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On a pairing heuristic
in binpacking

by

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October 1986
ON A PAIRING HEURISTIC IN BINPACKING

ABSTRACT

For the analysis of a pairing heuristic in binpacking an important result is used without proof in [1] and [2]. In this note we discuss this result and give a detailed proof of it.

Introduction
Let $n \in \mathbb{N}$ be given and suppose $(X_1, \cdots, X_n)$ is a $n$-dimensional stochastic vector with joint density $f(x_1, \cdots, x_n)$

Moreover assume

(i) $0 \leq X_i \leq 1 \quad i = 1, \cdots, n$
(ii) The stochastic vector $(X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(n)})$ is distributed as $(X_1, X_2, \cdots, X_n)$ for every permutation $\sigma$ on $\{1, \cdots, n\}$.
(iii) $f(x_1, x_2, \cdots, x_n) = f(1-x_1, \cdots, x_n)$

Remark
Condition (ii) states that we are dealing with a finite sequence of so-called exchangeable random variables (cf. [3]), while condition (iii) is a symmetry condition.
Note that by (ii) the symmetry in (iii) holds in every component.

Before stating the main result introduce the following notations

\[ I_A := \begin{cases} 1 & \text{if the event } A \text{ happens} \\ 0 & \text{otherwise} \end{cases} \]

\[ Y_i := (1-X_i)I_{\{X_i > \frac{1}{2}\}} + X_i I_{\{X_i \leq \frac{1}{2}\}} \quad i = 1, \cdots, n \]

\[ (i) := \begin{cases} +1 & \text{if } X_i > \frac{1}{2} \\ -1 & \text{if } X_i \leq \frac{1}{2} \end{cases} \quad i = 1, \cdots, n \]

If we order the random variables $Y_i$ in non-decreasing order, say $Y_{i_1} \leq Y_{i_2} \leq \cdots \leq Y_{i_n}$, we denote by $(i_k)$ the label of the $k$-order statistic of the sequence $\{Y_i\}_{i=1}^n$.

Now the main result reads as follows.
Theorem 1
Suppose the random variables \( \{X_i\}_{i=1}^n \) satisfy the conditions (i), (ii) and (iii).
Then the following results hold

a) \( \{X_i, i \in A\} \) and \( \{(j), i \in A\} \) are independent for every subset \( A \subset \{1, 2, \ldots, n\} \)

b) \[ P \left( \left\{ (i_j) = \pi(i_j), k \in A \right\} \right) = \prod_{k \in A} P \left( (i_j) = \pi(i_k) \right) = 2^{-|A|} \]

for every subset \( A \subset \{1, 2, \ldots, n\} \) and for every function \( \pi: \{1, 2, \ldots, n\} \rightarrow \{-1, 1\} \).

Proof For every sequence \( \{y_i\}_{i=1}^k \) with \( y_i \in (0, 1/2) \) and \( \sigma \) some permutation on \( \{1, \ldots, n\} \) we obtain

\[ P \left( X_{\sigma(i)} \leq y_{\sigma(i)}, \sigma(i)) = \pi(\sigma(i)) \right) = \]

\[ = P \left( 1 - X_{\sigma(i)} \leq y_{\sigma(i)} \left( i \in C \right) \land X_{\sigma(i)} \leq y_{\sigma(i)} \left( i \in \{1, \ldots, k\} - C \right) \right) \]

where \( 1 \leq k \leq n \) and \( C := \{ j: 1 \leq j \leq k \land \pi(\sigma(j)) = 1 \} \)

By (ii) and (iii) it follows easily

\[ P \left( \left\{ X_{\sigma(i)} \leq y_{\sigma(i)} ; \pi(\sigma(i)) \right\} = \pi(\sigma(i)) \right) = \]

\[ (1) \quad P \left( X_{\sigma(i)} \leq y_{\sigma(i)} ; i = 1, \ldots, k \right) = P \left( X_i \leq y_{\sigma(i)} ; i = 1, \ldots, k \right) \]

and this implies

\[ P \left( Y_{\sigma(i)} \leq y_{\sigma(i)} ; i = 1, \ldots, k \right) = \]

\[ = \sum_{\tau \in D} P \left( X_{\sigma(i)} \leq y_{\sigma(i)}, \sigma(i)) = \tau(\sigma(i)) \right) ; i = 1, \ldots, k \]

\[ (2) \quad = \sum_{\tau \in D} P \left( X_i \leq y_{\sigma(i)} ; i = 1, \ldots, k \right) = 2^|D| P \left( X_i \leq y_{\sigma(i)} ; i = 1, \ldots, k \right) \]

where \( D \) is the set of functions \( \tau: \{1, 2, \ldots, n\} \rightarrow \{-1, 1\} \) which are different on \( \{\sigma(1), \ldots, \sigma(k)\} \).

Moreover by (1)
\[ P \{ (\sigma(i)) = \pi(\sigma(i)); i = 1, \cdots, k \} = \]
\[ = P \{ X_{\sigma(i)} \leq \frac{1}{2}, (\sigma(i)) = \pi(\sigma(i)); i = 1, \cdots, k \} = \]
\[ = P \{ X_i \leq \frac{1}{2}; i = 1, \cdots, k \}. \]

Since the density \( f(x_1, \cdots, x_n) \) is symmetric it is easy to prove that for every \( 1 \leq l \leq n - 1 \)
\[ P \{ X_1 \leq \frac{1}{2}, \cdots, X_l \leq \frac{1}{2} \} = 2 \cdot P \{ X_1 \leq \frac{1}{2}, \cdots, X_{l+1} \leq \frac{1}{2} \} \]
and this implies using \( P \{ X_1 \leq \frac{1}{2} \} = \frac{1}{2} \) that
\[ P \{ X_1 \leq \frac{1}{2}, \cdots, X_l \leq \frac{1}{2} \} = 2^{-l} \]

Now by the relations (1), (2), (3) and (4)
\[ P \{ X_{\sigma(i)} \leq y_{\sigma(i)}; (\sigma(i)) = \pi(\sigma(i)); i = 1, \cdots, k \} = \]
\[ P \{ X_i \leq y_{\sigma(i)}; i = 1, \cdots, k \} = \]
\[ = 2^{-k} \cdot 2^k \cdot P \{ X_i \leq y_{\sigma(i)}; i = 1, \cdots, k \} = \]
\[ = P \{ (\sigma(i)) = \pi(\sigma(i)); i = 1, \cdots, k \}. P \{ X_{\sigma(i)} \leq y_{\sigma(i)}; i = 1, \cdots, k \} \]
and so we have proved the result in (a).

In order to prove the result in b) we note that for every subset \( A \subset \{ 1, 2, \cdots, n \} \) and every function \( \pi: \{ 1, 2, \cdots, n \} \to \{ -1, +1 \} \)
\[ P \{ (i_k) = \pi(i_k); k \in A \} = \]
\[ = \sum_{\sigma} P \{ X_{\sigma(1)} \leq \frac{1}{2}, X_{\sigma(1)} \leq \cdots \leq X_{\sigma(n)}; (\sigma(k)) = \pi(\sigma(k)); k \in A \} \]
\[ = \sum_{\sigma} P \{ X_{\sigma(1)} \leq \frac{1}{2}, X_{\sigma(1)} \leq \cdots \leq X_{\sigma(n)} \} \cdot P \{ (\sigma(k)) = \pi(\sigma(k)); k \in A \} \]
where we have used (a) to obtain the last equality.
Hence
\[ P \{ (i_k) = \pi(i_k); k \in A \} = \]
\[ = 2^{-|A|} \sum_{\sigma} P \left( Y_{\sigma(1)} \leq \cdots \leq Y_{\sigma(n)} \leq \frac{1}{2}; i = 1, \cdots, n \right) \]

\[ = 2^{-|A|} \prod_{k \in A} P \left( (i_k) = \pi(i_k) \right) \]

References


