Memorandum COSOR 89-34

On the optimality of a new class
of adjusted orthogonal designs

S. Bagchi
E.E.M. van Berkum

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Eindhoven, December 1989
The Netherlands
ON THE OPTIMALITY OF A NEW CLASS OF ADJUSTED ORTHOGONAL DESIGNS

S. Bagchi

Computer Science Unit, Indian Statistical Institute, 203, B.T. Road,
Calcutta-700035, India.

E.E.M. van Berkum

Department of Mathematics and Computing Science, University of Technology,
ABSTRACT

We define a new class of adjusted orthogonal row-column designs, termed lattice-LBD. These are shown to be E-optimal in the class of all connected row-column designs. Methods of construction of such designs are also provided.

AMS Subject classification: Primary 62K05; secondary 62K10.

Key words and phrases: lattice design, linked block design, adjusted orthogonality, E-optimality.
1. Introduction

The notion of adjusted orthogonality for row-column designs was introduced in Eccleston and Russel (1975). Subsequently interesting and useful properties of adjusted orthogonal designs were shown in Shah (1977) and Shah and Eccleston (1986). Anderson and Eccleston (1985) gave a class of adjusted orthogonal designs. These were proved to be E-optimal in the class of all connected row-column designs in Shah and Sinha (1989). They were also shown to be optimal within the equireplicate class with respect to a large class of optimality criteria in Bagchi and Shah (1989).

We consider a new class of row-column designs which are adjusted orthogonal. For such a design, the row-column design is a t-ple lattice design and the column design is a linked block design. These are shown to be E-optimal within the class of all connected designs. We construct several series of such designs.

2. Optimality of lattice-LBD

A row-column design with parameters $b$, $k$ and $v$ is a design for comparing $v$ treatments using $bk$ experimental units arranged in $b$ rows and $k$ columns. Let $M = ((m_{ij}))$ and $N = ((n_{ij}))$ denote the treatment-row and treatment-column incidence matrices respectively. Let $r_i$ denote the replication number of the $i$-th treatment, $1 \leq i \leq v$. It is well-known that the $C$-matrix for the reduced normal equation for the treatment-effects is given by

$$C = r^8 - k^{-1} MM' - b^{-1} NN' + (bk)^{-1} rr',$$

where

$$r^8 = \text{diag}(r_1, \cdots, r_v) \text{ and } r = (r_1, \cdots, r_v)' .$$

By a block design with parameters $b$, $k$ and $v$ we mean a design for comparing $v$ treatments using $b$ homogeneous blocks each of size $k$. Note that for a row-column design we can think of two block designs, one of them with the $b$ rows, each of size $k$, as blocks and the other one with the $k$ columns of size $b$ as blocks. We shall call the former the row design and the latter the column design of the row-column design.
Thus, if \( C_{1d} \) and \( C_{2d} \) denote the \( C \)-matrices of the row design and the column design of a row-column design \( d \) and if \( C_0 \) denotes the \( C \)-matrix of a completely randomized design with the replication numbers same as those of \( d \), then

\[
\begin{align*}
C_{1d} &= r^δ - k^{-1} M M', \\
C_{2d} &= r^δ - b^{-1} N N', \\
C_{0d} &= r^δ - (bk)^{-1} rr',
\end{align*}
\]

(2.2)

so that (2.1) implies

\[
C_d = C_{1d} + C_{2d} - C_{0d}
\]

The suffix \( d \) will be dropped, when there is no scope of ambiguity.

**Notation 2.1.** \( D(b,k,v) \) denotes the class of all connected row-column designs with given \( b \), \( k \) and \( v \). \( D_r(b,k,v) \) denotes the class of all equireplicate designs in \( D(b,k,v) \).

**Definition 2.2.** In view of Shah and Eccleston (1986), we define a row-column design to be *adjusted orthogonal* if it is binary and it satisfies

\[
M' r^δ N = J_{bk},
\]

where \( J \) is the all-one matrix.

It is easily seen that for an adjusted orthogonal design in \( D_r \) the matrices \( C \), \( C_1 \) and \( C_2 \) commute and hence have a set of common eigenvectors. Further, the eigenvalues are related by the following.

**Theorem 2.3.** [Shah(1977)]. If for an equireplicate and adjusted orthogonal design the corresponding eigenvalues of \( C \), \( C_1 \) and \( C_2 \) are denoted by \( \lambda_i \), \( \theta_i \) and \( \phi_i \) respectively, one of the following must hold:

\[
\begin{align*}
\lambda_i &= \theta_i < r, \quad \phi_i = r, \\
\lambda_i &= \phi_i < r, \quad \theta_i = r, \\
\lambda_i &= \theta_i = \phi_i = r.
\end{align*}
\]
We now define a new class of adjusted orthogonal designs. First we recall the definition of a t-ple lattice design and of a linked block design.

**Definition 2.4.** A *t*-ple lattice design of order $s$ is defined when there exists $t-2$ mutually orthogonal latin squares of order $s$. The treatments are the positions of an $s \times s$ square. Every block is a set of $s$ positions which are either on the same row, or on the same column or are occupied by the same symbol in any of the $t-2$ latin squares.

It is well-known that such a design is a PBIBD with $L_t$ association scheme and $\lambda_1 = 1$, $\lambda_2 = 0$. It is also well-known that the dual of this design is a group divisible design with $t$ groups each of size $s$ and with $\lambda_1 = 0$ and $\lambda_2 = 1$. [See Raghavarao(1971) for details].

**Definition 2.5.** A binary block design is said to be a linked block design (LBD) if the size of intersection of every pair of distinct blocks is a constant. [See Raghavarao(1971)].

**Definition 2.6.** A row-column design is said to be a lattice-LBD if its row design is a t-ple lattice design of order $s$ for some integer $t < s$, its column design is an LBD and it is adjusted orthogonal. Note that a necessary condition for its existence is

$$s-1 \text{ divides } t(t-1) \quad (2.3)$$

Using the well-known properties of the row design and the column design of a lattice-LBD $d^*$ with parameters $t$ and $s$ together with the properties of an adjusted orthogonal design as given by Theorem 2.3, we get the following.

**Lemma 2.7.**

(a) The replication number of $d^*$ is $t$.

(b) The positive eigenvalues of $C_{1d^*}$ are $\theta_1 = t-1$ and $\theta_2 = t$ with multiplicities $t(s-1)$ and $(s-t+1)(s-1)$ respectively.

(c) The positive eigenvalues of $C_{2d^*}$ are $\phi_1 = s(t-1)/(s-1)$ and $\phi_2 = t$ with multiplicities $s-1$ and $s^2 - s$ respectively.

(d) Those of $C_{d^*}$ are $\lambda_1 = \theta_1$, $\lambda_2 = \phi_1$ and $\lambda_3 = t$ with multiplicities $t(s-1)$, $s-1$ and $(s-1)(s-t)$ respectively.
Theorem 2.8. A lattice-LBD \( d^* \) is E-optimal within \( D(b,k,v) \).

**Proof.** Let \( \mu_1(A) \) denote the minimum positive eigenvalue of a nonnegative definite matrix \( A \). Let \( d \) be any design in \( D(b,k,v) \).

From (2.2) we get \( C_{1d} - C_d = C_{0d} - C_{2d} \) which is a nonnegative definite matrix. Thus the result 1f.2.1 of Rao(1965) gives

\[
\mu_1(C_d) \leq \mu_1(C_{1d}).
\]

But from Theorem 4.1(ii) of Cheng(1980) we obtain

\[
\mu_1(C_{1d}) \leq \mu_1(C_{1d^*}).
\]

Now the result follows from Lemma 2.7 (d) and (e).

3. Construction of the optimal designs

**Theorem 3.1.** If \( s \) is a prime power and there exists a difference set of size \( t \) in the additive group of the field of order \( s \), then there exists a lattice-LBD, with parameters \( t \) and \( s \).

**Proof.** Let \( F \) denote the set of elements of the field of order \( s \). Let

\[
D_t = \{ \alpha_0, \alpha_1, \cdots, \alpha_{t-1} \}
\]  \hspace{1cm} (3.1)

denote the given difference set and let \( d \) denote the BIBD generated from this set. Our treatment set is

\[
V = F \times F = \{ (i,j) \mid i,j \in F \}.
\]

Let \( \beta_0 = 0, \beta_1, \cdots, \beta_{t-1} \) be \( t \) distinct members of \( F \). For \( 0 \leq l \leq t-1 \), we construct a \( s \times s \) square \( S_l \) having rows and columns indexed by \( F \) and the \((i,j)\)-th entry as \((x,y)\) where

\[
x = \beta_i y + i, \quad y = j + \alpha_l,
\]  \hspace{1cm} (3.2)

with \( \alpha_l \) given in (3.1).

Then the \( ts \times ts \) array

(e) \( \lambda_1 < \lambda_2 < \lambda_3 \).
A = \begin{bmatrix} S_0 \\ S_1 \\ S_{t-1} \end{bmatrix} \quad (3.3)

is our required row-column design.

To see that $A$ satisfies the desired properties of definition (2.6), we make the following observations.

Let $T_j = \{ (i,j) \mid i \in F \}$; let $R_{il}$ and $C_{jl}$ denote the set of treatments in the $i$-th row and $j$-th column of $S_l$ respectively. Then from (3.2), we see that for $i,j \in F$ and $0 \leq l, m \leq t-1$,

$$C_{jl} = T_j + \alpha_l,$$  \hspace{1cm} (3.4)

$$|R_{il} \cap C_{jm}| = 1,$$ \hspace{1cm} (3.5)

$$|R_{il} \cap R_{jm}| = \delta_{lm},$$ \hspace{1cm} (3.6)

where $\delta_{lm}$ is the Kronecker $\delta$ symbol.

Now note that (3.6) says that the row design is a t-ple lattice. Again (3.1) means that the column design is the design obtained by replacing the treatment $j \in F$ of $d$ by the set $T_j$. Since $T_j$'s are disjoint, it follows that the column design is an LBD with $s$ blocks, each of size $ts$. Further (3.5) implies that the size of intersection of every row and every column of $A$ is $t$, so that $D$ is adjusted orthogonal. Hence the proof is complete.

Examples. [See Lander(1983) for details]: for prime powers $s$,

1. the trivial difference set $D_{s-1} = F - \{0\}$ always exists,

2. if $s = 3 \mod 4$, then $D_t$ exists with $t = (s \pm 1)/2$,

3. if $s = 4m^2$ where $m$ is a power of 2, then $D_t$ exists with $t = 2m^2 \pm m$,

4. if $s = 4x^2 + 1$, where $x$ is an odd integer, then $D_t$ exists with $t = (s-1)/4$,

5. if $s = 4x^2 + 9$, where $x$ is an odd integer, then $D_t$ exists with $t = (s+3)/4$. 
Acknowledgement. This work was done when the first author was visiting the department of Mathematics and Computing Science of the Technological University of Eindhoven. She thanks this department for its hospitality. She is also thankful to Professor J.J. Seidel and Professor R. Doornbos who made this visit possible.
References


List of COSOR-memoranda - 1989

<table>
<thead>
<tr>
<th>Number</th>
<th>Month</th>
<th>Author</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 89-01</td>
<td>January</td>
<td>D.A. Overdijk</td>
<td>Conjugate profiles on mating gear teeth</td>
</tr>
<tr>
<td>M 89-02</td>
<td>January</td>
<td>A.H.W. Geerts</td>
<td>A priori results in linear quadratic optimal control theory</td>
</tr>
<tr>
<td>M 89-03</td>
<td>February</td>
<td>A.A. Stoorvogel, H.L. Trentelman</td>
<td>The quadratic matrix inequality in singular $H_\infty$ control with state feedback</td>
</tr>
<tr>
<td>M 89-04</td>
<td>February</td>
<td>E. Willekens, N. Veraverbeke</td>
<td>Estimation of convolution tail behaviour</td>
</tr>
<tr>
<td>M 89-05</td>
<td>March</td>
<td>H.L. Trentelman</td>
<td>The totally singular linear quadratic problem with indefinite cost</td>
</tr>
<tr>
<td>M 89-06</td>
<td>April</td>
<td>B.G. Hansen</td>
<td>Self-decomposable distributions and branching processes</td>
</tr>
<tr>
<td>M 89-07</td>
<td>April</td>
<td>B.G. Hansen</td>
<td>Note on Urbanik's class $L_n$</td>
</tr>
<tr>
<td>M 89-08</td>
<td>April</td>
<td>B.G. Hansen</td>
<td>Reversed self-decomposability</td>
</tr>
<tr>
<td>M 89-09</td>
<td>April</td>
<td>A.A. Stoorvogel</td>
<td>The singular zero-sum differential game with stability using $H_\infty$ control theory</td>
</tr>
<tr>
<td>M 89-10</td>
<td>April</td>
<td>L.J.G. Langenhoff, W.H.M. Zijm</td>
<td>An analytical theory of multi-echelon production/distribution system</td>
</tr>
<tr>
<td>M 89-11</td>
<td>April</td>
<td>A.H.W. Geerts</td>
<td>The Algebraic Riccati Equation and Singular Optimal Control</td>
</tr>
<tr>
<td>Number</td>
<td>Month</td>
<td>Author</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>-----------------------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>M 89-12</td>
<td>May</td>
<td>D.A. Overdijk</td>
<td>De geometrie van de kroonwieloverbrenging</td>
</tr>
<tr>
<td>M 89-13</td>
<td>May</td>
<td>I.J.B.F. Adan</td>
<td>Analysis of the shortest queue problem</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J. Wessels</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>W.H.M. Zijm</td>
<td></td>
</tr>
<tr>
<td>M 89-14</td>
<td>June</td>
<td>A.A. Stoorvogel</td>
<td>The singular $H_{\infty}$ control problem with dynamic measurement feedback</td>
</tr>
<tr>
<td>M 89-15</td>
<td>June</td>
<td>A.H.W. Geerts</td>
<td>The output-stabilizable subspace and linear optimal control</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M.L.J. Hautus</td>
<td></td>
</tr>
<tr>
<td>M 89-16</td>
<td>June</td>
<td>P.C. Schuur</td>
<td>On the asymptotic convergence of the simulated annealing algorithm in the presence of a parameter dependent penalization</td>
</tr>
<tr>
<td>M 89-17</td>
<td>July</td>
<td>A.H.W. Geerts</td>
<td>A priori results in linear-quadratic optimal control theory (extended version)</td>
</tr>
<tr>
<td>M 89-18</td>
<td>July</td>
<td>D.A. Overdijk</td>
<td>The curvature of conjugate profiles in points of contact</td>
</tr>
<tr>
<td>M 89-19</td>
<td>August</td>
<td>A. Dekkers</td>
<td>An approximation for the response time of an open CP-disk system</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J. van der Wal</td>
<td></td>
</tr>
<tr>
<td>M 89-20</td>
<td>August</td>
<td>W.F.J. Verhaegh</td>
<td>On randomness of random number generators</td>
</tr>
<tr>
<td>M 89-21</td>
<td>August</td>
<td>P. Zwietering</td>
<td>Synchronously Parallel: Boltzmann Machines: a Mathematical Model</td>
</tr>
<tr>
<td></td>
<td></td>
<td>E. Aarts</td>
<td></td>
</tr>
<tr>
<td>M 89-22</td>
<td>August</td>
<td>I.J.B.F. Adan</td>
<td>An asymmetric shortest queue problem</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J. Wessels</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>W.H.M. Zijm</td>
<td></td>
</tr>
<tr>
<td>M 89-23</td>
<td>August</td>
<td>D.A. Overdijk</td>
<td>Skew-symmetric matrices in classical mechanics</td>
</tr>
<tr>
<td>M 89-24</td>
<td>September</td>
<td>F.W. Steutel</td>
<td>The gamma process and the Poisson distribution</td>
</tr>
<tr>
<td></td>
<td></td>
<td>J.G.F. Thiemann</td>
<td></td>
</tr>
<tr>
<td>M 89-25</td>
<td>September</td>
<td>A.A. Stoorvogel</td>
<td>The discrete time $H_{\infty}$ control problem: the full-information case</td>
</tr>
<tr>
<td>Number</td>
<td>Month</td>
<td>Author</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>-----------------------------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>M 89-26</td>
<td>October</td>
<td>A.H.W. Geerts, M.L.J. Hautus</td>
<td>Linear-quadratic problems and the Riccati equation</td>
</tr>
<tr>
<td>M 89-27</td>
<td>October</td>
<td>H.L. Trentelman, A.A. Stoorvogel</td>
<td>Completion of the squares in the finite Horizon $H^\infty$ control problem by measurement feedback</td>
</tr>
<tr>
<td>M 89-28</td>
<td>November</td>
<td>P.J. Zwietering, E.H.L. Aarts</td>
<td>A Note on the Convergence of a Synchronously parallel Boltzmann machine for the Knapsack Problem</td>
</tr>
<tr>
<td>M 89-29</td>
<td>November</td>
<td>P.C. Schuur</td>
<td>Classification of acceptance criteria for the simulated annealing algorithm</td>
</tr>
<tr>
<td>M 89-30</td>
<td>November</td>
<td>W.H.L. Neven, C. Praagman</td>
<td>Column reduction of polynomial matrices an iterative algorithm</td>
</tr>
<tr>
<td>M 89-31</td>
<td>November</td>
<td>A.A. Stoorvogel</td>
<td>The discrete time $H_\infty$ control problem with measurement feedback</td>
</tr>
<tr>
<td>M 89-32</td>
<td>November</td>
<td>J.H. van Geldrop, Shou Jilin, C.A.A.M. Withagen</td>
<td>Existence of general equilibria in economies with natural exhaustible resources and an infinite horizon</td>
</tr>
<tr>
<td>M 89-33</td>
<td>December</td>
<td>A.A. Stoorvogel, H.L. Trentelman</td>
<td>The finite horizon singular time-varying $H_\infty$ control problem with dynamic measurement feedback</td>
</tr>
<tr>
<td>M 89-34</td>
<td>December</td>
<td>S. Bagchi, E.E.M. van Berkum</td>
<td>On the optimality of a new class of adjusted orthogonal designs</td>
</tr>
</tbody>
</table>