Control aspects of linear discrete time-varying systems

Citation for published version (APA):

Published: 01/01/1987

Document status and date:

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 19. Mar. 2019
Memorandum COSOR 87-21

Control aspects of linear discrete
time-varying systems

by

J.C. Engwerda

Eindhoven, Netherlands

September 1987
Control aspects of linear discrete time-varying systems.

ABSTRACT

In this paper results for time-invariant linear systems concerning controllability, reachability, output controllability, stabilizability and target path controllability are extended to time-varying systems. For the extensions which are already known in literature elementary proofs are given. Moreover necessary and sufficient conditions are given for the (strong) admissibility of a target path.

I. Introduction

Exactly tracking a target path has attracted a lot of attention both in control theory and economics.

After the pioneering work of Tinbergen [21], Brockett and Mesarovic [5] formulated necessary and sufficient conditions for target path controllability in linear continuous time-invariant systems.

Since then a lot of papers appeared on this subject [1,3,4,16-19,25]. Recently Wohltmann [27] and Grasse [9] extended the results to continuous time-varying systems.

This paper is concerned with discrete time-varying systems. After introducing some notation and establishing some elementary lemma's in section II, we treat target point problems in section III. Although these problems were treated before (see e.g. Meditch [14] th.2.2 and th.2.4, Ludyck [13] th.2.2 and th.2.4, and Kwakernaak [11] th.6.7.), either the solutions there were incomplete, or the proofs were false. So for completeness sake we give here results for controllability, stabilizability, output and reachability problems, together with elementary proofs.

In section IV we turn to target path controllability problems. The aim is to track a certain predescribed target path for a given time interval. A preliminary question is whether there exists a number \( k \) such that any target path can be tracked for a period of length \( k \).

Subsequent questions concern the maximum of such numbers \( k \) and the reaction time, i.e. the time needed to reach the target path (also known as policy lead question). In [16,25] this problem was solved for time-invariant systems. Here we derive an algebraic algorithm for time-varying systems. This section is followed by a section in which the decoupling problem is
solved.

In section VI we take another point of view. Given a system we give a characterization of all target paths which can be tracked. Section VII gives a characterization of target paths which can be tracked asymptotically. The paper ends with a conclusion.
II. Preliminaries.

In this section we consider a system described by the following linear time-varying discrete time recurrence equation:

\[ x(k+1) = A(k) x(k) + B(k) u(k); \quad x(k_0) = x \]
\[ \Sigma: \quad y(k) = C(k) x(k) \]

Here \( x(k) \in \mathbb{R}^n \) is the state of the system, \( u(k) \in \mathbb{R}^m \) the applied control, and \( y(k) \in \mathbb{R}^r \) the output at time \( k \).

We use the following notation:

- \( y^*(i) \) denotes a reference value for variable \( y \) at time \( i \).
- \( y^T(i) \) denotes the transpose of \( y(i) \).
- \( y[k,l] := (y^T(k), \ldots, y^T(l))^T \).
- \( y[k,\cdot] := (y^T(k), y^T(k+1), \ldots)^T \).

**Lemma 1:**

Let \( \|A(k)\| \leq c \) for all \( k \), and let \( \{e(k)\} \) satisfy

\[ e(k+1) = A(k) e(k) + v(k) . \]

Then \( e(k) \to 0 \) implies \( v(k) \to 0 \).
Proof:
Follows directly from the equality $v(k) = e(k + 1) - A(k) e(k)$. 

The second lemma gives necessary and sufficient conditions for solvability of a linear equation. A proof can be found in [12], chap. 3.10.

Lemma 2:
Let $B$ be an $n \times m$ matrix.

Then the equation $B u = y$ is solvable if and only if (iff) rank $[B \mid y] = \text{rank } B$. Moreover, the solution is uniquely determined iff rank $B = m$. Then the solution is given by $u = (B^T B)^{-1} B^T y$. 

Note that if matrix $B$ in this lemma is not full column rank there always exists a transformation $S$ in the input space $U$ such that $BS$ equals $[B' \mid 0]$, where now matrix $B'$ is full column rank. So if in this case the problem is solvable, then there exist infinitely many solutions.

Our third lemma has a more set theoretic background. It tells us that $R^n$ can not be covered by a countable set of linear subspaces, unless one of them equals $R^n$.

Lemma 3:
Let $I$ be an index set and $S(i)$ and $S$ be linear subspaces of $R^n$ with dimension $S(i) < \text{dimension } S$. Then $\bigcup_{i \in I} S(i) = S$ implies that $I$ is uncountable.

This last lemma is used in the next section to derive a result for outputcontrollability. The next three lemma's are used in section IV to deduce an algorithm for target path controllability. Since the reader may not be familiar with the results short proofs are provided.

Lemma 4:
Let $A \in R^{n \times m}, B \in R^{n \times m}$.

Then $\begin{bmatrix} A \\ B \end{bmatrix}$ is full row rank iff i) $\text{Im } A = R^n$ and ii) $B \text{ Ker } A = R^n$ (or equivalently i) $\text{Im } B = R^n$ and ii) $A \text{ Ker } B = R^n$.

Proof:
We first note that $m > n + p$ is a necessary condition.

Now consider the following decomposition of $R^m : R^m = X \oplus \text{Ker } A$.

With respect to this basis matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ has the following structure:
\[
\begin{bmatrix}
A' & 0 \\
B' & B''
\end{bmatrix}, \text{ where } A' \text{ is full row rank.}
\]

Therefore \[
\begin{bmatrix}
A \\
B
\end{bmatrix}
\] is full row rank iff \( A' \) is invertible and \( B'' \) is full row rank.

\[\square\]

**Corollary 1:**

Let \( A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times q}, C \in \mathbb{R}^n \).

Then \[
\begin{bmatrix}
A & B \\
0 & C
\end{bmatrix}
\]

is full row rank iff

i) \( \text{Im } C = \mathbb{R}^n \)

ii) \( \text{Im } A + B \text{ Ker } C = \mathbb{R}^n \).

\[\square\]

**Lemma 5:**

Let \( s \) be a homomorphism from \( V \rightarrow W \).

Then \( s^{-1}(s(V)) = S + \text{ Ker } s \) (\( s^{-1} \) means inverse mapping).

**Corollary 2:**

Let \( A \in \mathbb{R}^{n \times m} \) and \( V \) be a linear subspace of \( \mathbb{R}^m \).

Then \( A \ V = \mathbb{R}^n \) is equivalent to

i) \( \text{Ker } A + V = \mathbb{R}^m \), and

ii) \( A \) is full row rank.

**Lemma 6:**

\( \text{Ker } C \cap A \text{ Ker } B = A \text{ Ker } \begin{bmatrix}
CA \\
B
\end{bmatrix}. \)

**Proof:**

\( x \in \text{Ker } C \cap A \text{ Ker } B \iff \)

there exists a vector \( b \) such that \( C x = 0, x = A \ b \) and \( B b = 0 \iff \)

there exists a \( b \in \text{Ker } \begin{bmatrix}
CA \\
B
\end{bmatrix} \)

such that \( x = A \ b \iff \)

\( x \in A \text{ Ker } \begin{bmatrix}
CA \\
B
\end{bmatrix}. \)

\[\square\]
III. Point controllability problems.

In this section several point controllability problems are analysed. To help the reader to notice clearly the differences between all the concepts that are treated, we place the most important ones together in one definition. Once we introduced this basic definition, subspaces which are closely related to these concepts are defined and remarks are made about these definitions.

Definition:
The initial state $x$ of the system $\Sigma$ is said to be
asymptotically stable at $k_0$: if $\lim_{k \to \infty} x(k, k_0, x, 0) = 0$;

stabilizable at $k_0$: if $\exists u[k_0, \ldots]$ such that $\lim_{k \to \infty} (k, k_0, x, u) = 0$;

zerocontrollable at $k_0$: if $\exists u[k_0, N - 1]$ with $k_0 < N < \infty$ such that $x(N, k_0, x, u) = 0$.

The state $x$ is said to be reachable at $k_0$ from zero if $\exists u[N, k_0 - 1]$ with $-\infty < N < k_0$ such that $x(k_0, N, 0, u) = x$.

The output $y$ is said to be outputcontrollable from zero at $k_0$ if $\exists u[k_0, N - 1]$ with $k_0 < N < \infty$ such that $y(N, k_0, 0, u) = y$.

Now let $L$ be a linear subspace of $\mathbb{R}^n$ (resp. $\mathbb{R}^r$).

Then $\Sigma$ is called $L$-zerocontrollable at $k_0$ if all states $x \in L$ are zerocontrollable at $k_0$. Complete analogous are $L$-stabilizability, $L$-zerocontrollability, $L$-reachability and $L$-outputcontrollability of $\Sigma$ defined.

For each of these concepts there is a maximal subspace $L$ having this property.

For these maximal subspaces we introduce some terminology which stems from time-invariant systems.

In the sequel the subspace consisting of all asymptotically stable states at $k_0$ is denoted by $X^-(A(k_0, \cdot))$. The stabilizability subspace at $k_0$ is the subspace consisting of all initial states at $k_0$ which are stabilizable and is abbreviated by $X_{stab}(k_0)$. The subspaces of all zerocontrollable states at $k_0$ and reachable states from zero at $k_0$ finally are denoted by respectively $Z_{k_0}$ and $R_{k_0}$.

That all the spaces defined here are indeed linear subspaces is easily shown.

In case the maximal subspaces equal $\mathbb{R}^n$ (resp. $\mathbb{R}^r$) we talk respectively about asymptotic stability, stabilizability, zerocontrollability, reachability and outputcontrollability of $\Sigma$ at $k_0$.

Remark 1: in the last defined system properties the subscript "from zero" is dropped for reachability and outputcontrollability.

This is due to the fact that reachability respectively outputcontrollability from zero implies reachability respectively outputcontrollability from any initial state. By considering the system $x(k) = 0$ for all $k$, we see that this implication does not hold for zerocontrollability.
Remark 2: the concepts of zerocontrollability and reachability are dual. This property is used in the proof of a result about reachability.

Remark 3: in literature often the concept of outputcontrollability of the system at \( k_0 \) to a prespecified target \( y^* \) is encountered. Usually this is defined as the property of \( \Sigma \) that for any initial state \( x \) at \( k_0 \) there exists a finite control sequence \( u[k_0, N - 1] \) such that \( y(N, k_0, x, u) = y^* \). Using this concept then also outputcontrollability of \( \Sigma \) at \( k_0 \) can be defined. We prefer however to define outputcontrollability as the property that the output can be controlled towards any reference point, given the initial state of \( \Sigma \).

As was already noted in the introduction many proofs have been given in the past concerning results for zerocontrollability and reachability which proved to be wrong for the most general system, as considered here. For this reason we will not only state here the results but also give elementary proofs of them.

The first result is about zerocontrollability.

**Theorem 1:**

\( \Sigma \) is \( L \)-zerocontrollable at \( k_0 \) iff there exists an integer \( M > k_0 \) such that \( A(M - 1, k_0) L \subseteq \text{Im} \, S(M - 1, k_0) \).

Proof:

Clearly \( A(M - 1, k_0) L \subseteq \text{Im} \, S(M - 1, k_0) \) is equivalent to: for all \( l \in L \), \( \exists \, u[k_0, M - 1] \) such that \( 0 = A(M - 1, k_0) l + S[M - 1, k_0] u[k_0, M - 1] \).

From this the sufficiency of the condition is clear.

To prove the necessity of the condition we let \( e_1, \ldots, e_k \) be a basis for \( L \).

Assume that \( u[k_0, N_i - 1] \) steers \( e_i \) to zero, and that \( M = \max N_i \).

If \( N_i < M \) for some index \( i \) then we define a new extended control sequence as follows: \( u[k_0, M - 1] := (u_i^T[k_0, N_i - 1], 0, \ldots, 0)^T \).

Let \( l \) be any element of \( L \), say \( l = \sum_{i=1}^k \alpha_i e_i \). Then the control sequence \( \sum_{i=1}^k \alpha_i u_i[k_0, M - 1] \) steers \( l \) to zero at \( M \).

So for \( l \in L \) the equation

\[
0 = A(M - 1, k_0) l + S[M - 1, k_0] u[k_0, M - 1]
\]

holds. From this the necessity of the inclusion is clear, too.

Note that as a special case of this theorem we obtain that \( \Sigma \) is zerocontrollable iff there exists an integer \( M > k_0 \) such that \( \text{Im} \, A(M - 1, k_0) \subseteq \text{Im} \, S[M - 1, k_0] \).
Our second result concerns reachability. The proof of it is a complete dualization of the proof of theorem 1.

**Theorem 2:**

\( \Sigma \) is \( L \)-reachable from zero at \( k_0 \) iff there exists an integer \( M < k_0 \) such that \( L \subset \text{Im} \ S[k_0 - 1, M] \).

**Proof:**

It is easily shown that the condition is sufficient. 

To prove the necessity of the condition let \( e_1, \ldots, e_k \) be a basis for \( L \) again. Then for each \( e_i \) there exists an input sequence \( u_i[N_i, k_0 - 1] \) such that this input sequence steers the state from zero at \( N_i \) to \( e_i \) at \( k_0 \).

Let \( M \) be the minimum of \( N_i, i = 1, \ldots, k \).

If \( N_i > M \) for some index \( i \) then we define the new control sequence \( u_i[M, k_0 - 1] \) as follows:

\[
u_i[M, k_0 - 1] := (0, \ldots, 0, u_i[N_i, k_0 - 1])^T .
\]

Now let \( l \) be any element of \( L \), say \( l = \sum_{i=1}^{k} \alpha_i \, e_i \). Then the control sequence \( \sum_{i=1}^{k} \alpha_i u_i[M, k_0 - 1] \) steers the initial state of the system from zero to \( l \) at \( k_0 \). In other words there exists an integer \( M \) such that any \( l \in L \) can be reached at \( k_0 \) from zero at \( M \). So for any \( l \in L \) there exists an input sequence \( u[M, k_0 - 1] \) such that the following equation holds:

\[
l = S[k_0 - 1, M] \, u[M, k_0 - 1],\]

which proves the theorem.

By taking \( L \) equal to \( R^n \) we obtain that \( \Sigma \) is reachable at \( k_0 \) iff there exists an integer \( M < k_0 \) such that \( \text{rank} \ S[k_0 - 1, M] = n \).

The next item we discuss is outputcontrollability.

A special case concerning this subject was stated by Kwakernaak [11] in th.6.7. However no proof was provided.

We give here a proof, though not constructive, of this generalization from his concept of complete controllability.

To give a constructive proof is difficult since an output which can be obtained at time \( k \) is in general unobtainable at time \( k+1 \).

**Theorem 3:**

\( \Sigma \) is \( L \)-outputcontrollable from zero at \( k_0 \) iff there exists an integer \( M > k_0 \) such that \( L \subset \text{Im} \ C(M) \ S[M - 1, k_0] \).
Proof:
The sufficiency of the condition is again easily shown.
The necessity of the condition is proved by contradiction.
Assume that for any integer \( M > k_0 \), \( L \not\subset \text{Im} \, C(M) \, S[M - 1, k_o] \).
Then \( L \cap \text{Im} \, C(M) \, S[M - 1, k_o] \) is for any \( M \) a linear subspace with dimension smaller than the dimension of \( L \).
Since any output \( y \in L \) can be obtained from zero at \( k_0 \) we know that the collection 
\[ \bigcup_{i \in I} L \cap \text{Im} \, C(M) \, S[i - 1, k_0] \] covers \( L \) where \( I = \{k_0, k_0 + 1, \ldots \} \).
So we have a countable collection of subspaces, all with lower dimension than the dimension of \( L \), which cover \( L \). This clearly contradicts lemma 3. 

A special case of this theorem is again obtained by taking \( L \) equal to \( \mathbb{R}^r \). The theorem states then that \( \Sigma \) is outputcontrollable from zero iff there exists an integer \( M > k_0 \) such that 
\[ \text{rank} \, C(M) \, S[M - 1, k_o] = r. \]

Another interesting aspect is that the rank condition given in the theorem is equivalent to the following statement: there exists an integer \( M > k_0 \) such that for all \( y \in L \) an input sequence \( u[k_0, M - 1] \) exists which steers the output from the initial state \( x(k_0) = 0 \) to \( y \) at \( M \).
This is a result which, due to the linearity of \( \Sigma \), also holds for \( L \)-zerocontrollability and \( L \)-reachability. So \( \Sigma \) has one of these properties iff there exists a uniform time \( M \) at which for each \( l \in L \) this property holds. In other words in the definition of these concepts the quantors \( \forall \, l \in L, \exists \, N \) such that etc. may be interchanged.
The last concept that is treated in this section is stabilizability.
For time-invariant systems we know that the stabilizability subspace is given by 
\[ X^{-}(A) + Z . \] (1)
For these systems it can be shown that if a system is stabilizable at \( k_0 \) then there exists a control such that from time \( k_0 + n \) on, no control is needed anymore to obtain convergence of all states towards zero.
This property does not hold any longer for time-varying systems. A simple example illustrates this phenomenon.

Example 1:
Take 
\[ A(k) = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} ; B(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; C(k) = I, k \neq 100, 200, \ldots \]
\[ A(k) = \begin{bmatrix} 2 & 1/k \\ 0 & 1/2 \end{bmatrix}; B(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; C(k) = I, k = 100, 200, \ldots \]

This system is stabilizable. At any point 100 \* k + 1 in time we must however control the system in order to achieve this property.

Moreover we see that \( X^{-1}(A(k_0, \cdot)) = 0 \) and \( Z_{k_0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). So property (1) ceases to hold.

In the rest of this section we embed the stabilizability subspace in a subspace from which one expects at a first glance that it equals the stabilizability subspace. Though this is not the case in general it turns out that for a very broad class of systems equality is obtained.

The advantage of this new introduced subspace, the potential stabilizibility subspace, is that it can be calculated as a stability subspace. The introduction of this subspace requires a state space decomposition. This state space decomposition is based on the property that the time dependent reachability subspace is \( A(k) \) invariant (i.e. \( A(k) R_k \subset R_{k+1} \)). This property is proved first now.

**Lemma 7:**
\[ A(k) \ R_k + \text{Im} \ B(k) = R_{k+1}. \]

**Proof:**
It is obvious that if \( x \) is reachable at \( k \), \( A(k) \ x \) is reachable at \( k + 1 \) (take \( u(k) = 0 \)). Furthermore is any element in the image of \( B(k) \) reachable at \( k + 1 \). So one inclusion is clear.

To prove the other inclusion we use the fact that for any \( k \) there exists an integer \( N(k) \) such that \( R_k = \text{Im} \ S[k, k - N(k)] \). By definition of \( R_k \) this implies that \( \text{Im} \ S[k, k - N] = R_k \) for any \( N > N(k) \).

Now let \( k \) be a given number.

We make a distinction between two cases, namely, \( N(k + 1) > N(k) \) and \( N(k + 1) < N(k) \).

First consider the case \( N(k + 1) > N(k) \). Then
\[
A(k) \ R_k + \text{Im} \ B(k) = A(k) \ \text{Im} \ S[k, k - N(k)] + \text{Im} \ B(k) \\
= A(k) \ \text{Im} \ S[k, k - N(k + 1)] + \text{Im} \ B(k) \\
= \text{Im} \ S[k + 1, k + 1 - N(k + 1)] , \text{ which proves one case } .
\]

On the other hand, if \( N(k + 1) < N(k) \), we have
\[
R_{k+1} = \text{Im} \ S[k + 1, k + 1 - N(k + 1)] \\
= A(k) \ \text{Im} \ S[k, k - N(k + 1)] + \text{Im} \ B(k)
\]
\[ A(k) S[k, k - N(k)] + \text{Im} B(k) \]

\[ = A(k) R_k + \text{Im} B(k) \], which completes the proof.

The required state space decomposition results immediately from this lemma. A proof of it can also be found in Ludyck [13] th.6.1.

Denote the dimension of the reachable subspace at time \( k \), \( R_k \), by \( r_k \) and let \( 0_{p,q} \) denote a zero matrix with \( p \) rows and \( q \) columns.

Corollary 3:

Let \( \mathbb{R}^n = R_k \oplus X_k \).

With respect to a basis adapted to this decomposition \( \Sigma \) is described by the following recurrence equation:

\[
\begin{bmatrix}
    x_1'(k + 1) \\
    x_2'(k + 1)
\end{bmatrix}
= \begin{bmatrix}
    A_{11}'(k) & A_{12}'(k) \\
    0 & A_{22}'(k)
\end{bmatrix}
\begin{bmatrix}
    x_1'(k) \\
    x_2'(k)
\end{bmatrix}
+ \begin{bmatrix}
    B_1'(k) \\
    0
\end{bmatrix} u(k)
\]

\[
y(k) = (C_1'(k) \mid C_2'(k)) \begin{bmatrix}
    x_1'(k) \\
    x_2'(k)
\end{bmatrix}
\]

where the subsystem \( x_1'(k + 1) = A_{11}'(k) x_1'(k) + B_1'(k) u(k) \) is reachable at any time \( k \), the state space dimension of \( x_1'(k) \) is \( r_k \) and that of \( x_2'(k) \) is \( n - r_k \).

Note that this state space decomposition can be obtained by a state transformation \( x'(k) = T(k) x(k) \).

We now define the potential stabilizability subspace at \( k_0 \).

Therefore we reconsider \( \Sigma \) for \( k > k_0 \), assuming that \( A(k) \) and \( B(k) \) are zero for all \( k < k_0 \). Furthermore we introduce \( A'(k) := A(k) \mod R_k \).

This is a mapping from \( \mathbb{R}^n \mod R_k \to \mathbb{R}^n \mod R_{k+1} \) which, due to lemma 7, makes sense. Note that for \( k = k_0 \) we have that \( A'(k_0) = A(k_0) \).

Definition:

The potential stabilizability subspace at \( k_0 \) is defined as follows:

\[ X^-(k_0) := \{ x \in \mathbb{R}^n \mid x(i + 1) = A'(i) x(i) \to 0, x(k_0) = x \} \].

To motivate the study of this potential stabilizability subspace we first characterize this subspace in two examples.
The first example we consider is the autonomous system, i.e. $B(k) = 0$ for all $k$. In that case is $R_k = 0$ for all $k$. Consequently the potential stabilizability subspace coincides with the stability and the stabilizability subspaces.

The second example concerns time-invariant systems, i.e. $A(k) = A$ and $B(k) = B$ for all $k$. In that case the potential stabilizability subspace coincides with the stabilizability subspace, as is shown later on in this section.

These two examples suggest that the potential stabilizability subspace is closely related to the stabilizability subspace. The exact relation is revealed in the next two theorems. The first theorem gives a justification of the chosen name for the potential stabilizability subspace.

**Theorem 4a:**

$X_{stab}(k_0) \subset X^-(k_0)$.

**Proof:**

Let $x$ be an element of the stabilizability subspace at $k_0$. By definition there exists then a control sequence $u[k_0, \ldots]$ such that in $x(k + 1) = A(k)x(k) + B(k)u(k)$;

$x(k_0) = x, \lim_{k \to \infty} x(k, k_0, x, u) = 0$.

But this is equivalent to the existence of a control sequence $u'[k_0, \ldots]$ such that, in the transformed system $x'(k + 1) = A'(k)x'(k) + B'(k)u'(k)$;

$x'(k_0) = x, \lim_{k \to \infty} x'(k, k_0, x, u) = 0$. Take the transformation now conform corollary 3. Then we observe that

\[
x'_{1}(k + 1) = A'_{11}(k)x'_{1}(k) + A'_{12}(k)x'_{2}(k) + B'_{1}(k)u'(k)
\]

\[
x'_{2}(k + 1) = A'_{22}(k)x'_{2}(k); x'(k_0) = x
\]

where the state space dimension of $x'_{1}(k)$ is time-varying and may even become zero.

So the matrices $A'_{11}(k)$ and $B'_{1}(k)$ may disappear (as happens e.g. already for $k = k_0$).

Anyhow, it is clear that if $x$ is stabilizable then necessarily in the above decomposed system $x'_{2}(k + 1) = A'_{22}(k)x'_{2}(k)$, with $x'(k_0) = x$, converges to zero when $k$ tends to infinity.

This implies that $x$ is an element of the potential stabilizability space, which had to be proved.

To prove the converse statement we have to construct a control sequence which steers incoming exogenous influences arising from the second state component (see corollary 3) to zero in the first state component. Systems which have this property will be called smooth controllable in the sequel. A proper definition is given below.

However, even in case the whole second state component of the system converges to zero it is in general not possible to construct such a sequence which results in the convergence of $x(k, k_0, x, u)$ to zero.
An example of such a system is example 1 with matrix $B(k)$ replaced by $\begin{bmatrix} (1/2)^{10k} \\ 0 \end{bmatrix}$. So the stabilizability subspace does in general not coincide with the potential stabilizability subspace.

The main problem which remains to solve is thus to find conditions from which easily smooth controllability of a system can be concluded.

A sufficient condition is when periodically these incoming exogenous influences can be steered to zero by means of a bounded input sequence. If in each period, at least once, the inverse of the controllability gramian of the reachable subsystem remains uniform bounded this condition is satisfied. This idea is reflected in the next definition of periodic stabilizability.

**Definition:**

$\Sigma$ is called smooth controllable if $X_{stb}(k_0) = X^-(k_0)$.

**Definition:**

Let $S'[k, k - N]$ be a matrix obtained from corollary 3 in the following way:

$$S'[k, k - N] := [B'(k) \mid A'(k)B'(k - 1) \mid \cdots \mid A'(k, k - N + 1) B'(k - N)].$$

$\Sigma$ is called periodically stabilizable at $k_0$ if i) $A(\cdot)$ and $B(\cdot)$ are bounded and ii) there exist constants $\varepsilon, k_1$ and $N$ such that for all $k > 0$ there exists an integer $k_2(k)$ in the interval $(k_0 + (k - 1) \ast k_1, k_0 + k \ast k_1)$ for which $S'[k_2 - N, k_2]S'^T[k_2 - N, k_2] > \varepsilon I$.

For time invariant systems we know that $S'[k, k - n]$ is full row rank at any time $k > k_0 + n$ (see theorem 2). So by taking $k_1 = N = n$, $k_2 = k \ast n$ and $\varepsilon = \sigma_{\min}$, where $\sigma_{\min}$ is the minimal singular value of $S'[k, k - n]$, we see that all time-invariant systems are periodically stabilizable.

**Theorem 4b:**

Let $\Sigma$ be periodically stabilizable at $k_0$. Then $X_{stb}(k_0) = X^-(k_0)$.

In other words: all periodically stabilizable systems are smooth controllable.

**Proof:**

Let $x$ be an element of $X^-(k_0)$.

Consider the state transformation from corollary 3. If we apply this transformation on the system, we see that $x \in X^-(k_0)$ iff the second state component, $x'_2(k)$, converges to zero.

This second component influences the first component of the transformed system via $A'_{12}(k)$ at any time $k$ (note that $A'_{12}(k)$ may be zero!).

Now consider the interval $(k_0, k_1)$ first.
Let $x_0$ be the initial state of the system at $k_0$. The flow of the initial state at $k_2(1)$ is then $x' := A(k_2(1), k_0)x_0$, and the sum of all components of the second state influencing the reachable subsystem by $e(k_2(1)) := \sum_{i=0}^{k_2(1)} A'_{12}(k_2(1), i) A'_{22}(i)x'(i)$. Consider the input

$$u[k_0, k_2(1) - N - 1] = 0,$$

and

$$u[k_2(1) - N, k_2(1)] = -S^T(S'S^T)^{-1}(e(k_2(1)) + x'_{1}),$$

where $S' = S'[k_2(1), k_2(1) - N]$. With this input the reachable part of the state at time $k_2(1) + 1, x'(k_2(1) + 1)$, becomes zero. We show now by induction that it is possible to regulate $x'(k_2(t) + 1)$ to zero for any $k$. Let therefore $t$ be any integer greater than one. Consider the interval $(k_0 + (t - 2) * k_1, k_0 + t * k_1)$ and the sum of all exogenous influences entering the reachable subsystem via matrix $A'(\cdot)$ from $k_2(t - 1) + 1$ until $k_2(t)$. Then

$$e(k_2(t)) := \sum_{i=k_2(t)}^{k_2(t)+1} A'_{12}(k_2(t), i) A'_{22}(i)x'(i).$$

Since by induction hypothesis $x'(k_2(t) + 1)$ is zero, application of the input

$$u[k_2(t) - 1, k_2(t) - N - 1] = 0, \\ u[k_2(t) - N, k_2(t)] = -S^T(S'S^T)^{-1} e(k_2(t)),$$

yields that $x'(k_2(t) + 1) = 0$. Here $S' = S'[k_2(t), k_2(t) - N]$. So the induction argument is complete.

Moreover we observe that $\|u(k)\| \leq M \|e(k_2(t))\|$, since $S'S^T > \varepsilon I$. Therefore $\|x(k)\| \leq M' \|e(k_2(t))\|$ for all $k \in (k_2(t-1), k_2(t))$. Due to the convergence of $e(k_2(t))$ to zero when $t$ tends to infinity, we conclude that $x(k)$ converges to zero when $k$ tends to infinity if this input is applied. Which completes the proof.

Another sufficient condition for smooth controllability is that the subsystem $x'(k + 1) = A'(k)x'(k) + B'(k)u(k)$ is uniformly stabilizable, in the sense of Anderson and Moore [2], for all $k > k_0$. This condition is however not necessary either as is seen by taking in example 1 matrix $B(k) = \begin{bmatrix} 1/k \\ 0 \end{bmatrix}$. We conclude this section by reconsidering example 1.

In this example is $A'(k_0) = \begin{bmatrix} 2 & a \\ 0 & 1/2 \end{bmatrix}$, where $a$ is either $1/k$ or $0$, $A'(k_0 + 1) = \begin{bmatrix} 0 & 1/2 \end{bmatrix}$ and $A'(k) = 1/2$ for all $k > k_0 + 1$. Furthermore is $B'(k) = 1$ for all $k$. So $S'[k, N] S^T(k, N) > 1$ for any $k$. Consequently is the system smooth controllable.

The potential stabilizability subspace is found now as $\{x \in \mathbb{R}^2 \mid (1/2)^T(0 1/2) x \to 0 \text{ if } k \text{ tends to infinity} \}$. This clearly equals $\mathbb{R}^2$. According to theorem 4b is the stabilizability subspace then equal to $\mathbb{R}^2$. 

\[\text{Another sufficient condition for smooth controllability is that the subsystem x'(k + 1) = A'(k)x'(k) + B'(k)u(k) is uniformly stabilizable, in the sense of Anderson and Moore [2], for all k > k_0. This condition is however not necessary either as is seen by taking in example 1 matrix B(k) = [1/k 0]. We conclude this section by reconsidering example 1.}

In this example is A'(k_0) = [[2 a] [0 1/2]], where a is either 1/k or 0, A'(k_0 + 1) = [0 1/2] and A'(k) = 1/2 for all k > k_0 + 1. Furthermore is B'(k) = 1 for all k. So S'[k, N] S^T(k, N) > 1 for any k. Consequently is the system smooth controllable.

The potential stabilizability subspace is found now as \{x \in \mathbb{R}^2 \mid (1/2)^T(0 1/2) x \to 0 \text{ if } k \text{ tends to infinity} \}. This clearly equals \mathbb{R}^2. According to theorem 4b is the stabilizability subspace then equal to \mathbb{R}^2.\]
IV. Target path controllability problems.

In section III various problems were solved which all dealt with the question whether it was possible to regulate an initial state at some starting point to another pre-described state in time. This kind of dynamic controllability is however unsatisfactory for the theory of economic policy. From the economic policy point of view the question whether it is possible to track any given target path over some time-interval from a prespecified point in time on is more interesting. This question has therefore gained a lot of attention in economics. People who were engaged in this subject are e.g. Tinbergen [21], Preston and Sieper [17], Aoki [3,4], Preston and Pagan [16] and Wohltmann [26]. The contribution of this paper is among others that results are generalized to time-varying systems.

Following Wohltmann [26] especially two questions are discussed: how long is the minimum policy lead (or reaction time) of the system, i.e. the minimum possible length of time needed to transfer the system from the initial state to the initial point of the desired target path; and how long is it possible to keep the system on some desired target path once achieved.

To answer these questions the notion of target path controllability is introduced.

Definition:
Let $p$ and $q$ be positive integers.
$\Sigma$ is said to be target path controllable at $k_0$ with lead $p$ and lag $q$ if for any initial state $x(k_0)$ and for any reference output trajectory $y^*[k_0+p, k_0+p+q-1]$ there exists a control sequence $u[k_0, k_0+p+q-2]$ such that $y(k, k_0, x(k_0), u) = y^*(k)$ for all $k_0+p \leq k \leq k_0+p+q-1$.

In the sequel we abbreviate this property by $TPC(k_0; p, q)$.
In case the system is $TPC(k_0; p, \infty)$ we say that the system is global target path controllable at $k_0$ with lead $p$.

Let $t = k + p$. The transfer matrix from $u(t+q-2, k)$ to $y(t+q-1, t)$:

\[
\begin{bmatrix}
C(t+q-1)B(t+q-2) & C(t+q-1)A(t+q-2)B(t+q-3) & \ldots & C(t+q-1)A(t+q-2,k+1)B(k) \\
0 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & C(t)B(t-1) & C(t)A(t-1)B(t-2) & \ldots & C(t)A(t-1,k+1)B(k)
\end{bmatrix}
\]

is denoted by $M(k; p, q)$.
From this matrix a necessary and sufficient condition for target path controllability is easily derived. The result reads as follows.
Lemma 8:
\[ \Sigma \text{ is TPC}(k_0; p, q) \iff \text{rank } M(k_0; p, q) = q \ast r. \]

Proof:
Let \( t_0 \) be equal to \( k_0 + p \).

By definition \( \Sigma \) is target path controllable iff \( y(k) = y^*(k) \) for all \( k \) satisfying \( t_0 \leq k \leq t_0 + q - 1 \). So the question is whether the following set of equations can be solved simultaneously for any \( y^*[t_0, t_0 + q - 1] \)

\[ y^*(k + 1) - C(k + 1) A(k, k_0) x(k_0) = C(k + 1) S[k, k_0] u[k, k_0], \quad t_0 - 1 \leq k \leq t_0 + q - 2. \]

This is possible iff the equation \( y'[t_0 + q - 1, t] = M(k_0; p, q) u[t_0 + q - 2, k_0] \) is solvable for any \( y'[t_0, t_0 + q - 1] \). According to lemma 2 this is equivalent to the requirement that the rank of matrix \( M(k_0; p, q) = q \ast r. \)

Since TPC\((k_0; p, q)\) implies TPC\((k_0; p, s)\) for any \( 0 < s \leq q \) the maximal obtainable lag \( q \) at time \( k_0 \) for a given lead \( p \) exists.

From the rank condition of lemma 8 we deduce immediately that the number of columns of matrix \( M(k_0; p, q) \), \((p + q - 1) \ast m\), must be at least equal to \( q \ast r \). So for a given lead \( p \) an upperbound for the maximal obtainable lag \( q \) at \( k_0 \) results from the inequality \((p - 1) \ast m \geq q \ast (r - m)\).

Conversely, one derives from this inequality for a predefined lag \( q \) a lowerbound for the lead needed to achieve TPC\((k_0; p, q)\). For Tinbergen models, i.e. \( r \leq m \), we see that the upperbound and lowerbound are respectively infinity and one.

To determine for a given lead \( p \) the maximal obtainable lag \( q \) we must check the rank of matrix \( M(k_0; p, s) \) for \( s = 1, 2, \ldots \). We give a recursive algorithm to do this.

The algorithm uses the following recursively defined subspaces.

**Definition:**
Let \( A(k) \) and \( B(k) \) be zero for \( k < k_0 \), and \( t = p + k_0 \).

Then the subspace \( R_1(k_0, p) \) is recursively defined as:

\[ R_1(k_0, p) = \text{Im } S[t - 2, k_0]; \]

\[ R_i(k_0, p) = \text{Ker } C(t + i - 2) \cap (A(t + i - 3) R_{i-1}(k_0, p) + \text{Im } B(t + i - 3)). \]

In the sequel the indices \( k_0 \) and \( p \) are omitted if it is clear from the context which indices are meant.
The next lemma gives a geometric interpretation of these subspaces.

**Lemma 9:**
\[ \mathbb{R}^{q+1} = S[t + q - 2, k_0] \ker M(k_0; p, q), q > 0. \]
Here \( M(k_0; p, 0) \) is defined as 0.

**Proof:**
By induction on \( q \).
For \( q = 0 \) the equality is clear, due to the definition of \( M(k_0; p, 0) \).
Now assume that the statement holds for \( i = q + 1 \).
Then by definition is \( \mathbb{R}^{q+2} = \ker C(t + q) \cap (A(t + q - 1) \mathbb{R}^{q+1} + \text{Im } B(t + q - 1)) \).
Using the induction step and the definition of \( S[k + q, q] \) this subspace can be rewritten as \( \ker C(t + q) \cap S[t + q - 1, k_0] \ker [0 \mid M(k_0; p, q)] \).
Applying lemma 6 yields that this subspace equals \( S[t + q - 1, k_0] \ker M(k_0; p, q + 1) \), which had to be proved.

The algorithm which obtains the maximal lag for a given lead is derived from the next theorem by induction on the lag \( q \).

**Theorem 5:**

Let \( p \) be any positive integer.
Then \( \Sigma \) is TPC \((k_0; p, q) \) iff the following two conditions are satisfied for \( i = t, \ldots, t + q - 1 : \)

i) \( C(i) \) is full row rank;

ii) \( A(i - 1) \mathbb{R}^{i+1} + \text{Im } B(i - 1) \ker C(i) = \mathbb{R}^* \).

**Proof:**
By induction on \( q \).
Let \( q = 1 \).
Then \( \Sigma \) is TPC \((k_0; p, 1) \) iff Im \( C(t) S[t - 1, k_0] = \mathbb{R}^* \). According to corollary 2 this holds iff \( \ker C(t) + \text{Im } S[t - 1, k_0] = \mathbb{R}^* \) and \( C(t) \) is full row rank.
Since \( \text{Im } S[t - 1, k_0] = A(t - 1) \mathbb{R}^1 + \text{Im } B(t - 1) \), this proves the first step.
Now assume that the theorem holds for \( q - 1 \).
In lemma 8 we proved that \( \Sigma \) is TPC \((k_0; p, q) \) iff \( M(k_0; p, q) \) is full rank. Now partition \( M(k_0; p, q) \) as follows:
Then according to corollary 1 $M(k_0; p, q)$ is full row rank iff the following two conditions are satisfied:

i) $\text{Im } C(t + q - 1)B(t + q - 2) + C(t + q - 1)A(t + q - 2)S[t + q - 3, k_0] \text{ Ker } M(k_0; p, q - 1) = \mathbb{R}^r$

ii) $M(k_0; p, q - 1)$ is full row rank.

Now the second condition is satisfied by induction hypothesis, while the first condition is equivalent to

$$C(t + q - 1)S[t + q - 2, k_0] \text{ Ker } [0 \ 1 \ M(k_0; p, q - 1)] = \mathbb{R}^r$$

Like for $q = 1$ we use corollary 2 to reformulate this as:

i) $C(t + q - 1)$ is full row rank and

ii) $\text{Ker } C(t + q - 1) + S[t + q - 2, k_0] \text{ Ker } [0 \ 1 \ M(k_0; p, q - 1)] = \mathbb{R}^n$.

Since ii) can be rewritten as

$$\text{Ker } C(t + q - 1) + \text{Im } B(t + q - 2) + A(t + q - 2)S[t + q - 3, k_0] \text{ Ker } M(k_0; p, q - 1) = \mathbb{R}^n$$

we can apply lemma 9 to obtain the final condition

$$\text{Ker } C(t + q - 1) + \text{Im } B(t + q - 2) + A(t + q - 2)R_q = \mathbb{R}^n$$

Which completes the induction argument.

Application of this theorem to the system

$$x(k + 1) = \begin{bmatrix} A(k) & B(k) \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(k), \quad y(k) = (C(k) \ D(k)) x(k)$$

with $v(k) = u(k + 1)$ yields necessary and sufficient conditions for $\text{TPC}(k_0; p, q)$ for the system

$$x(k + 1) = A(k) x(k) + B(k) u(k)$$

$$y(k) = C(k) x(k) + D(k) u(k).$$

This last system is studied for time-invariant systems extensively in Preston and Pagan [16]. As will be shown later on in this section their main conclusions concerning global TPC are easily rederived from theorem 5.

Another special case is obtained when $C(i)$ is invertible at any point in time. This case was studied by Wohltmann in [26] for time-invariant systems and $C$ equal to the identity matrix. In this paper he states that for this class of systems TPC with a lag greater than one can occur if the number of instrument variables is smaller than the number of output (=state) variables. He
pretends to illustrate this with an example. However, from theorem 5 it is immediately clear that for systems with complete state observation $TPC(k_0; p, q)$ for a lag greater than one occurs iff $\text{Im } B(i) = \mathbb{R}^n$ for $i = t, \ldots, t + q - 1$. So the Tinbergen condition is also a necessary condition to get $TPC$ for this class of systems. Wohltmann's statements concerning this subject in [26] are therefore incorrect.

Before considering the consequences of our algorithm for time-invariant systems we call attention to two algorithms developed for time-invariant systems which bear a great similarity with our algorithm.

One of them is the algorithm developed by Willems in [24], see also [23] for an extension, to calculate the controllable $L_\infty$-almost output nulling subspace. The difference between this algorithm and the one developed here is the initialization. Our algorithm in general starts not with $R_1 = \{0\}$.

Consequently the inclusion property of the algorithm, i.e. $R_{i-1} \subset R_i$, does not hold in our algorithm. Different subspaces will therefore be obtained ultimately. The other algorithm is the algorithm developed in the same papers to calculate the controllable $L_2$-almost output nulling subspace. Though at a first glance our algorithm seems to be essentially different, the following definitions show the contrary.

Define namely $S_1 = A(t - 1) R_1 + \text{Im } B(t - 1)$ and

$$S_{i+1} = A(t + i - 2) [S_i \cap \text{Ker } C(t + i - 1)] + \text{Im } B(t + i - 2).$$

Then it is easily seen by induction that $S_i = A(t + i - 2) R_i + \text{Im } B(t + i - 2)$. Therefore the equivalent condition resulting from theorem 5 for $TPC(k_0; p, q)$ is that $C(t + i - 1) S_i = \mathbb{R}^r$ for $i = 1, \ldots, q$.

So again the conclusion can be drawn that the difference between both algorithms is only slightly, namely the initialization. The advantage of the last algorithm is that the subspace $S_i$ can be interpreted as:

$$S_i = \{ x \in X | \exists \ w \in S_{i-1}, u \in U \text{ such that } A(t + i - 2) w + B(t + i - 2) u = x \text{ and } C(t + i - 1) w = 0 \}$$

(see e.g. Schumacher [20]).

For the derivation of results concerning global $TPC$ is the following lemma important.

**Lemma 10:**

Let $\Sigma$ be time-invariant.

If the lead $p$ is greater than $n$, then for all $i \geq 2$ is $R_i \subset R_{i-1}$. 

**Proof:**

By induction on $i$.

Let $i = 2$. By definition is

$$R_2 = \ker C \cap \text{Im} [B \mid \cdots \mid A^{p-1}B].$$

Due to Cayley Hamilton's theorem this space equals:

$$\ker C \cap \text{Im} [B \mid \cdots \mid A^{p-2}B].$$

Since the last subspace is contained in $R_1$ this proves the lemma for the first step.

Now assume that the lemma holds for $i = k$.

Then $R_{k+1} = \ker C \cap (A R_k + \text{Im} B)$ (definition)

\[ \subseteq \ker C \cap (A R_{k-1} + \text{Im} B) \] (induction argument)

\[ = R_k \] (definition).

So the proof is complete now.

---

From the second part of the proof of this lemma we see that if $R_{k+1} = R_k$ for some $k$ then $R_i = R_k$ for any $i > k$. This property is used in the proof of theorem 6. The theorem states that a time-invariant system is global TPC with a lead $p > n$ iff the system is global TPC with lead $n + 1$. Since Cayley Hamilton's theorem holds, the necessary and sufficient conditions derived in theorem 5 for checking TPC(0, n + 1) reduce to one condition (apart from $C$ being full row rank).

**Theorem 6:**

Let $\Sigma$ be time-invariant and $p > n$.

Then $\Sigma$ is global TPC with lead $p$ iff

i) $C$ is full row rank

ii) $A R_{k+1} + \text{Im} B + \ker C = \mathbb{R}^n$.

**Proof:**

$\Rightarrow$ From the remark stated above this theorem and lemma 10 we observe that either $R_{k+1} = R_n$ or $R_{n+1} = 0$. Therefore the condition $A R_{k+1} + \text{Im} B + \ker C = \mathbb{R}^n$ implies that $A R_i + \text{Im} B + \ker C = \mathbb{R}^n$ for any $i > n$.

Since, due to our assumption on $p$, $R_i$ is contained in $R_{i-1}$ we see that the inclusion $A R_i + \text{Im} B + \ker C \subseteq A R_{i-1} + \text{Im} B + \ker C$ holds. So the range condition $A R_{n+1} + \text{Im} B + \ker C = \mathbb{R}^n$ implies that the range of $A R_i + \text{Im} B + \ker C$ equals $\mathbb{R}^n$ for any $i < n + 1$.

Together with the assumption that matrix $C$ is full row rank we obtain now from theorem 5 that $\Sigma$ is TPC with lead $p$ for any lag.

$\Leftarrow$ This implication results as a special case from theorem 5.
A consequence of this theorem for time-invariant systems is that if \( \Sigma \) is not global TPC with lead \( n + 1 \), it will not be global TPC for any lead. For if the lead \( p > n \) we saw in theorem 6 that TPC(0; \( p, . \)) implies TPC(0; \( n + 1, . \)), while for a lead \( p \leq n \) this implication is trivially satisfied.

So the statement is proved with this by indirect demonstration.

A result derived in the proof of theorem 6 which we like to state separately is that TPC(0; \( p, n + 1 \)) implies TPC(0; \( p, k \)) for any \( k > n \) if \( p > n \).
V. The decoupled TPC problem.

So far we considered the general target path controllability problem. A special case appears if we additionally to this problem require that the \( i \)-th input channel affects only the \( i \)-th output channel, for \( i = 1, \ldots, r \). We call this the decoupled TPC\((k_0; p, q)\) problem. A proper definition is given now.

Definition: 
\( \Sigma \) is called decoupled TPC\((k_0; p, q)\) if there exist compensators \( F(k) \) and \( G(k) \) such that with 
\[ u(k) = F(k) x(k) + G(k) w(k) \]
the transferfunction on the interval \([k_0, k_0 + p + q]\) from \( w \to y \) is zero for \( i \neq j \), and moreover with this choice of control \( \Sigma \) is TPC\((k_0; p, q)\).

Of course this problem only makes sense if the number of outputs equals the number of inputs. 

Note that the transferfunction from \( w \to y \) can always be made diagonal on the interval \([k_0 + p, k_0 + p + q]\). Choose namely 
\[ u[k_0, k_0 + p + q - 1] = M^T(MM^T)^{-1}w[k_0 + p, k_0 + p + q], \]
where \( M = M(k_0; p, q) \).

However, this control makes in general the transferfunction on the whole interval \([k_0, k_0 + p + q]\) not diagonal. To solve this problem, we have to consider the decoupling problem first.

The decoupling problem, as defined here, was solved for time-invariant systems by Falb and Wolovich in [8]. A first attempt to generalize this result to time-varying systems was taken by Porter in [15]. In this paper a sufficient condition for the so called integrator decoupling problem was given. This problem deals with the question under which conditions compensators \( F(k) \) and \( G(k) \) exists such that the closed loop system \((D y)(t) = \Lambda(t) w(t) \) \( t \in \mathbb{R} \) results. Here \( \Lambda \) is a diagonal matrix, and \( D \) is an operator which is defined below. This result was generalized by Tzafestas and Pimenides in [22].

Due to the decoupled inputs influencing the outputs succeeding \((D y)(t)\) and which can not be controlled by \( u(t + 1) \), a solution to the integrator decoupling problem solves in general not the decoupling problem. We illustrate this with an example.

Example 2:
Take 
\[ A(1) = I, B^T(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]
\[ A(2) = I, B^T(2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C(2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]
This system can be integrator decoupled, but not decoupled.

We give now a (strong) condition under which the decoupling problem is solvable. From this condition then immediately a sufficient condition for the solvability of the decoupled TPC problem is obtained. Before we state the result we introduce some notation.

Notation:
In the sequel the \( i \)-th row of matrix \( C(k) \) is denoted by \( c_i(k) \). Using this notation \( a_i(k_0) \) is defined to be the minimum over all \( k > k_0 \) of the following set
\[
\{ k \mid c_i(k)A[k - 1, k_0 + 1]B(k_0) \neq 0 \}, \text{ for } i = 1, \ldots, r.
\]
Under the assumption that all \( a_i(k_0) \) exist (as finite integers) we define \( B^*(k_0) \) to be the matrix which has as its \( i \)-th row entry the row \( c_i(a_i)A[a_i - 1, k_0 + 1]B(k_0) \), for \( i = 1, \ldots, r \). Finally the operator \( (D \ y)(k_0) \) is defined as:
\[
(D \ y)(k_0) := (d_1y_1(k_0), \ldots, d_r y_r(k_0))
\]
\[
:= (y_1(a_1(k_0)), \ldots, y_r(a_r(k_0))).
\]
In the sequel the subscript \( k \) in \( a_i(k) \) is dropped.

Under the assumption that all \( a_i(\cdot) \) exist the following propositions hold.

Proposition 1:
Let \( B^*(k) \) be invertible, and \( d_i(k) = d_i \), \( i = 1, \ldots, r \).

Then \( \Sigma \) can be decoupled.

Proof:
Using the assumptions it is clear that \( \Sigma \) is also described by the following equations:
\[
x(k + 1) = A(k)x(k) + B(k)u(k); \ x(k_0) = x
\]
\[
y_1(k) = c_1(k)A(k - 1, k_0)x, \ k_0 < k < a_1(k_0)
\]
\[
\vdots
\]
\[
y_r(k) = c_r(k)A(k - 1, k_0)x, \ k_0 < k < a_r(k_0)
\]
Applying now the feedback \( u(k) = B^*(k)w(k) - A^*(k)B^*-1(k)x(k) \) yields that
\[
y_i(k) = w_i(k), \quad k > a_i(k_0).
\]
Here \( A^*(k) \) is the matrix which has as its \( i \)-th row entry the row \( c_i(a_i)A(a_i - 1, k) \).

Since \( y_i(k) \) is also not influenced by \( u(k) \) if \( k_0 < k < a_i(k_0) \) this proves the theorem.

From the theorem the following corollary is obvious:

**Corollary 4:**

\( \Sigma \) is decoupled TPC\((k_0; p, q) \) if

i) \( d_i(k) \) is constant on the interval \([k_0; k_0 + p + q] \), \( i = 1, \ldots, r \)

ii) \( B^*(k) \) is invertible on the interval \([k_0; k_0 + p + q] \)

iii) \( \max a_i(k_0) \leq p \).

Note that for time-invariant systems condition i) in the corollary is always satisfied and that the conditions ii) and iii) are then necessary too.
VI. The strongly admissible reference trajectories.

Strongly related to sections III and IV is the question whether a predescribed target path can be tracked exactly or at least ultimately. In the next two sections a characterization is given of all these trajectories.

A first, rather trivial, observation is that these trajectories are totally characterized by $\Sigma$. Though this seems to be a superfluous statement practise proves to be different.

In economics for example often conflicting goals appear. Policy is then mostly designed to make the best out of it on the short term. However, this policy can result in an instable closed loop system as was shown by Engwerda and Otter in [6]. So, despite the fact that a stabilizing policy may exist the intrinsic structure of the system is in such situations totally neglected in obtaining a short term optimal policy.

Important concepts that are used in the next two sections are defined now.

**Definition:**

Let $-\infty < k_0 \leq k < l \leq \infty$, and $x$ the initial condition of $\Sigma$ at $k_0$.

A reference trajectory $y^*[k, l]$ is called strongly admissible for $x$ at $k_0$ if $\exists u[k_0, l - 1]$ such that

$$y(i, k_0, x, u) = y^*(i), \quad k \leq i \leq l .$$

$\epsilon$-almost strongly admissible for $x$ at $k_0$ if $\exists u[k_0, l - 1]$ such that

$$\| y[k, l] - y^*[k, l] \| < \epsilon .$$

A reference trajectory $y^*[k_0, \infty)$ is called admissible (in the large) for $x$ at $k_0$ if $\exists u[k_0, \infty)$ such that $\| y(i) - y^*(i) \| \to 0$ for $i \to \infty$.

The "trivial" proposition reads then as follows.

**Proposition 2:**

A reference trajectory $y^*[l, l + 1]$ is strongly admissible for $x$ iff there exists a $u*[k_0, l]$ such that:

$$y^*(k) = C(k) x^*(k) , \quad t \leq k \leq l + 1 ,$$

$$x^*(k + 1) = A(k) x^*(k) + B(k) u^*(k); \quad x^*(k_0) = x , \quad k_0 \leq k \leq l .$$
Proof:
The sufficiency of the condition is trivial. That the condition is also necessary is seen by the following reasoning. We know that

\[ x(k + 1) = A(k) x(k) + B(k) u(k); x(k_0) = x \]
\[ y(k) = C(k) x(k). \]

So the following equations hold for random \( y^*(k) \):

\[ x(k + 1) = A(k) x(k) + B(k) u(k); x(k_0) = x \]
\[ y(k) - y^*(k) = C(k) x(k) - y^*(k). \]

Consider time \( t \). Let \( y^*(t) \) be strongly admissible. Consequently is \( y(t) - y^*(t) \) zero. So \( y^*(t) = C(t) x^*(t) \) for some \( x^*(t) \) generated by the system.

Since \( x^*(t) = A(t - 1, k_0) x + S[t - 1, k_0] u[k_0, t - 1] \), we conclude that there exists a \( u^*(k_0, t - 1) \) such that \( x(t) = x^*(t) \).

By induction on \( k \) it is now easily verified that the relation holds for any \( t \leq k \leq l \).

Input/output descriptions are sometimes easier to perform calculations with than state space descriptions. For time invariant systems it is always possible to derive from a state space representation an input/output representation. This is among other things due to Cayley Hamilton's theorem again. This property ceases to hold for time-varying systems. The attempt to generalize Cayley Hamilton's theorem in assuming that \( A(k_0 + N, k_0) \) is a linear function of \( A(k_0), \ldots, A(k_0 + N - 1, k_0) \) for any \( N > N_0 \) makes only sense if matrix \( C(k) \) is time-invariant.

So the problem is not trivial. We give here a sufficient condition for the existence of such a relationship.

This condition is reconstructibility.

This concept is defined e.g. by Ludyck in [12], chap. 2.4. In this chapter he also gives a necessary and sufficient condition for reconstructibility. The proof of this conjecture is however incorrect.

Therefore we provide a correct proof. The proof stems from a proof that Willems gives in [25] for reconstructibility of time-invariant systems.

Definition:
A state is reconstructible at \( k_0 \) if there exists a time \( k_0 - N \) such that \( x(k_0) \) is uniquely determined by \( u[k_0 - N, k_0 - 1] \) and \( y[k_0 - N + 1, k_0] \).

\( \Sigma \) is called \( \text{i/o-convertible} \) at \( k_0 \) if there exists an \( N > 0 \) and matrices \( P_k(i), Q_k(i) \) such that for all \( k > k_0 \) \( \Sigma \) is described by the input/output relation:

\[ y(k) = \sum_{i=k_0-N}^{k-1} P_k(i) y(i) + Q_k(i) u(i). \]
Proposition 3:

$\Sigma$ is reconstructible at $k_0$ iff there exists a positive integer $N$ such that $\text{Ker } W^T[k_0 - N, k_0] \subset \text{Ker } A(k_0 - 1, k_0 - N)$ (for the definition of $W$ see section 2).

Proof:

First note that $x(k_0) = A(k_0 - 1, k_0 - N)x(k_0 - N) + S[k_0 - N, k_0 - 1]u[k_0 - N, k_0 - 1]$ and $y(k) = C(k)A(k - 1, k_0 - N)x(k_0 - N) + C(k)S[k_0 - N, k - 1]u[k_0 - N, k - 1], \quad k = k_0 - N, ..., k_0 - 1.$

By definition is $\Sigma$ reconstructible at $k_0$ iff from the past observations $v(k) = C(k + 1)A(k, k_0 - N)x(k_0 - N)$ can be determined uniquely.

This is not possible for all states at $k_0$ iff there exist two states $x'(k_0 - N)$ and $x(k_0 - N)$ such that

$$C(k + 1)A(k, k_0 - N)x'(k_0 - N) = C(k + 1)A(k, k_0 - N)x(k_0 - N)$$

and $x'(k_0 - N) - x(k_0 - N) \notin \text{Ker } A(k_0 - 1, k_0 - N).$

Or equivalently $x'(k_0 - N) - x(k_0 - N) \notin \text{Ker } W[k_0 - N, k_0].$

Since $A(k_0 - 1, k_0 - N)x(k_0 - N)$ is uniquely determined by $v(k), k_0 - N \leq k \leq k_0 - 1,$ we can rewrite $A(k_0 - 1, k_0 - N)x(k_0 - N)$ as $[X(k_0 - N) \ | \cdots \ | X(k_0 - 1)]v[k_0 - N, k_0 - 1]$ for some matrix $[X(k_0 - N) \ | \cdots \ | X(k_0 - 1)].$ A direct consequence is

Proposition 4:

$\Sigma$ is i/o convertible at $k_0$ if $\Sigma$ is reconstructible at $k_0$.

The input/output relation is given by

$$y(k) = C(k)A(k - 1, k_0)([X(k_0 - N) \ | \cdots \ | X(k_0 - 1)]v[k_0 - N, k_0 - 1] +$$

$$+ S[k_0 - N, k_0 - 1]u[k_0 - N, k_0 - 1]) + C(k)S[k_0 - N, k_0 - 1]u[k_0 - N, k_0 - 1].$$

Since this input/output relation depends severely on the initial state $x(k_0)$ of the system we give sufficient conditions under which a dynamic input/output relation (i.e. a relation which is independent of the initial state of the system) is obtained. This is the contents of the next theorem.
Theorem 7:
Assume that there exists an $N > 0$ such that $W[k, k + N]$ is full column rank for any $k$. Then the following input/output relation holds for $\Sigma$:

$$y(k + N + 1) = C(k + N + 1)(W^T[k, k + N] W[k, k + N])^{-1}W^T[k, k + N] *$$

$$* \left[ y[k, k + N] - \begin{bmatrix} 0 \\ M(k; 1, n) u[k + N - 1, k] \end{bmatrix} \right]$$

Proof:
Obvious from the consideration that $y[k, N + k] = W[k, N + k] x(k) +$

$$+ \begin{bmatrix} 0 \\ M(k; 1, N - 1) u[k + N - 1, k] \end{bmatrix} \text{ for any } k .$$

Note that $W^T[k + 1, N + k + 1] W[k + 1, N + k + 1] = W^T[k, N + k] W[k, N + k] +$

$$+ A^T(k + N - 1, k) C^T(k + N) C(k + N) A(k + N - 1, k) - A^T(k) C^T(k + 1) C(k + 1) A(k).$$

From this equality we obtain easily a recursion formula for the calculation of

$$\{W^T[k, k + N] W[k, k + N]\}^{-1}.$$

In the previous part of this section we characterized all strongly admissible trajectories. However, the question how right away the strong admissibility of a prespecified reference trajectory can be checked was not answered. Moreover the question whether there exists an $\varepsilon$ such that the trajectory is $\varepsilon$-almost strongly admissible when it is not strong admissible arises. These questions are answered in the final theorems of this section.

Theorem 8:
A reference trajectory $y^*[k_0 + p, k_0 + p + q - 1]$ is strongly admissible at $k_0$ iff

$$\text{rank } [M(k_0; p, q) M^{+}(k_0; p, q) - I] z[k_0 + p, k_0 + p + q - 1] = \text{rank } M(k_0; p, q).$$

Here $z(i) = y^*(i) - C(i) A(i - 1, k_0) x(k_0)$.

Proof:
Obvious from lemma 2.}

Theorem 9:
A reference trajectory $y^*[k_0 + p, k_0 + p + q - 1]$ is $\varepsilon$-almost strongly admissible iff

$$\| (M(k_0; p, q) M^{+}(k_0; p, q) - I) z[k_0 + p, k_0 + p + q - 1] \| < \varepsilon .$$
Here $M^*(k_0; p, q)$ is the Moore Penrose inverse of $M(k_0; p, q)$ (see Lancaster [12], chapter 12.8), and $z$ is as defined in the previous theorem 8.

**Proof:**

Using the definition of $z$ from theorem 8 we write $z[k_0 + p, k_0 + p + q - 1]$ as $M(k_0; p, q) u[k_0, k_0 + p + q - 2] + y^*[k_0 + p, k_0 + p + q - 1] - y[k_0 + p, k_0 + p + q - 1]$. So

$$\| z[k_0 + p, k_0 + p + q - 1] - y^*[k_0 + p, k_0 + p + q - 1] \| < \varepsilon$$

iff

$$\| z[k_0 + p, k_0 + p + q - 1] - M(k_0; p, q) u[k_0, k_0 + p + q - 2] \| < \varepsilon$$

But

$$\min_u \| M(k_0; p, q) u[k_0, k_0 + p + q - 2] - z[k_0 + p, k_0 + p + q - 1] \|$$

is obtained by the least squares approximation $u[k_0, k_0 + p + q - 2] = M^*(k_0; p, q) z[k_0 + p, k_0 + p + q - 1]$ (see Lancaster [12], pp.436).

This proves the result.

From theorem 9 it is clear that for a given lead $p$ the minimal $\varepsilon$ for which a trajectory is $\varepsilon$-almost strongly admissible exists.

This $\varepsilon$ is given by (2) where the inequality has to be replaced by an equality.
VII. The reference trajectories admissible in the large

In section V we derived a criterion to check whether a certain reference trajectory could be tracked exactly or not.

Moreover an exact characterization was given how a reference trajectory has to be generated in order to be strongly admissible.

We shall now treat the problem of tracking a reference trajectory asymptotically. To tackle this problem, we assume that the input is chosen as a mixture of static/dynamic, state/output feedback.

That is:

\[ u(k) = E(k)w(k) + F(k)x(k) + H(k)z(k) + D(k)y(k) + g(k) \]

where

\[ w(k + 1) = M(k)w(k) + N(k)x(k) \]
\[ z(k + 1) = P(k)z(k) + Q(k)y(k) \]

Then, for random \( u^*(k), u^*(k - 1), w^*(k + 1), w^*(k), z^*(k + 1), z^*(k), x^*(k), y^*(k), y^*(k + 1) \) the following closed loop system results

\[
\begin{bmatrix}
I & 0 & 0 & -B(k) & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & -Q(k) \\
0 & 0 & 0 & I & -D(k) \\
0 & 0 & 0 & 0 & I \\
\end{bmatrix}
\begin{bmatrix}
A(k) & 0 & 0 & 0 & 0 \\
N(k) & M(k) & 0 & 0 & 0 \\
0 & 0 & P(k) & 0 & 0 \\
0 & 0 & 0 & F(k) & E(k) \\
0 & 0 & 0 & 0 & C(k) \\
\end{bmatrix}
\begin{bmatrix}
e(k) + 1 \\
e(k) \\
e(k) + 1 \\
e(k) + 1 \\
e(k) + 1 \\
\end{bmatrix}
\]

This error equation can be rewritten as

\[ S(k)e(k + 1) = A(k)e(k) + v(k). \]

To give some more insight in the properties of an admissible trajectory the next theorem, which immediately results from lemma 1, is stated.
Theorem 10:
An admissible reference trajectory is generated as follows:

\[ x^*(k + 1) = A(k)x^*(k) + B(k)u^*(k) + v_1(k) \]
\[ y^*(k) = C(k)x^*(k) + v_2(k) \]

with \( v_i(k) \to 0 \) when \( k \) tends to infinity.

This condition is also sufficient if the input stabilizes the system.

Using theorem 7, theorem 10 can be reformulated in the following way: a necessary condition for a reference trajectory to be admissible is that it is generated in the limit by the same input/output recurrence relation as the system \( \Sigma \).

The exact characterization for admissibility of a reference trajectory reads as follows.

Theorem 11:
A reference trajectory is admissible at \( k_0 \) iff there exist \( v_i[k_0, \ldots] \), \( i = 1, \ldots, 5 \), such that the following conditions are met:

i) \( e(k_0) := [(x(k_0)-x^*(k_0)^T, (w(k_0)-w^*(k_0))^T, (z(k_0)-z^*(k_0))^T] \) is stabilized by means of \( v_i[k_0, \ldots], i = 1, \ldots, 5 \), in the linear system

\[ e(k + 1) = \begin{bmatrix} A(k) + B(k)F(k) & B(k)E(k) & B(k)H(k) \\ N(k) & M(k) & 0 \\ Q(k)C(k) & 0 & P(k) \end{bmatrix} e(k) + \begin{bmatrix} v_1(k) + B(k)v_4(k) \\ v_2(k) \\ v_3(k) + Q(k)v_5(k) \end{bmatrix} \]

ii) \( x^*(k + 1) = A(k)x^*(k) + B(k)u^*(k) + v_1(k) \)
\( w^*(k + 1) = M(k)w^*(k) + N(k)x^*(k) + v_2(k) \)
\( z^*(k + 1) = P(k)z^*(k) + Q(k)y^*(k) + v_3(k) \)
\( g(k) = -F(k)x^*(k) - E(k)w^*(k) - H(k)z^*(k) + v_4(k) \)
\( y^*(k) = C(k)x^*(k) + v_5(k) \)

where \( v_i(k) \to 0 \) if \( k \) tends to infinity.

Proof:
"\( \Rightarrow \)" From lemma 1 we know that it is necessary for convergence of \( e(\cdot) \) in \( S(k)e(k + 1) = A'(k)e(k) + v(\cdot) \) that \( S^{-1}(k)v(\cdot) \) converges to zero.

Consequently \( v(\cdot) \) has to converge to zero. This implies the second condition in the theorem.

Straightforward multiplication shows that \( S^{-1}(k)A'(k) \) equals \( \begin{bmatrix} T(k) & 0 \\ W(k) & 0 \end{bmatrix} \) for some matrices \( T \) and \( W \).
Using this structure of the error equation matrix, the first condition of the theorem results now immediately too.

"⇐" By making use again of the structure of the error equation matrix as derived above this part is easily proved.

We conclude this section with an illustration of theorem 10.

In Engwerda [7], the infinite time quadratic tracking problem was solved under some conditions. As a special case the following problem was treated:

\[
\min_{u(\{0\} \to \infty} \lim_{N \to \infty} \sum_{k=0}^{N-1} \{(y(k)-y^*(k))^T \mathbf{Q} (y(k)-y^*(k)) + (u(k)-u^*(k))^T \mathbf{R} (u(k)-u^*(k))\}
\]

\[+ (y(N)-y^*(N))^T \mathbf{Q} (y(N)-y^*(N)),\]

where \( \mathbf{Q} \) and \( \mathbf{R} \) are positive definite symmetric matrices and \( y(k+1) \) is given by the system equation \( y(k+1) = Ay(k) + Bu(k) + Gd(k) \).

Here \( Gd(k) \) is a known exogenous noise component.

The optimal solution to this problem turned out to be

\[
u(i) = -(\mathbf{R} + B^T KB)^{-2} B^T (KAy(i) + KGd(i) - h(i+1) - B (B^T B)^{-2} Ru^*(i))
\]

where \( K \) is the positive definite solution of the Algebraic Riccati Equation

\[K = A^T [K - KB (\mathbf{R} + B^T KB)^{-1} B^T ] A + \mathbf{Q}, \]

and \( h(i) \) is given by the recurrence equation

\[
h(1) = \sum_{k=1}^{\infty} \{(A - BH)^T \}^{k-1} \{(Qy^*(k) - (RH)^T u^*(k) - (A - BH)^T KGd(k))\}
\]

\[h(i+1) = ((A - BH)^T)^{-1} \{h(i) - Qy^*(i) + (RH)^T u^*(i)\} + KGd(i).
\]

Here \( H \) denotes the matrix \( (\mathbf{R} + B^T KB)^{-1} B^T KA \).

In this paper was proved that this control stabilizes the closed loop system. So, application of theorem 10 yields now that the error \((y(k) - y^*(k))^T, (u(k) - u^*(k))^T\) converges to zero if and only if the following vector converges to zero when \( k \) tends to infinity:

\[
\begin{bmatrix}
v_1(k) \\
v_2(k)
\end{bmatrix} = \begin{bmatrix}
Ay^*(k) + Bu^*(k) + Gd(k) - y^*(k+1) \\
(R + B^T KB)^{-2} B^T (h(k+1) - KAy^*(k) - KGd(k) + B (B^T B)^{-2} Ru^*(k)) - u^*(k)
\end{bmatrix}
\]

Now substitution of \( v_1(k) \) into \( v_2(k) \) yields:

\[
v_2(k) = -(R + B^T KB)^{-2} B^T (K_y^*(k+1) - h(k+1) - v_1(k)).
\]

So we conclude that \( v_2(k) \) converges to zero if and only if \( B^T (K_y^*(k+1) - h(k+1)) \) converges to zero.
Summarizing, we have the following result:

**Theorem 12:**
A reference trajectory is admissible in the infinite time quadratic tracking problem iff the following two conditions hold for the reference trajectory:

i) \( y^*(k + 1) - Ay^*(k) - Bu^*(k) - Gd(k) \rightarrow 0 \) when \( k \rightarrow \infty \).

ii) \( B^T(Ky^*(k + 1) - h(k + 1)) \rightarrow 0 \) when \( k \rightarrow \infty \).
VIII. Conclusions.

In this paper we treat various target point and target path problems for time-varying linear discrete time systems.

For some of the target point problems new elementary proofs are given, while for the output-controllability and stabilizability problem new results are obtained.

For the general target path controllability problem a geometric recursive algorithm is derived to check the solvability of the problem. This algorithm appears to bear a great similarity with other algorithms derived in literature to calculate controllable almost output nulling subspaces for time-invariant systems.

An advantage of this algorithm is that easily (existing) results for time-invariant system concerning the TPC problem are reobtained. For a special problem, the decoupled TPC problem, a sufficient condition is given for solvability.

Finally a characterization is given of the trajectories which can be (almost or ultimately) tracked. A sufficient condition is given for obtaining a dynamic input/output relation from the state space representation. Furthermore a method to check the (almost) strong admissibility of a prespecified reference trajectory is given.
References


[7] Engwerda J.C., The solution of the infinite time reference trajectory tracking problem for discrete-time systems possessing an exogenous component; Memorandum Cosor 86-08, 1986, Eindhoven University of Technology


[25] Willems J.C., Dictaat Systeemtheorie, lecture notes course 1980 chap.4; University of Groningen


[27] Wohltmann H.W., Target path controllability of linear time-varying dynamical systems; I.E.E.E. AC-30, 1985, pp.84-87