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Published: 01/01/1989

Citation for published version (APA):
ESTIMATES FOR SPHERICAL HARMONICS

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Summary
In this paper we present $L_\infty$- and $L_2$-estimates for spherical harmonics in which partial differentiation operators and multiplication operators are involved. Our results are compared with classical results of Stein, Seeley and Calderón and Zygmund.

A.M.S. Classifications: 26D10, 33A45.
Key Words: spherical harmonics, homogeneous polynomials, reproducing kernels.
1. Introduction

The present discussion starts with the space $\mathbb{R}^q$, $q \geq 3$, provided with the Euclidean inner product $x \cdot y = x_1 y_1 + \cdots + x_q y_q$. By $\sigma^{q-1}$ we denote the surface measure on the unit sphere $S^{q-1}$ in $\mathbb{R}^q$, normalized by the relation

$$dx = r^{q-1} dr \, d\sigma^{q-1}(\omega)$$

Where $dx = dx_1 \cdots dx_q$. So we have

$$\omega_q := \int_{S^{q-1}} d\sigma^{q-1} = 2\pi^{q/2} \Gamma(q/2).$$

Consider the Hilbert space $L_2(S^{q-1})$ of square integrable functions on $S^{q-1}$ with the natural inner product

$$(f,g)_{L_2(S^{q-1})} := \int_{S^{q-1}} f(\xi) g(\xi) \, d\sigma^{q-1}(\xi), f,g \in L_2(S^{q-1}).$$

We introduce the following spaces of polynomials in $q$ variables

- $P^q$: the linear space of all polynomials
- $H^q_m$: the linear space of all $m$-homogeneous harmonic polynomials
- $Y^q_m$: the linear space of all restrictions of polynomials in $H^q_m$ to $S^{q-1}$.

All spaces can be endowed with the $L_2(S^{q-1})$-inner product. The vector spaces $H^q_m$ and $Y^q_m$ are finite dimensional each with dimension $d^q_m$ given by

$$d^q_m = \frac{(2m+q-2)(m+q-3)!}{(q-2)! \, m!}.$$  

And there is the orthogonal decomposition

$$L_2(S^{q-1}) = \bigoplus_{m=0}^\infty Y^q_m.$$  

In $P^q$ we introduce another inner product,

$$(p_1,p_2)_{P^q} := p_1(\nabla) p_2(x) \big|_{x=0}, \, p_1,p_2 \in P^q$$

Where $\nabla$ represents the gradient vector $\left[ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_q} \right]$.

For harmonic homogeneous polynomials this $P^q$-inner product is related to the $L_2(S^{q-1})$-inner product as follows. Let $h_1$ and $h_2$ be harmonic homogeneous polynomials of degree $m$ and $n$ respectively. Then
The Gegenbauer polynomials $C_m^\lambda$, $m \in \mathbb{N}_0$, $\lambda > -1$, introduced in [MOS, p. 218 ff.] obey the orthogonality relations
\[
\int_{-1}^{1} C_m^\lambda(t) C_n^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}} \, dt = \frac{\pi 2^{1-2\lambda} \Gamma(n+2\lambda)}{n! (\lambda+n)! [\Gamma(\lambda)]^2} \delta_{mn}.
\] (ix)

Now for $m \in \mathbb{N}_0$ and $q \geq 3$ we set
\[
P^q_m = \frac{m^! (q-3)^!}{(q+m-3)!} C_m^{q/2-1}.
\] (x)

Then $P^q_m(1) = 1$ and according to (ix)
\[
\int_{-1}^{1} P^q_m(t) P^q_n(t)(1-t^2)^{q-3} \, dt = \frac{\omega_k}{\omega_{q-1}} \delta_{nm}.
\] (xi)

It is a well-known result, cf. [Mil, Lemma 7, p.14], that for all $f \in \mathcal{Y}_m^q$ and $\xi \in S^{q-1}$
\[
f(\xi) = \frac{d_m^q}{\omega_q} \int_{S^{q-1}} P_m^q(\xi \cdot \omega) f(\omega) \, d\sigma^{q-1}(\omega).
\] (xii)

So $(\xi, \omega) \mapsto \frac{d_m^q}{\omega_q} P_m^q(\xi \cdot \omega)$ is the reproducing kernel of the Hilbert space $\mathcal{Y}_m^q$ and for any orthonormal basis $(e^q_{m,j} : 1 \leq j \leq d_m^q)$ of $\mathcal{Y}_m^q$ we have
\[
\sum_{j=1}^{d_m^q} e^q_{m,j}(\xi) e^q_{m,j}(\omega) = \frac{d_m^q}{\omega_q} P_m^q(\xi \cdot \omega).
\] (xiii)

Using the reproducing kernel property of the $P_m^q$, we obtain for all $h \in H_m^q$ and $\xi \in S^{q-1}$
\[
\|h(\xi)\| \leq \left[\frac{d_m^q}{\omega_q} \|P_m^q(\xi, \xi)\|^{\frac{1}{2}} \right] \|h\|_{L^2(S^{q-1})} = \left[\frac{d_m^q}{\omega_q} \right]^{\frac{1}{2}} \|h\|_{L^2(S^{q-1})},
\] (xiv)

In this paper we present various $L_\infty$- and $L_2$-estimates for spherical harmonics.

We conclude the introduction with some notations concerning multi-indices and partial derivatives. By $e^k$ we denote the multi-index with components $e_j^k = \delta_{kj}$, $k, j \in \{1, \ldots, q\}$, and $e = (1, \ldots, 1)$. Furthermore, for all $k \in \mathbb{N}_0$, $x \in \mathbb{R}^q$, and for all multi-indices $\alpha, \beta \in \mathbb{N}_0^q$
\[
x^\alpha = x_1^{\alpha_1} \cdots x_q^{\alpha_q}
\]
\[
|\alpha| = \alpha_1 + \cdots + \alpha_q
\]
\[
\alpha + k \beta = (\alpha_1 + k \beta_1, \ldots, \alpha_q + k \beta_q) \quad \text{(xv)}
\]
\[
\alpha \leq \beta \iff \alpha_1 \leq \beta_1, \ldots, \alpha_q \leq \beta_q
\]
\[
(\alpha)_{\beta} = (\alpha_1)_{\beta_1} \cdots (\alpha_q)_{\beta_q}
\]

Here \((\cdot)_j\) denotes the Pochhammer symbol, which is defined by
\[
(b)_j := \frac{\Gamma(b+j)}{\Gamma(b)}, \quad j = 0, 1, 2, \ldots \quad \text{(xvi)}
\]

Utilizing the inequalities
\[
\binom{n+m-k-l}{n-k} \leq \binom{n+m}{n}, \quad n \geq k \geq 0, \quad m \geq l \geq 0, \quad \text{(xvii)}
\]

it follows that for all multi-indices \(\alpha, \beta \in \mathbb{N}_0^q\) with \(\beta \geq \epsilon\),
\[
(\beta)_\alpha \leq (1/\beta_1 - q + 1)_{\alpha_1}. \quad \text{(xviii)}
\]

By \(\partial_k f\) we denote the partial derivative of \(f\) with respect to the \(k\)-th variable. As usual, for all multi-indices \(\alpha \in \mathbb{N}_0^q\) we write
\[
\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_q^{\alpha_q}. \quad \text{(xix)}
\]

Clearly \(\partial^\alpha\) maps \(H^q_n\) onto \(H^q_{n-\alpha_1}\), where \(H^q_n\) is taken to be the trivial space for \(n < 0\).

2. \(L_\infty\) and \(L_2\)-estimates for spherical harmonics

We start with integrating monomials \(\omega^\beta\) over the unit sphere \(S^{q-1}\).

**Lemma 1.**

Let \(\beta \in \mathbb{N}_0^q\) be a multi-index.

(i) If \(\beta_j\) is odd for some \(j \in \{1, \ldots, q\}\), then \(\int_{S^{q-1}} \omega^\beta \, d\sigma^{q-1}(\omega) = 0\).

(ii) For all multi-indices \(\alpha \in \mathbb{N}_0^q\) satisfying \(\alpha \leq \beta\), we have
\[
\int_{S^{q-1}} \omega^{2\beta} \, d\sigma^{q-1}(\omega) = (\alpha + 1/2, \epsilon)_{\beta - \alpha} \int_{S^{q-1}} \omega^{2\alpha} \, d\sigma^{q-1}(\omega).
\]

**Proof:**

Consider the integral \(\int_{S^{q-1}} \omega^\beta \, d\sigma^{q-1}(\omega)\). Substituting \(\omega = te_1 + \sqrt{1-t^2} \eta\), with \(-1 \leq t \leq 1\) and \(\eta \in \text{span} \{e_2, \ldots, e_q\}\) we derive
\[
\int_{S^{q+1}} \omega^2 \, d\sigma^{q-1}(\omega) = \frac{1}{\omega^{q+1}} \int_{S^1} t^\beta \left(1-t^2\right)^{\frac{q+1}{2}} \, dt \int_{S^{q-1}} \eta_2^b \cdots \eta_q^b \, d\sigma^{q-2}(\eta) = \\
= \left\{ \begin{array}{cl}
0 & \text{if } \beta_1 \text{ is odd} \\
\frac{\Gamma\left(\frac{1}{2} (\beta_1 + 1) \right) \Gamma\left(\frac{1}{2} (q + 1 \beta_1 - \beta_1 - 1) \right)}{\Gamma\left(\frac{1}{2} (q + 1 \beta_1) \right)} \int_{S^{q-1}} \eta_2^b \cdots \eta_q^b \, d\sigma^{q-2}(\eta) & \text{if } \beta_1 \text{ is even}.
\end{array} \right.
\]

Now induction arguments yield the first part of the lemma.

Furthermore, with the aid of the above formula and the relation \( z \Gamma(z) = \Gamma(z+1) \) we obtain

\[
\int_{S^{q+1}} \omega^{2(m+\epsilon)} \, d\sigma^{q-1}(\omega) = \frac{\beta_1 + \frac{1}{2} q}{1 \beta_2 + 1 \beta q} \int_{S^1} \omega^{2b} \, d\sigma^{q-1}(\omega).
\]

And from this formula we deduce for all \( m \in \mathbb{N}_0 \) and \( k \in \{1, \ldots, q\} \),

\[
\int_{S^{q+1}} \omega^{2(m+\epsilon_k)} \, d\sigma^{q-1}(\omega) = \frac{2 \pi^{q/2} (1 \epsilon)_{\beta_k}}{\Gamma\left(\frac{1}{2} q \right) \Gamma\left(\frac{1}{2} q + 1 \beta_k\right)} \int_{S^{q+1}} \omega^{2b} \, d\sigma^{q-1}(\omega).
\]

Using these integral relations the second part of the lemma follows by induction.

Taking \( \alpha = 0 \) in the preceding lemma, we get explicit expressions for integrals of monomials:

**Corollary 2.**

For all multi-indices \( \beta \in \mathbb{N}^q_0 \),

\[
\int_{S^{q+1}} \omega^{2b} \, d\sigma^{q-1}(\omega) = \frac{2 \pi^{q/2} (1 \epsilon)_{\beta_k}}{\Gamma\left(\frac{1}{2} q \right) \Gamma\left(\frac{1}{2} q + 1 \beta_k\right)}.
\]

**Theorem 3.**

Let \( p \) be an \( m \)-homogeneous polynomial and let \( \alpha \in \mathbb{N}^q_0 \) be a multi-index. Then

\[
\| x^\alpha p \|_{L_2(S^{q+1})} \leq \frac{(m+1/2)_{|\alpha|}}{(m+1/2 q)_{|\alpha|}} \| p \|_{L_2(S^{q+1})}.
\]

**Proof.**

The proof is by induction to the length of \( \alpha \in \mathbb{N}^q_0 \). Obviously equality holds for \( \alpha = 0 \). Suppose the estimates are valid for all multi-indices \( \alpha \in \mathbb{N}^q_0 \) with length \( |\alpha| \leq n \) for some \( n \geq 0 \). Let \( \beta \in \mathbb{N}^q_0 \) be a multi-index with length \( |\beta| = n + 1 \). Then there exists \( j \in \{1, \ldots, q\} \) such that \( \beta_j \geq 1 \). Since \( p \) is an \( m \)-homogeneous polynomial, \( x^{2(\beta - \epsilon_j)} \| p \|^2 \) is a \( 2(m+n) \)-homogeneous polynomial. Therefore, \( x^{2(\beta - \epsilon_j)} \| p \|^2 \) is a linear combination of monomials \( x^\gamma \) with \( |\gamma| = 2(m+n) \):
With the aid of Lemma 1 we estimate
\[ \int_{S^{q-1}} \omega^{m+q} |p(\omega)|^2 \, d\sigma^{q-1}(\omega) = \sum_{|\gamma| = 2(m+n)} c(\gamma) \int_{S^{q-1}} \omega^{m+q} \, d\sigma^{q-1}(\omega) = \]
\[ \sum_{|\gamma| = 2(m+n)} c(\gamma) \frac{\gamma_1 + 1}{2(m+n) + q} \int_{S^{q-1}} \omega^q \, d\sigma^{q-1}(\omega) \leq \]
\[ \frac{2(m+n) + 1}{2(m+n) + q} \int_{S^{q-1}} \omega^{2q} |p(\omega)|^2 \, d\sigma^{q-1}(\omega). \]

So using the induction hypothesis we obtain
\[ \int_{S^{q-1}} \omega^{m+q} |p(\omega)|^2 \, d\sigma^{q-1}(\omega) \leq \frac{(m+\frac{1}{2})_{q+1}}{(m+\frac{1}{2})_{q+1}} \int_{S^{q-1}} |p(\omega)|^2 \, d\sigma^{q-1}(\omega) \]
which completes the proof.

Note that equality in the above theorem holds if \( a = m e^k \) and \( p(x) = (x_k)^m \).

Next we pay attention to estimates for partial derivatives of spherical harmonics. But we gather some auxiliary results first.

**Lemma 4.**
Let \( p \) be a homogeneous polynomial of degree \( m \). Then for each \( k \in \{1, \ldots, q\} \)
\[ \int_{S^{q-1}} (\partial_k p)(\omega) \, d\sigma^{q-1}(\omega) = (m+q-1) \int_{S^{q-1}} \omega_k p(\omega) \, d\sigma^{q-1}(\omega). \]

**Proof:**
We may as well assume that \( k = 1 \). Let \( \psi \) denote a compactly supported differentiable function on \( R^+ \). Using the homogeneity of \( \partial_1 p \) we have
\[ \int_0^\infty \psi(r) r^{q-1} \, dr \int_{S^{q-1}} (\partial_1 p)(\omega) \, d\sigma^{q-1}(\omega) = \]
\[ = \int_{R^{q+1}} \{ \int_{R^q} |x_1|^{-m-1} \psi(|x|) (\partial_1 p)(x) \, dx \}_{dx_2 \cdots dx_q}. \]

Applying partial integration the latter integral equals
\[ = \int_0^\infty [(m-1) \psi(x) - x_1 \psi'(x)] x_1 p(x) \, dx_1 \, dx_2 \cdots dx_q = \]
\[ = \int_0^\infty [(m-1) \psi(r) - r \psi'(r)] r^{q-1} \, dr \int_{S^{q-1}} \omega_1 p(\omega) \, d\sigma^{q-1}(\omega) \]
and the result follows.
Corollary 5.
Let $f$ and $g$ be homogeneous polynomials of degree $m$ and $n$. Then for each $k \in \{1, \ldots, q\}$

$$(\partial_k f, g)_{L^2(S^{q-1})} + (f, \partial_k g)_{L^2(S^{q-1})} = (m + n + q - 1) (x_k f, g)_{L^2(S^{q-1})}.$$ 

Proof.
Putting $p = f \cdot \hat{g}$ in Lemma 4, we are done. 

Note that the above relation is a special case of the integral formula established in [Ma, Theorem 1.16]. However, our proof is based on simpler arguments.

Lemma 6.
Let $h \in H^q_m$ and let $k \in \{1, \ldots, q\}$. Then we have

$$\|\partial_k h\|_{L^2(S^{q-1})}^2 = (2m + q - 2)(m + \frac{1}{2}) \|x_k h\|_{L^2(S^{q-1})}^2 - \frac{1}{2} \|h\|_{L^2(S^{q-1})}^2.$$ 

Proof.
Let $f = h$ and $g = \partial_k h$ in the previous corollary. Then we derive, writing $(\cdot, \cdot)$ instead of $(\cdot, \cdot)_{L^2(S^{q-1})}$

$$\|\partial_k h\|_{L^2(S^{q-1})}^2 = (\partial_k h, \partial_k h) = (\partial_k h, \partial_k h) + (h, \partial_k^2 h) =$$

$$= (2m + q - 2)(x_k h, \partial_k h) =$$

$$= (2m + q - 2) \frac{1}{2} [(\partial_k(x_k h), h) + (x_k h, \partial_k h) - (h, h)] =$$

$$= (2m + q - 2) [(m + \frac{1}{2}) (x_k^2 h, h) - \frac{1}{2} (h, h)]$$

where in the last step Corollary 5 is used again. 

As a consequence of Lemma 6 and Theorem 3 we get immediately

Corollary 7.
Let $h \in H^q_m$ and let $k \in \{1, \ldots, q\}$. Then

$$\|\partial_k h\|_{L^2(S^{q-1})} \leq m(2m + q - 2) \|h\|_{L^2(S^{q-1})}.$$ 

We recall that for each multi-index $\alpha \in \mathbb{N}^d$, the operator $\partial^\alpha$ maps $H^q_m$ onto $H^q_{m-1|\alpha|}$. Now we arrive at the main theorem of this paper covering $L^\infty(S^{q-1})$ - and $L^2(S^{q-1})$ - estimates for partial derivatives of spherical harmonics.
Theorem 8.

(i) Let \( \alpha \in \mathbb{N}_0^q \). Then for all \( h \in H^q_m \)

\[
\|\partial^\alpha h\|_{L_2(S^{r-1})}^2 \leq \frac{m! (2m+q-2)!!}{(m-1\alpha)! (2m+q-2-2\alpha)!!} \|h\|_{L_2(S^{r-1})}^2.
\]

(ii) Let \( \alpha \in \mathbb{N}_0^q \). Then for all \( h \in H^q_m \) and \( \xi \in S^{r-1} \)

\[
\Gamma(\partial^\alpha h)(\xi) \leq \left[ \frac{d_{m-1\alpha} m! (2m+q-2)!!}{\omega_q (m-1\alpha)! (2m+q-2-2\alpha)!!} \right]^{1/2} \|h\|_{L_2(S^{r-1})}.
\]

Proof.
The first part of the theorem follows from Corollary 7 and a simple induction argument. The second part is a direct consequence of the inequality (xiv) presented in the introduction and the first part of this theorem.

We now give another, very elegant proof of Theorem 8(i) where the correspondence (viii) between the \( P^q \)-inner product and the \( L_2(S^{r-1}) \)-inner product is utilized.

Second proof of Theorem 8(i).

Let \( \alpha \in \mathbb{N}_0^q \) with \( |\alpha| \leq m \). Since \( h \in H^q_m \), \( h \) can be written as

\[
h(x) = \sum_{|\beta| = m} b(\beta) x^\beta
\]

and so

\[
(\partial^\alpha h)(x) = \sum_{|\beta| = m} (\beta - \alpha + \epsilon)_\alpha b(\beta) x^{\beta-\alpha}.
\]

From relation (viii) it follows that

\[
\pi^{-q/2} 2^{m-1} \Gamma(m + \frac{1}{2} q) (h,h)_{L_2(S^{r-1})} = (h,h)_{P^q} = \sum_{|\beta| = m} \beta! |b(\beta)|^2.
\]

The polynomial \( \partial^\alpha h \) belongs to \( H^q_{m-|\alpha|} \) and so, applying (viii) once more and using (xviii) thereafter, we estimate

\[
\pi^{-q/2} 2^{m-|\alpha|+1} \Gamma(m - |\alpha| + \frac{1}{2} q) (\partial^\alpha h, \partial^\alpha h)_{L_2(S^{r-1})} =
\]

\[
= (\partial^\alpha h, \partial^\alpha h)_{P^q} = \sum_{|\beta| = m} \beta! |(\beta - \alpha + \epsilon)_\alpha b(\beta)|^2 \leq
\]

\[
\leq (m - |\alpha| + 1)_\alpha \sum_{|\beta| = m} \beta! |b(\beta)|^2 =
\]

\[
= \frac{m!}{(m - |\alpha|)!} \cdot \pi^{-q/2} 2^{m-1} \Gamma(m + \frac{1}{2} q) (h,h)_{L_2(S^{r-1})},
\]

yielding the wanted result. \( \square \)
Using Stirling's formula we obtain from Theorem 8(ii) the following asymptotic formula.

**Corollary 9.**
For each multi-index \( \alpha \in \mathbb{N}_0^q \) there exists a constant \( C_{q, \alpha} \alpha_1 > 0 \) such that for all \( h \in H^2_m \) and \( \xi \in S^{q-1} \)

\[
| \langle \partial^\alpha h \rangle (\xi) | \leq C_{q, \alpha} \alpha_1 \cdot m \cdot q^{q-1+\alpha_1} \| h \|_{L^2(S^{q-1})}.
\]

Although the result of Corollary 9 is known (cf. [CZ, formula (4)] or [Se, Theorem 4b]) our proofs are more concise and based on elementary techniques.

3. **Comparison with corresponding results.**

Stein has proved the following pointwise estimate, cf. [St, p.276].

For any multi-index \( \alpha \in \mathbb{N}_0^q \) there exists a constant \( C_{q, \alpha} \) such that for all \( h \in H^2_m \) and for all \( \xi \in S^{q-1} \)

\[
| \langle \partial^\alpha h \rangle (\xi) | \leq C_{q, \alpha} \cdot m \cdot q^{q-1+\alpha_1} \| h \|_{L^2(S^{q-1})}.
\]  (xx)

From Corollary 9 we see that instead of \( m \cdot q^{q-1+\alpha_1} \), in Stein's estimate, we can take \( m \cdot q^{q-1+\alpha_1} \). This improvement by a factor \( m \) has consequences for the continuity and compactness of the differential operators \( \partial^\alpha \) on certain function spaces of harmonic functions (cf. [Li2, Theorem 7]).

Liu and Martens proved in [LM, Theorem 4] the following result.

Let \( k \in \{1, \ldots, q\} \), let \( \xi \in S^{q-1} \) and let \( h \in H^2_m \). Then

\[
| \langle \partial_k h \rangle (\xi) | \leq m \left( \frac{d_{\alpha_k}^m}{\omega_q} \right)^{\frac{q}{2}} \| h \|_{L^2(S^{q-1})}. 
\]  (xxi)

We shall show that this estimate is best possible. According to the estimate of Theorem 8(ii), we get

\[
| \langle \partial_k h \rangle (\xi) | \leq \left( \frac{2m+q-4}{m+q-3} \right)^{\frac{q}{2}} \cdot m \left( \frac{d_{\alpha_k}^m}{\omega_q} \right)^{\frac{q}{2}} \| h \|_{L^2(S^{q-1})}. 
\]  (xxii)

So the estimate (xxi), settled by Liu and Martens, is about a factor \( \sqrt{2} \) sharper than ours. In their proof they explore the Gegenbauer polynomials. Using an orthonormal basis \( \{ c_{m,j}^k : 1 \leq j \leq d_{\alpha_k}^m \} \) in \( H^2_m \), they write
\[ h(\xi) = \sum_{j=1}^{d_n} (h, e_{m,j}^n) L_2(S^{r-1}) e_{m,j}^n(\xi). \]  

Hence, by the Cauchy-Schwarz inequality,

\[ |(\partial_k h)(\xi)| \leq \|h\|_{L_2(S^{r-1})} \left( \sum_{j=1}^{d_n} \left| (\partial_k e_{m,j}^n)(\xi) \right|^2 \right)^{1/2}. \]

It turns out that

\[ \sum_{j=1}^{d_n} \left| (\partial_k e_{m,j}^n)(\xi) \right|^2 = \left[ m^2 \xi_k^2 P_q^m(1) + (1 - \xi_k^2) P_q^m(1) \right] \frac{d_m^q}{\omega_q} \]

from which the result can be obtained.

Take \( k = 1 \) and \( \xi = e_1 = (1, 0, \ldots, 0) \). We define \( h_0 \in H^q_m \) by

\[ h_0 = \sum_{j=1}^{d_n} (\partial_1 e_{m,j}^n)(e_1) e_{m,j}^n. \]

Then

\[ |(\partial_1 h_0)(e_1)| = m \left( \frac{d_m^q}{\omega_q} \right)^{1/2} \|h_0\|_{L_2(S^{r-1})}. \]

So their estimate is best possible. This answers a question raised in [LM]. We note that in [LM] no estimates for \(|(\partial^\alpha h)(\xi)|\) with \(|\alpha| > 1\) are given. In fact, their techniques are not appropriate for deriving these estimates. Finally we present another method, based on the reproducing kernel of \( \mathbb{Y}^q_{m-1} \), to obtain supremum norm estimates for the partial derivatives \( \partial_k h \). Let \( h \in H^q_m \) and let \( k \in \{1, \ldots, q\} \). Using the reproducing kernel of \( \mathbb{Y}^q_{m-1} \), the orthogonality relations of spherical harmonics, Corollary 5 and the Cauchy-Schwarz inequality we get for all \( \xi \in S^{q-1} \),

\[ |(\partial_k h)(\xi)| = (2m + q - 2) \frac{d_{m-1}^q}{\omega_q} \int_{S^{q-1}} \omega_k h(\omega) P_{m-1}^q(\xi \cdot \omega) d\sigma^{q-1}(\omega) \leq \]

\[ \leq (2m + q - 2) \frac{d_{m-1}^q}{\omega_q} \|h\|_{L_2(S^{q-1})} \left( \int_{S^{q-1}} \omega_k^2 P_{m-1}^q(\xi \cdot \omega)^2 d\sigma^{q-1}(\omega) \right)^{1/2}. \]

If we take \( \xi = e_k \) and substitute \( \omega = te_k + \sqrt{1 - t^2} \eta \) in the integral, a tedious calculation of the integral yields

\[ |(\partial_k h)(e_k)| \leq B_m^q \left( \frac{d_m^q}{\omega_q} \right)^{1/2} \|h\|_{L_2(S^{q-1})}, \]  

with
\[
B_m^q = \left[ 1 - \frac{(m+q-2)(m+q-3)(m+\frac{1}{2}q-3)+(m-1)(m-2)(m+\frac{1}{2}q-1)}{2(2m+q-4)(m+\frac{1}{2}q-1)(m+\frac{1}{2}q-3)} \right]^{\frac{1}{2}}.
\]

\[
\cdot \left[ \frac{(2m+q-2)(2m+q-4)m}{m+q-3} \right]^{\frac{1}{2}}.
\]

Since \( B_m^q > m \), an exact calculation of the integral on the right hand side of (xxviii) will yield an inequality which is less accurate than the inequality (xxi) of Liu and Martens. Hence equality in (xxviii) will never be attained. This leads to the following observation.

In his thesis [Li1, p.104], Liu proves that the function \( \rho_0 \rightarrow \rho_0 h(\rho) \), defined on the sphere \( S^{q-1} \), extends to a harmonic polynomial in the space \( H^q_{m-1} \oplus H^q_{m+1} \), i.e. there exist polynomials \( p_{m-1} \in H^q_{m-1} \) and \( p_{m+1} \in H^q_{m+1} \) such that

\[
p_{m-1}(\omega) + p_{m+1}(\omega) = \omega_k h(\omega), \quad \omega \in S^{q-1}.
\]

Since equality in (xxviii) will not occur it follows that neither \( p_{m-1} \) nor \( p_{m+1} \) is identically equal to zero.
References


[Li2] Liu, G.-Z., Hilbert spaces of harmonic functions in which differentiation operators are continuous or compact, RANA 89-09, June 1989.


