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Citation for published version (APA):

Sevat, M. F., Houben, S. H. M. J., & Maten, ter, E. J. W. (2005). *DRK methods for time-domain oscillator simulation*. (CASA-report; Vol. 0514). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/2005

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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CASA-Report 05-14
March 2005

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by

M.F. Sevat, S.H.M.J. Houben, E.J.W. ter Maten

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

DRK methods for time-domain oscillator simulation

M.F. Sevat¹, S.H.M.J. Houben², and E.J.W. ter Maten³

¹ Philips Research Laboratories, Eindhoven marcel.sevat@philips.com

² Magma Design-Automation, Eindhoven

³ Philips Research Laboratories and Eindhoven University of Technology

Summary. This paper presents a new Runge-Kutta type integration method that is well-suited for time-domain simulation of oscillators. A unique property of the new method is that its damping characteristics can be controlled by a continuous parameter.

Introduction

In the case of weakly non-linear circuit behaviour, oscillators can be simulated in the frequency domain, using e.g. the Harmonic Balance method. In the case of strongly non-linear circuit behaviour, they are to be simulated in the time domain, using e.g. the BDF-methods or the Trapezoidal Rule (TR). If the start-up behaviour of an oscillator is to be observed, a time domain method is even mandatory. However, the BDF methods exert a considerable damping on an oscillatory solution of the circuit equations. The TR method, when used on oscillators, does not exert any damping at all for all frequencies (which is also not wanted). To remedy this situation, an integration method would be preferred that has some damping to avoid numerical instability, but still so small that its effect on the oscillation can be neglected. In the next sections, DRK methods will be investigated as potential candidates for such methods.

1 DRK methods

We apply a Diagonal Runge-Kutta (DRK) method to a general DAE of the form

$$g(t, \dot{x}, x) = 0. \quad (1)$$

Given a step-size h and an initial value x_0 , the DRK method computes a sequence $\{x_n\}$, where x_n is an approximation to the solution at $t = nh$. Given (a_{ii}) and (b_i) , x_{n+1} is computed from x_n as follows

$$g\left(t_n + ha_{ii}, X_n^{(i)}, x_n + ha_{ii}X_n^{(i)}\right) = 0, \quad (2a)$$

$$x_{n+1} = x_n + h \sum_{i=1}^s b_i X_n^{(i)}. \quad (2b)$$

The quantities $X_n^{(i)}$ are called the stages of the DRK method, and s is called the stage count. Note that (2a) constitutes an Implicit Euler step with step-size ha_{ii} . So in essence, a DRK method is a linear combination of Implicit Euler steps.

1.1 Order conditions

For the ODE $\dot{x} = f(x)$, $f \in C^\infty(\mathbb{R}, \mathbb{R})$ we have $g(t, \dot{x}, x) = \dot{x} - f(x) = 0$, leading to the following DRK procedure:

$$X_n^{(i)} = f(x_n + ha_{ii}X_n^{(i)}), \quad \text{for } i = 1 \dots s, \quad (3a)$$

$$x_{n+1} = x_n + h \sum_{i=1}^s b_i X_n^{(i)}. \quad (3b)$$

The order conditions up to order k are now found by equating, for arbitrary f , the following terms at the point $h = 0$ (see [2]):

$$\frac{d^j x_{n+1}}{dh^j} = \frac{d^j x(t_n + h)}{dh^j}, \quad \text{for } j = 0 \dots k. \quad (4)$$

In [4] it is shown that this is only possible up to order 2. Then, the following order conditions should be satisfied:

$$\sum_{i=1}^s b_i = 1, \quad \sum_{i=1}^s b_i a_{ii} = \frac{1}{2}. \quad (5)$$

The fact that DRK methods are limited to such low orders, appears to make them quite unappealing. In contrast to the common approach in Runge-Kutta theory, as presented in e.g. [2], we do not aim for maximising of the order of the method. Rather we balance the desire for a high order against the goal of obtaining a method that does not damp out oscillations.

1.2 Stability conditions

To study stability we apply the DRK-methods to the Dahlquist test equation

$$\dot{x} = \lambda x, \quad \lambda \in \mathbb{C}$$

Let x_{n+1} be computed from x_n with the DRK method using stepsize $h > 0$. Then

$$\zeta(h\lambda) := \frac{x_{n+1}}{x_n} = 1 + \sum_{i=1}^s \frac{b_i h\lambda}{1 - a_{ii} h\lambda} \quad (6)$$

defines the amplification factor. For a DRK method to be usable as an integration method for oscillator problems, $\zeta(h\lambda)$ should satisfy the following conditions:

$$|\zeta(j\omega)| \lesssim 1, \quad \omega \in \mathbb{R}, \quad (7a)$$

$$|\zeta(z)| < 1 \text{ for } \operatorname{Re}(z) < 0, \quad (7b)$$

$$\lim_{\operatorname{Re}(z) \rightarrow -\infty} \zeta(z) = 0, \quad (7c)$$

It can be shown that, once condition (7a) is satisfied, and $\zeta(z)$ is analytic in the left half of the complex z -plane, then also condition (7b) is satisfied (see [3]). Assuming for the moment that condition (7a) is satisfied, then to also satisfy condition (7b) the poles of $\zeta(z)$ need to be in the right half of the complex z -plane. This leads to the following restriction on the coefficients a_{ii} :

$$a_{ii} > 0 \quad \text{for } i = 1, \dots, s \quad (8)$$

Note that even with this restriction satisfied, we still need to check on any proposed set of coefficients whether (7a) is satisfied. Applying condition (7c) to (6) leads to:

$$\sum_{i=1}^s \frac{b_i}{a_{ii}} = 1 \quad (9)$$

which embodies another restriction on the DRK coefficients.

2 Two-stage example

To have a DRK method suitable for use in oscillator simulation, the coefficients a_{ii} and b_i should satisfy order and stability conditions as derived in the preceding sections. For two stages already solutions with one degree of freedom exist. For this particular case, the set of equations to be solved is:

$$b_1 + b_2 = 1, \quad b_1 a_{11} + b_2 a_{22} = \frac{1}{2}, \quad \frac{b_1}{a_{11}} + \frac{b_2}{a_{22}} = 1, \quad (10a)$$

$$a_{11} > 0, \quad a_{22} > 0. \quad (10b)$$

In the sequel, we denote $\gamma := a_{22}$ as the degree of freedom. We then find the following solution to (10):

$$b_1 = \frac{2\gamma^2 - 3\gamma + 1}{2\gamma^2 - 4\gamma + 1}, \quad b_2 = \frac{-\gamma}{2\gamma^2 - 4\gamma + 1}, \quad a_{11} = \frac{2\gamma - 1}{2\gamma - 2}, \quad a_{22} = \gamma, \quad (11a)$$

$$\gamma \in (0, \frac{1}{2}) \cup (1, \infty), \quad \gamma \neq \frac{1}{2 \pm \sqrt{2}}. \quad (11b)$$

It remains to be checked if this set of coefficients satisfies the condition (7a). To that end, we investigate

$$\zeta(j\omega) = \frac{b_1/a_{11}}{1 - ja_{11}\omega} + \frac{b_2/a_{22}}{1 - ja_{22}\omega}, \quad (12)$$

from which we find (see [4])

$$|\zeta(j\omega)|^2 = 1 - \frac{\omega^4 \gamma^2 (1 - 2\gamma)^2}{(1 + \gamma^2 \omega^2)[4(1 - \gamma)^2 + \omega^2(1 - 2\gamma)^2]}, \quad (13)$$

showing that $|\zeta(j\omega)| \leq 1$, thereby satisfying condition (7a). Note that by making γ small enough, $|\zeta(j\omega)|$ can be brought as close to 1 as we want. For $\gamma \rightarrow 0$ the method approaches the midpoint rule.

The stability diagrams for various values of γ are shown in Figure 1. This figure clearly illustrates the fact that γ can be used to control the amount of damping on the imaginary axis. Furthermore, it shows that by sufficiently decreasing the value of γ the amount of damping can be brought as close to zero as we require.

3 Alternative formulation

Using the transformation $\widetilde{\mathbf{X}}_n^{(i)} = \mathbf{x}_n + ha_{ii}\mathbf{X}_n^{(i)}$ for $i = 1, \dots, s$ and assuming that (9) holds, we obtain the following alternative formulation of the DRK method:

$$g \left(t_n + ha_{ii}, \frac{\widetilde{\mathbf{X}}_n^{(i)} - \mathbf{x}_n}{ha_{ii}}, \widetilde{\mathbf{X}}_n^{(i)} \right) = 0, \quad (14a)$$

$$\mathbf{x}_{n+1} = \sum_{i=1}^s \frac{b_i}{a_{ii}} \widetilde{\mathbf{X}}_n^{(i)}, \quad (14b)$$

The numerical robustness of this alternative formulation is better than the one of the standard formulation, as it avoids the summation of relatively small quantities to the current approximation in the update equation (see (2b)). It thereby circumvents the unnecessary loss of accuracy. For the two-stage case, considered in the previous section, the coefficients (11) satisfy condition (9). So (14) holds for this case, with the following expressions for its coefficients:

$$\frac{b_1}{a_{11}} = \frac{2(\gamma - 1)^2}{2\gamma^2 - 4\gamma + 1}, \quad \frac{b_2}{a_{22}} = \frac{-1}{2\gamma^2 - 4\gamma + 1}, \quad (15)$$

with the coefficients a_{11} , a_{22} and the restrictions on γ the same as in (11).

4 Conclusions

We developed a Diagonal Runge-Kutta algorithm that is particularly suited for transient simulation of oscillators, in the sense that it does not damp out any oscillation present in the solution of the circuit equations. In fact it has been shown that its damping characteristics can be controlled by a dedicated parameter. The new algorithm allows designers to better simulate oscillators, or to detect unwanted oscillation earlier than would be the case with standard integration methods.

References

1. E. Griepentrog and R. März. *Differential-Algebraic Equations and Their Numerical Treatment*. Teubner, 1986.
2. E. Hairer, S.P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I, Non-stiff problems*. Second edition, Springer, 1993.
3. E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II, Stiff and Differential Algebraic problems*. Second edition, Springer, 1996.
4. S.H.M.J. Houben *Circuits in Motion*. PhD thesis, Eindhoven University of Technology, 2003.

Fig. 1. Stability diagram of the DRK method in the complex $h\lambda$ -plane.