Typ}^* PC \equiv PC''$. □

Theorem 8.17. If $PC \in \Delta$, $PP'C \in \Delta$, and for a permissible n $\text{Typ}^n PC \equiv PC'$, then n is permissible for $PP'C$, and $\text{Typ}^n PP'C \geq_\alpha PP'C'$.

Proof. It is sufficient to assume $\text{Tail } PC \neq \tau$ and $n = 1$.

Let $\text{Tail } PC \equiv x$ and $(PC)|x \equiv Q_1[x, B]Q_2x$, then $(PP'C)|x \equiv Q'_1[x, B]Q'_2x$, and $[x, B]$ appears in either P or C . The remainder follows. □

Theorem 8.18. If $PC \in \Delta$, $PP'C \in \Delta$ and for a permissible n $\text{Typ}^n PP'C \equiv PP'C'$, then n is permissible for PC and $\text{Typ}^n PC \geq_\alpha PC'$.

Proof. Similar to the previous proof. □

CHAPTER III, THE FORMAL SYSTEM Λ

§ 1. LEGITIMATE EXPRESSIONS

The "meaning" of $\{A\}B$ is the application of function B to argument A . So far this application was unrestricted: any expression could serve as an argument. Besides, it was of no interest whether B really was a function or not.

In the formal system Λ , which we shall introduce in this chapter, we only admit the expressions of Δ which obey the applicability condition. (For an informal introduction of the applicability condition: see Section I.4.) We call this kind of expressions *legitimate expressions*.

Since Λ is a part of Δ , we again provide expressions with abstractor chains Q , as we did with expressions in Δ (cf. the beginning of Section I.6). We begin with the definitions of function, domain and applicability with respect to an abstractor chain Q :

Definition 1.1. Let $QB \in \Delta$. We call QB a *Q-function* if there are x , K and L such that $\text{Typ}^* QB \geq Q[x, K]L$. The expression QK is called a *Q-domain* of QB . □

Definition 1.2. The expression QB is called *Q-applicable* to QA if QB is a Q -function with Q -domain QK , $\text{Deg } QA > 1$ and $\text{Typ } QA \geq QK$. In that case $Q\{A\}B$ is a *legitimate Q-application* of QB to QA . □

The formal system Λ is inductively defined by:

Definition 1.3.

- (1) $\tau \in \Lambda$.
- (2) If $QA \in \Lambda$ and if x does not occur in QA , then $Q[x, A]x \in \Lambda$ and $Q[x, A]\tau \in \Lambda$.
- (3) If QA and $Qy \in \Lambda$, if x does not occur in QA and if $x \neq y$, then $Q[x, A]y \in \Lambda$.
- (4) If QA and $QB \in \Lambda$, if the binding variables in A and B are distinct and if QB is Q -applicable to QA , then $Q\{A\}B \in \Lambda$. □

The only difference to the (second) definition of Δ as given by Th. II.3.10 lies in the *applicability condition* in (4): QB must be Q-applicable to QA, i.e. $\text{Typ}^* \text{QB} \geq \text{Q}[y, K]L$ and $\text{Typ} \text{QA} \geq \text{QK}$. These reductions are defined for expressions in Δ (cf. the following Th. 1.4 and Th. II.8.2). Note that the applicability condition does not state that the reductions mentioned concern expressions in Λ only.

The applicability condition has the powerful consequence that all expressions in Λ normalize (cf. Section I.2), which we shall prove later in this chapter, whereas in the wider system Δ normalization is not guaranteed.

Theorem 1.4. If $A \in \Lambda$, then $A \in \Delta$.

Proof. Induction on the length of proof of $A \in \Lambda$. □

Restricting ourselves to α - and β -reductions, we can weaken the applicability condition in the sense that we replace $\geq$ by $\sim$:

Theorem 1.5. If QA and $\text{QB} \in \Lambda$, $\text{Q}[y, K]L \in \Delta$, $\text{Typ}^* \text{QB} \sim_{\beta} \text{Q}[y, K]L$, $\text{Typ} \text{QA} \sim_{\beta} \text{QK}$ and if the binding variables in A and B are distinct, then $\text{Q}\{A\}B \in \Lambda$.

Proof. Let $\text{Typ}^* \text{QB} \equiv \text{QB}'$ (Th. II.8.16). Since $\text{QB}' \sim_{\beta} \text{Q}[y, K]L$, there is an M such that $\text{QB}' \geq_{\beta} \text{QM}$ and $\text{Q}[y, K]L \geq_{\beta} \text{QM}$ (Th. II.5.12 and Th. II.5.16). From Th. II.5.16 and Th. II.5.20: $\text{QM} \equiv \text{Q}[y, K']L'$ such that $\text{QK} \geq_{\beta} \text{QK}'$.

Let $\text{Typ} \text{QA} \equiv \text{QA}'$. Since $\text{QA}' \sim_{\beta} \text{QK}$, there is a K'' such that $\text{QA}' \geq_{\beta} \text{QK}''$ and $\text{QK} \geq_{\beta} \text{QK}''$. Hence (Church-Rosser theorem for β -reduction, Th. II.6.43) $\text{QK}' \sim \text{QK}''$, so there is a K''' : $\text{QK}' \geq_{\beta} \text{QK}'''$ and $\text{QK}'' \geq_{\beta} \text{QK}'''$. Also $\text{Q}[y, K']L' \geq_{\beta} \text{Q}[y, K''']L'$.

Resuming: $\text{Typ}^* \text{QB} \geq_{\beta} \text{Q}[y, K''']L'$ and $\text{Typ} \text{QA} \geq \text{QK}'''$. So $\text{Q}\{A\}B \in \Lambda$. □

Note that the above theorem does not hold if we use lambda-equivalence ($\sim$) instead of β -equivalence ($\sim_{\beta}$). Let $\text{QA} \in \Lambda$ and

Typ $QA \equiv QA'$. Let $QB \in \Lambda$ for some B . Then $\text{Typ}^* QB \equiv QB' \sim Q[y, A']\{y\}B'$ for some fresh y , since $Q[y, A']\{y\}B' \geq_{\eta} QB'$. If the above theorem were to hold with $\sim$ instead of $\sim_{\beta}$, it would follow that $Q\{A\}B \in \Lambda$. Note that A and B are arbitrary. This can clearly not generally be the case. As a counterexample, take $Q \equiv [x, \tau], A \equiv B \equiv x$. Then $Q\{A\}B \equiv [x, \tau]\{x\}x$, which does not belong to Λ .

We shall prove a number of theorems concerning Λ .

Theorem 1.6. If $A \in \Lambda$ and $A \geq_{\alpha} B$, then $B \in \Lambda$. □

As with Δ , it holds for Λ that, given $K \in \Lambda$, only one of the derivation steps in Def. 1.3 can yield $K \in \Lambda$ as a conclusion (unique Λ -constructibility).

Theorem 1.7. If $Q\{A\}B \in \Lambda$, then QA and $QB \in \Lambda$.

Proof. Follows from the unique Λ -constructibility. □

Theorem 1.8. If $Q[x, A]B \in \Lambda$, then $QA \in \Lambda$.

Proof. Induction on $|B|$, using the unique Λ -constructibility.

Let $B \equiv [y_1, B_1] \dots [y_k, B_k]Ps$, where $P \neq [z, E]P'$, and $s \equiv \tau$, $s \equiv y \neq x$ or $s \equiv x$.

case 1. $P \equiv \emptyset$, $k = 0$. Then $QA \in \Lambda$ from rule (2) of Def. 1.3 for all possible s .

case 2. $P \equiv \emptyset$, $k \geq 1$. Then $Q[x, A][y_1, B_1] \dots [y_{k-1}, B_{k-1}]B_k \in \Lambda$ from rule (2) or (3), so $QA \in \Lambda$ by induction.

case 3. $P \equiv \{E\}P'$. Then $Q[x, A][y_1, B_1] \dots [y_k, B_k]E \in \Lambda$ by Th. 1.7, hence $QA \in \Lambda$ by induction. □

Theorem 1.9. If $QA \in \Lambda$, then $Q\tau \in \Lambda$.

Proof. Induction on $|A|$. If $A \equiv \tau$, there is nothing to prove. If $A \equiv x$, then $Q \equiv Q_1[y, B]$ or $Q \equiv Q_1[x, B]$. In both cases $Q_1B \in \Lambda$, so also $Q\tau \in \Lambda$. If $A \equiv \{B\}C$ or $A \equiv [z, B]C$, then $QB \in \Lambda$ by Th. 1.7 or by Th. 1.8, so by induction $Q\tau \in \Lambda$. □

Theorem 1.10. If $A \in \Lambda$ and $B \subset A$, then $A|B \in \Lambda$.

Proof. Induction on $|A|$. If $A \equiv \tau$ then the proof is trivial. Let $A \equiv [x_1, A_1] \dots [x_k, A_k]Ps$, where $P \neq [z, E]P'$.

- (1) If $B \equiv [x_j, A_j] \dots [x_k, A_k]Ps$ or $B \equiv Ps$, then $A|B \equiv A \in \Lambda$.
- (2) If $B \subset A_i$, then $A|B \equiv [x_1, A_1] \dots [x_{i-1}, A_{i-1}](A_i|B) \equiv ([x_1, A_1] \dots [x_{i-1}, A_{i-1}]A_i)|B$ and $[x_1, A_1] \dots [x_{i-1}, A_{i-1}]A_i \in \Lambda$ by Th. 1.8, so by induction $A|B \in \Lambda$.
- (3) Let $B \subset Ps$, $B \neq Ps$. If $P \equiv \emptyset$ then $B \equiv s$ and $A|B \equiv A \in \Lambda$. So assume $P \equiv \{K\}P'$. Distinguish the cases $B \subset K$ and $B \subset P's$. In both cases we may conclude $A|B \in \Lambda$ by a similar reasoning as in (2). □

Corollary 1.11. If $A \in \Lambda$ and $x \subset A$, then $A|x \in \Lambda$. □

Theorem 1.12. If QA and $QB \in \Lambda$, $Q[x, A]B \in \Delta$, then $Q[x, A]B \in \Lambda$.

Proof. Induction on $|B|$. Let $B \equiv [y_1, B_1] \dots [y_k, B_k]Ps$, where $P \neq [z, E]P'$.

case 1. $P \equiv \emptyset$, $k = 0$. Then $Q[x, A]B \in \Lambda$ by Def. 1.3 (2) or (3).

case 2. $P \equiv \emptyset$, $k \geq 1$. Call $[y_1, B_1] \dots [y_{k-1}, B_{k-1}] \equiv Q'$.

- (1) Assume $s \equiv y_k$. Then $QQ'B_k \in \Lambda$ by Th. 1.8, and $Q[x, A]Q'B_k \in \Delta$ (Th. II.3.8 and Th. II.3.9), so by induction $Q[x, A]Q'B_k \in \Lambda$, hence $Q[x, A]B \in \Lambda$.
- (2) Assume $s \neq y_k$. Then $QQ'B_k$ and $QQ's \in \Lambda$ (by the unique Λ -constructibility), $Q[x, A]Q'B_k$ and $Q[x, A]Q's \in \Delta$ (Th. II.3.8 and Th. II.3.9), so by induction $Q[x, A]Q'B_k$ and $Q[x, A]Q's \in \Lambda$. It follows that $Q[x, A]B \in \Lambda$.

case 3. $P \equiv \{E\}P'$. Call $[y_1, B_1] \dots [y_k, B_k] \equiv Q''$. Then $QQ''E$ and $QQ''P's \in \Lambda$ by Th. 1.7, $\text{Typ}^* QQ''P's \equiv QQ''F' \geq QQ''[z, K]L$ and $\text{Typ} QQ''E \equiv QQ''E' \geq QQ''K$. It follows from Th. II.7.33, Th. II.8.9 and Th. II.8.17 that $\text{Typ}^* Q[x, A]QQ''P's \equiv Q[x, A]Q''F' \geq Q[x, A]Q''[z, K]L$ and $\text{Typ} Q[x, A]Q''E \equiv Q[x, A]Q''E' \geq Q[x, A]Q''K$. By Th. II.3.8 and Th. II.3.9 $Q[x, A]Q''P's$ and $Q[x, A]Q''E \in \Delta$, so by induction they also belong to

Λ , hence $Q[x, A]B \in \Lambda$. □

Theorem 1.13. If $Q[x, A]B \in \Lambda$ and $QB \in \Delta$, then $QB \in \Lambda$.

Proof. Induction on $|B|$. The proof is similar to the proof of Th. 1.12, with the use of Th. II.3.10 instead of Th. II.3.11. □

We shall use the following theorem as a lemma for the important Th. 1.15.

Theorem 1.14. Let $PP'K, PL \in \Lambda$, $PP'L \in \Delta$ and $\text{Typ}^* PP'K \geq_{\alpha} \text{Typ}^* PP'L$. Then $PP'L \in \Lambda$.

Proof. Induction on $\|PP'\|$. If $PP' \equiv \emptyset$, the proof is trivial.

case 1. Assume $P \equiv Q\{E\}P''$. Then $QP''P'K \in \Lambda$, $QE \in \Lambda$, $\text{Typ}^* QP''P'K \geq Q[y, M]N$ and $\text{Typ}^* QE \geq QM$. Also: $QP''L \in \Lambda$ and $QP''P'L \in \Delta$. We now prove that $\text{Typ}^* QP''P'K \geq_{\alpha} \text{Typ}^* QP''P'L$. Let $\text{Typ}^* PP'K \equiv PP'K'$ and $\text{Typ}^* PP'L \equiv PP'L'$, then by hypothesis $Q\{E\}P''P'K' \equiv PP'K' \geq_{\alpha} PP'L' \equiv Q\{E\}P''P'L'$, so also $QP''P'K' \geq_{\alpha} QP''P'L'$ (Th. II.4.6). But $\text{Typ}^* QP''P'K \equiv QP''P'K'$ and $\text{Typ}^* QP''P'L \equiv QP''P'L'$ by Th. II.8.6 and Th. II.8.18. It follows by induction that $QP''P'L \in \Lambda$. Also $\text{Typ}^* QP''P'L \geq Q[y, M]N$, so $Q\{E\}P''P'L \equiv PP'L \in \Lambda$.

case 2. Assume $P \equiv Q$ and $P' \equiv Q'\{E\}P''$. Then $QQ'P'K \in \Lambda$, $QQ'E \in \Lambda$, $\text{Typ}^* QQ'P'K \geq QQ'[y, M]N$ and $\text{Typ}^* QQ'E \geq QQ'M$. Also: $QQ'P''L \in \Delta$ and $\text{Typ}^* QQ'P'K \geq_{\alpha} \text{Typ}^* QQ'P''L$ (which can be proved as in case 1), so by induction $QQ'P''L \in \Lambda$. Since $\text{Typ}^* QQ'P''L \geq QQ'[y, M]N$ it follows that $QQ'\{E\}P''L \equiv PP'L \in \Lambda$.

case 3. Assume $P \equiv Q$ and $P' \equiv Q'$. If $Q' \equiv \emptyset$ there is nothing to prove. Let $Q' \equiv [x_1, M_1] \dots [x_n, M_n]$ for $n \geq 1$. Since $QL \in \Delta$, x_i cannot occur in QL (Th. II.3.8) or in $Q[x_1, M_1] \dots [x_{i-1}, M_{i-1}]M_i$. It follows from $QQ'K \in \Lambda$ (Th. 1.8) that $QM_1, Q[x_1, M_1]M_2, \dots, Q[x_1, M_1] \dots [x_{n-1}, M_{n-1}]M_n \in \Lambda$. So also $Q[x_1, M_1]L, Q[x_1, M_1][x_2, M_2]L, \dots, QQ'L \in \Lambda$ by Th. II.3.11 and Th. 1.12. □

Theorem 1.15. If $A \in \Lambda$, then $\text{Typ}^n A \in \Lambda$ for all permissible n .

Proof. Let $A \equiv P_1[x, B]P_2x$, then $\text{Typ } A \equiv P_1[x, B]P_2\text{Fr}B$. Since $A \in \Lambda$: $P_1B \in \Lambda$ (Th. 1.8), so $P_1\text{Fr}B \in \Lambda$ (Th. 1.6). Also $\text{Typ } A \in \Delta$ (Th. II.8.2) and $\text{Typ}^* A \geq_\alpha \text{Typ}^*(\text{Typ } A)$. Now, applying Th. 1.14, we obtain $\text{Typ } A \in \Lambda$. The theorem follows directly. $\square$

§ 2. THE NORMALIZATION THEOREM

In this section we shall prove the normalization theorem: if $A \in \Lambda$, there is a B in normal form such that $A \geq B$ (B is said to be in normal form if there are no reductions $B \geq'_\beta B'$ or $B \geq'_\eta B'$). We do this by the aid of a norm ρ , which is a partial function from expressions in Δ to expressions in Δ , and which has the following powerful properties with relation to Λ :

- (1) If $A \in \Lambda$, then $\rho(A)$ is defined.
- (2) If $A \in \Lambda$ and $A \geq B$, then $\rho(A) \geq_\alpha \rho(B)$.
- (3) If $A \in \Lambda$ and $\text{Deg } A > 1$, then $\rho(A) \geq_\alpha \rho(\text{Typ}(A))$.

Hence this norm is invariant (apart from α -reduction) with respect to reduction and typing.

We first define ρ_A for every $A \in \Delta$. This ρ_A is a partial function from subexpressions of A to expressions. It is rather in contradiction to our philosophy to define the norm with respect to subexpressions, which need not belong to Δ . We could have avoided this by giving a definition of the norm in the line of our second definition of Δ , only considering norms of expressions in Δ . This, however, would have impaired understanding of the following and would have led to laborious descriptions. On the other hand, in this section the context of a subexpression will always be clear, so that no confusion can arise.

In the following inductive definition of ρ_A we do not explicitly indicate which occurrence of a subexpression in an expression is meant, since this will be clear from the context.

Definition 2.1. Let $A \in \Delta$.

- (1) If $\tau \subset A$ then $\rho_A(\tau) \equiv \tau$.
- (2) If $x \subset A$, $A|x \equiv Q_1[x, B]Q_2x$ and if $\rho_A(B)$ is defined, then $\rho_A(x) \equiv \rho_A(B)$.
- (3) If $[x, B]C \subset A$, and if both $\rho_A(B)$ and $\rho_A(C)$ are defined, then $\rho_A([x, B]C) \equiv [x, \rho_A(B)]\rho_A(C)$.
- (4) If $\{B\}C \subset A$, if both $\rho_A(B)$ and $\rho_A(C)$ are defined and $\rho_A(C) \equiv [y, D]E$ where $D \geq_\alpha \rho_A B$, then $\rho_A(\{B\}C) \equiv E$. □

From this definition it can easily be seen that, if $\rho_A A$ is defined for $A \in \Delta$, $\rho_A A$ contains no bound variables.

The following theorem is obvious:

Theorem 2.2. If $A \in \Delta$ and if $\rho_A(A)$ is defined, then $\rho_A(A) \in \Delta$; if, moreover, $A \geq_\alpha B$, then $\rho_A(A) \geq_\alpha \rho_B(B)$ and $|\rho_A(A)| = |\rho_B(B)|$. □

The binding variables in $\rho_A A$ will be irrelevant to our purposes. We might as well do without them. Our reason for retaining them is personal taste: we find the property $\rho_A(A) \in \Delta$ agreeable.

In trying to calculate $\rho_A(A)$ for a certain $A \in \Delta$, we apply the four rules of Def. 2.1; the only event in which this calculation can break down prematurely (before $\rho_A(A)$ is obtained), is when we encounter a subexpression $\{B\}C \subset A$ for which the conditions stated in Def. 2.1 (4) are not fulfilled.

These conditions may be considered as a weaker form of the applicability condition (cf. Section I.6, where this is explained in an informal manner): (1) C must have a norm with a functional character: $\rho_A C \equiv [y, D]E$, and (2) B has a norm which behaves as an appropriate argument for the "function" $\rho_A C$: $\rho_A B \geq_\alpha D$. If these conditions are fulfilled, the norm of $\{B\}C$ is defined as the result of the application of the "function" $\rho_A C$ to the "argument" $\rho_A B$: $\rho_A(\{B\}C) \equiv E$; if these conditions are not fulfilled, the norm of $\{B\}C$ is not defined, and neither is the norm of A .

Note that the norm of a bound variable is defined as the norm of its "type": if $[x, B]$ is the binding abstractor of x , then $\rho_A(x) \equiv \rho_A(B)$.

The existence of $\rho_A A$ for a certain $A \in \Delta$ indicates that some weak functional condition is fulfilled. Surprisingly enough, the existence of $\rho_A A$ already guarantees that there is a normal form for A . We shall prove this in Th. 2.17. We are especially interested in normalization properties of expressions in Λ . We note that expressions in Λ have, so to say, a much stronger functional character than is required for the existence of the norm of expressions. Th. 2.7, stating that $\rho_A A$ exists for $A \in \Lambda$, is not hard to prove.

If in the following we speak of the norm of a subexpression B of a certain expression A , it will be clear which A we mean, even if we do not state this explicitly. In such cases we shall write $\rho(B)$ instead of $\rho_A(B)$.

If $A \in \Delta$, $B \subset A$ and $\rho_A(B)$ is defined, we call B ρ_A -normable. Here, too, we speak of " ρ -normable B " if it is clear which A (with $B \subset A \in \Delta$) we mean. If $Q\tau \in \Delta$, if $Q\tau$ is ρ -normable and $\rho(Q\tau) \equiv Q'\tau$, we call Q ρ -normable, and we abbreviate $\rho(Q) \equiv Q'$.

Theorem 2.3. If $A \in \Delta$, A is ρ -normable and $B \subset A$, then B is ρ -normable.

Proof. Induction on $|A|$, with the use of the definition of $B \subset A$ (Def. II.2.5). □

Theorem 2.4. If $QA \in \Delta$ and QA is ρ -normable, then Q and A are ρ -normable and $\rho(QA) \equiv (\rho Q)\rho A$; if $QA \in \Delta$, and if Q and A are ρ -normable, then QA is ρ -normable and $\rho(QA) \equiv (\rho Q)\rho A$.

Proof. Induction on $\|Q\|$. □

Theorem 2.5. If $A \in \Delta$, if A is ρ -normable and $A \geq B$, then B is ρ -normable and $\rho A \geq_\alpha \rho B$.

Proof. First assume that $A \geq' B$. We proceed by induction on the length of proof that $A \geq' B$. If $A \geq'_\alpha B$ then the proof is trivial; this case is expressed in Th. 2.2.

Ia. $A \geq' B$ is $Q\{C\}[x, D]E \geq'_\beta Q(x := C)E$. Since A is ρ -normable: $\rho([x, D]E) \equiv [x, \rho D]\rho E$ and $\rho C \geq_\alpha \rho D \equiv \rho x$ (see Th. 2.3 and Th. 2.4). Moreover, $\rho A \equiv \rho(Q\{C\}[x, D]E) \equiv (\rho Q)\rho E$. We now prove for this x , C and E :

Lemma. If $K \subset E$, then $(x := C)K$ is ρ -normable and $\rho(x := C)K \geq_\alpha \rho K$.

Proof of the lemma. Induction on $|K|$.

(1a) If $K \equiv x$, then $(x := C)K \equiv \text{Fr}C$ and $\rho(x := C)K \equiv \rho \text{Fr}C \geq_\alpha \rho C \geq_\alpha \rho x \equiv \rho K$.

(1b) If $K \equiv y \neq x$ or $K \equiv \tau$, then $(x := C)K \equiv K$ and $\rho(x := C)K \equiv \rho K$.

(2) If $K \equiv [y, F]G$, then $\rho K \equiv [y, \rho F]\rho G$. Note that $y \neq x$. By induction: $(x := C)F$ and $(x := C)G$ are ρ -normable, $\rho(x := C)F \geq_\alpha \rho F$ and $\rho(x := C)G \geq_\alpha \rho G$. So $(x := C)K$ is ρ -normable and $\rho(x := C)K \equiv [y, \rho(x := C)F]\rho(x := C)G \geq_\alpha [y, \rho F]\rho G \equiv \rho K$.

(3) If $K \equiv \{F\}G$, then $\rho G \equiv [y, L]H$ and $L \geq_\alpha \rho F$, and $\rho K \equiv H$. By induction: $(x := C)F$ and $(x := C)G$ are ρ -normable, $\rho(x := C)F \geq_\alpha \rho F$ and $\rho(x := C)G \geq_\alpha \rho G$. It follows that $\rho(x := C)G \equiv [z, L']H'$ and $L' \geq_\alpha L \geq_\alpha \rho F \geq_\alpha \rho(x := C)F$, so $(x := C)K$ is ρ -normable and $\rho(x := C)K \equiv H' \geq_\alpha H \equiv \rho K$. =

It follows that B is ρ -normable (since $E \subset E$), and

$$\rho B \equiv (\rho Q)\rho(x := C)E \geq_\alpha (\rho Q)\rho E \equiv \rho A.$$

Ib. $A \geq' B$ is $Q[x, C]\{x\}D \geq'_\eta QD$. Since A is ρ -normable: QC is ρ -normable, $\rho x \equiv \rho C$, $\rho D \equiv [y, L]H$ and $L \geq_\alpha \rho x \equiv \rho C$, so $\rho A \equiv (\rho Q)[x, \rho C]H \geq_\alpha (\rho Q)[y, \rho C]H \geq_\alpha (\rho Q)\rho D \equiv \rho QD \equiv \rho B$.

II. $A \geq' B$ is a direct consequence of a monotony rule. It depends on the monotony rule which of the following three cases applies:

IIa. $A \geq' B$ is $Q\{C\}E \geq' Q\{C\}F$ as a direct consequence of $QE \geq' QF$. Since A is ρ -normable: QE is ρ -normable. By induction: QF is ρ -normable, and $\rho QE \equiv (\rho Q)\rho E \geq_\alpha (\rho Q)\rho F \equiv \rho QF$, so $\rho E \geq_\alpha \rho F$. Moreover,

$\rho E \equiv [y, L]H$ and $\rho C \geq_{\alpha} L$, so also $\rho F \equiv [z, L']H'$, where $L' \geq_{\alpha} L$ and $H' \geq_{\alpha} H$. It follows that $\{C\}F$ is ρ -normable, and $\rho\{C\}F \equiv H' \geq_{\alpha} H$. So $\rho B \geq_{\alpha} (\rho Q)\rho H \equiv \rho A$.

IIb. $A \geq' B$ is $Q[x, C]E \geq' Q[x, D]E$ as a direct consequence of $QC \geq' QD$.

Then $\rho A \equiv (\rho Q)[x, \rho C]\rho E$. By induction: QD is ρ -normable and $\rho QC \equiv (\rho Q)\rho C \geq_{\alpha} (\rho Q)\rho D \equiv \rho QD$, so $\rho C \geq_{\alpha} \rho D$. Hence B is ρ -normable and $\rho A \geq_{\alpha} (\rho Q)[x, \rho D]\rho E \equiv \rho B$.

IIc. $A \geq' B$ is $Q\{C\}E \geq' Q\{D\}E$ as a direct consequence of $QC \geq' QD$. Then $\rho E \equiv [x, L]H$ and $L \geq_{\alpha} \rho C$. By induction: QD is ρ -normable and $\rho QC \equiv (\rho Q)\rho C \geq_{\alpha} (\rho Q)\rho D \equiv \rho QD$, so $\rho C \geq_{\alpha} \rho D$. Hence B is ρ -normable and $\rho B \equiv (\rho Q)H \equiv \rho A$.

Finally, if $A \geq B$ is a multiple-step reduction, decompose the reduction and apply the above. $\square$

Theorem 2.6. If $A \in \Delta$, $\text{Deg } A > 1$ and A is ρ -normable, then $\text{Typ } A$ is ρ -normable and $\rho A \geq_{\alpha} \rho \text{Typ } A$.

Proof. Let $A \equiv P_1[x, B]P_2x$, then $\text{Typ } A \equiv P_1[x, B]P_2\text{Fr}B$. It is not hard to show that $\rho\text{Fr}B \geq_{\alpha} \rho B \equiv \rho x$. Let $P_1[x, B]P_2 \equiv P''P''$. Next prove by induction on $\|P''\|$ that $P''\text{Fr}B$ is ρ -normable, and $\rho(P''\text{Fr}B) \geq_{\alpha} \rho(P''x)$. $\square$

Theorem 2.7. If $A \in \Lambda$, then A is ρ -normable (i.e. ρ is a total function on Λ).

Proof. Induction on the length of proof of $A \in \Lambda$.

- (1) $A \equiv \tau$: trivial.
- (2) $A \equiv Q[x, B]x$ or $Q[x, B]\tau \in \Lambda$ as a direct consequence of $QB \in \Lambda$.
Then by induction QB is ρ -normable, hence Q is ρ -normable and $\rho B \equiv \rho x$. Hence A is ρ -normable.
- (3) $A \equiv Q[x, B]y \in \Lambda$ as a direct consequence of $QB \in \Lambda$ and $Qy \in \Lambda$.
By induction: QB and Qy are ρ -normable, hence Q , B and y are ρ -normable, so A is ρ -normable.

(4) $A \equiv Q\{B\}C \in \Lambda$ as a direct consequence of $QB \in \Lambda$, $QC \in \Lambda$, and the Q -applicability of QC to QB . Then QB and QC are ρ -normable (induction), and so are Q , B and C . The Q -applicability implies that $\text{Typ}^* QC \geq Q[x, K]L$ and $\text{Typ} QB \geq QK$. From Th. 2.5 and Th. 2.6: $\text{Typ}^* QC$ and $Q[x, K]L$ are ρ -normable, $\rho QC \geq_{\alpha} \rho \text{Typ}^* QC \geq_{\alpha} (\rho Q)[x, \rho K]\rho L$ (so $\rho C \geq_{\alpha} [x, \rho K]\rho L$) and $\rho QB \geq_{\alpha} \rho \text{Typ} QB \geq_{\alpha} \rho QK$ (so $\rho B \geq_{\alpha} \rho K$). Hence $\{B\}C$ is ρ -normable and so is A . $\square$

Instead of: ρ is total on Λ , we also say: Λ is ρ -normable.

From Th. 2.5 we derive:

Theorem 2.8. If $A, B \in \Lambda$ and $A \sim B$, then $\rho A \geq_{\alpha} \rho B$. $\square$

We shall now prove the normalization theorem for ρ -normable expressions.

Definition 2.9. $A \in \Delta$ is *normal* (or *in normal form*) if there are no reductions $A \geq'_{\beta} B$ or $A \geq'_{\eta} B$; A is *normalizable* (or A has a normal form) if there exists a normal C such that $A \geq C$ (C is called a *normal form* of A). $\square$

Definition 2.10. $A \in \Delta$ is β -*normal* if there is no reduction $A \geq'_{\beta} B$; A is β -*normalizable* (or A has a β -normal form) if there exists a β -normal C such that $A \geq_{\beta} C$ (C is called a β -*normal form* of A). $\square$

Hence A is normal if A admits of neither β -, nor η -reductions; A is β -normal if A admits of no β -reductions (except trivial ones).

Theorem 2.11. If $A \in \Delta$ is normal and $A \geq_{\alpha} B$, then B is normal. If $A \in \Delta$ is β -normal and $A \geq_{\alpha} B$ or $A \geq_{\eta} B$, then B is β -normal.

Proof. The only non-trivial statement is that B is β -normal if A is β -normal and $A \geq_{\eta} B$. It can, however, easily be seen that a single-step η -reduction of a β -normal expression cannot introduce the possibility of a single-step β -reduction. $\square$

We restate the following well-known theorem:

Theorem 2.12. If $A \in \Delta$ is β -normalizable, then A is normalizable.

Proof. As a result of Th. 2.11, η -reductions of A do not cancel the β -normal character. But the possible number of single-step η -reductions applicable to A is finite, since the expression becomes shorter with each step. $\square$

Theorem 2.13. If $A \in \Delta$ is β -normalizable, the β -normal form of A is unique but for α -reductions.

Proof. Let C and D be β -normal, $A \geq_{\beta} C$ and $A \geq_{\beta} D$. Then, by the Church-Rosser theorem for β -reduction (Th. II.6.43): there is an E such that $C \geq_{\beta} E$ and $D \geq_{\alpha, \beta} E$. Hence $C \geq_{\alpha} D$. $\square$

Theorem 2.14. Assume that $Q\{A_k\} \dots \{A_1\}B$ is in Δ and ρ -normal.

Then $|\rho B| > \sum_{i=1}^k |\rho A_i|$.

Proof. Induction on k . If $k = 0$ the proof is trivial. Let $k > 0$.

Then $\{A_1\}B$ is ρ -normal, hence $\rho B \equiv [x, M]N$ and $\rho A_1 \geq_{\alpha} M$, so

$|\rho A_1| = |M|$. Moreover, $\rho\{A_1\}B \equiv N$, hence $|N| > \sum_{i=2}^k |\rho A_i|$ by induction. It follows that

$$|\rho B| = |[x, M]N| > |M| + \sum_{i=2}^k |\rho A_i| = \sum_{i=1}^k |\rho A_i|. \quad \square$$

Definition 2.15. Assume that A is in Δ and ρ -normal, $A \equiv Q\{C_n\} \dots$

$\dots \{C_1\}F$ for some $n \geq 1$ and $F \neq \{M\}N$. Then $\sigma(A) = \sum_{i=1}^n |\rho C_i|$. If

$A \equiv Qs$ (with $s \equiv x$ or $s \equiv \tau$), then $\sigma(A) = 0$. $\square$

Theorem 2.16. Assume that A is in Δ and ρ -normal, $A \equiv Q\{C_n\} \dots$

$\{C_1\}F$, $F \neq \{M\}N$, and let QC_i (for $1 \leq i \leq n$) and QF be β -normal.

Then A is β -normalizable.

Proof. Induction on $\sigma(A)$.

(1) If $\sigma(A) = 0$, then $n = 0$ and $A = QF$ in β -normal form.

(2) Let $\sigma(A) > 0$. A is β -normal if $n = 0$. Let $n \geq 1$. We proceed by induction on $|F|$. If $F \equiv y$, then A is in β -normal form ($F \equiv \tau$ cannot occur since A is ρ -normable). So let $|F| > 1$. Then $F \equiv [x, D]E$.

(i) Assume that $E \equiv \{H_m\} \dots \{H_1\}y$ where $y \neq x$ and $m \geq 0$. Then $A \geq Q\{C_n\} \dots \{C_2\}\{(x := C_1)H_m\} \dots \{(x := C_1)H_1\}y$. Now note that $Q\{C_1\}[x, D]H_i \in \Delta$, ρ -normable, and $\sigma(Q\{C_1\}[x, D]H_i) = |\rho C_1| \leq \sigma(A)$. Moreover, $|H_i| < |E|$, so by induction $Q\{C_1\}[x, D]H_i$ is β -normalizable. Since QC_1 , QD and QH_i are β -normal, the β -normalization of $Q\{C_1\}[x, D]H_i$ must commence with $Q\{C_1\}[x, D]H_i \geq'_\beta Q(x := C_1)H_i$, so $Q(x := C_1)H_i$ is β -normalizable; say $Q(x := C_1)H_i \geq QK_i$ in β -normal form. It follows that $A \geq Q\{C_n\} \dots \{C_2\}\{K_m\} \dots \{K_1\}y$ in β -normal form.

(ii) Assume that $E \equiv \{H_m\} \dots \{H_1\}\tau$ where $m \geq 0$. Again, analogously to (i), A is β -normalizable. (Moreover, m must be 0 since A is ρ -normable.)

(iii) Assume that $E \equiv \{H_m\} \dots \{H_1\}x$ where $m \geq 0$. Then $A \geq A' \equiv Q\{C_n\} \dots \{C_2\}\{K_m\} \dots \{K_1\}FrC_1$, where we obtain the β -normal QK_i as in (i). If now FrC_1 is a variable or if FrC_1 begins with an applicator, we have obtained a β -normal form. If $FrC_1 \equiv [y, M]N$, then $\sigma(A') = \sum_{i=2}^n |\rho C_i| + \sum_{j=1}^m |\rho K_j| < |\rho FrC_1|$ (by Th. 2.14 and Th. 2.5) = $|\rho C_1| \leq \sigma(A)$, so by the induction hypothesis: A' is β -normalizable, so A is β -normalizable too, or $n = 1$ and $m = 0$, whence A' is β -normal.

(iv) Assume that $E \equiv [y, H_1]H_2$. Then

$A \geq Q\{C_n\} \dots \{C_2\}[y, (x := C_1)H_1](x := C_1)H_2 \geq Q\{C_n\} \dots \{C_2\}[y, K_1]K_2$, where we again obtain the β -normal $Q[y, K_1]K_2$ as in (i). Since $\sigma(Q\{C_n\} \dots \{C_2\}[y, K_1]K_2) = \sum_{i=2}^n |\rho C_i| < \sigma(A)$, it follows by induction that A is β -normalizable, or $n = 1$ and $Q[y, K_1]K_2$ is β -normal. $\square$

Theorem 2.17 (*β -normalization theorem*). If $A \in \Delta$ is ρ -normalable, then A is β -normalizable.

Proof. Induction on the length of proof of $A \in \Delta$.

- (1) $A \equiv \tau$; trivial.
- (2) $A \equiv Q[x, B]x$ or $A \equiv Q[x, B]\tau \in \Delta$ as a direct consequence of $QB \in \Delta$. Then by induction QB is β -normalizable, so $QB \geq_{\beta} Q'B'$ in β -normal form, and $A \geq_{\beta} Q'[x, B']x$ or $A \geq_{\beta} Q'[x, B']\tau$ in β -normal form.
- (3) $A \equiv Q[x, B]y \in \Delta$ as a direct consequence of $QB \in \Delta$ and $Qy \in \Delta$. Then by induction $QB \geq_{\beta} Q'B'$ in β -normal form, so $A \geq Q'[x, B']y$ in β -normal form.
- (4) $A \equiv Q\{B\}C \in \Delta$ as a direct consequence of $QB \in \Delta$ and $QC \in \Delta$. Then by the induction hypothesis: $QB \geq_{\beta} Q'B'$ in β -normal form (with $\|Q\| = \|Q'\|$) and $QC \geq_{\beta} Q''C'$ in β -normal form (with $\|Q\| = \|Q''\|$). From Th. 2.13 and induction on $\|Q\|$ it follows that $Q''C' \geq_{\alpha} Q'C'$. Also $Q\{B\}C \geq_{\beta} Q'\{B'\}C'$, which is in β -normal form if $C' \neq [x, D]E$. So all that is left to prove is that $Q'\{B'\}[x, D]E$ is β -normalizable. But this follows from Th. 2.16 and Th. 2.5. □

Theorem 2.18 (*normalization theorem for Λ*). If $A \in \Lambda$ then A is β -normalizable and normalizable.

Proof. Follows from Th. 2.17, Th. 2.12 and Th. 2.7. □

In fact we proved that $A \in \Lambda$ is *effectively* normalizable, since all our proofs are constructive, which implies that the normal form of $A \in \Lambda$ is effectively computable.

§ 3. STRONG NORMALIZATION

In the previous section we have proved normalization for Λ . This guarantees that for every $A \in \Lambda$ there is a reduction which leads to a normal form. However, we do not yet know whether an arbitrary sequence of single-step reductions, beginning with A , terminates (in a normal form). We shall prove this in this section. The property that an arbitrary sequence of single-step reductions, beginning with some A , terminates, will be called the property of *strong normalization*.

In the proof we shall use β_1 -reduction and β_2 -reduction, introduced in Section II.6. A feature of β_1 -reduction is that "scars" of old β_1 -reductions are retained. We shall first prove strong normalization for Λ as to β_1 -reduction, and derive strong normalization for Λ as to β -reduction; finally, we shall incorporate η -reductions.

Definition 3.1. $A \in \Delta$ is β_1 -normal (or in β_1 -normal form) if there is no B such that $A \geq'_{\beta_1} B$; A is β_1 -normalizable (or A has a β_1 -normal form) if there exists a β_1 -normal C such that $A \geq_{\beta_1} C$ (C is then called a β_1 -normal form of A). The concepts β_2 -normal, β_2 -normal form and β_2 -normalizable are defined analogously. $\square$

Theorem 3.2. If $A \in \Delta$, A is β_1 -normal and $A \geq_{\alpha} B$, then B is β_1 -normal. $\square$

Theorem 3.3. If $A \in \Delta$ is β_1 -normalizable, the β_1 -normal form is unique but for α -reduction.

Proof. This follows from CR for β_1 -reductions (Th. II.6.38). $\square$

Theorem 3.4. If $A \in \Delta$, A is ρ -normable and $A \geq_{\beta_1} B$, then B is ρ -normable and $\rho A \geq_{\alpha} \rho B$.

Proof. First assume that $A \geq'_{\beta_1} B$. We proceed with induction on the length of proof of $A \geq'_{\beta_1} B$.

I. Let $A \equiv Q\{C\}\hat{P}[x,D]E \geq_{\beta_1} Q\{C\}\hat{P}[\overset{\circ}{x},D](x := C)E \equiv B$. Then $\hat{P}[x,D]E$ is ρ -normalable. It is easy to see (induction on $\|P\|$) that $\rho(\hat{P}[x,D]E) \equiv \rho([x,D]E) \equiv [x,\rho D]\rho E$. So $\rho C \geq_{\alpha} \rho D$. As in the lemma occurring in the proof of Th. 2.5 we can prove that $(x := C)E$ is ρ -normalable and that $\rho E \geq_{\alpha} \rho((x := C)E)$. It follows that B is ρ -normalable and $\rho A \geq_{\alpha} \rho B$.

II. $A \geq'_{\beta_1} B$ is a direct consequence of a monotony.

In all three cases the proof is identical to that given in part II of the proof of Th. 2.5.

Finally, if $A \geq_{\beta_1} B$ is a multiple-step reduction, decompose the reduction into single-step β_1 -reductions and apply the above. $\square$

We shall now prove the β_1 -normalization theorem. We do this in quite the same manner in which we proved the β -normalization theorem. In fact, if we had begun by proving the β_1 -normalization theorem, the β -normalization theorem would have been a corollary. We have not chosen this order because in the following proof the main lines are obscured by the presence of a number of β -chains $\hat{P}_i$. In contrast to this, the line of thought in the proof of the β -normalization theorem, given in the previous section, is much more lucid.

Definition 3.5. Let $A \in \Delta$, $A \equiv P_1\hat{P}B$ and let $\hat{P}$ be such that, for each $[x,C]$ for which $\hat{P} \equiv P_2[x,C]P_3$, it holds that $x \notin P_3B$. Then we call $\hat{P}$ an *ineffective* β -chain, and write $A \equiv P_1\overset{\circ}{P}B$. $\square$

Theorem 3.6. If $A \in \Delta$ is β_1 -normal and $B \subset A$, then B has the form $B \equiv \overset{\circ}{P}_0[x_1, A_1]\overset{\circ}{P}_1 \dots [x_n, A_n]\overset{\circ}{P}_n \{B_1\}\overset{\circ}{P}'_1 \dots \{B_\ell\}\overset{\circ}{P}'_\ell s$, with $s \equiv x_i$ or $s \equiv \tau$.

Proof. Induction on $|B|$. $\square$

Theorem 3.7. Let $Q\hat{P}_{k+1}\{A_k\}\hat{P}_k\{A_{k-1}\}\hat{P}_{k-1} \dots \{A_1\}\hat{P}_1B$ belong to Δ and be ρ -normable. Then $|\rho B| > \sum_{i=1}^k |\rho A_i|$.

Proof. Analogous to the proof of Th. 2.14. Note again that for ρ -normable $\hat{P}C$ it holds: $\rho(\hat{P}C) \equiv \rho C$. □

Definition 3.8. Let $A \in \Delta$ and assume that A is ρ -normable. Let $A \equiv Q\hat{P}_{n+1}^{\circ}\{C_n\}\hat{P}_n^{\circ} \dots \{C_1\}\hat{P}_1^{\circ}F$, where $F \equiv \tau$, $F \equiv x$ or $F \equiv [y, M]N$ with $y < N$. Then $\sigma_1(A) = \sum_{i=1}^n \rho|C_i|$ if $n \geq 1$, and $\sigma_1(A) = 0$ in case $n = 0$ and $\hat{P}_{n+1}^{\circ} \neq \emptyset$. Moreover, $\sigma_1(A) = 0$ if $A \equiv Q\tau$ or Qx . □

Theorem 3.9. Let A belong to Δ and be ρ -normable, let $A \equiv Q\hat{P}_{n+1}^{\circ}\{C_n\}\hat{P}_n^{\circ} \dots \{C_1\}\hat{P}_1^{\circ}F$, where $F \equiv \tau$, $F \equiv x$ or $F \equiv [x, D]E$ with $x < E$. Let QC_1 , QF and $Q\hat{P}_1^{\circ}\tau$ be β_1 -normal. Then A is β_1 -normalizable.

Proof. Induction on $\sigma_1(A)$. The proof is analogous to that of Th. 2.16. However, some modifications are required due to the $\hat{P}_1^{\circ}$. We shall briefly comment on this: As to (2)(i): $E \equiv \hat{P}_m^{\circ}\{H_m\} \dots \hat{P}_1^{\circ}\{H_1\}\hat{P}_0^{\circ}y$, and $A \geq Q\hat{P}_{n+1}^{\circ}\{C_n\}\hat{P}_n^{\circ} \dots \{C_2\}\hat{P}_2^{\circ}\{C_1\}\hat{P}_1^{\circ}[x, D]\hat{P}_m^{\circ}\{K_m\} \dots \hat{P}_1^{\circ}\{K_1\}\hat{P}_0^{\circ}y$, where the K_i are obtained as in the proof of Th. 2.16 and where $Q\hat{P}_1^{\circ}\tau$ are the β_1 -normal forms of $Q(x := C_1)\hat{P}_1^{\circ}\tau$. These can be obtained since either (1) if $x \neq \hat{P}_1^{\circ}$: $\hat{P}_1^{\circ} \equiv \hat{P}_1^{\circ}$, or (2) if $x < \hat{P}_1^{\circ}$: $\sigma_1(Q\{C_1\}[x, D]\hat{P}_1^{\circ}\tau) \equiv |\rho C_1| \leq \sigma(A)$ and $|\hat{P}_1^{\circ}\tau| < |F|$ (apply the induction on $|F|$). Note that $\hat{P}_2^{\circ}\{C_1\}\hat{P}_1^{\circ}[\hat{x}, D]\hat{P}_m^{\circ}$ is an ineffective β -chain.

As to (2)(iii): $\text{Fr}C_1$ can be $\hat{P}_0^{\circ}[x, M]N$. If $x < N$, then the proof is similar to that of Th. 2.16. If $x \neq N$, we can take $[\hat{x}, M]$ as part of an ineffective β -chain $\{K_1\}\hat{P}_0^{\circ}\hat{P}_0^{\circ}[\hat{x}, M]$, and look at the structure of N instead of that of $\text{Fr}C_1$.

This amounts to looking for the first "effective" abstractor in N . If there is such an abstractor and the obtained expression is not yet in β_1 -normal form, induction is applicable as in Th. 2.16. If not, we have already obtained β_1 -normal form.

As to (2)(iv): $E \equiv \overset{\circ}{P}_0[y, H_1]H_2$. If $y \subset H_2$ (so $y \subset K_2$), the proof is obvious. If not, look at K_2 instead of E , as in the previous case. $\square$

Theorem 3.10 (β_1 -normalization theorem). If $A \in \Delta$ is ρ -normal, then A is β_1 -normalizable.

Proof. Induction on the length of proof of $A \in \Delta$, analogously to the proof of Th. 2.17. As to case (4) of this proof: the only case worth mentioning is $C' \equiv \overset{\circ}{P}[x, D]E$. If $x \subset E$, then Th. 3.9 yields the desired result, and if $x \not\subset E$ we have already obtained β_1 -normal form. $\square$

Theorem 3.11 (β_1 -normalization theorem for Λ). If $A \in \Lambda$, then A is β_1 -normalizable.

Proof. Follows from Th. 2.7 and Th. 3.10. $\square$

In fact we have proved that $A \in \Lambda$ is *effectively* β_1 -normalizable.

Theorem 3.12. Let $A \in \Delta$ and $A \geq'_{\beta_1} B$. Then $|A| \leq |B|$.

Proof. Induction on the length of proof of $A \geq'_{\beta_1} B$. $\square$

Definition 3.13. Let $A \in \Lambda$. We write β_1 -nf A for the β_1 -normal form of A which we obtain from the effective computation as suggested by Th. 3.10 and used in Th. 3.11. $\square$

Note: this β_1 -normal form is unique (Th. 3.3).

Definition 3.14. We call $K \in \Delta$ *strongly β -normalizable* if there is an upper bound for the length ℓ of reduction sequences $K \equiv K_1 \geq'_\beta K_2 \geq'_\beta \dots \geq'_\beta K_\ell$. Analogously we define the concepts *strong β_1 -*, β_2 - or η -normalizability of K . $\square$

Theorem 3.15 (strong β_2 -normalization theorem for Δ). If $A \in \Delta$, then A is strongly β_2 -normalizable.

Proof. Induction on $|A|$. □

Definition 3.16. Let $A \in \Delta$, and let $A \equiv A_1 \geq'_{\beta_2} A_2 \geq'_{\beta_2} \dots \geq'_{\beta_2} A_p$ be the longest possible sequence of single-step β_2 -reductions beginning with A . Then $\theta_2(A) = p$. □

Note that $p \leq |A|$.

Theorem 3.17. If $A \in \Delta$ and $A \geq'_{\beta_1} B$, then $\theta_2(\beta_1 - \text{nf } A) = \theta_2(\beta_1 - \text{nf } B)$.

Proof. Follows from Th. 3.3. □

Theorem 3.18. If $A \geq'_{\beta_1} B$, then $\theta_2(A) < \theta_2(B)$.

Proof. Induction on the length of proof of $A \geq'_{\beta_1} B$. The only interesting case is $A \equiv Q\{C\}\hat{P}[x, D]E \geq'_{\beta_1} Q\{C\}\hat{P}[x, D](x := C)E \equiv B$, where, indeed, we have at least one single-step β_2 -reduction more on the right hand side. The rest of the proof is easy. □

Corollary 3.19. If $A \in \Delta$, then $\theta_2(A) \leq \theta_2(\beta_1 - \text{nf } A)$. □

Theorem 3.20 (strong β_1 -normalization theorem for Δ). If $A \in \Delta$, then A is strongly β_1 -normalizable.

Proof. Follows from Th. 3.17, Th. 3.18 and Cor. 3.19. □

Definition 3.21. Let $A \in \Delta$, and let $A \equiv A_1 \geq'_{\beta_1} A_2 \geq'_{\beta_1} \dots \geq'_{\beta_1} A_p$ be the longest possible sequence of single-step β_1 -reductions beginning with A . Then $\theta_1(A) = p$. □

Theorem 3.22. Let $A \in \Lambda$, let A be in β_1 -normal form and let $A \geq_{\beta_2} B$. Then B is also in β_1 -normal form.

Proof. If B were not in β_1 -normal form, then $B \geq'_{\beta_1} C$ for some C . In that case there would be a reduction $A \geq'_{\beta_1} B' \geq_{\beta_2} C$ according to Th. II.6.17. Contradiction. $\square$

Theorem 3.23. If $A \in \Lambda$, then there is an upper bound for the length ℓ of reduction sequences $A \equiv A_1 \geq' A_2 \geq' \dots \geq' A_\ell$, where each $A_i \geq' A_{i+1}$ is a single-step β_1 - or β_2 -reduction.

Proof. Induction on $\theta_1(A)$. If $\theta_1(A) = 0$, then A is in β_1 -normal form. If we can apply β_2 -reductions on A such that $A \geq_{\beta_2}^n B$ ($n \geq 1$), then B is also in β_1 -normal form (Th. 3.22). The number of possible single-step β_2 -reductions applicable is finite ($\leq \theta_2(A)$; cf. Th. 3.15).

So let $\theta_1(A) = p > 0$, and assume that the theorem holds for all K with $\theta_1(K) < p$. Let $A \geq D$ be a reduction sequence consisting of single-step β_1 - and β_2 -reductions. If no β_1 -reductions occur in the reduction sequence, the length of the reduction sequence can be at most $\theta_2(A)$. Else, let $A \geq D$ be $A \geq_{\beta_2}^n B \geq'_{\beta_1} C \geq D$. Then by Th. II.6.17 there is also a reduction sequence $A \geq'_{\beta_1} B' \geq_{\beta_2} C \geq D$. Each B'' such that $A \geq'_{\beta_1} B''$ has by induction (since $\theta_1(B'') < \theta_1(A)$) an upper bound for the length of reduction sequences $B'' \geq' \dots \geq' E$ in which each single-step reduction is either a β_1 - or a β_2 -reduction. Let m be the maximum of these upper bounds. Then the length of the reduction sequence $B' \geq_{\beta_2} C \geq D$, hence of $C \geq D$, cannot be more than m . It follows that the length of any reduction sequence $A \geq D$ can be at most $\theta_2(A) + m + 1$. $\square$

Theorem 3.24 (strong β -normalization theorem for Λ). If $A \in \Lambda$, then A is strongly β -normalizable.

Proof. Each β -reduction sequence of A can be decomposed into single-step β_1 - and β_2 -reductions by Th. II.6.15. So Th. 3.23 yields the desired result. $\square$

Definition 3.25. Let $A \in \Lambda$, and let $A \equiv A_1 \geq'_\beta A_2 \geq'_\beta \dots \geq'_\beta A_p$ be the longest possible sequence of single-step β -reductions beginning with A . Then $\Theta(A) = p$. $\square$

Theorem 3.26 (*strong η -normalization theorem for Δ*). If $A \in \Delta$, then A is strongly η -normalizable.

Proof. Induction on $|A|$. $\square$

Definition 3.27. We call $K \in \Delta$ *strongly normalizable* if there is an upper bound for the length ℓ of reduction sequences $K \equiv K_1 \geq' K_2 \geq' \dots \geq' K_\ell$ where each reduction $K_i \geq' K_{i+1}$ is a single-step β - or η -reduction. $\square$

Theorem 3.28 (*strong normalization theorem for Λ*). If $A \in \Lambda$, then A is strongly normalizable.

Proof. Induction on $\Theta(A)$. The proof is similar to that of Th. 3.23. Use Th. 3.26 instead of Th. 3.22, and instead of Th. II.6.17, use the theorem: If $K \in \Delta$, $K \geq_\eta L \geq'_\beta M$, then $K \geq'_\beta L' \geq M$. The latter theorem is easy to prove, since each reduction $A \geq'_\eta B \geq'_\beta C$ can be replaced either by a reduction $A \geq'_\beta B' \geq^r_\eta C$ (where $r \geq 0$) or by a reduction $A \geq'_\beta B' \geq'_\beta C$ (see p. 65; see also Th. 7.25). $\square$

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SAMENVATTING

Aan de lambda-calculus ligt een formele generalisatie van de wiskundige begrippen functie en functietoepassing ten grondslag. In een getypeerde lambda-calculus worden de expressies uit de calculus van typen voorzien om de interpreteerbaarheid in gangbare wiskundige termen te vergemakkelijken. De toepasbaarheid van een functie op een argument wordt in een getypeerde lambda-calculus vaak op een natuurlijke manier beperkt in overeenstemming met de typetoekenning (de *toepasbaarheidsvoorwaarde*).

In dit proefschrift worden twee systemen van getypeerde lambda-calculus onderzocht, die Δ en Λ worden genoemd. Het systeem Λ is een deelsysteem van Δ . In Δ wordt de toepasbaarheidsvoorwaarde nog niet gesteld, in Λ gebeurt dit wel. De typen in de systemen Δ en Λ zijn algemeen in die zin, dat ze dezelfde structuur hebben als de expressies. Het systeem Λ is voortgekomen uit de wiskundige taal Automath.

Hoofdstuk I van dit proefschrift geeft een inleiding in de lambda-calculus en beschrijft de typetoekenning. Voorts wordt het verband met Automath besproken. Een uitgebreide samenvatting van het proefschrift besluit dit hoofdstuk.

In Hoofdstuk II wordt het systeem Δ geïntroduceerd en besproken. De uit de lambda-calculus bekende relaties worden ook in het systeem Δ ingevoerd. Daarnaast worden andere relaties gedefinieerd en onderzocht als voorbereiding op Hoofdstuk III.

In Hoofdstuk III wordt het systeem Λ ingevoerd. Er wordt aangetoond dat de toepasbaarheidsvoorwaarde voldoende is om voor alle expressies in Λ normaliseerbaarheid te bewerkstelligen: we noemen een expressie normaliseerbaar als men functietoepassingen binnen de expressie in eendusdanige volgorde kan uitvoeren, dat er na een eindig aantal stappen een expressie ontstaat waarin geen functietoepassing meer mogelijk is. In het laatste geval zegt men dat de *normaalvorm* is bereikt.

In het systeem A blijkt zelfs *sterke* normaliseerbaarheid te gelden: *iedere* volgorde van functietoepassingen moet in een eindig aantal stappen tot de normaalvorm leiden. Dit wordt eveneens bewezen in Hoofdstuk III, met gebruikmaking van de hierboven genoemde (gewone) normaliseerbaarheid.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 19 juli 1942 geboren in Den Haag. Van 1953 tot 1959 doorliep hij de gymnasiumafdeling van het Nederlands Lyceum in deze stad. Daarna begon hij een studie aan de Rijksuniversiteit te Leiden. Na een studie-onderbreking legde hij hier in 1965 het candidaatsexamen sterre- en wiskunde af en in juni 1969, cum laude, het doctoraalexamen wiskunde met bijvak Russisch. In het studiejaar 1966-1967 was hij als assistent werkzaam bij de rekenafdeling van het Mathematisch Centrum te Amsterdam. Sinds zijn afstuderen is hij als wetenschappelijk medewerker werkzaam bij de Onderafdeling der Wiskunde van de Technische Hogeschool te Eindhoven, met een wetenschappelijke taak in het onderzoek van de wiskundige taal Automath, onder leiding van Prof.dr. N.G. de Bruijn, en met een onderwijstaak in het basisonderwijs.

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STELLINGEN

1. Noem een rechthoek met afmetingen $l \times p$ een *strook* van lengte p . Definieer een k -set als een verzameling stroken $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$, waarbij σ_i een strook van lengte i is. Men kan trachten om bij zekere k met alle stroken uit n k -sets een rechthoek op te vullen; hierbij mogen de stroken in de lengte- en in de breedterichting worden gelegd. Als er binnen een zo gevulde rechthoek geen rechte lijn is die twee overstaande zijden verbindt, zeggen we dat met de n k -sets een *hechte* rechthoek is geconstrueerd.

Als bij gegeven k met n k -sets een hechte rechthoek kan worden geconstrueerd, terwijl dit met minder dan n k -sets niet mogelijk is, definiëren we $m_k = n$. Er geldt:

- (1) m_k bestaat voor alle k en $m_k \leq 2$ als $k \neq 2$.
- (2) $m_2 = 8$.
- (3) $k = 14$ is het kleinste getal (afgezien van 1) waarvoor $m_k = 1$.
- (4) Voor oneindig veel waarden van k geldt $m_k = 1$.

2. Zij een verzameling V van termen gegeven door de volgende inductieve definitie:

- (1) $()$ is een term van V .
- (2) Als A en B termen van V zijn, dan is (AB) een term van V .

Definieer de *lengte* $|A|$ van een term A inductief door:

- (1) $|()| = 1$.
- (2) $|(AB)| = |A| + |B|$.

Zij $\sim$ de equivalentierelatie op V gegenereerd door:

- (1) $((AB)C) \sim ((AC)B)$.
- (2) Als $A \sim B$ en $C \sim D$, dan $(AC) \sim (BD)$.
(A, B, C en D zijn termen van V .)

Termen uit een equivalentieklasse K hebben gelijke lengte; noteer deze lengte als $\ell(K)$. Er is nu een natuurlijke één-éénduidige afbeelding T van de verzameling van equivalentieklassen K op de verzameling van eindige "rooted trees", met de eigenschap dat voor iedere K $\ell(K)$ gelijk is aan het aantal knooppunten van $T(K)$.

(Opmerking: Theorem 2.2 uit [1] is hiervan een direct gevolg.)

- [1] F. Göbel en R.P. Nederpelt, The number of numerical outcomes of iterated powers, Amer. Math. Monthly, Vol. 78, No. 10, december 1971.

3. We nemen aan dat op elk kruispunt van de stad X. precies v wegen samenkomen en dat elk huizenblok wordt begrensd door tenminste drie straten; verbindingswegen van X. met de rest van de wereld bestaan niet. Voorts heeft de stad een circuit van wegen met een bijzondere eigenschap: als men dit circuit eenmaal rondrijdt, blijkt het dat men bij opvolgende kruispunten afwisselend de meest linkse en de meest rechtse afslag neemt, terwijl geen kruispunt tweemaal wordt gepasseerd. Het circuit telt een even aantal $(2k)$ kruispunten.

Over het aantal kruispunten $f_v(2k)$ dat X. tenminste moet hebben kunnen we het volgende zeggen:

$$(1) \quad f_3(4k) = 4k, \quad f_3(4k+2) = 4k+4 \quad (k = 1, 2, \dots).$$

$$(2) \quad f_4(4) = 9, \quad f_4(2k) = 2k \quad (k = 3, 4, \dots).$$

(3) Als $v = 5$ kan een bovengrens $b(2k)$ worden aangegeven voor $f_v(2k)$, die voor voldoende grote k voldoet aan $b(2k) \leq 2k + 2\left[\frac{2k}{10}\right] + 4$.

4. De laatste paragraaf (§ 3.5.5) van Wegners beschrijving van de lambda-calculus in [2] bevat een aantal onnauwkeurigheden.

[2] P. Wegner, Programming languages, information structures, and machine organization, McGraw-Hill Book Company, New York, 1968.

5. Het zou goed zijn als men bij de vernieuwing van het wiskunde-onderwijs op middelbare school niveau ook aandacht zou schenken aan de combinatoriek. Wiskundeleraren kunnen uit [3] een goed inzicht krijgen in de achtergronden van dit vak.

[3] G. Bergman en K.D. Fryer, Introduction to combinatorics, Academic Press, Inc., New York, 1972.

6. Een notatie van functies die aansluit bij de lambda-notatie van A. Church verdient uit didactisch en uit wetenschappelijk oogpunt de voorkeur boven de traditionele notatie. Ook in de theorie van de programmeertalen kan de lambda-notatie zijn nut bewijzen.