Rapid variation with remainder and rates of convergence

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Rapid variation with remainder
and rates of convergence

by

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Rapid variation with remainder and rates of convergence

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ABSTRACT. The remainder term of the class of rapidly varying functions is considered. Some probabilistic applications to limit laws of extreme value theory and to the estimation of the index parameter of a regularly varying tail are considered.

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1. INTRODUCTION

Let $U : \mathbb{R} \to \mathbb{R}^+$ be a measurable function such that

$$\lim_{x \to \infty} U(tx)/U(x) = t^\alpha \quad \text{for every } t > 0.$$ 

Then $U$ is called regularly varying with index $\alpha$ ($\aleph_\alpha$). If $\alpha = 0$ we say that $U$ is slowly varying, while if $\alpha = \infty$ $U$ is called rapidly varying.

An important class of rapidly varying functions is the so-called class $\Gamma$, introduced by de Haan (1970):

Let $f : \mathbb{R} \to \mathbb{R}^+$ be a measurable function, then $f \in \Gamma$ if and only if there exists a measurable function $\varphi : \mathbb{R} \to \mathbb{R}^+$ such that

$$\lim_{x \to \infty} f(x + u \varphi(x))/f(x) = \exp(u)$$

locally uniformly (l.u.) in $u \in \mathbb{R}$.

If (1.1) holds, we call $\varphi$ an auxiliary function of $f$ (notation $f \in \Gamma(\varphi)$) and it is known that in this case $\varphi$ is self-neglecting (see de Haan (1970)):

$$\lim_{x \to \infty} \varphi(x + u \varphi(x))/\varphi(x) = 1$$

l.u. in $u \in \mathbb{R}$.

At this point, notice that our definition of $\Gamma$ is somewhat more general than the one given by de Haan (1970) as he restricts the class $\Gamma$ to monotone functions which satisfy (1.1) pointwise in $u \in \mathbb{R}$.

By far the most important probabilistic application of $\Gamma$ is the characterization of the domain of attraction of the double exponential law in the maximum-scheme: let $X_{1;n} \leq X_{2;n} \leq \ldots \leq X_{n;n}$ denote the order statistics of a sample of size $n$ from a distribution function (df) $F$. We denote $F = 1 - F$. Then one can find normalizing constants $a_n > 0$ and $b_n$ such that for all $x \in \mathbb{R}$,

$$P(X_{n;n} - b_n \leq a_n x) \to \exp(-\exp(-x)) =: \Lambda(x), \quad n \to \infty$$

iff

$$1/F \in \Gamma$$
Another characterizing property of $\Gamma$ concerns the Hill estimator (Hill (1975), Beirlant and Teugels (1987)): if $F\in C_2:=\{F|F(0)=0$, $F$ continuous and eventually strictly increasing$\}$, then Hill's estimate

$$H_{m,n} := m^{-1} \sum_{i=1}^{m} \log X_{n-i+1:n} - \log X_{n-m:n}$$

is attracted as $n \to \infty$ to the gamma law of $m^{-1} \sum_{i=1}^{m} E_i$ (see (1975), Beirlant and Teugels (1987)) if $F$ is exponential random variables with mean one). We therefore consider the following asymptotic relations:

- let $r$ be a measurable function from $\mathbb{R}$ to $\mathbb{R}$ such that $r(x) \to 0$ as $x \to \infty$. Then

$$f(x + u\phi(x))/f(x) = \phi^d(t + O(r(x))) \quad (x \to \infty) \quad \text{i.u. in } u \in \mathbb{R}$$

- if $f$ satisfies one of the relations (GR$_1$) ($i=1,2,3$), with auxiliary functions $\phi$ and $r$, we denote it as $f \in \Gamma R_i(\phi,r)$.

It is well-known that $\Gamma$ is strongly connected with the class $\Pi$ of slowly varying functions (de Haan (1970)): if $f$ is non-decreasing, $f \in \Gamma(\phi)$ iff

$$\lim_{x \to \infty} \left( \frac{f^d(tx) - f^d(x)}{\phi(f^d(x))} \right) = \log(t) \quad \text{for every } t > 0$$

where $f^d$ is the inverse of $f$. We denote (1.3) as $f^d \in \Pi(\phi(f^d))$.

Similarly as for $\Gamma$, we can define remainder versions of $\Pi$-variation (see Omey and Willekens (1987)):

for positive measurable functions $a$ and $b$, consider
\((\text{PIR}_1)\) \[ f(x) - f(x) - a(x) \log(t) = O(b(x)) \quad (x \to \infty) \]

\((\text{PIR}_2)\) \[ f(x) - f(x) - a(x) \log(t) \sim h(u)b(x) \quad (x \to \infty) \]

\((\text{PIR}_3)\) \[ f(x) - f(x) - a(x) \log(t) = o(b(x)) \quad (x \to \infty). \]

Similarly as above, we use the notation \(\mathcal{F} \text{PIR}_i(a, b), i=1,2,3.\)

As one might expect and as was shown by de Haan and Dekkers (1987), the stated relationship between \(\Gamma\) and \(\Pi\) (see (1.3)) maintains (under appropriate conditions) for the remainder versions, i.e.

\[ \mathcal{F} \text{PIR}_i(\phi, r) \iff \mathcal{F} \text{PIR}_i(\phi(f^1), \phi(f^1)r(f^1)), \quad i=1,2,3. \]

In the next section we define a transform which also relates the classes \(\mathcal{F} \Gamma R_i\) and \(\text{PIR}_i\), but which is also valid for non-monotone functions. The analytic results of section 2 are then applied in section 3 to establish rates of convergence in the previously mentioned examples.

Before starting with section 2, we notice that \(\text{PIR}_i\) is closely related to the concept of slow variation with remainder (SR\(_i\)) as defined in Goldie and Smith (1987). Indeed, if \(b(x) \to 0\) \((x \to \infty)\), we have for any function \(f\) that \(\mathcal{F} \text{PIR}_i(0, b)\) \(\iff \exp f \in \text{SR}_i(\phi).\)

2. SOME ANALYTIC RESULTS

As in Goldie and Smith (1987) and Omey and Willekens (1987) it will be appropriate to impose some conditions on the remainder term \(r\) in \(\Gamma R_i\) \((i=1,2,3)\). Unless otherwise stated, we will assume that

\[ (2.1) \quad \lim_{x \to \infty} r(x+u\phi(x))/r(x) = \exp(\gamma u) \quad \text{for every } u \in \mathbb{R} \text{ and some } \gamma \leq 0. \]

Clearly the limit in (2.1) can only be of the stated form. In the proof of our theorems we will frequently use the following proposition, due to Bingham and Goldie (1983).
Proposition. Let $\phi$ be self-neglecting, $g$ satisfy
\begin{equation}
(g(x+u\phi(x)) - g(x))/z(x) \to 0 \quad (x \to \infty)
\end{equation}
with $z$ a measurable function satisfying
\[ z(x+u\phi(x))/z(x) \to \exp(yu) \quad (x \to \infty), \gamma \leq 0, \ u \in \mathbb{R}. \]
Then (2.2) holds uniformly on compact $u$-sets.

We now define the transform which will be considered in the forthcoming theorem: suppose $\phi$ is bounded away from zero on any finite interval, and let
\begin{equation}
\Phi(x) := \int_0^x dt/\phi(t), \quad x \in \mathbb{R}.
\end{equation}
Then $\Phi$ is a strictly increasing continuous function whose inverse is well-defined. Define for any $f \in \Gamma(c\phi), c \in \mathbb{R}_+$,
\[ A: f \to A_f = f \circ \Phi \circ \log. \]
It follows from de Haan (1973) that any function $f$ in $\Gamma(\phi)$ can be represented as
\begin{equation}
f(x) = U(\exp\Phi(x)) \quad \text{with } U \in \mathbb{R}_f.
\end{equation}
Clearly with the definition of $A$, (2.4) implies that $A_f = U$ whence $\log A_f \in \Pi(1)$. So the operator $A$ provides an obvious relation between $\Gamma$ and $\Pi$, and it is not hard to imagine that we can expect a similar relation between $\Gamma R_f$ and $\Pi R_f$.

Before stating the main theorem of this section, we first consider the function $\Phi$ somewhat closer.

By local uniformity in (1.2), we have for any $u \in \mathbb{R}$,
\[
\Phi(x+u\phi(x)) - \Phi(x) = \int_x^{x+u\phi(x)} dt/\phi(t)
\]
\[ = \int_0^u \phi(x)/\phi(x+u\phi(x)) \ dv \]
\[ = u + o(1) \quad (x \to \infty). \]
Conversely, fixing any number $u \in \mathbb{R}$, one can find $t = t(x)$ with $t(x) \to u$ as $x \to \infty$ such that (cfr. Bingham and Goldie (1983))

\[(2.5) \quad \Phi(x+t(x)\phi(x)) = \Phi(x) + u.\]

This relation is very useful and will be used throughout in the sequel of the paper. We now state our main theorem.

Theorem 2.1. Let $\phi$ be self-neglecting and let $r$ satisfy (2.1). For any $i \in \{1,2,3\}$ the following assertions are equivalent:

1. $f \in \mathcal{GR}_i(\phi,r)$
2. $\log A_f \in \mathcal{II}_1(1,A_r)$ and $A_\phi \in \mathcal{SR}_1(A_r)$
3. $f(x) = \exp(\Phi(x)) \, \nu(\exp(\phi(x)))$ with $A_\phi \in \mathcal{SR}_1(A_r)$ and $\nu \in \mathcal{ESR}_1(A_r)$.

Proof. We first prove the theorem for $i = 2$.

(i) $\Rightarrow$ (ii). From the definition of $A_f$ we have that $f(x) = A_f(\exp(x))$.

Therefore,

\[
\forall x \in \mathcal{GR}_2(\phi,r) \quad \text{iff} \quad A_f(\exp(x+\nu(x))) = A_f(\exp(x)) - e^u \mu(u) \, r(x) \quad (x \to \infty)
\]

Now with $t(x)$ defined as in (2.5), it follows from local uniformity that

\[(2.6) \quad A_f(e^{\Phi(x)+u})/A_f(e^{\Phi(x)}) - e^u - e^u(t(x)-u) \sim e^u \mu(u) \, r(x) \quad (x \to \infty).\]

We first determine the order of $t(x) - u$. Defining

\[(2.7) \quad R_u(x) := (\log(x+u\phi(x)) - \log(x-u))/r(x),\]

we have that

\[
R_{u+v}(x) = R_{v(x)}(x+u\phi(x)) \, r(x+u\phi(x))/r(x)
+ R_u(x) + v(-1 + \phi(x)/\phi(x+u\phi(x)))/r(x).
\]
with $v(x)=\psi(x)/\phi(x+u\phi(x))$.
Then by $\Gamma R_2$ and (2.1),

\[
\lim_{x \to \infty} \left( \phi(x+u\phi(x)))/\phi(x) \right) \text{ exists.}
\]

Denoting the limit in (2.8) as $k(u)$, it is not hard to show that

\[
k(u) = a \int_0^t \theta^{-1} \, d\theta.
\]

with $a$ a real constant and $\gamma$ determined by (2.1).
Using Proposition one can show that convergence in (2.8) holds l.u. in $u \in \mathbb{R}$, so that

\[
\phi(x+u\phi(x))-\phi(x) = (r(x)) \int_0^t \left( (\phi(x)/\phi(x+u\phi(x))) - 1 \right) \, du
\]

\[
\rightarrow h(t) = -\int k(u) \, du,
\]

l.u. in $t \in \mathbb{R}$.
This implies that the function $t(x)$ in (2.5) is of the form

\[
t(x) = u-h(u)r(x)+o(r(x)) \quad (x \to \infty).
\]

Then clearly from (2.6), after a change of variables ($y=\exp\phi(x), \lambda = e^u$)

\[
\log A_p(y) - \log A_p(y) - \log \lambda \sim (m(\log \lambda)-h(\log \lambda)) A_p(y) \quad (y \to \infty)
\]

showing that $\log A_p \in \Pi R_2(1, A_p)$.
The fact that $A_p \in \Pi R_2(A_p)$ follows immediately from (2.8), local uniformity and the definition of $t(x)$.

(1) $\Rightarrow$ (iii). Obviously $\log A_p \in \Pi R_2(1, A_p)$ iff $V(x) = \log(A_p(x))/x \in \Pi R_2(A_p)$.
The representation theorem follows then immediately.

(iii) $\Rightarrow$ (1). Immediate.
The proof of the theorem for $i=1$ or 3 follows exactly the same lines.
Only the limit relations have to be changed in $O$- or $o$-versions. □

Remarks.

1. It follows from (2.5) that $r$ satisfies (2.1) iff $A_r \in E\gamma$. Clearly for proving Theorem 2.1 if $i = 1$ ($i = 3$), the assumption on $r$ in (2.1) can be relaxed to $r(x+u\phi(x)) = O(r(x)) \langle o(r(x)) \rangle$ as $x \to \infty$. This then implies that $A_r$ is $O$-regularly varying (see Goldie and Smith (1987)).

2. Theorem 2.1 implies that if $\gamma < 0$ in (2.1), any function $f$ satisfying $\Gamma R_1$ is essentially an exponential function. Indeed, if we consider $\Gamma R_2$ it follows from $\text{VESR}_2(A_r)$ and Seneta (1976) (pp. 73-74) that there exists constants $c$ and $d \neq 0$ such that

$$V(\exp \phi(x)) = d + cr(x) + o(r(x)) \quad (x \to \infty).$$

For the same reason, there exists constants $c_0 \neq 0$ and $c_1$ such that

$$c_0 \phi(x) = \exp (c_1 r(x) + o(r(x))) \quad (x \to \infty),$$

from which

$$\Phi(x) = c_2 + x c_0^{-1} + c_1 c_0^{-1} \int c_3 r(u)(1+o(1)) du.$$ (2.10)

Combining (2.9) and (2.10) we have from Theorem (2.1) that

$$f(x) = (d + cr(x) + o(r(x))) \exp (c_2 + x c_0^{-1} + c_1 c_0^{-1} \int c_3 r(u)(1+o(1)) du) \quad (x \to \infty).$$

3. The proof of Theorem 2.1 shows that from $\Gamma R_2$ and (2.1)

$$m(u+v) = m(v) \exp (\gamma u) + m(u) - o^\phi \gamma^{-1} d \theta \quad (\phi \in R).$$

Hence
\( m(u) = \begin{cases} 
\alpha y^{-1}(\exp(yu) - 1) - \alpha y^{-1}\int_0^u (\exp(yv) - 1) \, dv & \text{if } \gamma \neq 0 \\
\alpha u - \alpha u^2/2 & \text{if } \gamma = 0.
\end{cases} \)

(2.11)

3. APPLICATIONS IN EXTREME VALUE THEORY

a. Rate of convergence for maxima in domain of attraction of the double exponential distribution.

Let \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) be an ordered sample from a df \( F \) with

\[ 1/(-\log F) \in \Gamma(\phi). \]

Take \(-\log F(b_n) \leq n^{\phi/2} \leq -\log F(b_n)\) and let \( a_n = \phi(b_n) \).

Then it is well-known that (see de Haan (1970))

\[ P(X_{n:n} \leq a_n u + b_n) \rightarrow \Lambda(u) := \exp(-e^{-u}), \quad u \in \mathbb{R}. \]

(3.1)

As was mentioned in the Introduction, strengthening the condition

\[ 1/(-\log F) \in \Gamma(\phi) \text{ to } 1/(-\log F) \in \Gamma R_1(\phi, r) \]

for some \( r \in (1, 2, 3) \) and \( r(x) \rightarrow 0 \), will allow us to study the rate of convergence in (3.1). Indeed, it is easily seen that if \( f \in \Gamma R_1(\phi, r) \),

\[ \Lambda_n(u) := P(X_{n:n} \leq a_n u + b_n) - \Lambda(u) = O(r(b_n)) \quad (n \rightarrow \infty) \]

(3.2)

while if \( f \in \Gamma R_2(\phi, r) \),

\[ \Lambda_n(u) = \Lambda(u)m(u)r(b_n) + o(r(b_n)) \quad (n \rightarrow \infty). \]

(3.3)

Whereas (3.2) and (3.3) give pointwise rates of convergence, the main problem is to show that they hold uniformly in \( u \in \mathbb{R} \).

Although many papers have been devoted to the uniform rate of convergence in (3.1), (see e.g. Anderson (1971), Cohen (1982), Omey and Rachev (1987), Resnick (1986)), it is still an open problem to give the most general conditions that imply the right rate; nearly all of the existing results work by
specifying the Von Mises conditions.
The contrary is true when attraction to the Frechet or Weibull law is concerned. Indeed in these cases Smith (1982) formulated best possible conditions in terms of slow variation with remainder.
In the theorem below, we prove that (3.2) ((3.3)) holds uniformly in $x \in \mathbb{R}$ under $\Gamma R_1$ ($\Gamma R_2$). Using the theory of section 2, we show that $\Gamma R$ can be reduced to slow variation with remainder so that we end up exactly with the same problem as was tackled by Smith (1982).
We believe that the present way of proof is properly motivated from the concept of $\Gamma$-variation with remainder and that it generalizes the approaches used in the references mentioned above.
The only minor drawback is the following assumption which will be used in the theorem:

$$(3.4) \quad F^n(b_n \cdot \Phi(b_n) \phi(b_n)) = o(r(b_n)) \quad (n \to \infty).$$

Condition (3.4) holds in most instances and may not be satisfied if $\phi$ is slowly varying with a specified remainder term. The following lemma ensures this statement.

Lemma 3.1. If any subsequence $(b'_n)_n$ of $(b_n)_n$ for which

(i) $b'_n - \Phi(b'_n) \phi(b'_n) \to \infty$

and

(ii) $\Phi(b'_n) \phi(b'_n)/b'_n \to 1 \quad (n \to \infty)$

satisfies $F^n(b'_n \cdot \Phi(b'_n) \phi(b'_n)) = o(r(b'_n)) \quad (n \to \infty)$

then (3.4) holds in case $F$ is concentrated on an interval of the form $[z, +\infty)$.

Proof. In case $F$ is concentrated on intervals of the specified type the only subsequences we have to consider are those for which (i) holds.

If furthermore $\limsup_{n \to \infty} \Phi(b'_n) \phi(b'_n)/b'_n < 1$ (3.4) follows from Davis and Resnick (1986) and the fact that $A_p$ is $O$-regularly varying. In case the $\limsup$ equals $1$ (3.4) follows from the assumptions. $\square$

The theorem reads as follows.
Theorem 3.1. Suppose $\phi$ is self-neglecting

a. If $\frac{1}{\log F} \in \Gamma_{1}(\phi,r)$ and (3.4) is satisfied, and if there exist constants $x_0, \theta, b, c$ all positive such that

$$bx^{-\theta} \leq A_{r}(x_{t})/A_{r}(x) \leq c$$

for all $x \geq x_0, t \geq 1$,

then

$$\sup_{u \in \mathbb{R}} |\Lambda_{n}(u)| = O(r(b_{n})) \quad (n \to \infty).$$

b. If $\frac{1}{\log F} \in \Gamma_{1}(\phi,r)$ and (3.4) is satisfied and if $A_{r} \in \Gamma_{1,1}, \gamma < 0$, then

$$\Lambda_{n}(u) = \Lambda'(u) m(u) r(b_{n}) + o(r(b_{n})) \quad (n \to \infty)$$

uniformly in $u \in \mathbb{R}$. Here $m(u)$ is given as in (2.11).

Proof. First notice that we may assume that $F$ is supported on $[z, \infty)$ for some $z \in \mathbb{R}$ Indeed, putting $Y_{i} := \max(z, X_{i}), i = 1, \ldots, n$, it is clear that for some $n$ large enough

$$\sup_{u \in \mathbb{R}} |P(Y_{mn} \leq q_{n} u + b_{n}) - P(X_{mn} \leq q_{n} u + b_{n})| = P(X_{mn} \leq z) = o(r(b_{n}))$$

uniformly in $u \in \mathbb{R}$, where the last inequality follows from the definition of $b_{n}$ and (3.5).

We now estimate

$$\Delta_{n}(x) := | - \log( - n \log F(a_{n} \log x + b_{n})) - \log x | \quad \text{for some } x > 0.$$  

Denoting $- \log F := f$ and $exp \Phi(b_{n}) := \nu_{n}$, we have from (2.5) and the definition of $b_{n}$ that

$$\Delta_{n}(x) = | - \log f(a_{n} \log x + b_{n}) + \log F(b_{n}) - \log x + o(r(b_{n}))|$$

$$= | - \log A_{r}(x \exp(\Phi(b_{n})) + c(b_{n})) + \log A_{r}(\exp(\Phi(b_{n})) - \log x$$

$$+ o(r(b_{n})))|$$

$$\leq | - \log (A_{r}(x)/x \nu_{n}) + \log (A_{r}(x)/x \nu_{n}) + (c(b_{n}) - \log x) + o(r(b_{n}))|$$
Now by Theorem 2.1, \( L(x) = A_r(x)/x \in SR_1(A_r) \) with \( i=1 \) or \( 2 \) depending on whether a. or b. is satisfied. Using the estimation in (3.6), we can copy the proofs of Theorems 1 and 2 of Smith (1982), implying uniform convergence in (3.2) and (3.3) over the region \( u = \log x \geq -\log v_n + c \), where \( c \) is some constant. Hence the proof is finished if we can show that both \( \Lambda(-\log v_n) \) and 
\( F_n(-\log v_n + b_n) \) are \( o(r(b_n)) \) as \( n \to \infty \). Under the conditions of the theorem, \( A_r \) is \( 0 \)-regularly varying such that \( \Lambda(-\log v_n) = \exp(-\nu_n) = o(A_r(\nu_n)) = o(r(b_n)) \) \( (n \to \infty) \). As to \( F_n(-\log v_n + b_n) \), notice that \( -a_n \log v_n + b_n = b_n \cdot \Phi(b_n) \). Lemma 3.1 applies now. □

b. Rate of convergence of Hill's estimate.

Beirlant and Teugels (1987) showed that if \( 1/(1-F) \) \( \exp \) belongs to \( \Gamma \), and \( F \) is continuous and strictly increasing in a neighborhood of \( \infty \), Hill's estimate \( H_{m,n} \) given in (1.2) is attracted as \( n \to \infty \) to the distribution of \( m^{-1} \sum_{i=1}^m E_i =: E_m, E_i \) being \( i \) iid exponential rv's with mean one.

Let now \( \phi \) be the auxiliary function corresponding to \( 1/F \exp \),
\( l(u) = \phi(\log F^{-1}(1-u^{-1})), u \in (0,1); \ q_n = m/n, \) and \( p_n^2 = m(n-m-1)/(n-1)^3 \).

Then it was also shown that \( n \to \infty, m_n \to \infty, m_n = o(n), \) and
\( (3.7a) \quad (m_n)^{1/2} \left\{ -1 + \int_0^\infty (q_n + p_n z)^{-1} (1 - F(-1 - \Phi(z))) \, dz \right\} \to 0 \)
\( \text{L.u. in } z > 0 \)

to entail that
\( (m_n)^{1/2} (H_{m,n}/(n/m) - 1) \overset{d}{\to} N(0,1). \)

If we assume that \( 1/F \exp \in \Gamma R_1(\phi,r) \) then it is clear that condition (3.7a) can be replaced by the more attractive condition
(3.7b) \( (m_n)^{1/2} r(\log^2 (1 - q_n, p_n z)) \to 0 \) as \( n \to \infty, m_n \to \infty, m_n = o(n) \)

l.u. in \( z > 0 \).

With the help of the Berry-Esseen theorem

\[
\sup_x | P((m_n)^{1/2} (H_{m,n} - I(n/m) - 1)) \leq x | - \Phi(x) |
\]

\[ \leq \tau_n + Cm^{-1/2} \]

where \( \tau_n = \sup_x | P((m_n)^{1/2} (H_{m,n} - I(n/m) - 1)) \leq x | - P((m_n)^{1/2} (E_m - I) \leq x) | \).

To bound \( \tau_n \) we apply the well-known smoothing inequality:

\[
\tau_n \leq \pi^{-1} \int_{-1}^{1} | \psi_{m,n}(t) - \kappa_{m}(t) | \, dt + K\, t^{-1},
\]

where \( \psi_{m,n} \) resp. \( \kappa_{m} \), denote the characteristic functions of the standardized versions of \( H_{m,n} \) resp. \( E_m \), given in Beirlant and Teugels (1987).

We get by choosing \( T = m^{1/2} \) that

\[
\tau_n \leq K m^{-1/2}
\]

\[ + A(n,m) \pi^{-1} (m!)^{-1} \int_{-1}^{1} t^{-1} \, dt \int_{-m^{-1/2}}^{m^{-1/2}} (1 - v/n) \, n^{-1/2} \, dv \]

\[ + (1+t) e^{-w} (wm^{1/2}/t) dw m \, m \]

where \( A(n,m) = O(1) \) as \( n \to \infty, m = o(n) \) and
First
\[ \max \{ |K_n(v, u/l(n/m))|, |(1 + \int_0^\infty e^{iw/w/u} dw)| \} \leq 1 \]
and if \( 1/F_0 \exp \) satisfies \( (\Gamma R_1) \) we get as \( n \to \infty \)
\[ |K_n(v, u/l(n/m)) - (1 + \int_0^\infty e^{iw/w/u} dw)| \leq e^{-w/u} O(r \log F'(1-v/n)) \]
so that by substituting \( t/m^{1/2} \) by \( u \) we find
\[ \tau_n \leq K n^{-1/2} + L \int_0^1 du \int (ml) (1-v/n) \frac{1}{u} \]
\[ \times O(r \log F'(1-v/n)) \] \( dv \)
for a certain constant \( L \).
So we have derived the following theorem.

**Theorem 3.2.** Suppose \( F \) is continuous and strictly increasing in a neighborhood of \( \infty \). Moreover assume that \( 1/F_0 \exp \in \Gamma R_1(\phi, r) \) and that (3.1b) holds.

Then there exists a positive constant \( C \) such that
\[ \sup_{x \in \mathbb{R}} |P(m^{1/2}H_{m,n}^{-1}(n/m) - 1) \leq x| - \Phi(x) | \]
\[ \leq C m^{-1/2} + m E( \psi_{m,n,F}(m n^{-1} F_{m+1})) \]
as \( n \to \infty, m_n \to \infty, m_n = o(n) \), where \( \psi_{m,n,F}(x) = O(r \log F'(1-x^{-1})) \) as \( x \to \infty \).

The above result generalizes results of Falk (1985), who derives rates of convergence for \( H_{m,n}^{-1} \) in more specific models.
References,