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Stabilizability and detectability of discrete-time time-varying systems
by

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Eindhoven, The Netherlands
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ABSTRACT

In this paper we introduce some notions of stabilizability and detectability for discrete time-varying systems. These concepts are introduced by making state-space decompositions of the system. This allows an intuitive interpretation and shows the difficulties which occur if one tries to derive general stabilizability and detectability properties.

Moreover, we show by means of a counterexample that the notions of uniform stabilizability and uniform detectability, as defined by Anderson and Moore, do not imply stabilizability respectively detectability of the system.

I. Introduction

In the theory of linear time-varying difference equations the concept of "uniform asymptotic stability" plays an important role. This, since according to a theorem of Poincaré-Bendixson uniform asymptotic stability of the linearized system implies uniform asymptotic stability of the non-linear system. Now, J.L. Willems proved in (1970), theorem 7.5.2, that the flow of a linear discrete time-varying system is uniformly asymptotically stable if and only if it is exponentially stable.

Therefore, a natural question is under which conditions such a system is exponentially stabilizable.

Stated differently, under which conditions does there exist a control sequence in the form of a state feedback, such that the resulting closed-loop system becomes exponentially stable.

For time-invariant systems these conditions are well known. For time-varying systems, however, this question is more complicated, and a general theory about it is lacking.

Hager and Horowitz (1976), and Anderson and Moore (1981) took a lead with the introduction of sufficient conditions for detectability and stabilizability of discrete-time time-varying systems. Moreover, they used these to solve some control and filtering problems. However, unfortunately the claim of Anderson and Moore that the time-varying discrete-time Kalman filter is exponentially stable under the conditions of uniform stabilizability and uniform detectability, is incorrect.

We provide a counterexample to this claim here. Furthermore, we believe that as well the definitions which Hager and Horowitz give as those of Anderson and Moore, do not give a clear
insight into the basic underlying structural problems.

To obtain a better insight in these problems, the state-space decomposition approach given by Ludyck in (1981) seems to be a more promising one. Therefore, we extend that analysis in this paper.

Based on two state-space decompositions we discern several types of stabilizability and detectability. These concepts can be used to solve e.g. the Linear Quadratic regulator problem, as will be reported elsewhere.

Since the proofs given in Ludyck (1981) to obtain the state-space decompositions are not entirely correct, we also provide correct proofs of them.

The outline of the paper is as follows.

First, in section 2 we introduce some definitions and provide the counter example. Then, by making a decomposition of the state-space at any time into three orthogonal subspaces, we obtain in section 3 an equivalent system representation from which easily various stabilizability properties of the original system can be deduced. The decomposition originates from considering the reachability and exponential stability subspaces. In section 4 an analogous analysis is performed for detectability. Here the decomposition of the state-space into the unobservable subspace and its complement plays an important role. At last we combine the results in section 5 in which we give a state-space description based on a simultaneous decomposition of the space into the reachable and unobservable parts. The paper ends with some concluding remarks.
II. Stabilizability

In this paper we consider a system described by the following linear discrete-time time-varying recurrence equation:

\[ x(k+1) = A(k) x(k) + B(k) u(k); \quad x(k_0) = x \]

\[ y(k) = C(k) x(k). \]

Here \( x(k) \in \mathbb{R}^n \) is the state of the system, \( u(k) \in \mathbb{R}^m \) the applied control, and \( y(k) \in \mathbb{R}^p \) the output at time \( k \). Moreover we assume that all matrices \( A(\cdot), B(\cdot) \) and \( C(\cdot) \) are bounded.

We use the following notation:

**Notation 0**

- \( v^T(i) \) denotes the transpose for \( v(i) \).
- \( v[k, l] := (v^T(k), \ldots, v^T(l))^T \).
- \( v[k, \cdot] := (v^T(k), v^T(k+1), \ldots)^T \).

\( \text{Im} \; A \) denotes the image of the mapping defined by matrix \( A \); \( \text{Ker} \; A \) denotes its kernel.

\[ A(k+i, k) := A(k+i-1) \ast \cdots \ast A(k), \text{ if } i \geq 1, \text{ and } A(k, k) := I. \]

\[ S[k, k-N] := [B(k) \mid A(k+1, k) B(k-1) \mid \cdots \mid A(k+1, k+1-N) B(k-N)] . \]

\[ W[N, N+i] := [C^T(N) \mid (C(N+1)A(N+1, N))^T \mid \cdots \mid (C(N+i)A(N+i, N))^T] . \]

\( O_{p,q} := \text{zero matrix with } p \text{ rows and } q \text{ columns.} \)

\( x(k, k_0, x_0, u) \) is the state of the system at time \( k \) resulting from the initial state \( x_0 \) at time \( k_0 \) when the input \( u[k_0, k-1] \) is applied.

\[ y(k, k_0, x_0, u) := C(k) x(k, k_0, x_0, u) . \]

If in \( \Sigma_y \), \( C(k) \) equals the identity matrix at any time \( k \) (i.e. we have full state observations), the subscript \( y \) is dropped.

We start our analysis by giving formal definitions of several notions of stability and stabilizability. In these definitions we use the concept of exponential convergence of a sequence \( u[k_0, \cdot] \).

This is defined as follows. We say that \( u(\cdot) \) converges exponentially fast to zero if there exist positive constants \( \alpha \) and \( M \) such that \( \| u(k) \| < Me^{-\alpha(k-k_0)} \) for all \( k > k_0 \).
Definition 1

The initial state $x$ of the system $\Sigma_y$ is said to be
stable at $k_0$ if $\lim_{k \to \infty} x(k, k_0, x, 0) = 0$.

exponentially stable at $k_0$ if there exist positive constants $\alpha$ and $M$ such that
$\| x(k, k_0, x, 0) \| \leq M e^{-\alpha (k-k_0)} \| x \|$ for any $k > k_0$.

stabilizable at $k_0$ if there exists a control sequence $u[k_0, \cdot ]$, with the property that $u(\cdot) \to 0$, such that $\lim_{k \to \infty} x(k, k_0, x, u) = 0$.

exponentially stabilizable at $k_0$ if there exists a control sequence $u[k_0, \cdot ]$, with the property that $u(\cdot)$ converges exponentially fast to zero, and positive constants $\alpha$ and $M$ such that
$\| x(k, k_0, x, u) \| \leq M e^{-\alpha (k-k_0)} \| x \|$ for any $k > k_0$.

The system $\Sigma_y$ is called stable (respectively exponentially stable, stabilizable, exponentially stabilizable) at $k_0$ if any initial state of $\Sigma_y$ possesses the corresponding property at $k_0$.

As announced in the introduction, we give in this section a counterexample for a result obtained by Anderson and Moore in (1981).

To that end we first introduce their concepts of uniform stabilizability and uniform detectability and quote their corollary 5.4.

Definition 2

$\Sigma_y$ is uniformly stabilizable if there exist integers $s, t \geq 0$ and constants $d, b$ with $0 \leq d < 1, 0 < b < \infty$, such that whenever
$\| A(k+1, k+1-t) v \| \geq d \| v \|$
for some $v, k$, then

$v^T S[k, k-s] S^T[k, k-s] v \geq bv^T v$.

$\Sigma_y$ is uniformly detectable if there exist integers $s, t \geq 0$ and constants $d, b$ with $0 \leq d < 1, 0 < b < \infty$, such that whenever
$\| A(k+1, k+1-t) v \| \geq d \| v \|$
for some $v, k$, then

$v^T W[k, k-s] W^T[k, k-s] v \geq bv^T v$.

"Corollary 3" (Corollary 5.4 in Anderson and Moore (1981))

If $\Sigma_y$ is uniformly detectable, there exists a bounded sequence $K(k)$ such that
$x(k+1) = (A(k) - K(k) C(k)) x(k)$ is exponentially stable.
This corollary is an immediate consequence of theorem 5.3 in the above mentioned paper.
In this theorem it is claimed that if the system is uniformly stabilizable and uniformly detectable, then the Kalman filter is exponentially stable (under the usual system noise assumptions).
Now, consider the following example:

**Counterexample 4**

Let

\[ A(2k) = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}; \quad C(2k) = (0 \ 1); \quad A(2k+1) = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}; \quad C(2k+1) = (0 \ 0), \quad k = 0, 1, 2, \cdots. \]

Then, with \( s = t = 0, d = \frac{3}{4} \) and \( b = \frac{1}{4} \), we see that \( \Sigma \) is uniformly detectable.

However, if we consider the initial state \( x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) at time zero, we see that for any sequence \( K(\cdot), x(k+1) = (A(k) - K(k)C(k))x(k), x(0) = x \) is not stable. Which contradicts corollary 3. \[ \square \]

A direct implication of this example is that the above mentioned theorem 5.3 is incorrect too. By dualizing this example, i.e. take \( A(\cdot) = A^T(\cdot) \) and \( B(\cdot) = C^T(\cdot) \), we see that the uniform stabilizability condition is not sufficient either to conclude that there exists a state feedback such that the closed-loop system becomes (exponentially) stable.

Since the major reason for introducing the uniform stabilizability and detectability condition is to have a criterion from which exponential stabilizability respectively exponential detectability of the system can be concluded, we concentrate ourselves in the rest of this paper on finding such criteria.
III. Sufficient conditions for exponential stabilizability

In this section we derive sufficient conditions for exponential stabilizability of \( \Sigma \).

To that end we first consider the exponential stability subspace at \( k_0 \), denoted by \( X_e(\cdot, k_0) \).

This subspace consists of all initial states at \( k_0 \) which are exponentially stable. Similarly, we define the stability (or modal) subspace at \( k_0 \), denoted by \( X^-(\cdot, k_0) \).

That \( X_e(\cdot, k_0) \) is indeed a linear subspace is easily verified.

The next lemma tells us, moreover, that the exponential stability subspace is \( A(k) \)-invariant. This property is used later on in this section to make an appropriate state-space decomposition.

Lemma 5
\[
A(k)X_e(\cdot, k) \subset X_e(\cdot, k+1).
\]

Proof:
Consider \( \Sigma \) at time \( k \). Let \( x \) be an exponentially stable state.

Then, we have by definition that for some positive constants \( \alpha \) and \( M \)
\[
\| x(t+1, k, x, 0) \| \leq Me^{-\alpha(t+1-k)} \| x \|
\]
at any time \( t \geq k \). But, since \( x(t+1, k, x, 0) = x(t+1, k+1, A(k)x, 0) \) we obtain immediately that \( A(k)x \in X_e(\cdot, k+1) \). Which proves the lemma.

Another property that plays an important role in our analysis concerns the reachability subspace. Its definition reads as follows.

Definition 6
The state \( x \) is said to be reachable at \( k_0 \) from zero if there exists a control sequence \( u [N, k_0 - 1] \) with \( -\infty < N < k_0 \) such that \( x(k_0, N, 0, u) = x \).

The subspace consisting of all reachable states from zero at \( k_0 \) is called the reachability subspace at \( k_0 \), and denoted by \( R_{k_0} \). Its dimension is denoted by \( r_{k_0} \).

The reachability property we are interested in is stated in the next lemma. A formal proof can be found in Engwerda (1987).

Lemma 7
Consider time \( k_0 \), and define \( A(k) = 0 \) and \( B(k) = 0 \) for \( k < k_0 \).

Then at any time \( k \geq k_0 \) we have:
This lemma tells us in particular that the reachability subspace is also $A(k)$-invariant.

Now consider the following state-space decomposition:

$$X_1(k) = X_{k} \cap X_{k}(A(\cdot, k)) ;$$
$$X_2(k) \oplus X_1(k) = R^k ;$$
$$X_3(k) \oplus X_2(k) \oplus X_1(k) = R^n ,$$

where $X_1, X_2$ and $X_3$ are chosen orthogonal.

With respect to a basis adapted to this state-space decomposition the next important corollary holds (see lemmas 5 and 7).

**Corollary 8**

There exists an orthogonal state transformation $T'(\cdot)$ which does not affect the boundedness property of the system parameters such that with $x(k) = T'(k)x(k)$, $\Sigma$ is described by

$$\Sigma'_1: \begin{bmatrix} x'_1(k+1) \\ x'_2(k+1) \\ x'_3(k+1) \end{bmatrix} = \begin{bmatrix} A'_{11}(k) & A'_{12}(k) & A'_{13}(k) \\ 0 & A'_{22}(k) & A'_{23}(k) \\ 0 & 0 & A'_{33}(k) \end{bmatrix} \begin{bmatrix} x'_1(k) \\ x'_2(k) \\ x'_3(k) \end{bmatrix} + \begin{bmatrix} B'_1(k) \\ B'_2(k) \\ B'_3(k) \end{bmatrix} u(k)$$

where

$$\Sigma'_1: x'_1(k+1) = A'_{11}(k)x'_1(k) + B'_1(k) u(k) \text{ is exponentially stable at any time } k \geq k_0 .$$

$$\Sigma'_2: x'_2(k+1) = A'_{22}(k)x'_2(k) + B'_2(k) u(k) \text{ is reachable at any time } k \geq k_0 , \text{ and is not exponentially stable .}$$

$$\Sigma'_3: x'_3(k+1) = A'_{33}(k)x'_3(k) .$$

In the remainder of this section we derive sufficient conditions in terms of the transformed system for exponential stabilizability of $\Sigma$.

From corollary 8 we have immediately the following result:

**Lemma 9**

Consider the transformed system (1).

If $\Sigma$ is exponentially stabilizable at $k_0$, then $\Sigma'_2$ has to be exponentially stabilizable at $k_0$ and $\Sigma'_3$ exponentially stable at $k_0$. 

$\square$
In fact we can conclude much more from the state-space decomposition.

We see namely that all disturbances entering the system $\Sigma_2$ at any time $k \geq k_0$ in a specific way (namely via $\text{Im}\ A'_{23}(k)A'_{33}(k, k_0)$) can be controlled exponentially fast to zero.

We investigate this phenomenon in more detail now.

**Definition 10**

Consider $\Sigma_d: x(k+1)=A(k)x(k)+B(k)u(k)+G(k)d(k)$, where $d(\cdot)$ is a known disturbance and $x(k_0)=x$.

Then $\Sigma_d$ is called exponentially disturbance stabilizable at $k_0$ if for any initial state at $k_0$ and for any disturbance there exists a control sequence $u[k_0, \cdot]$ converging exponentially fast to zero, such that $x(k, k_0, x, u, d)$ converges exponentially fast to zero. Here $x(k, k_0, x, u, d)$ is defined similar to $x(k, k_0, x, u)$.

From the above considerations we have the following theorem.

**Theorem 11**

$\Sigma$ is exponentially stabilizable at $k_0$ iff the following two conditions are satisfied at $k_0$:

i) $\Sigma'_3$ is exponentially stable at $k_0$.

ii) $\Sigma'_{2d}: x'_{2}(k+1)=A'_{22}(k)x'_{2}(k)+B'_{2}(k)u(k)+A'_{23}(k)A'_{33}(k, k_0)d$ is exponentially disturbance stabilizable at $k_0$.

**Proof:**

That both the conditions are necessary was argued in lemma 9 and the ensuing remark.

That they are also sufficient is seen as follows. Due to assumption ii) we know that for any $x'_3(k_0)$ there exists a control sequence $\bar{u}(\cdot)$, which converges exponentially fast to zero, such that the second state component of $x'(k, k_0, x', \bar{u})$ converges exponentially fast to zero. Here $x'=(x'_1, x'_2, x'_3)^T$.

But, since $\bar{\Sigma}_1: x'_1(k+1)=A'_{11}(k)x'_1(k)$ is exponentially stable at any time $k \geq k_0$, and matrix $B$ is bounded, this implies that the first state component of $x'(k, k_0, x', \bar{u})$ converges also exponentially fast to zero. As the third state component of $x'(k, k_0, x', u)$ converges exponentially fast to zero irrespective of what $u$ is, it is clear now, that with $u=\bar{u}$, we have found an appropriate control sequence which stabilizes $\Sigma'$ exponentially fast.

So, the main problem left to be solved is to give conditions under which $\Sigma'_{2d}$ is exponentially disturbance stabilizable.

We provide sufficient conditions. Therefore, we introduce the concept of periodic smooth controllability, as defined by Engwerda in (1987). Roughly spoken, a system is called periodically
smoothly controllable if there exists a finite time period such that whenever such a time period has passed, the system has been at least once controllable during that period. Formally its definition reads as follows (for the definition of $S$ see notation 0).

Definition 12

$\Sigma$ is called periodically smoothly controllable at $k_0$ if there exist constants $\varepsilon$, $k_1$ and $N$ such that for all $k > 0$ there exists an integer $k_2(k)$ in the interval $(k_0+(k-1)k_1, k_0+k_1)$ for which $S[k_2-N, k_2] S^T[k_2-N, k_2] \geq \varepsilon I$.

Note that without loss of generality we can take $N=2k_1$ in this definition since, whenever $N < 2k_1$, we have that $S[k_2-N, k_2] S^T[k_2-N, k_2] \geq \varepsilon I$ implies that the same inequality holds for $S[k_2-2k_1, k_2] S^T[k_2-2k_1, k_2]$.

Theorem 13

Consider $\Sigma'_{2d}$ from theorem 11.

Let $\Sigma'_2$ be periodically smoothly controllable at $k_0$ and $\Sigma'_3$ exponentially stable at $k_0$.

Then, $\Sigma'_{2d}$ is exponentially disturbance stabilizable at $k_0$.

Proof:

First of all we note that due to the exponential stability assumption on $\Sigma'_3$, $A'_{23}(k) A'_{33}(k, k_0) d$ converges exponentially fast to zero.

Now consider the time interval $(k_0, k_1)$.

Let $e(k_2(1))$ denote the sum of all disturbances entering $\Sigma'_{2d}$ during this time period, i.e. $e(k_2(1)) = \sum_{i=k_2(1)}^{k_2(t)} A'_{22}(i) A'_{23}(i) A'_{33}(i, k_0) d$.

Consider the input

$u[k_0, k_2(1)-N-1]=0$, and

$u[k_2(1)-N, k_2(1)] = -S'_2 (S'_2 S'_2)^{-1} (e(k_2(1)) + x'_2)$,

where $S'_2 := S[k_2(1), k_2(1)-N]$ w.r.t. $\Sigma'_2$ and $x'_2$ is the initial state of $\Sigma'_{2d}$.

With this input, $x'_2(k_2(1)+1)$ becomes zero.

We show now by induction that it is possible to regulate $x'_2(k_2(k)+1)$ to zero for any $k$. Let therefore $t$ be any integer greater than one.

Consider the interval $(k_0+(t-2)k_1, k_0+t k_1)$. The sum of all exogenous influences entering the reachable subsystem via matrix $A'(\cdot)$ from $k_2(t-1)+1$ until $k_2(t)$ on is then
\[ e(k_2(t)) := \sum_{i=k_2(t-1)+1}^{k_2(t)} A'_{22}(k_2(t), i) A'_{23}(i) x'_3(i) . \]

Since by induction hypothesis \( x'_2(k_2(t)+1) \) is zero, application of the input

\[ u[k_2(t-1)+1, k_2(t)-N-1]=0, \text{ and} \]
\[ u[k_2(t)-N, k_2(t)]=S_2^{T}(S_2^{T}S_2^{T})^{-1} e(k_2(t)) , \]

yields that \( x'_2(k_2(t)+1)=0 \). Here \( S'_2 := S[k_2(t), k_2(t)-N] \) w.r.t. \( \Sigma'_2 \).

This completes the induction argument.

Moreover we observe that \( \|u(k)\| \leq M \| e(k_2(t))\| \) for some constant \( M \), since \( S'_2S'_2^{T} \geq \varepsilon I \) and \( S'_2 \) is bounded. Therefore \( \| x'_2(k)\| \leq M \| e(k_2(t))\| \) for all \( k \in (k_2(t-1), k_2(t)) \). Due to the exponential convergence of \( e(k_2(t)) \) to zero when \( t \) tends to infinity, we conclude that both \( x'_2(k) \) and \( u(k) \) converge exponentially fast to zero when \( k \) tends to infinity. This completes the proof. \( \square \)

**Corollary 14**

\( \Sigma \) is exponentially stabilizable at \( k_0 \) if

i) \( \Sigma'_3 \) is exponentially stable at \( k_0 \)

ii) \( \Sigma'_2 \) is periodically smoothly controllable at \( k_0 \). \( \square \)
IV. Sufficient conditions for exponential detectability.

We now give a necessary and sufficient condition for exponential detectability of $\Sigma_y$. To that end we first introduce the notion of observability.

**Definition 15**
The initial state $x$ of the system $\Sigma_y$ is said to be unobservable at $k_0$ if $y(k, k_0, x, 0) = 0$ for any $k \geq k_0$.
The set of all unobservable states at $k_0$ is denoted by $U_{k_0}$ and called the unobservable subspace.
$\Sigma_y$ is said to be observable at $k_0$ if $x = 0$ is the only unobservable state of $\Sigma_y$ at $k_0$.

**Remark:**
Note that $\Sigma_y$ is observable at $k_0$ iff $U_{k_0} = 0$.

Analogous to the lemmas 4 and 5 we have that $U_k$ is $A(k)$-invariant.
This is the content of lemma 16.

**Lemma 16**

$A(k)U_k \subset U_{k+1}$.

**Proof:**
Let $x_1$ be an element of $A(k)U_k$.
By definition there exists then a $x_0$ such that
i) $x_1 = A(k_0)x_0$, and
ii) $y(k, k_0, x_0, 0) = 0$ for all $k \geq k_0$.
The rest of the proof follows now from the observation that $y(k, k_0 + 1, x_1, 0) = y(k, k_0, x_0, 0)$.

We define now the notion of (exponential) detectability.

**Definition 17**
The initial state $x$ of $\Sigma_y$ is said to be detectable at $k_0$ if there exists a finite integer $N > 0$ such that $x(k_0)$ modulo $X^\sim(A(\cdot, k_0))$ is determined from any $y[k_0, k_0 + N - 1]$ and $u[k_0, k_0 + N - 2]$.
$\Sigma_y$ is called detectable at $k_0$ if all states, $x$, are detectable at $k_0$. 
$\Sigma_y$ is called exponentially detectable at $k_0$ if in the above definition of detectability $X^-(A(\cdot, k_0))$ is replaced by $X^-_r(A(\cdot, k_0))$. 

Next, consider the state-space decomposition:

\[ X_1(k) = X_1^*(k) \cap U_k, \]
\[ X_2(k) \oplus X_1(k) = U_k, \]
\[ X_3(k) \oplus X_2(k) \oplus X_1(k) = \mathbb{R}^n, \]

where $X_1, X_2$ and $X_3$ are chosen orthogonal.

With respect to a basis adapted to this decomposition the following analogue of corollary 8 holds.

**Corollary 17**

There exists an orthogonal state-space transformation $x(\cdot) = T''(\cdot) x'(\cdot)$ which does not affect the boundedness property of the system parameters such that $\Sigma_y$ is described by the recurrence equation:

\[
\Sigma''_y: \begin{bmatrix}
x''_1(k+1) \\
x''_2(k+1) \\
x''_3(k+1)
\end{bmatrix} = \begin{bmatrix}
A''_{11}(k) & A''_{12}(k) & A''_{13}(k) \\
0 & A''_{22}(k) & A''_{23}(k) \\
0 & 0 & A''_{33}(k)
\end{bmatrix} \begin{bmatrix}
x''_1(k) \\
x''_2(k) \\
x''_3(k)
\end{bmatrix} + \begin{bmatrix}
B''_{1}(k) \\
B''_{2}(k) \\
B''_{3}(k)
\end{bmatrix} u(k)
\]

\[ y(k) = (0 \ 0 \ C''_3(k)) x''(k), \]

where

\[ \Sigma''_1: x''_1(k+1) = A''_{11}(k) x''_1(k), \] is exponentially stable at any time $k \geq k_0$.

\[ \Sigma''_3: x''_3(k+1) = A''_{33}(k) x''_3(k) + B''_3(k) u(k); \ y(k) = C''_3(k) x''(k), \]

is observable at any time $k \geq k_0$.

With the notation from the previous corollary we have:

**Theorem 18**

$\Sigma_y$ is exponentially detectable at $k_0$ iff.

\[ \Sigma''_2: x''_2(k+1) = A''_{22}(k) x''_2(k) \] is exponentially stable at $k_0$.

**Proof:**

"$\Rightarrow$" Consider the transformed system $\Sigma''_y$.

Since $\Sigma_y$ is exponentially detectable, the inclusion $X_2 \subseteq X^-_r(A(\cdot, k_0))$ must hold. Consequently, $\Sigma''_2$ has to be exponentially stable at $k_0$. 

"$\Leftarrow$" Since $\Sigma_y$ is exponentially detectable, the inclusion $X_2 \subseteq X^-_r(A(\cdot, k_0))$ must hold. Consequently, $\Sigma''_2$ has to be exponentially stable at $k_0$.
"\leq" From corollary 17 we know that $\Sigma'_1$ is exponentially stable at any time $k \geq k_0$. Due to the assumption that $\Sigma''_2$ is exponentially stable at $k_0$ we have that the following system

$$
\begin{bmatrix}
x''_1(k+1) \\
x''_2(k+1)
\end{bmatrix} = \begin{bmatrix} A''_{11}(k) & A''_{12}(k) \\ 0 & A''_{22}(k) \end{bmatrix} \begin{bmatrix} x''_1(k) \\
x''_2(k)\end{bmatrix}
$$

is also exponentially stable at $k_0$.

So, $X_1 \oplus X_2 \subset X''_N(A(\cdot, k_0))$.

Since $\Sigma''_3$ is observable at $k_0$, it is clear now that $\Sigma''_y$, and therefore $\Sigma_y$ is exponentially detectable at $k_0$. \qed


V. Sufficient conditions for simultaneously exponentially stabilizable and detectable systems.

In the previous two sections we gave necessary and sufficient conditions for exponentially stabilizable and exponentially detectable systems, respectively. In the present section we combine these results.

We derive now sufficient conditions to conclude that the system \( \Sigma_y \) is both exponentially stabilizable and exponentially detectable.

To that end we make again a state-space decomposition.

Consider

\[
\begin{align*}
X_1(k) &= R_k \cap U_k ; \\
X_2(k) \oplus X_1(k) &= R_k ; \\
X_3(k) \oplus X_2(k) \oplus X_1(k) &= \mathbb{R}^n ,
\end{align*}
\]

where \( X_1, X_2 \) and \( X_3 \) are chosen again orthogonal.

Then, analogous to the corollaries 8 and 17 we have:

Corollary 19

There exists an orthogonal state-space transformation \( x(\cdot) = T(x') x'(\cdot) \) which does not affect the boundedness property of the system parameters such that \( \Sigma_y \) is described by the recurrence equation

\[
\Sigma_y : \begin{bmatrix} x_1'(k+1) \\ x_2'(k+1) \\ x_3'(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}(k) & A_{12}(k) & A_{13}(k) \\ 0 & A_{22}(k) & A_{23}(k) \\ 0 & 0 & A_{33}(k) \end{bmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \\ x_3'(k) \end{bmatrix} + \begin{bmatrix} B_1'(k) \\ B_2'(k) \\ B_3'(k) \end{bmatrix} u'(k)
\]

\[
y(k) = (0 \ C_1'(k) C_2'(k)) x'(k) ,
\]

where

\[
\Sigma_1 : x_1'(k+1) = A_{11}(k) x_1'(k) + B_1'(k) u'(k) \text{ is reachable at any time } k \geq k_0 ;
\]

\[
\Sigma_2 : x_2'(k+1) = A_{22}(k) x_2'(k) + B_2'(k) u'(k) + A_{23}(k) A_{33}(k+1, k_0) d
\]

\[
y(k) = C_2'(k) x_2'(k) \text{ is both reachable and observable at any time } k \geq k_0 ;
\]

\[
\Sigma_3 : x_3'(k+1) = A_{33}(k) x_3'(k) .
\]
Theorem 20

With the notation as in corollary 19 we have that \( \Sigma_{\gamma} \) is both exponentially stabilizable and exponentially detectable at \( k_0 \) if the following three conditions are satisfied:

i) \( \Sigma_{\gamma 1} \) is exponentially stable for all \( k \geq k_0 \);

ii) \( \Sigma_{\gamma 2} \) is exponentially disturbance stabilizable and observable at \( k_0 \);

iii) \( \Sigma_{\gamma 3} \) is exponentially stable at \( k_0 \).

Proof:

That \( \Sigma_{\gamma} \) is exponentially stabilizable at \( k_0 \) under these conditions is proved similarly to the proof of theorem 11.

To prove exponential detectability of \( \Sigma_{\gamma} \) at \( k_0 \), we note that the conditions i) and iii) imply that \( X_1 \oplus X_3 \subset X_{\gamma}(A(\cdot, k_0)) \).

Using this and the property that any state from \( X_2 \) can be observed, we have that consequently any element of the factor space \( \mathbb{R}^n \) modulo \( X_{\gamma}(A(\cdot, k_0)) \) can be observed. So, \( \Sigma_{\gamma} \) is exponentially detectable at \( k_0 \).

\[ \square \]

Note that condition ii) in this theorem is also a necessary condition, and that, moreover, exponential stability of \( \Sigma_{\gamma 1} \) and \( \Sigma_{\gamma 3} \) at \( k_0 \) are also necessary requirements.
VI. Concluding remarks

In this paper we showed that exponential stabilizability and exponential detectability properties of a system can be analyzed like in the time-invariant case by using appropriate state-space decompositions.

On the one hand this is due to our choice of the definitions of stabilizability and detectability. In our definition of stabilizability we required, namely, that additional to the property that the closed-loop system must be stable after a well-chosen input has been applied, the input itself must be stable too.

On the other hand this is due to our choice of the state-space decompositions. They are all chosen in such a way that the convergence properties of the transformed and original system remain the same.

A direct consequence of this last mentioned prerequisite was that, when we analyzed systems which are both stabilizable and detectable, we did not choose the state-space decomposition which seems at a first glance to be the most appropriate one for analyzing these systems.

Taking in regard the several attempts which have been taken in the past to analyze stabilizability and detectability aspects of time-varying systems and the relative ease by which results are obtained when using this analysis, it seems worth while to deepen this analysis in the future.
References:


