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An infinite number of infinite hierarchies of conserved quantities of the Federbush model

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The construction of two Lie–Bäcklund transformations is given, which are Hamiltonian vector fields leading to an infinite number of hierarchies of conserved functionals and associated Lie–Bäcklund transformations.

I. INTRODUCTION AND GENERAL

In two recent papers$^1,2$ we constructed eight [in effect four, $Y_0^+, Y_0^-, Z_0^+, Z_0^-$ (iεZ)] infinite hierarchies of Lie–Bäcklund transformations of the Federbush model.$^3$ We conjectured that the hierarchies $Y_i^+, Y_i^- (i \in \mathbb{Z})$ are $(x,t)$ independent, while the hierarchies $Z_i^+, Z_i^- (i \in \mathbb{Z})$ are linear in $x$ and $t$. These Lie–Bäcklund transformations turned out to be Hamiltonian vector fields$^4,5$ and the corresponding Hamiltonian densities were given. In this way we obtained $r$-independent and $t$-dependent conserved functionals for the Federbush model.

Now we shall construct two $(x,t)$-dependent Lie–Bäcklund transformations of degree 0, with respect to the grading, which are polynomial in $x,t$ of degree 2 and from which we can obtain the creating and annihilating Lie–Bäcklund transformations $Z_{\pm}^\frac{1}{2}$, by taking the Lie bracket with the $(x,t)$-independent vector fields $Y_{\pm}^\frac{1}{2}$ (cf. the Appendix). Moreover these two vector fields turn out to be Hamiltonian vector fields and the associated Hamiltonian densities are given. This will be done in Sec. II. In Sec. III we prove a theorem from which we obtain an infinite number of infinite hierarchies of Hamiltonian vector fields, where the $Y_i^+, Y_j^- (i,j \in \mathbb{Z})$ are just the first four of this infinite number of hierarchies. The Hamiltonian densities of the vector fields $Z_i^\pm (i = -1,0,1)$, $Y_j^\pm (j = -2, -1,0,1,2)$ are surveyed in an Appendix at the end of this paper for reasons of completeness. In this section we shall introduce the notions needed in Secs. II and III. All computations have been carried through on a DEC-system computer, using the symbolic language REDUCE$^6$ and software packages$^7,8$ to do the huge computations at hand.

Lie–Bäcklund transformations are vector fields $V$ defined on the infinite jet bundle of $M,N$, $J^\infty(M,N)$, where $M$ is the space of independent variables and $N$ the space of the dependent variables. A Lie–Bäcklund transformation of a differential equation is a vector field $V$ defined on $J^\infty(M,N)$ satisfying the condition

$$\mathcal{L}_V (D^J) \subset D^{-I},$$

where $I$ denotes a differential ideal associated to the differential equation at hand, while $D^J$ denotes its infinite prolongation to $J^\infty(M,N)$; $\mathcal{L}_V$ is the Lie derivative with respect to the vector field $V$. Since the vector field $V$ is supposed to depend only on a finite number of variables, condition (1.1) reduces to

$$\mathcal{L}^r V \subset D^J$$

for some $r$. Using this method we computed Lie–Bäcklund transformations of the Federbush model.$^1$ It can be shown that the Lie–Bäcklund transformations in this setting are just symmetries in the works of Magri$^4$ and Ten Eikelder$^5$ where (generators of) symmetries of partial differential equations of evolutionary type are described as transformations on special types of infinite-dimensional spaces. Suppose that

$$\frac{du}{dt} = \Omega^{-1} dH$$

is an infinite-dimensional Hamiltonian system, where $\Omega$ is the symplectic operator, $H$ is the Hamiltonian, and $dH$ is the Fréchet derivative of $H$. Then to each Hamiltonian symmetry (also called canonical symmetry) $Y$ there corresponds by definition a Hamiltonian $F(Y)$ such that

$$Y = \Omega^{-1} dF(Y)$$

and the Poisson bracket of $F$ and $H$ vanishes.$^4,5$ Suppose that $Y_1, Y_2$ are two Hamiltonian symmetries, then $[Y_1, Y_2]$ is a Hamiltonian symmetry and

$$F([Y_1, Y_2]) = \{ F(Y_1), F(Y_2) \},$$

where $\{,\}$ is the Poisson bracket defined by

$$\{ F(Y_1), F(Y_2) \} = \langle dF(Y_1), Y_2 \rangle,$$

where $\langle,\rangle$ denotes the contraction of a one-form and a vector field.

II. CONSTRUCTION OF TWO NEW LIE–BÄCKLUND TRANSFORMATIONS OF THE FEDERBUSH MODEL

We construct two Lie–Bäcklund transformations of the Federbush model. This model is described by

$$\left( \begin{array}{cc} i(\partial_x + \partial_t) & -m(s) \\ -m(s) & i(\partial_x - \partial_t) \end{array} \right) \begin{pmatrix} \psi_{s,1} \\ \psi_{s,2} \end{pmatrix} = 4\pi\lambda \left( \begin{array}{c} |\psi_{s,1}|^2 \\ -|\psi_{s,2}|^2 \end{array} \right) \begin{pmatrix} \psi_{s,1} \\ \psi_{s,2} \end{pmatrix} (s = \pm 1),$$

where the $\psi_s(x,t)$ are two-component complex valued func-
tions. Suppressing the factor $4\pi(\lambda' = 4\pi\lambda)$ and introducing eight real variables $u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4$ by
\[
\psi_{1,1} = u_1 + iv_1, \quad \psi_{-1,1} = u_3 + iv_3, \quad m(1 + 1) = m_1, \tag{2.2}
\]
\[
\psi_{1,2} = u_2 + iv_2, \quad \psi_{-1,2} = u_4 + iv_4, \quad m(-1 + 1) = m_2.
\]
Eq. (2.1) is rewritten as a system of eight nonlinear partial differential equations for the functions $u_1, \ldots, u_4$, i.e.,
\[
\begin{align*}
    u_{1t} + u_{1x} - m_1v_2 &= \lambda R_1 v_1, \\
    -v_{1t} + v_{1x} - m_1u_2 &= \lambda R_4 u_1, \\
    u_{2t} + u_{2x} - m_1v_1 &= -\lambda R_3 v_2, \\
    -v_{2t} + v_{2x} - m_1u_1 &= -\lambda R_3 u_2, \\
    u_{3t} + u_{3x} - m_2v_1 &= -\lambda R_2 v_3, \\
    -v_{3t} + v_{3x} - m_2u_4 &= -\lambda R_2 u_3, \\
    u_{4t} - u_{4x} - m_2v_2 &= \lambda R_4 u_4, \\
    -v_{4t} + v_{4x} - m_2u_3 &= \lambda R_4 u_4,
\end{align*}
\tag{2.3}
\]
where in (2.3)
\[
Y^+(2,0) = x^2(\alpha_1 Y_1^+ + \alpha_2 m_1 Y_1^- + \alpha_3 m_1 Y_0^- + \alpha_4 m_1 Y_0^+ + \alpha_5 Y_1 + \alpha_6 Y_2 - 2(\beta_1 Y_2^+ + \beta_2 m_1 Y_1^- + \beta_3 m_1 Y_0^- + \beta_4 m_1 Y_0^+ + \beta_5 Y_1 + \beta_6 Y_2) + 2t(\gamma_1 Y_1^+ + \gamma_2 m_1 Y_1^- + \gamma_3 m_1 Y_0^- + \gamma_4 m_1 Y_0^+ + \gamma_5 Y_1 + \gamma_6 Y_2) + t^2(\delta_1 Y_1^+ + \delta_2 m_1 Y_1^- + \delta_3 m_1 Y_0^- + \delta_4 m_1 Y_0^+ + \delta_5 Y_1 + \delta_6 Y_2) + xC_1^+ + t(C_2^+ + C_0^-),
\tag{2.6}
\]
where the $Y_i^+$ (i = -2, -1, 0, 1, 2) are the vector fields associated with the conserved functionals $F(Y_i^+)$ surveyed in the Appendix; $\alpha_i, \beta_i, \gamma_i$ (i = 1, ..., 5) being constant, while $C_1^+, C_2^+, C_0^-$ are vector fields of degree 2, 2, and 0, respectively. Substituting (2.6) into the Lie–Bäcklund condition (1.2),
\[
\mathcal{L}_Y IC D^2 f,
\tag{2.7}
\]
and solving the resulting overdetermined system of partial differential equations for the coefficients $\alpha_i, \beta_i, \gamma_i$ (i = 1, ..., 5) and the vector fields $C_0^+, C_1^+, C_2^+$ using (2.4), we obtained the following result:
\[
Y^+(2,0) = x^2(\alpha_1 Y_1^+ + \alpha_2 m_1 Y_1^- + \alpha_3 m_1 Y_0^- + \alpha_4 m_1 Y_0^+ + \alpha_5 Y_1 + \alpha_6 Y_2) + 2t(\beta_1 Y_2^+ + \beta_2 m_1 Y_1^- + \beta_3 m_1 Y_0^- + \beta_4 m_1 Y_0^+ + \beta_5 Y_1 + \beta_6 Y_2) + t^2(\gamma_1 Y_1^+ + \gamma_2 m_1 Y_1^- + \gamma_3 m_1 Y_0^- + \gamma_4 m_1 Y_0^+ + \gamma_5 Y_1 + \gamma_6 Y_2) + xC_1^+ + tC_2^+ + C_0^-,
\tag{2.8}
\]
where in (2.8)
\[
C_{1}^{\tau} = \begin{pmatrix}
-2v_{1x} - m_1u_2 - \lambda R_{34} u_1 \partial_{u_1} + (2u_{1x} - m_1v_2 - \lambda R_{34} v_1) \partial_{v_1} + (-2v_{2x} + m_1u_1 - \lambda R_{34} u_2) \partial_{u_2} + (2u_{2x} + m_1v_1 - \lambda R_{34} v_2) \partial_{v_2} \\
+ (2u_{3x} + m_2v_3 + \lambda R_{12} u_3) \partial_{u_3} + (2u_{3x} + m_2v_3 + \lambda R_{12} v_3) \partial_{v_3} + (-2v_{4x} + m_2u_4 + \lambda R_{12} u_4) \partial_{u_4} + (2u_{4x} + m_2v_4 + \lambda R_{12} v_4) \partial_{v_4}
\end{pmatrix}
\tag{2.9}
\]
while in (2.6)
\[
C_{0}^{\sigma} = 0.
\tag{2.10}
\]
In a similar way,1,2 motivated by the structure of the Lie algebra, we obtain another Lie–Bäcklund transformation, i.e.,
\[
Y^-(2,0) = x^2(\alpha_1 Y_1^- - \alpha_2 m_1 Y_0^- + \alpha_3 m_1 Y_0^+ + \alpha_4 m_1 Y_1^- + \alpha_5 Y_2 - 2(\beta_1 Y_2^- + \beta_2 m_1 Y_1^- + \beta_3 m_1 Y_0^- + \beta_4 m_1 Y_0^+ + \beta_5 Y_1^- + \beta_6 Y_2) + 2t(\gamma_1 Y_1^- + \gamma_2 m_1 Y_1^- + \gamma_3 m_1 Y_0^- + \gamma_4 m_1 Y_0^+ + \gamma_5 Y_1^- + \gamma_6 Y_2) + t^2(\delta_1 Y_1^- + \delta_2 m_1 Y_1^- + \delta_3 m_1 Y_0^- + \delta_4 m_1 Y_0^+ + \delta_5 Y_1^- + \delta_6 Y_2) + xC_1^- + tC_2^- + C_0^-,
\tag{2.11}
\]
where in (2.11)
\[
C_{1}^{-} = \begin{pmatrix}
-2v_{3x} - m_2u_4 - \lambda R_{12} u_3 \partial_{u_3} + (2u_{3x} - m_2v_3 - \lambda R_{12} v_3) \partial_{v_3} + (-2v_{4x} + m_2u_3 + \lambda R_{12} u_4) \partial_{u_4} + (2u_{4x} + m_2v_4 + \lambda R_{12} v_4) \partial_{v_4}
\end{pmatrix}
\tag{2.12}
\]
To give an idea of the action of the vector fields \( Y^+(2,0) \) and \( Y^-(2,0) \), we compute the Lie bracket with the vector fields

\[
Y_t, \quad Y_0^+, \quad Y_{\sim 1}, \quad Y_1^-, \quad Y_0^-, \quad Y_1 = 1
\]

yielding the following results:

\[
\begin{align*}
\{ Y^+(2,0), Y_t \} & = 2Z_t, \\
\{ Y^-(2,0), Y_t \} & = 2Z_0, \\
\{ Y^+(2,0), Y_0^+ \} & = 0, \\
\{ Y^-(2,0), Y_0^- \} & = 0, \\
\{ Y^+(2,0), Y_{\sim 1} \} & = -2Z_{\sim 1}, \\
\{ Y^-(2,0), Y_{\sim 1} \} & = -2Z_1, \\
\{ Y^+(2,0), Y_1^- \} & = 0, \\
\{ Y^-(2,0), Y_1^- \} & = 0
\end{align*}
\]

These results suggest setting \( Y^\pm(i,1) = Z_{\pm} \) and \( Y^\pm(i,0) = Y_{\pm}(i) \) (\( i \in \mathbb{Z} \)).

Now we arrive at the following remarkable fact: the vector fields \( Y^+(2,0) \) and \( Y^-(2,0) \) are again Hamiltonian vector fields, the corresponding Hamiltonian densities being given by

\[
\begin{align*}
\{ Y^+(2,0), Y^+ \} & = (x + t)^2 \mathcal{F}(Y^+) - \mathcal{F}(Y^+), \\
\{ Y^-(2,0), Y^+ \} & = (x + t)^2 \mathcal{F}(Y^+) - \mathcal{F}(Y^+),
\end{align*}
\]

and

\[
\begin{align*}
\{ Y^+(2,0), Y^- \} & = (x + t)^2 \mathcal{F}(Y^-) - \mathcal{F}(Y^-), \\
\{ Y^-(2,0), Y^- \} & = (x + t)^2 \mathcal{F}(Y^-) - \mathcal{F}(Y^-),
\end{align*}
\]

where the densities \( \mathcal{F}(Y^+) \) (\( i = -2, 0, 2 \)) are given in the Appendix.

This result shows a remarkable resemblance to the results for the Benjamin–Ono equation. 9

III. PROOF OF THE EXISTENCE OF AN INFINITE NUMBER OF HIERARCHIES

In this section we shall first prove a generalization of a lemma proved in Ref. 2. The main theorem of this section is a direct application of Lemma 3.1 to the special cases at hand and leads to the existence of an infinite number of infinite hierarchies of algebraically independent conserved functionals for the Federbush model. The associated Lie–Bäcklund transformations are obtained from these results by application of formula (1.4).

We state the following lemma.

**Lemma 3.1:** Let \( H^+_n(u,v), K^+_n(u,v) \) be defined by

\[
\begin{align*}
H^+_n(u,v) & = \int_{-\infty}^{\infty} x^n(u^n_2 + v^n_2), \\
K^+_n(u,v) & = \int_{-\infty}^{\infty} x^n(u^n_2 + v^n_2)
\end{align*}
\]

(3.1)

where in (3.1)

\[
\begin{align*}
u_n & = \left( \frac{d}{dx} \right)^n u, \\
v_n & = \left( \frac{d}{dx} \right)^n v,
\end{align*}
\]

and \( r,n \) such that the degree of \( H^+_n, K^+_n \) is positive. Define the Poisson bracket of functionals \( F,L \) by

\[
\{ F,L \} = \int_{-\infty}^{\infty} \left( \frac{\delta F}{\delta u} \frac{\delta L}{\delta v} - \frac{\delta L}{\delta u} \frac{\delta F}{\delta v} \right),
\]

(3.2)

then

\[
\begin{align*}
\{ H^+_1, H^+_n \} & = 4(n-r)K^+_n, \\
\{ H^+_1, K^+_n \} & = 4(n-r)H^+_n + r(r-1)(r-n-1)H^+_n^{-2}, \\
\{ H^+_1, H^+_1 \} & = 4(2n-r)K^+_n^{-1}, \\
\{ H^+_1, K^+_1 \} & = (2n+1-r)(4H^+_n^{-1} - r^2H^+_n^{-2}) \quad (r,n = 0,1,...).
\end{align*}
\]

(3.3)

**Proof:** Relations (3.3a) and (3.3b) are generalizations of formulas given in Ref. 2 and can be proved in a similar way. We now prove (3.3c) and (3.3b). Calculation of the Fréchet derivatives of \( H^+_n, K^+_n \) yields

\[
\begin{align*}
\frac{\delta H^+_n}{\delta u} & = \left( - \frac{d}{dx} \right)^n (2^n u^n_2), \\
\frac{\delta K^+_n}{\delta u} & = \left( - \frac{d}{dx} \right)^n (2^n u^n_2),
\end{align*}
\]

(3.4a)

Substitution of (3.4a) into (3.2) results in
\{H_1^+, H_2^-\} = \int_{-\infty}^{\infty} - \frac{d}{dx} (2x^2v_1) \cdot (-1)^n \left( \left( \frac{d}{dx} \right)^n (2x^u_u) + \frac{d}{dx} (2x^u_u) \cdot (-1)^n \left( \left( \frac{d}{dx} \right)^n (2x^v_v) \right) \right)

= (-1)^n (-1)^{n-1} \int_{-\infty}^{\infty} \left( \left( \frac{d}{dx} \right)^n (2x^u_u) \cdot \frac{d}{dx} (2x^v_v) \right) - \left( \left( \frac{d}{dx} \right)^n (2x^u_u) \right) \frac{d}{dx} (2x^v_v)

= -4 \int_{-\infty}^{\infty} \left( x^u_{n+1} + 2nxu_n + n(n-1)u_{n-1} \right) (x^u_{n+1} + rx^{-1}v_n)

= -4 \int_{-\infty}^{\infty} \left( x^v_{n+1} + 2nxv_n + n(n-1)v_{n-1} \right) (x^v_{n+1} + rx^1u_n)

= -4 \int_{-\infty}^{\infty} \left( x^u_{n+1} + v_n - v_{n+1} \right) - 2nx'_{n+1} (u_n + v_n)

+ n(n-1)x'_{n+1} (u_n - v_{n+1}) + n(n-1)rx'_{n+1} (v_n - u_{n+1}) = 4(2n-r)K'_{r+1}, (3.5)

which proves relation (3.3c). The last equality in (3.5) results from the fact that the last two terms are just a total derivative of \( n(n-1)x'_{n+1} (u_n - v_{n+1}) \).

In order to prove (3.3d) we substitute (3.4a) and (3.4b) into (3.2), which results in

\{H_2^-, K'_+\} = \int_{-\infty}^{\infty} - \frac{d}{dx} (2x^v_v) \cdot \left( (-1)^n (x^v_v) - (-1)^n \left( \left( \frac{d}{dx} \right)^n (x^v_v) \right) \right)

= \int_{-\infty}^{\infty} \left( \left( \frac{d}{dx} \right)^n (x^v_v) \right) \cdot \left( (-1)^n \left( \left( \frac{d}{dx} \right)^n (x^v_v) \right) \right)

= \int_{-\infty}^{\infty} \left( x^v_{n+2} + 2(n+1)xv_n + n(n+1)v_n \right) (2x^v_{n+1} + rx^1v_n)

+ x^v_{n+2} + 2(n+1)xv_n + n(n+1)v_n) (2x^v_{n+1} + rx^1v_n).

(3.7)

Expanding the expressions in (3.7), we arrive after a short computation at

\{H_2^-, K'_+\} = (2n + 1 - r) (4H'_{r+1} - \gamma^2 H_{r-1}^-), (3.8)

which proves (3.3d).

We are now in a position to prove the main theorem of this section.

**Theorem 3.1** The conserved functionals \( F(Y \pm (2,0)) \) associated to the Lie-Bäcklund transformations \( Y \pm (2,0) \) generate an infinite number of hierarchies, starting at the \( F(Y_+)^{\pm k} \), \( F(Y_-)^{\pm k} \) hierarchies by repeated action of the Poisson bracket.

The \( F(Y_+)^{\pm k} \), \( F(Y_-)^{\pm k} \) hierarchies are obtained by the first step of this procedure [cf. (2.13)]. Moreover, the \( F(Y_+)^{\pm k} \), \( F(Y_-)^{\pm k} \) hierarchies are obtained from \( F(Y_+)^{\pm} \) by repeated action of the conserved functional

\( F \left( Z_{\pm}^{+} \right) = \pm \frac{1}{2} F \left[ \left( Y \pm (2,0), Y \pm (2,0) \right) \right] \) (3.9)

(cf. Table 1).

**Proof**: The proof of theorem 3.1 is a straightforward application of Lemma 3.1 and the observation that the \((\lambda, m_1, m_2)\)-independent parts of the conserved densities associated to \( Y_{\pm 1} \), \( Y \pm (2,0) \), \( Y - (2,0) \), \( (A_3) \), \( (A_4) \), \( (2.14a) \), \( (2.14b) \) are given by

\( Y_{\pm 1} = -\frac{1}{2} (u_{5_2}v_3 - u_3v_5), \quad Y \pm (2,0) = -\left( u_5v_3 + v_5u_3 \right), \) (3.10)

\( Y \pm (k,0) = \alpha_k \left[ \left( Z_{\pm}^{\pm 1} \right) \right]. \)

**IV. Conclusion**

By the construction of two Hamiltonian vector fields \( Y \pm (2,0) \) and \( Y \pm (2,0) \) we construct an infinite number of infinite hierarchies, the elements of which are all Hamiltonian vector fields. The associated conserved functionals are obtained by the action of the Poisson bracket.
APPENDIX: CONSERVED FUNCTIONALS FOR THE FEDERBUSH MODEL

We summarize here some of the results obtained in Ref. 2 that are of interest in Sec. II. We derived the following conserved functionals:

\[ F(Y) = \int_{-\infty}^{\infty} \tilde{F}(\ast) dx, \]  

where the densities \( \bar{F}(\ast) \) are given by

\[ \bar{F}(Y^+_{2}) = \frac{1}{2}(R_1 + R_2), \quad \bar{F}(Y^-_{2}) = \frac{1}{2}(R_3 + R_4), \]  

\[ \bar{F}(Y^+_1) = -\frac{1}{2}(u_{2x}v_2 - u_{2x}v_2) + (\lambda/4)R_{34}R_2 \]  

- \( m_1(u_{x,u_{x}} + v_{v_{x}}) \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{1x}v_1 - u_{1x}v_1) + (\lambda/4)R_{34}R_1 \]  

+ \( m_1(u_{x,u_{x}} + v_{v_{x}}) \),

\[ \bar{F}(Y^+_1) = -\frac{1}{2}(u_{4x}v_4 - u_{4x}v_4) - (\lambda/4)R_{12}R_4 \]  

- \( m_2(u_{3x}v_3 + v_{3x}v_3) \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{3x}v_3 - u_{3x}v_3) - (\lambda/4)R_{12}R_3 \]  

+ \( m_2(u_{3x}v_3 + v_{3x}v_3) \),

and

\[ \bar{F}(Y^+_2) = -\frac{1}{2}(u_{2x}^2 + v_{2x}^2) + (\lambda/2)R_{34}(u_{2x}v_2 - u_{2x}v_2) \]  

- \( m_1(u_{x,u_{x}} - u_{x,v_{x}}) - \frac{1}{2}R_{34}R_2 \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{1x}^2 + v_{1x}^2) + (\lambda/2)R_{34}(u_{1x}v_1 - u_{1x}v_1) \]  

+ \( m_1R_{34}(u_{x,u_{x}} + v_{v_{x}}) - \frac{1}{2}m_1R_{12} \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{4x}^2 + v_{4x}^2) - (\lambda/2)R_{12}(u_{4x}v_4 - u_{4x}v_4) \]  

- \( m_2R_{12}(u_{x,u_{x}} + v_{v_{x}}) - \frac{1}{2}m_2R_{34} \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{3x}^2 + v_{3x}^2) - (\lambda/2)R_{12}(u_{3x}v_3 - u_{3x}v_3) \]  

+ \( m_2R_{12}(u_{x,u_{x}} + v_{v_{x}}) - \frac{1}{2}m_2R_{34} \).

The \( t \)-dependent conserved functionals are

\[ \bar{F}(Y^+_2) = -\frac{1}{2}(u_{2x}^2 + v_{2x}^2) + (\lambda/2)R_{34}(u_{2x}v_2 - u_{2x}v_2) \]  

- \( m_1(u_{x,u_{x}} - u_{x,v_{x}}) - \frac{1}{2}R_{34}R_2 \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{1x}^2 + v_{1x}^2) + (\lambda/2)R_{34}(u_{1x}v_1 - u_{1x}v_1) \]  

+ \( m_1R_{34}(u_{x,u_{x}} + v_{v_{x}}) - \frac{1}{2}m_1R_{12} \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{4x}^2 + v_{4x}^2) - (\lambda/2)R_{12}(u_{4x}v_4 - u_{4x}v_4) \]  

- \( m_2R_{12}(u_{x,u_{x}} + v_{v_{x}}) - \frac{1}{2}m_2R_{34} \),

\[ \bar{F}(Y^-_{1}) = -\frac{1}{2}(u_{3x}^2 + v_{3x}^2) - (\lambda/2)R_{12}(u_{3x}v_3 - u_{3x}v_3) \]  

+ \( m_2R_{12}(u_{x,u_{x}} + v_{v_{x}}) - \frac{1}{2}m_2R_{34} \).
\[ \begin{align*}
\bar{F}(Z^+_0) &= (x + t)\bar{F}(Y^+_1) - (x - t)\bar{F}(Y^-_1), \\
\bar{F}(Z^-_0) &= (x + t)\bar{F}(Y^-_1) - (x - t)\bar{F}(Y^+_1), \\
\end{align*} \]

(A5)

and

\[ \begin{align*}
\bar{F}(Z^+_1) &= (x + t)\bar{F}(Y^+_2) - \frac{1}{4}m_1^2(x - t)\bar{F}(Y^+_0), \\
\bar{F}(Z^-_1) &= -(x - t)\bar{F}(Y^-_2) + \frac{1}{4}m_1^2(x + t)\bar{F}(Y^-_0), \\
\bar{F}(Z^+_2) &= (x + t)\bar{F}(Y^+_3) - \frac{1}{4}m_2^2(x - t)\bar{F}(Y^+_0), \\
\bar{F}(Z^-_2) &= -(x - t)\bar{F}(Y^-_3) + \frac{1}{4}m_2^2(x + t)\bar{F}(Y^-_0).
\end{align*} \]

(A6)

The vector fields \( Y_j^\pm \) \((i = -2, -1, 0, 1, 2)\) and \( Z_j^\pm \) \((j = -1, 0, 1)\) obtained from (A2)-(A6) by

\[ Y = \Omega^{-1} dF(Y). \]

(A7)