The robustness of multi-echelon service models under autocorrelated demands

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Models under Autocorrelated Demands**

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The Robustness of Multi-Echelon Service Models under Autocorrelated Demands

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Abstract

A common assumption underlying the development of multi-echelon service models is that successive demands are independent. In this paper we study the robustness of two models, developed for a two-echelon serial inventory network, when the demand process is autocorrelated. Both stockpoints operate a periodic review order-up-to- S echelon inventory policy. Using an analytic approach, we extend the original models so as to also apply exactly for any autocorrelated period demand process and explore the behaviour of the extended models under different scenarios. In order to study the effects of different autocorrelation characteristics, an MA(1) and an AR(1) process are analysed in detail. Our results demonstrate that, depending on the overall system parameters setting, models which ignore the effects of autocorrelation may provide very unsatisfactory estimates of the system performance. Hence, extreme caution is required when deciding the use of multi-echelon service models in practice.

Keywords: Autocorrelation, Inventory, Multi-echelon, Service Level, Stochastic models

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Introduction

The development of multi-echelon models giving the service performance of multi-level inventory networks is a well established research area. By encapsulating all major interactions occurring within a network, multi-echelon models not only allow its study, but also provide the means to evaluate the system parameters which ensure the desired performance. Clark¹ and Nahmias² give reviews of early work, while only a few more recent examples are Rosenbaum³, Deurmeyer and Schwarz⁴, Jönsson and Silver^{5,6}, van Donselaar and Wijngaard⁷, Svoronos and Zipkin^{8,9}, Schwarz¹⁰, van Donselaar¹¹, Axsäter¹², de Kok¹³ and Lagodimos^{14,15}. As a result of this research, models for systems with several network forms and control disciplines are now available.

One issue that, to our knowledge, has hardly been studied is the behaviour of multi-echelon models under demand conditions differing from those originally assumed. This becomes particularly important since existing models invariably assume simple stationary demand processes (members of the Poisson family or normal) exhibiting no time correlations. Bearing in mind the complexity of real demand processes as well as the difficulty in exactly determining their form and parameters (see Wagner¹⁶), only if these models proved to be robust, one could suggest their general use. It is the robustness of two multi-echelon service models when faced with autocorrelated demand processes that we explore in this paper. We should note that, for single-echelon service models, a number of similar studies have appeared (see Ray¹⁷, Fotopoulos *et al.*¹⁸, Eppen and Martin¹⁹ and An *et al.*²⁰).

The multi-echelon models we consider here were proposed by van Donselaar and Wijngaard⁷ and Lagodimos¹⁵ and represent the α and γ measures of service defined in Schneider²¹. The former gives the fraction of periods with no stockouts while the latter relates to the average system backorders. Both models assume independent identically distributed normal demands and were developed for two-echelon serial networks using echelon-based periodic review order-up-to- S policies at both echelons. The demand is imposed at the upper echelon stockpoint, which is supplied by the lower echelon, and all unsatisfied demand is backordered. In the event of material shortages at the lower echelon, an order placed by the upper echelon is split, to be satisfied entirely when material becomes available. There are no capacity constraints and lead times at both echelons are fixed. A detailed discussion of all operating assumptions is given elsewhere^{7,15}. We should note, however, that the backordering assumption (effectively) permits analytical tractability and has been used by all previously cited papers and most theoretical research in the area.

Despite their simplicity, the ordering policies assumed by the models are of considerable practical interest, since they correspond to those implemented by the well established *base stock control* (BSC) system (van Donselaar and Wijngaard⁷ developed their model while studying a major consumer electronics manufacturer, operating under BSC). As it has been recently shown (see Lagodimos²² and Axsäter and Rosling²³), under certain conditions these policies are also identical to those implemented by *material requirements planning* (MRP) systems, using lot-for-lot ordering at both echelons. Therefore, all the results in this paper also apply to BSC and MRP systems.

When the demand process is autocorrelated, the accuracy of the above models will inevitably deteriorate. The extent of this phenomenon, together with the factors which affect it, is one of the issues we will study. Another issue relates to the demand process specification. It has been reported (see Ehrhardt *et al.*²⁴) that first order correlations between successive period demands are sometimes the only reliable correlation estimates that may be available in practice. It is therefore important, when exploring the robustness of multi-echelon models, to study processes with a different autocorrelation structure in order to determine its impact on the models behaviour.

The approach we use to study these issues is analytic. By a simple redefinition of their parameters, we show that both models can be extended to apply for any stationary autocorrelated normal demand process and present them in a form which allows us to compute the resulting service by using standard computer routines or available graphs. In order to study the effects of autocorrelation, the demand process needs to be entirely specified. For this purpose we consider two simple processes, a *moving average* MA(1) and an *autoregressive* AR(1) process, having different autocorrelation characteristics. For both processes, we evaluate the induced correlations between the model parameters and report numerical results demonstrating the behaviour of α and γ under different parameter settings. Our analysis is concluded with a discussion of our findings and recommendations for further research.

General service models

We start by presenting models for α and γ which are valid for any stationary period demand process $\mathbf{d} = \{d_t, t \in N\}$, where N is the set of natural numbers and $d_t \sim N(\mu, \sigma^2)$. We use the following notation to represent the network parameters:

- l : upper echelon lead time
- L : lower echelon lead time
- S_1 : echelon order-up-to level at the upper echelon
- S_2 : echelon order-up-to level at the lower echelon
- SS_1 : planned upper echelon safety stock
- SS_2 : planned lower echelon safety stock

Let I_t be the upper echelon *net inventory* (inventory on hand minus backorders) at the end of period t . As shown in Schneider²¹, both α and γ are functions of I_t . For the system we are considering, van Donselaar and Wijngaard⁷ proposed a distribution-free expression for I_t :

$$\begin{aligned} I_t &= \min \left\{ SS_1 + SS_2 + (L + l + 1)\mu - Y - X, SS_1 + (l + 1)\mu - X \right\} \\ &= \min \{ S_2 - Y - X, S_1 - X \}, \end{aligned} \quad (1)$$

where $Y = \sum_{i=0}^{L-1} d_{t-L-l+i}$ and $X = \sum_{i=0}^l d_{t-l+i}$. For any stationary normal demand process, the random variables X and Y , being sums of normal variables, are normally distributed. Depending on the demand process, X and Y are generally correlated, with correlation coefficient ρ . Obviously:

$$E\{X\} = (l + 1)\mu \quad \text{and} \quad E\{Y\} = L\mu,$$

while $\text{Var}(X)$, $\text{Var}(Y)$ and ρ depend on the specifications of $\mathbf{d}$.

Although (1) is sufficient to develop models for α and γ , it is convenient to use non-dimensional ratios in our analysis (see Brown²⁵ and Lagodimos²²). Define:

$$Z_1 = \frac{S_1 - (l + 1)\mu}{\text{Var}^{1/2}(X)}, \quad Z_2 = \frac{S_2 - (L + l + 1)\mu}{\text{Var}^{1/2}(X)}, \quad V = \frac{\text{Var}^{1/2}(Y)}{\text{Var}^{1/2}(X)}, \quad M = \frac{\mu}{\text{Var}^{1/2}(X)}. \quad (2)$$

Introducing these ratios and the standardising transformations:

$$X = (l + 1)\mu + x\text{Var}^{1/2}(X), \quad Y = L\mu + y\text{Var}^{1/2}(Y),$$

in expression (1), after some algebra, we obtain:

$$I_t = \text{Var}^{1/2}(X) \min \{ Z_2 - Vy - x, Z_1 - x \}, \quad (3)$$

where x and y are normal variables $N(0, 1)$ with correlation coefficient ρ .

Other than the fact that Z_1 , Z_2 , V and M have been generalised and that x and y are now correlated, (3) is identical to the expression for I_t obtained by assuming uncorrelated

period demands. As a result, the corresponding models for α and γ will also be identical, being (see Lagodimos¹⁵):

$$\begin{aligned}\alpha &= \Pr\{x < Z_1 \text{ and } x + Vy < Z_2\}, \\ (1 - \gamma)M &= E\{x - Z_1 | \mathcal{A}_1\} \Pr\{\mathcal{A}_1\} + E\{x + Vy - Z_2 | \mathcal{A}_2\} \Pr\{\mathcal{A}_2\},\end{aligned}\quad (4)$$

where:

$$\mathcal{A}_1 = \{x > Z_1 \text{ and } Vy < Z_2 - Z_1\}, \quad \mathcal{A}_2 = \{x + Vy > Z_2 \text{ and } Vy \geq Z_2 - Z_1\}.$$

Both models represent integrals over the space of the standardised bivariate normal variable $(x, y; \rho)$. Observe that, for independent period demands:

$$\text{Var}^{1/2}(X) = \sigma\sqrt{l+1}, \quad \text{Var}^{1/2}(Y) = \sigma\sqrt{L} \quad \text{and} \quad \rho = 0, \quad (5)$$

so both models reduce to those in Lagodimos¹⁵. It is the robustness of these models that we will explore.

Closed-form service expressions

In the form they have been presented, several numerical integrations are necessary in order to evaluate α and γ . In this section we outline how it is possible to obtain closed-form expressions which drastically reduce (or even eliminate) the need for numerical integrations. A result of more general interest is given in an appendix.

We start with α . From (4) α is a function of $(x, y; \rho)$. Define a new variable:

$$w = (x + Vy)A^{-1/2} \quad \text{where:} \quad A = V^2 + 2\rho V + 1. \quad (6)$$

It is straightforward to show that $w \sim N(0, 1)$ and that x and w form a standardised bivariate normal variable $(x, w; (\rho V + 1)A^{-1/2})$. Substitution of $w\sqrt{A}$ for $x + Vy$ in (4) gives:

$$\begin{aligned}\alpha &= \Pr\{x < Z_1 \text{ and } w < Z_2 A^{-1/2}\} \\ &= \Phi\left(Z_1, \frac{Z_2}{\sqrt{A}}; \frac{\rho V + 1}{\sqrt{A}}\right),\end{aligned}\quad (7)$$

where $\Phi(\cdot, \cdot; \zeta)$ is the standardised bivariate normal probability function with correlation coefficient ζ . This may be found in mathematical handbooks (see, for example, Abramowitz and Stegun²⁶) and is also available in certain computer packages.

Consider now γ . Introducing, as before, w from (6) in (4), we can rewrite γ as:

$$(1 - \gamma)M = E\{x | \mathcal{A}_1\} \Pr\{\mathcal{A}_1\} + \sqrt{A}E\{w | \mathcal{A}_2\} \Pr\{\mathcal{A}_2\} - Z_1 \Pr\{\mathcal{A}_1\} - Z_2 \Pr\{\mathcal{A}_2\},$$

where:

$$\mathcal{A}_1 = \{x > Z_1 \text{ and } y < \frac{Z_2 - Z_1}{V}\}, \quad \mathcal{A}_2 = \{w > \frac{Z_2}{\sqrt{A}} \text{ and } y \geq \frac{Z_2 - Z_1}{V}\},$$

and where y and w form a standardised bivariate normal variable $(y, w; (V + \rho)A^{-1/2})$. Using the general result in the Appendix with the appropriate limits, we can directly obtain closed-form expressions for the two partial expectations above. We can also express the two probability expressions in terms of the standardised normal probability function. After some cumbersome algebra, we finally obtain:

$$\begin{aligned} (1 - \gamma)M &= \sqrt{A}\phi\left(\frac{Z_2}{\sqrt{A}}\right)\left[1 - \Phi\left(\frac{Z_2(\rho V + 1) - Z_1 A}{V\sqrt{1 - \rho^2}\sqrt{A}}\right)\right] \\ &\quad + V\phi\left(\frac{Z_2 - Z_1}{V}\right)\left[1 - \Phi\left(\frac{Z_1(V + \rho) - \rho Z_2}{V\sqrt{1 - \rho^2}}\right)\right] \\ &\quad + \phi(Z_1)\Phi\left(\frac{Z_2 - Z_1(\rho V + 1)}{V\sqrt{1 - \rho^2}}\right) - Z_2(1 - \alpha) \\ &\quad + (Z_2 - Z_1)\left[\Phi\left(\frac{Z_2 - Z_1}{V}\right) - \Phi\left(Z_1, \frac{Z_2 - Z_1}{V}; \rho\right)\right], \end{aligned} \quad (8)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the univariate standardised normal probability density and function respectively, A is defined in (6) and α is the service measure in (7). We can easily show that, for $\rho = 0$, (8) reduces to the expression for γ derived in Lagodimos¹⁵ for independent period demands.

The proposed closed-form expressions for α and γ are general and apply for any normal demand process $\mathbf{d}$. Therefore, provided that reliable estimates for the ratios Z_1 , Z_2 , V , M and the correlation coefficient ρ are available, (7) and (8) can be used irrespective of the exact specifications of $\mathbf{d}$, which do not need to be known.

Tables 1 and 2 give values of α and γ for different scenarios. By considering the linear expression $(1 - \gamma)M$ instead of γ , the variables involved were reduced. Moreover, by introducing the ratio $p = Z_1/Z_2$, we can also observe the impact of different safety stock allocation policies (using the definitions of Z_1 and Z_2 we immediately see that $p = SS_1/(SS_1 + SS_2)$, hence p represents the fraction of the total available safety stock allocated at the upper echelon). As it has been theoretically shown elsewhere (see Lagodimos and Anderson²⁷), for all serial systems α and γ are increasing function of p , irrespective of either the demand process characteristics or the available safety stock. We further discuss the results in Tables 1 and 2 in a later section.

In all cases, we evaluated the bivariate probability function using an exact procedure proposed by Owen²⁸ which, by appropriate transformations, reduces the problem to a one-

dimensional numerical integration, which may be performed very accurately. Our results have been exhaustively tested using both Owen's tables and the Abramowitz and Stegun²⁶ graphs, and found very satisfactory.

Specific demand processes

In this section we focus on two stationary processes and propose exact expressions for $\text{Var}(X)$, $\text{Var}(Y)$ and ρ . The processes we consider are a *moving average* MA(1) and an *autoregressive* AR(1) process. An interesting feature of these processes is that they exhibit entirely different autocorrelation structures. This is one of the reasons (together with their simplicity) that these are often used for studying the impact of demand autocorrelation on inventory models (see Fotopoulos *et al.*¹⁸ and An *et al.*²⁰).

In the following, we use the notation $\epsilon = \{\epsilon_t, t \in N\}$ for a *random noise* process, consisting of a stream of independent random variables $\epsilon_t \sim N(0, \sigma_\epsilon^2)$. We also use θ_k for the autocorrelation coefficient between any terms d_t and d_{t+k} of $\mathbf{d}$. We start with some definitions which may be found in any standard textbook (see, for example, Kendall and Stuart²⁹).

When $\mathbf{d}$ is a stationary MA(1) process, d_t is given by:

$$d_t = a - b\epsilon_{t-1} + \epsilon_t, \quad (9)$$

where each $\epsilon_t \in \epsilon$, a and b are constants, and $|b| < 1$. It can be shown that:

$$\mu = a, \quad \sigma^2 = (1 + b^2)\sigma_\epsilon^2, \quad \theta_1 = -\frac{b}{b^2 + 1}, \quad (10)$$

while $\theta_k = 0$ for $k > 1$; hence, only consecutive terms of MA(1) are correlated. For this reason it is said that MA(1) has a short memory. We can easily show that the maximum obtainable first order correlation coefficient for the MA(1) process is $|\theta_1| = 1/2$.

When $\mathbf{d}$ is a stationary AR(1) process, d_t is given by:

$$d_t = a + bd_{t-1} + \epsilon_t, \quad (11)$$

where each $\epsilon_t \in \epsilon$, a and b are constants, and $|b| < 1$. It can be shown that:

$$\mu = \frac{a}{1 - b}, \quad \sigma^2 = \frac{1}{1 - b^2}\sigma_\epsilon^2, \quad \theta_k = b^k \text{ for } k \geq 1. \quad (12)$$

Observe the long autocorrelation structure of AR(1), which spans an infinite number of terms. Also note that, for AR(1), any $|\theta_1| < 1$ can be obtained. Omitting the cumbersome and lengthy algebra required for its derivation, we now state the major result of this section (see Verrijdt and Lagodimos³⁰ for derivation details).

Proposition 1 Consider a stationary normal stochastic process $\mathbf{d}$. Let X and Y be the consecutive sums of consecutive terms of $\mathbf{d}$ defined in (1) and let ρ be the correlation coefficient between X and Y . When $\mathbf{d}$ is MA(1) defined by (9), then:

$$\text{Var}(X) = [(1-b)^2 l + (1+b^2)]\sigma_\epsilon^2, \quad \text{Var}(Y) = [(1-b)^2(L-1) + (1+b^2)]\sigma_\epsilon^2,$$

$$\rho = -\frac{b}{\sqrt{[(1-b)^2 l + (1+b^2)][(1-b)^2(L-1) + (1+b^2)]}}.$$

When $\mathbf{d}$ is AR(1) defined by (11), then:

$$\text{Var}(X) = \frac{2b^{l+2} - (l+1)b^2 - 2b + l + 1}{(1-b^2)(1-b)^2}\sigma_\epsilon^2, \quad \text{Var}(Y) = \frac{2b^{L+1} - Lb^2 - 2b + L}{(1-b^2)(1-b)^2}\sigma_\epsilon^2,$$

$$\rho = \frac{b(1-b^L)(1-b^{l+1})}{\sqrt{[2b^{l+2} - (l+1)b^2 - 2b + l + 1][2b^{L+1} - Lb^2 - 2b + L]}}.$$

For any system parameters setting and for either demand process, this proposition can be used to evaluate the non-dimensional ratios in (2) and, subsequently, the resulting α and γ service. Observe that, for $b = 0$, all expressions reduce to those in (5), which correspond to the uncorrelated demand case.

In a previous study, An *et al.*²⁰ proposed expressions for determining $\text{Var}(X)$ or $\text{Var}(Y)$, which are slightly different from those given in Proposition 1. We should stress that the latter have been exhaustively tested and found to perform as expected for all particular cases examined.

Analysis and discussion

We can now test the robustness of the two multi-echelon models under consideration. Having extended the original models, however, we are able to explore the service impact of demand autocorrelation in a more general context. This is, in fact, one of the major advantages in using analytic models, instead of computer simulation, in exploratory studies.

Useful insight into the effects of demand autocorrelation can be obtained from Tables 1 and 2. Each line of the tables gives the corresponding service for systems with identical V , Z_2 and $p(= Z_1/Z_2)$ ratios, but facing demand processes resulting in different ρ values. Recall that, for any given network, ρ only depends on the demand autocorrelation structure. In this context, these results highlight the impact of demand autocorrelation, effectively isolated from all other parameters which may influence service.

Provided that the available safety stock allocation allows the system to operate as a real two-echelon network (we further discuss this point later), both α and γ depend on ρ . The nature of this dependence, however, is quite complex. Hence, although γ always decreases with increasing $\rho \in [-0.8, 0.8]$, the behaviour of α is contingent on the entire system setting. In this context, α may either decrease with ρ or even obtain a minimum value for different parameter combinations.

Both tables reveal that, when the demand process is autocorrelated ($\rho \neq 0$), errors in estimating the system performance may be incurred when using models which ignore autocorrelation (i.e. assume that $\rho = 0$). For a given demand process, the extent of this phenomenon depends on the system design (through V), the available safety stock (through Z_2) and its allocation between stock points (through p). In general, the original models become quite inadequate in situations where: the lower echelon demand variability over its lead time is much higher than that of the upper echelon (high V values); there is not sufficient total safety stock (low Z_2 values); the lower echelon is relatively understocked (high p values).

It is also interesting to notice a few cases for which service appears quite indifferent to changes in ρ (for example $V = 0.5$, $Z_2 = 2.5$ and $p = 0.5$ in both tables). These effectively demonstrate that, for certain parameter combinations, the two-echelon network may degenerate into a single-echelon network due to relative overstocking at the lower echelon. We can easily show that, when this happens:

$$\alpha \rightarrow \Phi(Z_1) \quad \text{and} \quad (1 - \gamma)M \rightarrow \phi(Z_1) - Z_1[1 - \Phi(Z_1)],$$

which are the corresponding single-echelon service models, and which only depend on the magnitude of Z_1 . Observe that, in general, α approaches its limiting value quicker than γ .

Our analysis, so far, has ignored the demand autocorrelation characteristics. In order to explore them, we considered identical systems facing MA(1) and AR(1) demand processes. In all cases, both processes were selected to have identical μ , σ and θ_1 . This was easily achieved by first determining the appropriate a , b and σ_e values (solving the corresponding algebraic system of three equations with three unknowns). Subsequently, using the results of Proposition 1 together with expressions (2), the ratios V , Z_1 , Z_2 and M were evaluated and introduced in (7) and (8) to give α and γ .

Tables 3 and 4 give numerical results showing the behaviour of α and γ as functions of θ_1 for both MA(1) and AR(1) demand processes. Since $|\theta_1| < 1/2$ for MA(1), only such values are shown. All results correspond to $\mu = 1000$ and $\sigma = 300$, values which ensure that $\Pr\{d_t < 0\}$ is sufficiently small (less than 0.0005). Aiming to encapsulate the effects of

different system designs, we considered systems with different cumulative lead times ($L + l$) and different lead time allocations between stock points ($q = l/L + l$). In order to obtain some common terms of reference, for each system we evaluated the total safety stock so as to ensure a desired α service for a single-echelon system with lead time equal to $L + l$. The cumulative lead time, as well as the total safety stock used for each system, are shown in Table 5.

Tables 3 and 4 reveal that, for all system settings, both α and γ are decreasing functions of θ_1 . In general, under identical conditions, the service level for AR(1) is always higher than for MA(1). This happens for both measures, although γ appears more sensitive to changes in θ_1 , particularly for $\theta_1 > 0$. More important, however, is to observe the difference between the service performance under AR(1) and MA(1). Both tables indicate that, for not extreme autocorrelation values (i.e. $|\theta_1| < 0.3$), the service difference between the two processes is not significant. Otherwise, the difference may increase considerably and this depends on the magnitude of the available safety stock, as well as the lead time and safety stock allocation between stock points. It appears that, in general, the difference increases with decreased amount of total safety stock in the system (low $SS_1 + SS_2$).

A final issue relates to the substantial errors that may result when models ignoring demand autocorrelation are used in conjunction with lead time demand variance estimates, not measured directly, but evaluated from the variance of the demand process using expressions (5). In Tables 3 and 4, all entries with $\theta_1 = 0$ may effectively be interpreted as service estimates obtained in this manner (recall that all values in the tables correspond to demand processes with identical mean and variance). We can directly observe that such service estimates may deviate dramatically from the true system performance. This becomes particularly important for processes which may exhibit high $|\theta_1|$ values, such as the AR(1) process.

Numerical Example

An example will hopefully clarify some of the issues we discussed. Consider a two-echelon serial network with $l = 2$ and $L = 8$, having $SS_1 = 1637$ and $SS_2 = 0$, and facing an $N(1000, 300^2)$ period demand. From Tables 3 and 4 ($SS_1 + SS_2 = \text{high}$, $L + l = 10$, $p = 1$, $q = 0.2$, $\theta_1 = 0$) we note that, if period demands were independent, then $\alpha = 0.95$ and $\gamma = 0.979$. However, if successive period demands were correlated according to an AR(1) process with $\theta_1 = 0.9$, then the service levels achieved by the system would differ considerably (to obtain the required demand process characteristics, the AR(1) model coefficients are easily

evaluated to be: $a = 100$, $b = 0.9$ and $\sigma_\epsilon = 130.8$). From the same tables ($SS_1 + SS_2 = \text{high}$, $L + l = 10$, $p = 1$, $q = 0.2$, $\theta_1 = 0.9$) we see that, in this case, $\alpha = 0.722$ and $\gamma = 0.524$. Hence, in the presence of demand autocorrelation, the system exhibits considerably worse performance than the one estimated by models which entirely ignore autocorrelation. In this example, we observe a 456 % increase in stockout occasions, while the average system backorders rise from 21 to 476 units, an increase of 2167 % (!). Average backorders are directly evaluated from the expression: $\text{backorders} = (1 - \gamma)\mu$ (see Schneider²¹).

Improved service estimates may be obtained, even with models that assume independent demands, provided that accurate estimates of $\text{Var}(X)$ and $\text{Var}(Y)$ are available (either through the knowledge of the exact demand process specifications or through direct measurements). To demonstrate this we consider the previous example and assume that the demand process is known exactly. For this AR(1) process, from Proposition 1 we obtain $\text{Var}(X) = 740173.79$ and $\text{Var}(Y) = 4455819.03$ and using the definitions in (2), we can easily evaluate the relevant ratios to be: $Z_1 = 1.903$, $Z_2 = 1.903$ and $V = 2.454$. Introducing these values in (7) and (8) with $\rho = 0$ (to obtain the models that assume uncorrelated demands in van Donselaar and Wijngaard⁷ and Lagodimos¹⁵) we find $\alpha = 0.751$ and $\gamma = 0.680$. Direct comparison with the actual system performance given earlier ($\alpha = 0.722$, $\gamma = 0.524$) shows that the estimated increase in stockout occurrences and average backorders is now 12 % and 49 % (from 320 to 476 units) respectively, a considerable improvement over the previous estimates.

Conclusions

In this paper we have tested the robustness of two different multi-echelon service models under conditions where the demand process is autocorrelated. For this purpose, we extended the original models to allow them to hold exactly for any autocorrelated demand process, and numerically explored the behaviour of the extended models for two simple demand process exhibiting different autocorrelation characteristics.

Our results demonstrate that, depending on the overall system setting, service models which ignore demand autocorrelation may provide α and γ estimates which deviate substantially from their true values. This phenomenon becomes more significant for systems with comparatively high lower echelon lead time demand variance and not sufficient safety stock. In cases, however, where such models are used in conjunction with lead time demand variance estimates obtained indirectly only from (estimates of) the demand process variance, the errors in estimating the resulting service may be dramatic.

Another issue relates to the effects of the demand autocorrelation characteristics. Our results demonstrate that, other than for moderate autocorrelation values, knowledge of the entire demand autocorrelation structure is essential. In this context, the knowledge of only the first order autocorrelation coefficient appears insufficient either to obtain accurate service estimates or to determine the safety stocks which will ensure a desired level of service.

Several related issues are open for further research. In this context, it may be worth exploring the behaviour of other service measures used in practice, such as the fill rate. Although γ is considered to be a good approximation for the fill rate (see, for example, Brown²⁵), there are cases where the fill rate deviates considerably from γ (see de Kok¹³). Using the methodology presented here, it would be straightforward to develop an exact model for the fill rate under autocorrelated demands and subsequently study its behaviour.

The analysis we have presented here has exclusively considered the simplest possible network configuration: two-echelon serial networks. This allowed us to concentrate on exact models demonstrating the pure effects of demand autocorrelation, avoiding additional complications and interactions, that arise in more complex networks. It is important, however, also to extend this analysis to other networks and particularly to networks with divergent form. This will allow a study of the impact of demand autocorrelation on the system design and the determination of its effects on the optimal allocation of safety stocks within these networks.

Appendix: Partial expectation integrals

In this appendix, we derive a closed-form expression for partial expectations of the form:

$$\begin{aligned} I &= \mathbb{E}\{x \mid l < x < L \text{ and } q < y < Q\} \Pr\{l < x < L \text{ and } q < y < Q\} \\ &= \int_q^Q \int_l^L x \phi(x, y; \tau) dx dy, \end{aligned} \quad (13)$$

where:

$$\phi(x, y; \tau) = \frac{1}{2\pi\sqrt{1-\tau^2}} \exp\left\{-\frac{x^2 - 2\tau xy + y^2}{2(1-\tau^2)}\right\},$$

is the bivariate normal probability density of $(x, y; \tau)$.

Introducing the notation:

$$a = \frac{1}{1-\tau^2}, \quad b = 2a\tau \quad \text{and} \quad A(x, y) = ax^2 - bxy + ay^2,$$

we have:

$$\begin{aligned} I &= \int_q^Q \int_l^L \frac{x e^{-\frac{A(x,y)}{2}}}{2\pi\sqrt{1-\tau^2}} dx dy \\ &= -\frac{1}{a} \int_q^Q \int_l^L \frac{e^{-\frac{A(x,y)}{2}}}{2\pi\sqrt{1-\tau^2}} d\left(-\frac{A(x,y)}{2}\right) dy + \frac{b}{2a} \int_q^Q \int_l^L \frac{y e^{-\frac{A(x,y)}{2}}}{2\pi\sqrt{1-\tau^2}} dx dy \\ &= -\frac{1}{a} I_1 + \frac{b}{2a} I_2. \end{aligned} \quad (14)$$

First consider I_1 . After evaluation of the internal integral we obtain:

$$\begin{aligned} I_1 &= \int_q^Q \frac{1}{2\pi\sqrt{1-\tau^2}} \left[e^{-\frac{A(x,y)}{2}} \right]_{x=l}^{x=L} dy \\ &= \int_q^Q \frac{e^{-\frac{A(L,y)}{2}}}{2\pi\sqrt{1-\tau^2}} dy - \int_q^Q \frac{e^{-\frac{A(l,y)}{2}}}{2\pi\sqrt{1-\tau^2}} dy \\ &= \frac{e^{-\frac{aL^2}{2}} e^{\frac{b^2 L^2}{8a}}}{\sqrt{2\pi}\sqrt{1-\tau^2}} \int_q^Q \frac{e^{-\frac{a}{2}(y-\frac{bL}{2a})^2}}{\sqrt{2\pi}} dy - \frac{e^{-\frac{al^2}{2}} e^{\frac{b^2 l^2}{8a}}}{\sqrt{2\pi}\sqrt{1-\tau^2}} \int_q^Q \frac{e^{-\frac{a}{2}(y-\frac{bl}{2a})^2}}{\sqrt{2\pi}} dy \\ &= \frac{e^{-\frac{aL^2}{2}} e^{\frac{b^2 L^2}{8a}}}{\sqrt{2\pi}} \int_{\sqrt{a}(q-\frac{bL}{2a})}^{\sqrt{a}(Q-\frac{bL}{2a})} \phi(v) dv - \frac{e^{-\frac{al^2}{2}} e^{\frac{b^2 l^2}{8a}}}{\sqrt{2\pi}} \int_{\sqrt{a}(q-\frac{bl}{2a})}^{\sqrt{a}(Q-\frac{bl}{2a})} \phi(w) dw, \end{aligned} \quad (15)$$

where $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the standardised univariate normal probability density.

Straightforward integration and introduction of the original variables gives:

$$I_1 = \phi(L) \left\{ \Phi\left(\frac{Q-\tau L}{\sqrt{1-\tau^2}}\right) - \Phi\left(\frac{q-\tau L}{\sqrt{1-\tau^2}}\right) \right\} - \phi(l) \left\{ \Phi\left(\frac{Q-\tau l}{\sqrt{1-\tau^2}}\right) - \Phi\left(\frac{q-\tau l}{\sqrt{1-\tau^2}}\right) \right\}, \quad (16)$$

where $\Phi(\cdot)$ is the standardised univariate normal probability function.

We also need to evaluate I_2 in (14). Observe that I_2 has exactly the same form as the original integral I except that it represents the partial expectation of y rather than x . So, since $A(x, y)$ is symmetric with respect to x and y , by reversing the order of integration and using identical steps as before, we can write:

$$\begin{aligned}
I_2 &= \int_l^L \int_q^Q \frac{y e^{-\frac{A(x,y)}{2}}}{2\pi\sqrt{1-\tau^2}} dy dx \\
&= -\frac{1}{a} \int_l^L \int_q^Q \frac{e^{-\frac{A(x,y)}{2}}}{2\pi\sqrt{1-\tau^2}} d\left(-\frac{A(x,y)}{2}\right) dx + \frac{b}{2a} \int_l^L \int_q^Q \frac{x e^{-\frac{A(x,y)}{2}}}{2\pi\sqrt{1-\tau^2}} dy dx \\
&= -\frac{1}{a} I_{21} + \frac{b}{2a} I_{22}.
\end{aligned} \tag{17}$$

We can evaluate I_{21} using the same transformations we used to evaluate I_1 . After some algebra:

$$I_{21} = \phi(Q) \left\{ \Phi\left(\frac{L-\tau Q}{\sqrt{1-\tau^2}}\right) - \Phi\left(\frac{l-\tau Q}{\sqrt{1-\tau^2}}\right) \right\} - \phi(q) \left\{ \Phi\left(\frac{L-\tau q}{\sqrt{1-\tau^2}}\right) - \Phi\left(\frac{l-\tau q}{\sqrt{1-\tau^2}}\right) \right\}. \tag{18}$$

Observe now that, by reversing the corresponding order of integration I_{22} becomes identical to I ; so, $I_{22} = I$. Therefore, using (14) and (17) and solving the resulting algebraic equation, we obtain the following exact expression for I :

$$I = -(I_1 + \tau I_{21})$$

where I_1 and I_{21} are given by (16) and (18) respectively. Using this expression (taking the appropriate values for the limits l , L , q , Q and the correlation coefficient τ), any partial expectation of the form in (13) can be directly evaluated, without resorting to numerical techniques.

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V	Z ₂	p	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
0.5	1.5	0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
		0.5	0.773	0.773	0.772	0.770	0.768	0.768	0.786	0.768	0.771
		1	0.933	0.926	0.916	0.905	0.894	0.883	0.873	0.863	0.852
	2.5	0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
		0.5	0.894	0.894	0.894	0.894	0.894	0.894	0.894	0.894	0.894
		1	0.994	0.993	0.992	0.990	0.985	0.980	0.974	0.967	0.960
	3.5	0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
		0.5	0.960	0.960	0.960	0.960	0.960	0.960	0.960	0.960	0.960
		1	1.000	1.000	1.000	0.999	0.999	0.998	0.997	0.995	0.993
1.5	1.5	0	0.465	0.453	0.452	0.456	0.462	0.470	0.479	0.488	0.497
		0.5	0.727	0.696	0.681	0.675	0.674	0.676	0.682	0.691	0.704
		1	0.883	0.836	0.806	0.786	0.771	0.761	0.752	0.744	0.736
	2.5	0	0.497	0.492	0.489	0.489	0.490	0.493	0.496	0.498	0.500
		0.5	0.891	0.878	0.864	0.853	0.844	0.839	0.836	0.835	0.838
		1	0.990	0.975	0.954	0.934	0.915	0.897	0.881	0.867	0.853
	3.5	0	0.500	0.499	0.498	0.498	0.498	0.499	0.500	0.500	0.500
		0.5	0.960	0.958	0.954	0.948	0.942	0.936	0.931	0.927	0.924
		1	1.000	0.998	0.992	0.984	0.974	0.963	0.951	0.940	0.930
2.5	1.5	0	0.335	0.354	0.372	0.390	0.407	0.425	0.443	0.463	0.484
		0.5	0.579	0.572	0.575	0.582	0.592	0.603	0.616	0.631	0.649
		1	0.731	0.707	0.694	0.687	0.683	0.680	0.678	0.676	0.672
	2.5	0	0.427	0.425	0.431	0.440	0.450	0.461	0.473	0.485	0.496
		0.5	0.812	0.787	0.771	0.762	0.758	0.756	0.757	0.760	0.764
		1	0.911	0.881	0.857	0.837	0.820	0.806	0.794	0.782	0.772
	3.5	0	0.476	0.469	0.468	0.471	0.476	0.483	0.489	0.495	0.499
		0.5	0.934	0.916	0.899	0.886	0.875	0.867	0.860	0.855	0.850
		1	0.974	0.955	0.936	0.919	0.903	0.888	0.875	0.863	0.851

Table 1: α as a function of ρ for different ratio combinations.

V	Z_2	p	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
0.5	1.5	0	0.399	0.399	0.399	0.399	0.399	0.399	0.399	0.399	0.399
		0.5	0.131	0.131	0.132	0.134	0.135	0.138	0.140	0.143	0.145
		1	0.030	0.032	0.037	0.045	0.054	0.066	0.079	0.093	0.109
	2.5	0	0.399	0.399	0.399	0.399	0.399	0.399	0.399	0.399	0.399
		0.5	0.050	0.050	0.050	0.050	0.051	0.051	0.051	0.052	0.052
		1	0.002	0.002	0.002	0.003	0.005	0.009	0.013	0.018	0.024
	3.5	0	0.399	0.399	0.399	0.399	0.399	0.399	0.399	0.399	0.399
		0.5	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0.016
		1	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003
1.5	1.5	0	0.415	0.437	0.456	0.473	0.488	0.500	0.511	0.519	0.523
		0.5	0.150	0.183	0.217	0.249	0.279	0.309	0.337	0.364	0.391
		1	0.049	0.088	0.131	0.174	0.216	0.258	0.299	0.340	0.381
	2.5	0	0.400	0.404	0.410	0.415	0.420	0.424	0.426	0.428	0.429
		0.5	0.051	0.058	0.071	0.086	0.104	0.122	0.142	0.162	0.183
		1	0.003	0.010	0.025	0.045	0.069	0.095	0.122	0.150	0.179
	3.5	0	0.399	0.399	0.400	0.401	0.403	0.403	0.404	0.404	0.404
		0.5	0.016	0.016	0.019	0.024	0.031	0.040	0.051	0.062	0.076
		1	0.000	0.000	0.003	0.009	0.018	0.029	0.043	0.058	0.074
2.5	1.5	0	0.593	0.641	0.680	0.712	0.740	0.764	0.785	0.803	0.816
		0.5	0.334	0.401	0.459	0.511	0.558	0.602	0.645	0.686	0.727
		1	0.234	0.308	0.377	0.440	0.500	0.557	0.612	0.666	0.720
	2.5	0	0.464	0.496	0.522	0.544	0.563	0.578	0.591	0.601	0.606
		0.5	0.119	0.162	0.205	0.247	0.288	0.328	0.367	0.406	0.445
		1	0.070	0.114	0.161	0.209	0.257	0.305	0.352	0.398	0.443
	3.5	0	0.417	0.432	0.446	0.459	0.469	0.478	0.484	0.488	0.490
		0.5	0.035	0.055	0.079	0.106	0.134	0.163	0.194	0.225	0.257
		1	0.018	0.038	0.063	0.091	0.122	0.155	0.188	0.222	0.256

Table 2: $(1-\gamma)M$ as a function of ρ for different ratio combinations.

L+1	SS ₁ +SS ₂	p	q	-0.90	-0.60	-0.45	-0.30	-0.15	0	0.15	0.30	0.45	0.60	0.90
10	low	0.5	0.2	0.915	0.851	0.807	0.762	0.720	0.682	0.648	0.616	0.589	0.567	0.563
				0.885	0.789	0.725	0.682	0.651	0.626	0.607				
		0.8		0.901	0.797	0.754	0.719	0.689	0.664	0.641	0.621	0.603	0.587	0.564
				0.846	0.741	0.693	0.664	0.643	0.628	0.616				
		1	0.2	0.992	0.935	0.896	0.856	0.816	0.778	0.741	0.706	0.673	0.643	0.607
				0.972	0.884	0.822	0.778	0.745	0.718	0.697				
	high	0.5	0.8	0.992	0.925	0.880	0.838	0.799	0.763	0.731	0.701	0.673	0.647	0.605
				0.964	0.866	0.804	0.763	0.734	0.712	0.694				
		0.8	0.2	0.996	0.990	0.980	0.963	0.938	0.907	0.871	0.833	0.793	0.753	0.695
				0.993	0.973	0.941	0.907	0.875	0.846	0.821				
		1	0.8	0.994	0.951	0.918	0.884	0.850	0.817	0.785	0.753	0.722	0.691	0.636
				0.979	0.908	0.855	0.817	0.788	0.766	0.747				
20	low	0.5	0.2	1.000	0.999	0.994	0.985	0.970	0.950	0.923	0.892	0.855	0.814	0.722
				1.000	0.993	0.973	0.950	0.927	0.906	0.886				
		0.8	0.2	1.000	0.998	0.993	0.982	0.965	0.941	0.912	0.879	0.841	0.800	0.716
				1.000	0.991	0.968	0.941	0.916	0.893	0.872				
	high	0.5	0.8	0.969	0.889	0.834	0.782	0.733	0.689	0.648	0.610	0.575	0.543	0.512
				0.939	0.818	0.740	0.689	0.653	0.625	0.604				
		0.8	0.2	0.943	0.809	0.762	0.724	0.692	0.663	0.638	0.614	0.593	0.573	0.542
				0.882	0.752	0.696	0.663	0.641	0.624	0.611				
		1	0.2	0.998	0.945	0.905	0.864	0.823	0.782	0.742	0.701	0.661	0.622	0.565
				0.986	0.896	0.829	0.782	0.746	0.718	0.695				
	low	0.5	0.8	0.998	0.934	0.888	0.843	0.802	0.763	0.726	0.692	0.660	0.629	0.574
				0.982	0.877	0.808	0.763	0.731	0.706	0.687				
	high	0.5	0.2	1.000	0.997	0.989	0.973	0.948	0.915	0.876	0.830	0.781	0.728	0.634
				0.999	0.985	0.953	0.915	0.880	0.849	0.822				
		0.8	0.2	0.999	0.959	0.925	0.889	0.853	0.818	0.783	0.748	0.713	0.678	0.606
				0.990	0.917	0.859	0.818	0.787	0.763	0.744				
		1	0.2	1.000	0.999	0.995	0.986	0.971	0.950	0.922	0.888	0.848	0.800	0.683
				1.000	0.994	0.974	0.950	0.926	0.904	0.884				
	low	0.5	0.8	1.000	0.999	0.994	0.984	0.966	0.942	0.910	0.872	0.829	0.781	0.671
				1.000	0.993	0.970	0.942	0.914	0.889	0.867				
	high	0.5	0.2	1.000	0.997	0.989	0.973	0.948	0.915	0.876	0.830	0.781	0.728	0.634
				0.999	0.985	0.953	0.915	0.880	0.849	0.822				
		0.8	0.2	0.999	0.959	0.925	0.889	0.853	0.818	0.783	0.748	0.713	0.678	0.606
				0.990	0.917	0.859	0.818	0.787	0.763	0.744				
		1	0.2	1.000	0.999	0.995	0.986	0.971	0.950	0.922	0.888	0.848	0.800	0.683
				1.000	0.994	0.974	0.950	0.926	0.904	0.884				
	low	0.5	0.8	1.000	0.999	0.994	0.984	0.966	0.942	0.910	0.872	0.829	0.781	0.671
				1.000	0.993	0.970	0.942	0.914	0.889	0.867				
	high	0.5	0.2	1.000	0.997	0.989	0.973	0.948	0.915	0.876	0.830	0.781	0.728	0.634
				0.999	0.985	0.953	0.915	0.880	0.849	0.822				
		0.8	0.2	0.999	0.959	0.925	0.889	0.853	0.818	0.783	0.748	0.713	0.678	0.606
				0.990	0.917	0.859	0.818	0.787	0.763	0.744				
		1	0.2	1.000	0.999	0.995	0.986	0.971	0.950	0.922	0.888	0.848	0.800	0.683
				1.000	0.994	0.974	0.950	0.926	0.904	0.884				

Table 3: The behaviour of α for AR(1) and MA(1) as a function of θ_1 for systems with different lead times and safety stocks.

L+1	SS ₁ +SS ₂	p	q	-0.90	-0.60	-0.45	-0.30	-0.15	0	0.15	0.30	0.45	0.60	0.90
10	low	0.5	0.2	0.988	0.971	0.951	0.925	0.893	0.853	0.804	0.744	0.669	0.570	0.242
				0.981	0.943	0.898	0.853	0.810	0.769	0.730				
		0.8		0.985	0.944	0.914	0.880	0.842	0.797	0.745	0.683	0.605	0.505	0.188
				0.968	0.904	0.848	0.797	0.752	0.709	0.670				
		1	0.2	0.999	0.985	0.970	0.948	0.920	0.884	0.839	0.782	0.707	0.607	0.259
				0.996	0.964	0.924	0.884	0.844	0.807	0.770				
	high	0.5	0.8	0.999	0.983	0.964	0.939	0.908	0.869	0.821	0.762	0.687	0.586	0.248
				0.994	0.957	0.913	0.869	0.827	0.788	0.750				
		0.8		1.000	0.999	0.997	0.992	0.984	0.969	0.947	0.915	0.867	0.796	0.516
				0.999	0.996	0.985	0.969	0.950	0.929	0.906				
		1	0.2	0.999	0.990	0.978	0.961	0.939	0.910	0.873	0.826	0.764	0.679	0.400
				0.997	0.974	0.942	0.910	0.878	0.847	0.817				
20	low	0.5	0.2	1.000	1.000	0.999	0.996	0.990	0.979	0.961	0.933	0.888	0.818	0.524
				1.000	0.998	0.991	0.979	0.964	0.946	0.927				
		0.8		1.000	1.000	0.999	0.995	0.988	0.976	0.956	0.924	0.877	0.805	0.519
				1.000	0.998	0.990	0.976	0.958	0.939	0.918				
	high	0.5	0.2	0.996	0.973	0.946	0.909	0.862	0.804	0.730	0.637	0.515	0.344	-0.389
				0.990	0.938	0.870	0.804	0.740	0.680	0.622				
		0.8		0.991	0.931	0.890	0.842	0.786	0.721	0.642	0.544	0.418	0.244	-0.479
				0.972	0.880	0.796	0.721	0.653	0.589	0.530				
		1	0.2	1.000	0.984	0.962	0.932	0.892	0.841	0.775	0.689	0.572	0.405	-0.347
				0.998	0.957	0.900	0.841	0.784	0.730	0.677				
	low	0.5	0.8	1.000	0.980	0.955	0.920	0.876	0.819	0.747	0.655	0.533	0.362	-0.383
				0.997	0.948	0.884	0.819	0.757	0.698	0.642				
	high	0.5	0.2	1.000	1.000	0.998	0.993	0.981	0.961	0.929	0.878	0.800	0.674	0.033
				1.000	0.997	0.984	0.961	0.934	0.903	0.870				
		0.8		1.000	0.989	0.974	0.950	0.919	0.877	0.823	0.750	0.650	0.504	-0.149
				0.999	0.969	0.924	0.877	0.830	0.785	0.740				
		1	0.2	1.000	1.000	0.999	0.995	0.987	0.971	0.945	0.902	0.832	0.715	0.064
				1.000	0.998	0.988	0.971	0.949	0.924	0.896				
	low	0.5	0.8	1.000	1.000	0.999	0.994	0.984	0.966	0.936	0.888	0.811	0.685	0.034
				1.000	0.998	0.987	0.966	0.941	0.912	0.881				
	high	0.5	0.2	1.000	1.000	0.998	0.993	0.981	0.961	0.929	0.878	0.800	0.674	0.033
				1.000	0.997	0.984	0.961	0.934	0.903	0.870				
		0.8		1.000	0.989	0.974	0.950	0.919	0.877	0.823	0.750	0.650	0.504	-0.149
				0.999	0.969	0.924	0.877	0.830	0.785	0.740				
		1	0.2	1.000	1.000	0.999	0.995	0.987	0.971	0.945	0.902	0.832	0.715	0.064
				1.000	0.998	0.988	0.971	0.949	0.924	0.896				
	low	0.5	0.8	1.000	1.000	0.999	0.994	0.984	0.966	0.936	0.888	0.811	0.685	0.034
				1.000	0.998	0.987	0.966	0.941	0.912	0.881				
	high	0.5	0.2	1.000	1.000	0.998	0.993	0.981	0.961	0.929	0.878	0.800	0.674	0.033
				1.000	0.997	0.984	0.961	0.934	0.903	0.870				
		0.8		1.000	0.989	0.974	0.950	0.919	0.877	0.823	0.750	0.650	0.504	-0.149
				0.999	0.969	0.924	0.877	0.830	0.785	0.740				
		1	0.2	1.000	1.000	0.999	0.995	0.987	0.971	0.945	0.902	0.832	0.715	0.064
				1.000	0.998	0.988	0.971	0.949	0.924	0.896				

Table 4: The behaviour of γ for AR(1) and MA(1) as a function of θ_1 for systems with different lead times and safety stocks.

L+1	10	20
<u>low</u> $\Phi \left[\frac{SS_1+SS_2}{\sigma\sqrt{L+l+1}} \right] = 0.80$	836	1155
<u>high</u> $\Phi \left[\frac{SS_1+SS_2}{\sigma\sqrt{L+l+1}} \right] = 0.95$	1637	2262

Table 5: The total safety stock values used for different cumulative lead times and service requirements.

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Number	Month	Author	Title
93-01	January	P. v.d. Laan C. v. Eeden	Subset selection for the best of two populations: Tables of the expected subset size
93-02	January	R.J.G. Wilms J.G.F. Thiemann	Characterizations of shift-invariant distributions based on summation modulo one.
93-03	February	Jan Beirlant John H.J. Einmahl	Asymptotic confidence intervals for the length of the shortt under random censoring.
93-04	February	E. Balas J. K. Lenstra A. Vazacopoulos	One machine scheduling with delayed precedence constraints
93-05	March	A.A. Stoorvogel J.H.A. Ludlage	The discrete time minimum entropy H_∞ control problem
93-06	March	H.J.C. Huijberts C.H. Moog	Controlled invariance of nonlinear systems: nonexact forms speak louder than exact forms
93-07	March	Marinus Veldhorst	A linear time algorithm to schedule trees with communication delays optimally on two machines
93-08	March	Stan van Hoesel Antoon Kolen	A class of strong valid inequalities for the discrete lot-sizing and scheduling problem
93-09	March	F.P.A. Coolen	Bayesian decision theory with imprecise prior probabilities applied to replacement problems
93-10	March	A.W.J. Kolen A.H.G. Rinnooy Kan C.P.M. van Hoesel A.P.M. Wagelmans	Sensitivity analysis of list scheduling heuristics
93-11	March	A.A. Stoorvogel J.H.A. Ludlage	Squaring-down and the problems of almost-zeros for continuous-time systems
93-12	April	Paul van der Laan	The efficiency of subset selection of an ε -best uniform population relative to selection of the best one
93-13	April	R.J.G. Wilms	On the limiting distribution of fractional parts of extreme order statistics

Number	Month	Author	Title
93-14	May	L.C.G.J.M. Habets	On the Genericity of Stabilizability for Time-Day Systems
93-15	June	P. van der Laan C. van Eeden	Subset selection with a generalized selection goal based on a loss function
93-16	June	A.A. Stoorvogel A. Saberi B.M. Chen	The Discrete-time H_∞ Control Problem with Strictly Proper Measurement Feedback
93-17	June	J. Beirlant J.H.J. Einmahl	Maximal type test statistics based on conditional processes
93-18	July	F.P.A. Coolen	Decision making with imprecise probabilities
93-19	July	J.A. Hoogeveen J.K. Lenstra B. Veltman	Three, four, five, six or the Complexity of Scheduling with Communication Delays
93-20	July	J.A. Hoogeveen J.K. Lenstra B. Veltman	Preemptive scheduling in a two-stage multiprocessor flow shop is NP-hard
93-21	July	P. van der Laan C. van Eeden	Some generalized subset selection procedures
93-22	July	R.J.G. Wilms	Infinite divisible and stable distributions modulo 1
93-23	July	J.H.J. Einmahl F.H. Ruymgaart	Tail processes under heavy random censorship with applications
93-24	August	F.W. Steutel	Probabilistic methods in problems of applied analysis
93-25	August	A.A. Stoorvogel	Stabilizing solutions of the H_∞ algebraic Riccati equation
93-26	August	R. Perelaer J.K. Lenstra M. Savelsbergh F. Soumis	The Bus Driver Scheduling Problem of the Amsterdam Transit Company
93-27	August	J.M. van den Akker C.P.M. van Hoesel M.W.P. Savelsbergh	Facet Inducing Inequalities for Single-Machine Scheduling Problems
93-28	September	H.J.C. Huijberts H. Nijmeijer	Dynamic disturbance decoupling of nonlinear systems and linearization

Number	Month	Author	Title
93-29	September	L.C.G.J.M. Habets	Testing Reachability and Stabilizability of Systems over Polynomial Rings using Gröbner Bases
93-30	September	Z. Liu J.A.C. Resing	Duality and Equivalencies in Closed Tandem Queueing Networks
93-31	November	A.G. de Kok J. van der Wal	Assigning identical operators to different machines
93-32	November	M.J.A. van Eenige J.A.C. Resing J. van der Wal	A Matrix-Geometric Analysis of Queueing Systems with Periodic Service Interruptions
93-33	November	A.G. de Kok	Backorder lead time behaviour in (b, Q) -inventory models with compound renewal demand
93-34	November	J. van der Wal	The lead time shift theorem for the continuous review (s, Q) and (s, S) inventory systems
93-35	November	J.H.C.M. Verrijdt A.G. de Kok	Distribution Planning for a Divergent 2-echelon Network without Intermediate Stocks under Service Restrictions
93-36	November	A.G. Lagodimos A.G. de Kok J.H.C.M. Verrijdt	The Robustness of Multi-Echelon Service Models under Autocorrelated Demands