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On a pairing heuristic
in binpacking

by

J.B.G. Frenk

Eindhoven, the Netherlands
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ON A PAIRING HEURISTIC IN BINPACKING

ABSTRACT

For the analysis of a pairing heuristic in binpacking an important result is used without proof in [1] and [2].
In this note we discuss this result and give a detailed proof of it.

Introduction
Let \( n \in \mathbb{N} \) be given and suppose \((X_1, \cdots, X_n)\) is a \( n \)-dimensional stochastic vector with joint density \( f(x_1, \cdots, x_n) \).
Moreover assume

(i) \( 0 \leq X_i \leq 1 \quad i = 1, \cdots, n \)
(ii) The stochastic vector \((X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(n)})\) is distributed as \((X_1, X_2, \cdots, X_n)\) for every permutation \(\sigma\) on \(\{1, \cdots, n\}\).
(iii) \( f(x_1, x_2, \cdots, x_n) = f(1-x_1, \cdots, x_n) \)

Remark
Condition (ii) states that we are dealing with a finite sequence of so-called exchangeable random variables (cf. [3]), while condition (iii) is a symmetry condition.
Note that by (ii) the symmetry in (iii) holds in every component.

Before stating the main result introduce the following notations

\[
I_A := \begin{cases} 
1 & \text{if the event } A \text{ happens} \\
0 & \text{otherwise}
\end{cases}
\]

\[
Y_i := (1-X_i)I_{\{X_i > \frac{1}{2}\}} + X_i I_{\{X_i \leq \frac{1}{2}\}}. \quad i = 1, \cdots, n
\]

\[
(\iota):= \begin{cases} 
+1 & \text{if } X_i > \frac{1}{2} \\
-1 & \text{if } X_i \leq \frac{1}{2}
\end{cases} \quad i = 1, \cdots, n
\]

If we order the random variables \(Y_i\) in non-decreasing order, say \(Y_{i_1} \leq Y_{i_2} \leq \cdots \leq Y_{i_n}\), we denote by \((\iota_k)\) the label of the \(k\)-order statistic of the sequence \(\{Y_i\}_{i=1}^{n}\).

Now the main result reads as follows.
Theorem 1

Suppose the random variables \( \{X_i, i \in A\} \) satisfy the conditions (i), (ii) and (iii). Then the following results hold

a) \( \{X_i, i \in A\} \) and \( \{(j), i \in A\} \) are independent for every subset \( A \subset \{1,2, \cdots, n\} \)

b) \( P \{(j_k) = \pi(j_k), k \in A\} = \prod_{k \in A} P \{(j_k) = \pi(j_k)\} = 2^{-|A|} \)

for every subset \( A \subset \{1,2, \cdots, n\} \) and

for every function \( \pi: \{1,2, \cdots, n\} \rightarrow \{-1,1\} \).

Proof For every sequence \( \{y_i\}_{i=1}^n \) with \( y_i \in (0, \frac{1}{2}) \) and \( \sigma \) some permutation on \( \{1, \cdots, n\} \) we obtain

\[
P \{X_{\sigma(i)} \leq y_{\sigma(i)}, (\sigma(i)) = \pi(\sigma(i)), i=1, \cdots, k\} =
\]

\[
= P \{1 - X_{\sigma(i)} \leq y_{\sigma(i)} (i \in C) \land X_{\sigma(i)} \leq y_{\sigma(i)} (i \in \{1, \cdots, k\} - C)\}
\]

where \( 1 \leq k \leq n \) and \( C := \{j: 1 \leq j \leq k \land \pi(\sigma(j)) = 1\} \)

By (ii) and (iii) it follows easily

\[
P \{X_{\sigma(i)} \leq y_{\sigma(i)}, (\sigma(i)) = \pi(\sigma(i)), i=1, \cdots, k\} =
\]

(1) \( P \{X_{\sigma(i)} \leq y_{\sigma(i)}; i = 1, \cdots, k\} = P \{X_i \leq y_{\sigma(i)}; i = 1, \cdots, k\} \)

and this implies

\[
P \{Y_{\sigma(i)} \leq y_{\sigma(i)}; i = 1, \cdots, k\} =
\]

\[
= \sum_{\tau \in D} P \{X_{\sigma(i)} \leq y_{\sigma(i)}, (\tau(i)) = \pi(\sigma(i)); i = 1, \cdots, k\} =
\]

(2) \( = \sum_{\tau \in D} P \{X_i \leq y_{\sigma(i)}; i = 1, \cdots, k\} = 2^k \cdot P \{X_i \leq y_{\sigma(i)}; i = 1, \cdots, k\} \)

where \( D \) is the set of functions \( \tau: \{1,2, \cdots, n\} \rightarrow \{-1,1\} \) which are different on \( \{\sigma(1), \cdots, \sigma(k)\} \).

Moreover by (1)
\( IP \{ (\sigma(i)) = \pi(\sigma(i)); i = 1, \ldots, k \} = \)

\( = IP \{ X_{\sigma(i)} \leq \frac{1}{2}, (\sigma(i)) = \pi(\sigma(i)); i = 1, \ldots, k \} = \)

(3) \( = \{ X_i \leq \frac{1}{2}; i = 1, \ldots, k \} \).

Since the density \( f(x_1, \ldots, x_n) \) is symmetric it is easy to prove that for every \( 1 \leq l \leq n - 1 \)

\( IP \{ X_1 \leq \frac{1}{2}, \ldots, X_l \leq \frac{1}{2} \} = 2 IP \{ X_1 \leq \frac{1}{2}, \ldots, X_{l+1} \leq \frac{1}{2} \} \)

and this implies using \( IP \{ X_1 \leq \frac{1}{2} \} = \frac{1}{2} \) that

(4) \( IP \{ X_1 \leq \frac{1}{2}, \ldots, X_l \leq \frac{1}{2} \} = 2^{-l} \)

Now by the relations (1), (2), (3) and (4)

\( IP \{ X_{\sigma(1)} \leq \gamma_{\sigma(1)}, (\sigma(i)) = \pi(\sigma(i)); i = 1, \ldots, k \} = \)

\( IP \{ X_i \leq \gamma_{\sigma(i)}; i = 1, \ldots, k \} = \)

\( 2^{-k} \cdot 2^k \cdot IP \{ X_i \leq \gamma_{\sigma(i)}; i = 1, \ldots, k \} = \)

\( IP \{ (\sigma(i)) = \pi(\sigma(i)); i = 1, \ldots, k \} \cdot IP \{ X_{\sigma(i)} \leq \gamma_{\sigma(i)}; i = 1, \ldots, k \} \)

and so we have proved the result in (a)

In order to prove the result in (b) we note that for every subset \( A \subset \{ 1, 2, \ldots, n \} \) and every function \( \pi: \{ 1, 2, \ldots, n \} \to \{-1, +1\} \)

\( IP \{ (i_k) = \pi(i_k); k \in A \} = \)

\( = \sum_{\sigma} IP \{ X_{\sigma(1)} \leq \frac{1}{2}, X_{\sigma(1)} \leq X_{\sigma(2)} \leq \cdots \leq Y_{\sigma(n)}, (\sigma(k)) = \pi(\sigma(k)); k \in A \} \)

\( = \sum_{\sigma} IP \{ X_{\sigma(1)} \leq \frac{1}{2}, X_{\sigma(1)} \leq \cdots \leq Y_{\sigma(n)} \} \cdot IP \{ (\sigma(k)) = \pi(\sigma(k)); k \in A \} \)

where we have used (a) to obtain the last equality.

Hence

\( IP \{ (i_k) = \pi(i_k); k \in A \} = \)
\begin{align*}
&= 2^{-|A|} \sum_{\sigma} P\{ \sum_{i=1}^{n} Y_{\sigma(i)} \leq \cdots \cdots \leq \sum_{i=1}^{n} Y_{\sigma(n)} \leq \frac{1}{2} : i = 1, \cdots, n\} \\
&= 2^{-|A|} \prod_{k \in A} P\{ (i_k) = \pi(i_k) \}
\end{align*}

References


