Propagation law for the generating function of Hermite-Gaussian-type modes in first-order optical systems

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Abstract: Based on the common Hermite-Gaussian modes, a general class of orthonormal sets of Hermite-Gaussian-type modes is introduced. Such modes can most easily be defined by means of their generating function. It is shown that these modes remain in their class of orthonormal Hermite-Gaussian-type modes, when they propagate through first-order optical systems. A propagation law for the generating function is formulated.

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1. Introduction

In this paper we introduce a general class of orthonormal sets of Hermite-Gaussian type modes, by generalizing the quadratic form that arises in the generating function of the common Hermite-Gaussian modes. We study how these modes propagate through first-order optical systems and express the generating function of the set of output modes in terms of the generating function of the set of input modes.

References and links

1. M. Abramowitz and I. A. Stegun, eds., Pocketbook of Mathematical Functions (Deutsch, Frankfurt am Main, Germany, 1984).
The requirement of orthonormality yields some additional conditions for the quadratic form of the generating function. As a result of that, we will be able to express the elements of this quadratic form in terms of four matrices that can be combined into a symplectic matrix.

The main result of this paper is that this symplectic matrix propagates through a first-order optical system by a mere multiplication with the system’s ray transformation matrix. From this simple propagation law we can easily derive how different members from the class of the Hermite-Gaussian-type modes (such as the common Hermite-Gaussian and Laguerre-Gaussian modes) can be converted into each other.

2. Hermite-Gaussian-type modes

The complex field amplitude of the common Hermite-Gaussian modes takes the form

\[ \mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y) = \mathcal{H}_n(x; w_x) \mathcal{H}_m(y; w_y) \]  

with

\[ \mathcal{H}_n(x; w) = 2^{1/4} (2^n n! w)^{-1/2} H_n \left( \sqrt{2} \pi x/w \right) \exp \left( -\pi x^2/w^2 \right), \]

where \( H_n(\cdot) \) are the Hermite polynomials [1, Section 22] and where the column vector \( \mathbf{r} = (x, y)^t \) is a short-hand notation for the spatial variables \( x \) and \( y \), with the superscript \(^t\) denoting transposition. Note that \( \mathcal{H}_n(x; w) \) has been defined such that we have the orthonormality relationship

\[ \int \mathcal{H}_n(x; w) \mathcal{H}_m(x; w) \, dx = \delta_{nm}, \]

where \( \delta_{nm} \) is the Kronecker delta. (All integrals in this paper extend from \(-\infty\) to \(+\infty\).) From the generating function of the Hermite polynomials [1, Eq. (22.9.17)],

\[ \exp \left( -s^2 + 2sz \right) = \sum_{n=0}^{\infty} H_n(z) \frac{s^n}{n!}, \]

we can easily find the generating function of the Hermite-Gaussian modes \( \mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y) \):

\[ 2^{1/2} (w_x w_y)^{-1/2} \exp \left[ -\left( s_x^2 + s_y^2 \right) + 2 \sqrt{2} \pi s_x (s_x/w_x + s_y y/w_y) - \pi (s_x^2/w_x^2 + y^2/w_y^2) \right] \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y) \left( \frac{2^{n+m}}{n! m!} \right)^{1/2} s_x^n s_y^m. \]  

The general class of orthonormal sets of Hermite-Gaussian-type modes \( \mathcal{H}_{n,m}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \) that we propose, is most easily defined by the generating function

\[ 2^{1/2} (\det \mathbf{K})^{1/2} \exp \left( -s^T \mathbf{M} s + 2 \sqrt{2} \pi s^T \mathbf{K} \mathbf{r} - \pi s^T \mathbf{L} \mathbf{r} \right) \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{H}_{n,m}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \left( \frac{2^{n+m}}{n! m!} \right)^{1/2} s_x^n s_y^m, \]

cf. Eq. (5), where we have introduced the column vector \( s = (s_x, s_y)^t \) and three (possibly complex) \( 2 \times 2 \)-matrices \( \mathbf{K}, \mathbf{L} = \mathbf{L}^t \), and \( \mathbf{M} = \mathbf{M}^t \). For the common Hermite-Gaussian modes \( \mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y) \) we have, see Eq. (5),

\[ \mathbf{K} = \begin{pmatrix} w_x & 0 \\ 0 & w_y \end{pmatrix}^{-1} = \mathbf{W}^{-1}, \quad \mathbf{L} = \mathbf{W}^{-2}, \quad \mathbf{M} = \mathbf{L}. \]

We will observe later that the matrix \( \mathbf{M} \) is completely determined by \( \mathbf{K} \), see Eq. (19), and therefore does not have to be included as a parameter in \( \mathcal{H}_{n,m}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \).
3. Propagation through first-order optical systems

We let Hermite-Gaussian-type modes \( \mathcal{H}_{n,m}(r; K_L) \) propagate through a lossless, first-order optical system – also called an ABCD-system – and determine the generating function of the set of modes to which the beam that appears at the output of this system belongs. Any lossless, first-order optical system can be described by its ray transformation matrix [2], which relates the position \( r_i \) and direction \( q_i \) of an incoming ray to the position \( r_o \) and direction \( q_o \) of the outgoing ray:

\[
\begin{pmatrix}
  r_o \\
  q_o
\end{pmatrix} =
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\begin{pmatrix}
  r_i \\
  q_i
\end{pmatrix}.
\]

The ray transformation matrix is real and symplectic, yielding the relations

\[
AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I,
\]

\[
A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I.
\]

Using the matrices \( A, B, \) and \( D \), and assuming that \( B \) is a non-singular matrix, we can represent the first-order optical system by the Collins integral [3]

\[
f_o(r_o) = \frac{\exp(i\phi)}{\sqrt{\det BD}} \int f_i(r_i) \exp \left[ i\pi \left( r_i^T B^{-1} A r_i - 2 r_i^T B^{-1} r_o + r_o^T D B^{-1} r_o \right) \right] \, dr_i,
\]

where the output amplitude \( f_o(r) \) is expressed in terms of the input amplitude \( f_i(r) \). The phase factor \( \exp(i\phi) \) in Eq. (10) is rather irrelevant and can often be chosen arbitrarily. We remark that the signal transformation \( f_i(r) \rightarrow f_o(r) \) that corresponds to a lossless, first-order optical system, is unitary, i.e.

\[
\int f_{i,1}(r) f_{o,1}^*(r) \, dr = \int f_{o,1}(r) f_{o,1}^*(r) \, dr,
\]

where \( * \) denotes complex conjugation.

With a Hermite-Gaussian-type mode \( \mathcal{H}_{n,m}(r; K_L) \) at the input of an ABCD-system, we denote the output beam by \( \mathcal{H}_{n,m}(r; K_o, L_o) \). To find the generating function of the set of modes to which this output beam belongs, we write

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{H}_{n,m}(r; K_o, L_o) \left( \frac{2n+m}{n!m!} \right)^{1/2} \exp(i\phi) \sqrt{\det BD} \int f_i(r_i) \exp \left[ i\pi \left( r_i^T B^{-1} A r_i - 2 r_i^T B^{-1} r_o + r_o^T D B^{-1} r_o \right) \right] \, dr_i,
\]

where Collins integral (10) has been used. After substituting from the generating function (6) and proceeding along the same lines as in [4], we get for the generating function

\[
\exp(i\phi) 2^{1/2} \left[ \frac{\det K_i}{\det BD} \frac{\det(L_i - iB^{-1} A)}{\det BD} \right]^{1/2} \exp \left( -s^T M_o s + i\pi r_o^T D B^{-1} r_o \right) \times \exp \left[ -\pi \left( B^{-1} r_o + i\sqrt{2/\pi} K_o s \right) \right]
\]

\[
= \exp(i\phi) 2^{1/2} (\det K_o)^{1/2} \exp \left( -s^T M_o s + 2 \sqrt{2/\pi} s^T K_o r_o - \pi r_o^T L_o r_o \right)
\]

with

\[
K_i = K_i(A + B/L_o)^{-1},
\]

\[
i L_o = (C + D L_o)(A + B/L_o)^{-1},
\]

\[
M_o = M_o - 2i K_i(A + B/L_o)^{-1} B K_i.
\]

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We conclude that the generating function (6) keeps its form when the associated Hermite-Gaussian-type modes propagate through a first-order optical system; we only have to replace the input matrices \( \mathbf{K}_i, \mathbf{L}_i, \) and \( \mathbf{M}_i \) by the output matrices \( \mathbf{K}_o, \mathbf{L}_o, \) and \( \mathbf{M}_o, \) respectively, in accordance with the input-output relationships (12–14). Note that Eq. (13) is in fact the well-known \( \text{ABCD} \)-law, and that Eqs. (12) and (13) can be combined into

\[
\left( \frac{\mathbf{I}}{i\mathbf{L}_o} \right) \mathbf{K}_o^{-1} = \left( \begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array} \right) \left( \frac{\mathbf{I}}{i\mathbf{L}_i} \right) \mathbf{K}_i^{-1}.
\]

(15)

4. Conditions resulting from orthonormality

From the requirement that the Hermite-Gaussian-type modes are orthonormal,

\[
\int \mathcal{H}_{n,m}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \mathcal{H}_{l,k}^{*}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \, d\mathbf{r} = \delta_{nl} \delta_{mk},
\]

(16)

we get additional conditions for the three matrices \( \mathbf{K}, \mathbf{L}, \) and \( \mathbf{M}. \) To derive these, we consider the expression

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2n+m)!}{n!m!} \frac{1}{t^{l+k}} \int \mathcal{H}_{n,m}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \mathcal{H}_{l,k}^{*}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \, d\mathbf{r}
\]

\[
= 2 |\text{det} \mathbf{K}| \exp(-s^t \mathbf{M} s - t^t \mathbf{M}^t t) \int \exp \left[ 2\sqrt{2\pi} (s^t \mathbf{K} + t^t \mathbf{K}^*) \mathbf{r} - \pi r^t (\mathbf{L} + \mathbf{L}^*) \mathbf{r} \right] \, d\mathbf{r},
\]

where \( t = (t_x, t_y)^t \) on the analogy of \( s = (s_x, s_y)^t, \) and where we have substituted from the generating function (6). We note that the integral in this expression equals

\[
|\text{det} (\mathbf{L} + \mathbf{L}^*)|^{-1/2} \exp \left\{ s^t \left[ \mathbf{K} \left( \frac{\mathbf{L} + \mathbf{L}^*}{2} \right)^{-1} \mathbf{K}^t \right] s + t^t \left[ \mathbf{K} \left( \frac{\mathbf{L} + \mathbf{L}^*}{2} \right)^{-1} \mathbf{K}^t \right] t \right\}
\]

\[
\times \exp \left\{ 2s^t \left[ \mathbf{K} \left( \frac{\mathbf{L} + \mathbf{L}^*}{2} \right)^{-1} \mathbf{K}^t \right] s \right\}
\]

and we get

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2n+m)!}{n!m!} \frac{1}{t^{l+k}} \int \mathcal{H}_{n,m}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \mathcal{H}_{l,k}^{*}(\mathbf{r}; \mathbf{K}, \mathbf{L}) \, d\mathbf{r}
\]

\[
= 2 |\text{det} \mathbf{K}| |\text{det} (\mathbf{L} + \mathbf{L}^*)|^{-1/2} \exp \left\{ 2s^t \left[ \mathbf{K} \left( \frac{\mathbf{L} + \mathbf{L}^*}{2} \right)^{-1} \mathbf{K}^t \right] s \right\}
\]

\[
\times \exp \left\{ -s^t \left[ \mathbf{M} - \mathbf{K} \left( \frac{\mathbf{L} + \mathbf{L}^*}{2} \right)^{-1} \mathbf{K}^t \right] s - t^t \left[ \mathbf{M} - \mathbf{K} \left( \frac{\mathbf{L} + \mathbf{L}^*}{2} \right)^{-1} \mathbf{K}^t \right] t \right\}.
\]

To get to the orthonormality condition (16), we have to require

\[
\mathbf{M} - \mathbf{K}[(\mathbf{L} + \mathbf{L}^*)/2]^{-1} \mathbf{K}^t = \mathbf{0},
\]

(17)

\[
\mathbf{K}[(\mathbf{L} + \mathbf{L}^*)/2]^{-1} \mathbf{K}^t = \mathbf{I},
\]

(18)

leading to the conditions

\[
\mathbf{M}^{-1} = \mathbf{M}^* = \mathbf{K}^* \mathbf{K}^{-1} = (\mathbf{K}^* \mathbf{K}^{-1})^t,
\]

(19)

\[
(\mathbf{L} + \mathbf{L}^*)/2 = \mathbf{K}^* \mathbf{K} = (\mathbf{K}^* \mathbf{K})^t,
\]

(20)
where we have also expressed the symmetry of the matrices \( L \) and \( M \) once again. Note that \( M = KK^{-1} \) is completely determined by \( K \), see Eq. (19), which is the reason why we did not include \( M \) as a parameter in \( H_{n,m}(r; K, L) \).

If we express \( K^{-1} \) in its real and imaginary parts, \( K^{-1} = a + ib \), we immediately get from the realness of \((K^*K)^{-1} \), see Eq. (20), that the matrix \( ab^{\dagger} \) is symmetric. If we then express \( L \) as \( L = (d - ic)K = (d - ic)(a + ib)^{-1} \), the symmetry of \( L \) leads to

\[
\text{a}'\text{d} + \text{b}'\text{c} = \text{d}'\text{a} + \text{c}'\text{b} \quad \text{and} \quad \text{a}'\text{c} - \text{b}'\text{d} = \text{c}'\text{a} - \text{d}'\text{b},
\]

while Eq. (20) leads to the requirements

\[
\text{a}'\text{d} - \text{b}'\text{c} + \text{d}'\text{a} - \text{c}'\text{b} = 2\text{I} \quad \text{and} \quad \text{a}'\text{c} + \text{b}'\text{d} = \text{c}'\text{a} + \text{d}'\text{b}.
\]

From these four conditions we conclude that the \( 4 \times 4 \)-matrix

\[
\begin{pmatrix}
\text{a} & \text{b} \\
\text{c} & \text{d}
\end{pmatrix}
\]

is symplectic and thus satisfies relations of the form (9).

The results in this paper resemble those derived by Wünsche [5]. The main difference is that we use as the Gaussian part \( \exp(-\pi r^2 L r) \), with a matrix \( L \) that can be chosen freely if we would only require Eq. (17) and not necessarily Eq. (18), whereas Wünsche uses a fixed expression of the form \( \exp(-\pi r^2 r) \). Wünsche’s results arise indeed from ours for the special choice \( L = \text{I} \), in which case Eq. (17) leads to \( M = KK^\dagger \), yielding the generating function

\[
\exp[-sKK's + 2sK(\sqrt{2\pi} r) - (\sqrt{2\pi} r^2)/(\sqrt{2\pi})]/2],
\]

which is compatible to [5, Eq. (8.4)]. Eq. (18) would yield the additional condition \( KK^{\dagger} = \text{I} \).

Special cases of Hermite-Gaussian-type modes can easily be recognized. We mention the (separable) Hermite-Gaussian modes (with curvatures in the \( x \) and \( y \) directions determined by \( \gamma_x \) and \( \gamma_y \), respectively), for which the matrices \( \text{a}, \text{b}, \text{c}, \) and \( \text{d} \) are given by

\[
W^{-1}(\text{a} + \text{ib}) = \begin{pmatrix}
\cos \gamma_x/\exp(i\gamma_x) & 0 \\
0 & \cos \gamma_y/\exp(i\gamma_y)
\end{pmatrix} \quad \text{W}(\text{d} - \text{ic}) = \begin{pmatrix}
\exp(i\gamma_x) & 0 \\
0 & \exp(i\gamma_y)
\end{pmatrix};
\]

(21)

the common Hermite-Gaussian modes with which we started this paper [see Eqs. (1-5)], arise for the special choice \( \gamma_x = \gamma_y = \gamma = 0 \). Note that for Hermite-Gaussian modes the matrix

\[
\text{L} = \begin{pmatrix}
(1 + i\tan \gamma_x)w^{-2} & 0 \\
0 & (1 + i\tan \gamma_y)w^{-2}
\end{pmatrix}
\]

is a diagonal matrix, and that the \( \text{ABCD} \)-law (13) is useful when such modes propagate through separable systems (for which \( \text{A}, \text{B}, \text{C}, \) and \( \text{D} \) are diagonal matrices). For (rotationally symmetric) Laguerre-Gaussian modes (with its curvature determined by \( \gamma \)) we have

\[
\text{w}^{-1}(\text{a} + \text{ib}) = \frac{\cos \gamma}{\exp(i\gamma)}\text{w}(\text{d} - \text{ic}) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\exp(i\gamma_x) & -i\exp(i\gamma_y) \\
-i\exp(i\gamma_x) & \exp(i\gamma_y)
\end{pmatrix};
\]

(22)

the special case \( \gamma = \gamma_x = \gamma_y = 0 \) has been reported, for instance, in [6]. Note that for Laguerre-Gaussian modes the matrix \( \text{L} = (1 + i\tan \gamma)w^{-2} \text{I} \) is a scalar matrix, and that the \( \text{ABCD} \)-law (13) is useful when such modes propagate through isotropic systems (for which \( \text{A}, \text{B}, \text{C}, \) and \( \text{D} \) are scalar matrices). We remark that the discriminating parameters in the above expressions are the widths \( (w_x, w_y, w) \) and the curvatures \( (\gamma_x, \gamma_y, \gamma) \) of the modes; the parameters \( \gamma_x \) and \( \gamma_y \) lead to a mere multiplication of the complex field amplitude by a phase factor that depends on the mode-number \((n, m)\) but not on the space variables \( r \).
5. An alternate propagation law

Now that we know that the input matrices \( K_i \), \( L_i \), \( M_i \) and the output matrices \( K_o \), \( L_o \), \( M_o \) can be expressed in the special forms

\[
K_{i,o} = (a_{i,o} + i b_{i,o})^{-1}, \\
L_{i,o} = (d_{i,o} - i c_{i,o})(a_{i,o} + i b_{i,o})^{-1}, \\
M_{i,o} = (a_{i,o} + i b_{i,o})^{-1}(a_{i,o} - i b_{i,o}),
\]

(23)

where the real matrices \( a_i, b_i, c_i, d_i \) and the real matrices \( a_o, b_o, c_o, d_o \) constitute two real symplectic matrices, we can – after some straightforward but rather lengthy calculations, in which we make extensive use of the symplecticity properties, cf. Eqs. (9) – bring the input-output relationships (12–14) into a different form and formulate the elegant propagation law

\[
\begin{pmatrix}
  a_o & b_o \\
  c_o & d_o \\
\end{pmatrix} = \begin{pmatrix}
  A & B \\
  C & D \\
\end{pmatrix} \begin{pmatrix}
  a_i & b_i \\
  c_i & d_i \\
\end{pmatrix}. 
\]

(24)

This propagation law resembles Eqs. (12) and (29) in Ref. [7], split up into their real and imaginary parts, where \( i(a + ib), d - ic \) correspond to the ‘matricial rays’ \( Q, P \) [7, Eq. (11)]: \( Q \sqrt{\pi} = i K^{-1} = i(a + ib) \) and \( P \sqrt{\pi} = \lambda L K^{-1} = \lambda(d - ic) \), with \( \lambda \) the wavelength of the light.

The treatment in [7] is based on a so-called ‘mode-generating system,’ which is excited by an off-axis point source at its input plane; in our case of lossless, first-order optics, this system is determined by the matricial rays \( Q, P \) that are associated with the mode that is to be generated. The modes in [7] then arise by expanding the resulting output field in a power series of the coordinates of the point source at the input. The kernel in the Collins integral that describes the mode-generating system thus plays the role of a generating function. The present treatment is directly based on the general form (6) of the generating function, and we get the additional result that the parameters \( a, b \) and \( c, d \) that characterize this generating function correspond to the real and imaginary parts of \( Q \) and \( P \), and constitute a real, symplectic matrix. With \( r = x \) and \( \lambda s = \pi y \sqrt{2} \), there is indeed a one-to-one correspondence between our generating function (6) and the one used in [7], see in particular [7, Eq. (22)].

We remark that all sets of Hermite-Gaussian-type modes can be converted into each other by means of properly chosen first-order optical systems, and we conclude that knowledge of the generating function and in particular its propagation law (24) may be valuable in the design of mode converters. Further work is in progress, see for instance [8], and future papers may also take lossy mode converters [7] into account.

6. Conclusion

A general class of orthonormal sets of Hermite-Gaussian-type modes has been introduced by formulating a generalized version of the generating function that yields the common Hermite-Gaussian modes. These sets of Hermite-Gaussian-type modes remain in their class when they propagate through first-order optical systems, and a propagation law for their generating function has been formulated. The propagation law is in a form that suits itself for the design of mode converters.

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