An input recursive minimal realization algorithm

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by

J.W. van der Woude

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An input recursive minimal realization algorithm

J. W. van der Woude

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Abstract:

Given a minimal realization realizing the impulse response of a number of input channels, and given the impulse response of an additional input channel, a method will be derived for obtaining a minimal realization realizing the impulse response of this larger set of input channels.

It will be shown that the originally given minimal realization may form part of the new obtained minimal realization. This construction will be the basis of the minimal realization algorithm derived in this paper. It will be shown that besides the determination of the rank of Hankel matrices the algorithm only requires the solving of linear equations.

Keywords: Multi-input systems, Minimal realization theory, Input channel recursive approach, Hermite canonical forms.

1. Introduction

Given the impulse response of a linear finite dimensional time invariant discrete time system there are several methods for obtaining a minimal realization (cf.[1],[2],[4]).

In this paper we will approach the minimal realization problem in a way that differs from existing methods. The result of this approach will be a minimal
realization algorithm that is recursive with respect to the number of input channels.

The approach will be as follows:

Suppose that the system has \( m > 1 \) input channels and that these are numbered 1 to \( m \). Furthermore suppose that for a given integer \( k \), with \( 1 \leq k < m \), a minimal realization with respect to the first \( k \) input channels is known and that the impulse response of the \((k + 1)\)-th input channel is available.

In this paper (Section 4) we shall describe a method for obtaining a minimal realization with respect to the first \( k + 1 \) input channels such that the existing realization with respect to the first \( k \) input channels is an actual part of this new minimal realization. This part, therefore, need not to be computed again.

In Section 3 we recall some results concerning the computation of a minimal realization from the impulse response for systems with only one input channel. Section 5 contains the description of our minimal realization algorithm. The algorithm basically consists of the application of the results of Sections 3 and 4 and will be illustrated by means of an example. Section 6 contains some concluding remarks. In Section 2 we will recall some preliminary results.

2. Preliminaries

In this section we shall give some notation and recall some facts concerning Hankel matrices and realizations.

Throughout this paper we shall denote the set of real vectors with \( s \) components by \( R^s \), and the set of real \( s \times t \) matrices by \( R^{s \times t} \).

For certain integers \( s \) and \( t \), let \( T := \{ T_i \}_{i=1}^{\infty} \) be an infinite sequence of matrices in \( R^{s \times t} \).

Define the two-sided infinite block Hankel matrix \( Ha(T) \) and, for given integers \( k, l \geq 1 \), the finite block Hankel matrix \( Ha(T,k,l) \) as

\[
Ha(T) := \begin{pmatrix}
T_1 & T_2 & T_3 & \ldots \\
T_2 & T_3 & \ldots & \ldots \\
T_3 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\[
Ha(T,k,l) := \begin{pmatrix}
T_1 & T_2 & \ldots & T_l \\
T_2 & T_3 & \ldots & T_{l+1} \\
\ldots & \ldots & \ldots & \ldots \\
T_k & T_{k+1} & \ldots & T_{k+l-1}
\end{pmatrix}.
\]

We shall make use of the following well-known theorem (cf.\([1],[2]\)).
Theorem 1.

\[ \text{Rank } H_a(T) = n \text{ if and only if} \]
\[ \text{for every } i \geq n : \text{rank } H_a(T, i, n) = \text{rank } H_a(T, i, n + 1) = n. \]

In this paper we shall consider linear finite dimensional time invariant discrete time systems. We assume that the systems have \( m \) input channels and \( p \) output channels. This means that if \( u \) represents the input and \( y \) represents the output of the system then \( u \) is in \( \mathbb{R}^m \) and \( y \) in \( \mathbb{R}^p \).

In this paper we let \( Q := \{ Q_i \}_{i=1}^\infty \) be the impulse response of a system with \( Q_i \) in \( \mathbb{R}^{p \times m} \) denoting the \( i \)-th Markov parameter.

If, for some integer \( n \), there exist matrices \( A \) in \( \mathbb{R}^{n \times n}, B \) in \( \mathbb{R}^{n \times m} \) and \( C \) in \( \mathbb{R}^{p \times n} \) such that for every \( i \geq 1 : CA^{i-1}B = Q_i \) then we shall call the triple \( (A, B, C) \) a realization of \( Q \) of order \( n \).

The triple \( (A, B, C) \) is called a minimal realization of \( Q \) if \( n \) is as small as possible. In the sequel we shall use \( R_n(A, B) \) for the compound matrix \( [B, AB, ..., A^{n-1}B] \) and \( Q_n(A, C) \) for the compound matrix

\[
\begin{bmatrix}
  CA \\
  CA^2 \\
  \vdots \\
  CA^{n-1}
\end{bmatrix}
\]

Define the two-sided infinite block Hankel matrix \( M \) as \( M := Ha(Q) \) and, for given \( k, l \geq 1 \), the finite block Hankel matrix \( M_{k, l} \) as \( M_{k, l} := Ha(Q, k, l) \). Now we mention the following well-known results (cf. [1],[2],[4]).

Theorem 2.

(i) \( \text{Rank } M = n \text{ if and only if any minimal realization of } Q \text{ is of order } n. \)

If the triple \( (A, B, C) \) is a realization of \( Q \) of order \( n \) then we have

(ii) \( \text{the triple } (A, B, C) \text{ is a minimal realization of } Q \text{ if and only if} \)
\[ \text{rank } R_n(A, B) = n \text{ and rank } Q_n(A, C) = n. \]
3. Single input systems \((m = 1)\)

In this section we shall review some results concerning the construction of a minimal realization from the impulse response for systems with only one input channel. In that case the Markov parameters \(Q_i\) as defined in Section 2 are matrices in \(R^{p \times 1} \cong R^p\). Let the matrices \(M\) and, for given \(k, l \geq 1\), \(M_{k, l}\) be defined as in Section 2. We note that \(M_{k, l}\) is a matrix in \(R^{p \times k \times l}\).

Assume that \(\text{rank } M = n\), then by Theorem 1 we know that for every \(i \geq n\) \(\text{rank } M_{i, n} = \text{rank } M_{i, n+1} = n\). Therefore there exists a uniquely determined vector \(x\) in \(R^n\) such that for every \(i \geq 1\):

\[
M_{i, n+1} \begin{bmatrix} x \\ -1 \end{bmatrix} = 0.
\]

Write \(x = [x_1, x_2, \ldots, x_n]^T\) where \(^T\) denotes transposition. Then we have the following recurrence relation.

For every \(i \geq 1\) \(Q_1 x_1 + Q_{i+1} x_2 + \cdots + Q_{i+n-1} x_n = Q_{i+n}\).

Define matrices \(A\) in \(R^{nxn}\), \(B\) in \(R^{nx1}\) and \(C\) in \(R^{pxn}\) by

\[
A := \begin{bmatrix}
0 & 0 & \cdots & x_1 \\
1 & 0 & \cdots & x_2 \\
0 & 1 & \cdots & x_3 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 1 & x_{n-1} \\
0 & 0 & 0 & 1 & x_n
\end{bmatrix},
B := \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix},
C := [Q_1, Q_2, \ldots, Q_n].
\]

By induction we may now prove that for every \(i \geq 1\):

\[
CA^{i-1} = [Q_1, Q_{i+1}, \ldots, Q_{i+n-1}]
\]

and therefore we have for every \(i \geq 1\) \(CA^{i-1}B = Q_i\).

Consequently the triple \((A, B, C)\) is a realization of \(Q\). The triple \((A, B, C)\) is even a minimal realization of \(Q\) since its order is \(n\) which by Theorem 2 is the order of any minimal realization of \(Q\).

From the above-mentioned facts we may conclude that in case of one input channel, once the rank of the infinite Hankel matrix is determined, the construction of a minimal realization can be achieved by solving a finite number of linear equations.

In the following section we shall describe a method for obtaining a minimal realization with respect to the first \(j + 1\) input channels based on an existing minimal realization with respect to the first \(j\) input channels and the impulse response of
the \((j + 1)\)-th input channel. The minimal realization obtained will be such that the existing minimal realization with respect to the first \(j\) input channels forms part of it.

We will show that after the determination of the rank of an infinite Hankel-like matrix, the construction of the minimal realization with respect to the first \(j + 1\) input channels also can be achieved by solving a finite number of linear equations.

Before describing this construction we shall give some necessary notation.

4. Multi-input systems \((m > 1)\)

In this section we let \(j\) be a fixed integer with \(1 \leq j < m\). We define matrices \(T_i\) in \(R^{p \times j}\) and \(T'\) in \(R^{p \times (j + 1)}\) to be the matrices formed of the first \(j\) and the first \(j + 1\) columns of \(Q_i\), respectively, where the matrices \(Q_i\) are the Markov parameters as defined in Section 2.

Furthermore, in this section we shall assume that a minimal realization with respect to the first \(j\) input channels is available. Hence, we know the minimal \(n\) and matrices \(A\) in \(R^{n \times n}\), \(B\) in \(R^{n \times j}\) and \(C\) in \(R^{p \times n}\) such that for every \(i \geq 1\):

\[
CA_{i-1}B = T_i.\]

Define \(q := \{q_i\}_i\) to be the impulse response of the \((j + 1)\)-th input channel, where \(q_i\) is the \((j + 1)\)-th column of \(Q_i\). Note that we can write \(T_i' = [T_i,q_i]\).

Define the two-sided infinite block Hankel matrices \(H\) and \(H'\) as \(H := Ha(T)\) and \(H' := Ha(T')\). And, for given \(k, l \geq 1\), define the finite block Hankel matrices \(H_{k,l}, H'_{k,l}\) and \(G_{k,l}\) as \(H_{k,l} := Ha(T,k,l)\), \(H'_{k,l} := Ha(T',k,l)\) and \(G_{k,l} := Ha(q,k,l)\).

Note that \(H'_{k,l} = [H_{k,l},G_{k,l}]P_{k,l}\) where \(P_{k,l}\) is some permutation matrix. From this we can conclude that for every \(k, l \geq 1\):

\[
\text{rank} H'_{k,l} = \text{rank} [H_{k,l},G_{k,l}] = n'.
\]

In the remainder of this section we will assume that \(\text{rank} H' = n'\). By the definitions of \(H\) and \(H'\) it is clear that \(\text{rank} H' \geq \text{rank} H\), i.e. \(n' \geq n\). Write \(n' = n + r\) with \(r \geq 0\).

We may now state the following theorem.
Theorem 3.

\[ \text{Rank} H' = n' = n + r \text{ if and only if } \]

for every \( i \geq n + r : \text{rank} [Q(A,C), G_{i,r}] = \text{rank} [Q(A,C), G_{i,r+1}] = n + r. \] (2)

The proof of this theorem is deferred to the appendix.

Note that the compound matrices \([Q(A,C), G_{i,r}]\) and \([Q(A,C), G_{i,r+1}]\) consist of \(n + r\) and \(n + r + 1\) columns, respectively.

From Theorem 3 we can conclude that if \( \text{rank} H' = n' \) then there exists a uniquely determined vector in \( \mathbb{R}^{n+r} \), written as \( \begin{bmatrix} x \\ y \end{bmatrix} \) with \( x \) in \( \mathbb{R}^n \) and \( y \) in \( \mathbb{R}^r \) such that for every \( i \geq 1 \):

\[ [Q(A,C), G_{i,r+1}] \begin{bmatrix} x \\ y \\ -1 \end{bmatrix} = 0. \] (3)

We shall now distinguish two cases.

1. \( n' = n \ (r = 0) \).
2. \( n' > n \ (r > 0) \).

1. If \( n' = n \) then the vector \( y \) in (3) does not exist and by the definitions of \( Q(A,C) \) and \( G_{i,1} \) we can conclude that the vector \( x \) in \( \mathbb{R}^n \) in (3) satisfies for every \( i \geq 1 \):

\( CA_{i-1}^{-1} x = q_i \).

Let \( A', B' \) and \( C' \) be matrices in \( \mathbb{R}^{n \times n}, \mathbb{R}^{n \times (j+1)} \) and \( \mathbb{R}^{p \times n} \), respectively, defined by \( A' := A, B' := [B, x] \) and \( C' := C \). We now have for every \( i \geq 1 \):

\( C' A_{i-1}^{-1} B' = CA_{i-1}^{-1} [B, x] = [T_i, q_i] = T_i' \). Hence, the triple \((A', B', C')\) is a realization of \( T \). In fact, it is even a minimal realization since its order \( n' \) by Theorem 2 equals the order of any minimal realization of \( T \).

2. If \( n' > n \) then by the definitions of \( Q(A,C) \) and \( G_{i,r+1} \) the vectors \( x \) in \( \mathbb{R}^n \) and \( y \) in \( \mathbb{R}^r \) in (3) satisfy for every \( i \geq 1 \):

\( CA_{i-1} x + [q_i, q_{i+1}, \ldots, q_{i+r-1}] y = q_{i+r} \).

Write \( x = [x_1, x_2, \ldots, x_n]^T \) and \( y = [y_1, y_2, \ldots, y_r]^T \) and define matrices \( A' \) in \( \mathbb{R}^{n \times n} \), \( B' \) in \( \mathbb{R}^{n \times (j+1)} \) and \( C' \) in \( \mathbb{R}^{p \times n} \) as

\[ A' := \begin{bmatrix} A & \hat{A} \\ 0 & \hat{A} \end{bmatrix}, \quad B' := \begin{bmatrix} B & 0 \\ 0 & \tilde{B} \end{bmatrix}, \quad C' := [C, \tilde{C}], \]

with \( \tilde{A} \) in \( \mathbb{R}^{r \times r} \), \( \hat{A} \) in \( \mathbb{R}^{n \times r} \), \( \tilde{B} \) in \( \mathbb{R}^{r \times 1} \) and \( \tilde{C} \) in \( \mathbb{R}^{p \times r} \) given by
\[
\tilde{A} = \begin{bmatrix}
0 & 0 & \cdots & y_1 \\
1 & 0 & \cdots & y_2 \\
0 & 1 & \cdots & 0 \\
. & . & \cdots & \cdot \\
0 & 0 & \cdots & 1
\end{bmatrix}, \quad \hat{A} = \begin{bmatrix}
0 & 0 & \cdots & x_1 \\
0 & 0 & \cdots & x_2 \\
. & . & \cdots & . \\
0 & 0 & \cdots & x_n
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
1 \\
0 \\
. \\
0
\end{bmatrix}
\]

\[\tilde{C} = [q_1, q_2, \ldots, q_r].\]

Then we have the following:

\[C' A^{d-1} = C' A' A^{d-2} = \begin{bmatrix} C, [q_1, q_2, \ldots, q_r] \end{bmatrix} \begin{bmatrix} A & \hat{A} \\ 0 & \hat{A} \end{bmatrix} A^{d-2} = \begin{bmatrix} CA, [q_2, q_3, \ldots, q_{r+1}] \end{bmatrix} A^{d-2}.\]

Using induction we may prove that for every \( i \geq 1 \):

\[C' A^{d-1} = [CA^{i-1}, [q_i, q_{i+1}, \ldots, q_{i+r-1}]].\]  \hspace{1cm} (4)

From (4) it is clear that for every \( i \geq 1 \):

\[C' A^{d-1} B' = C' A^{d-1} \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} = [CA^{i-1} B, q_i] = [T_i, q_i] = T'.\]

Therefore the triple \((A', B', C')\) is a realization of \( T' \) of order \( n' \). Again, this realization even is a minimal one since its order, by Theorem 2, equals the order of any minimal realization of \( T' \).

From (4) it follows that

for every \( i \geq 1 \): \(Q_i(A', C') = [Q_i(A, C), G_{i,r}]\). \hspace{1cm} (5)

5. A minimal realization algorithm

In this section we shall combine the results of the previous sections in order to obtain a minimal realization algorithm. First we shall introduce some necessary notation.

Let \( Q = \{Q_i\}_{i=1}^{m} \) be the impulse response of the system as defined in Section 2. We shall denote the \( j \)-th column of \( Q \) by \( q_i^j \) and define \( q^j := \{q_i^j\}_{i=1}^{m} \). Note that the
The latter represents the impulse response of the \( j \)-th input channel. For given \( k,l \geq 1 \) we define the finite block Hankel matrix \( G_{k,l} \) to be \( Ha(q^j,k,l) \).

We shall denote a minimal realization with respect to the first \( j \) input channels of order \( n_j \) by \( (A_j,B_j,C_j;n_j) \). The matrices \( A_j, B_j \) and \( C_j \) are in \( \mathbb{R}^{n_j \times m_j} \), \( \mathbb{R}^{m_j \times p} \) and \( \mathbb{R}^{p \times m_j} \), respectively.

Combination of the results of Sections 3 and 4 yields the following minimal realization algorithm.

(i) Determine a minimal realization \( (A_1,B_1,C_1;n_1) \) with respect to the first input channel based on \( q^1 \) as described in Section 3.

(ii) Let \( j \) take values from 1 to \( m-1 \) and for each value of \( j \) apply the results of Section 4 in order to obtain \( (A_{j+1},B_{j+1},C_{j+1};n_{j+1}) \) based on \( (A_j,B_j,C_j;n_j) \) and \( q^{j+1} \).

Note that \( n_1 \leq n_2 \leq \cdots \leq n_m \), and write \( r_j = n_{j+1} - n_j \) with \( r_0 = n_1 \). By writing Expression (5) in the notation of this section and applying the result to the algorithm we obtain for any \( j \) with \( 1 \leq j < m \) and for every \( i \geq 1 \):

\[
Q_i(A_j,C_{j+1}) = [Q_i(A_j,C_j),G_{i,r_j}^{i+1}] = [G_i^1, G_i^2, \ldots, G_i^{i,r_j}] .
\]  

(6)

And therefore, since the indices \( r_0,r_1,...,r_{j-1} \) are known, there is no need to actually evaluate \( Q_i(A_j,C_j) \).

The rank condition of Theorem 3 can now be written as:

For every \( i \geq n + r \) :
\[
\text{rank}[G_i^1, G_i^2, \ldots, G_i^{i,r_j} \mid G_i^{i+1,r_j}] = \\
\text{rank}[G_i^1, G_i^2, \ldots, G_i^{i,r_j} \mid G_i^{i+1,r_j+1}] = n_j + r_j .
\]  

(7)

With these observations the algorithm may be described in more detail as follows.

(i) Determine a minimal realization \( (A_1,B_1,C_1;n_1) \) with respect to the first input channel based on \( q^1 \) and set \( r_0 = n_1 \).

(ii) Let \( j \) take values from 1 to \( m-1 \) and for each value of \( j \)

(a) determine \( r_j \) such that Condition (7) is satisfied and set \( n_{j+1} = n_j + r_j \).

(b) apply the construction of Section 4 to obtain \( (A_{j+1},B_{j+1},C_{j+1};n_{j+1}) \) from \( (A_j,B_j,C_j;n_j) \) and \( q^{j+1} \).

We shall illustrate the algorithm by means of an example.
Example.

\( m = 3, p = 1 \). Let

\[
Q = \{ Q_i \}_{i=1}^n = \{ [-1,-1],[2,2,3],[0,9,-4],[9,7,-12],[2,-34,-16],[36,19,-79],[17,-129,-111] \}.
\]

Application of the algorithm yields the following.

Step (i) results in

\[
\begin{align*}
n_1 &= r_0 = 3, \\
G_{i,r_0+1} &= \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 9 \\ 0 & 9 & 2 \\ 9 & 2 & 36 \\ 0 & 2 & 17 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = [1 \ 2 \ 0].
\end{align*}
\]

Step (ii) successively yields:

\[
\begin{align*}
j &= 1, r_1 = 0, n_2 = 3, \\
\begin{bmatrix} G_{i,r_0}, G_{i,r_1} \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 9 & 2 \\ 0 & 9 & 2 & -9 \\ 9 & 2 & 36 & 7 \end{bmatrix}, \\
A_2 &= A_1, \quad C_2 = C_1, \quad B_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\end{align*}
\]

\[
j &= 2, r_2 = 2, n_3 = 5, \\
\begin{bmatrix} G_{i,r_0}, G_{i,r_1}, G_{i,r_2} \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 0 & -1 & 3 & 4 \\ 2 & 0 & 9 & 3 & 4 & -12 \\ 0 & 9 & 2 & 4 & -12 & -16 \\ 9 & 2 & 36 & -12 & -16 & -79 \\ 2 & 36 & 17 & -16 & -79 & -111 \end{bmatrix}
\]

Note that \( G_{i,r_1}^2 \) does not exist.

\[
A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_3 = [1 \ 2 \ 0 \ -1 \ 3].
\]
Finally, we may conclude that \((A_3,B_3,C_3)\) is a minimal realization of the impulse response \(\{Q_i\}_{i=1}^\infty\).

### 6. Conclusions and remarks

a) After termination of the algorithm, we have obtained a state space representation of the system in what is known as a controllable Hermite form. For more details concerning canonical Hermite forms we refer to [3], [4]. The nice structure of the state space representation obtained may be profitable for several control problems.

b) In the realization algorithm presented no matrix inversion is required. It is only required to solve a finite number of linear equations. Furthermore, by renumbering the input channels alternative minimal state space representations of the system can be obtained.

c) The algorithm presented simultaneously determines structure parameters (the Hermite indices \(r_0, r_1, \ldots, r_{m-1}\)) and corresponding system parameters. Note that the number of system parameters is \(m r_0 + (m - 1) r_1 + (m - 2) r_2 + \cdots + r_{m-1}\).

Renumbering the input channels may give rise to a different set of structure indices and may reduce the number of system parameters.

d) The recursive feature of the algorithm may be helpful for programming the algorithm.

e) Analogously to the previous sections, a minimal realization algorithm that is recursive with respect to the number of outputs can be derived. This algorithm applied to the impulse response of the example would then yield the following state space representation.

\[
A'_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -3 & 4 \end{bmatrix}, \quad B'_1 = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & 3 \\ 0 & -9 & 4 \\ 9 & 7 & -12 \end{bmatrix}, \quad C'_1 = [1 \ 0 \ 0 \ 0 \ 0].
\]
References


Appendix

Proof of Theorem 3.

From Theorem 1 and (1) it follows that rank $H' = n' = n + r$ if and only if

for every $i \geq n'$: rank $[H_{i,n'}^i, G_{i,n'}^i] = \text{rank} [H_{i,n'+1}^i, G_{i,n'+1}^i] = n'$.

(A1)

Note that for every $i,j \geq 1$: $H_{i,j} = Q_i(A,C) R_j(A,B)$. Therefore we have for every $i,j \geq 1$:

$$[H_{i,j}, G_{i,j}] = [Q_i(A,C), G_{i,j}] \begin{bmatrix} R_j(A,B) & 0 \\ 0 & I_j \end{bmatrix}.$$  

Since for every $j \geq n$: rank $R_j(A,B) = n$, it follows that for every $j \geq n$: $\begin{bmatrix} R_j(A,B) & 0 \\ 0 & I_j \end{bmatrix}$ is right-invertible and has rank $j + n$. Therefore we may conclude that for every $i,j \geq n$:

$$\text{rank} [H_{i,j}, G_{i,j}] = \text{rank} [Q_i(A,C), G_{i,j}].$$

Consequently (A1) is equivalent to

for every $i \geq n'$: rank $[Q_i(A,C), G_{i,n'}^i] = \text{rank} [Q_i(A,C), G_{i,n'+1}^i] = n'$.

(A2)

Since for $i \geq n$: rank $Q_i(A,C) = n$, we have that for $i \geq n' \geq n$ the $n$ columns of $Q_i(A,C)$ are linearly independent. And so the remaining $n' - n = r$ linearly independent columns in $[Q_i(A,C), G_{i,n'}^i]$ are contained in $G_{i,n}$. From this it follows that rank $G = i \geq r$. Therefore by Theorem 1 we have that for every $i \geq t$: rank $G_{i,r} = r$. So for $i$ sufficiently large, the first $r$ columns of $G_{i,n}$ are linearly independent. Furthermore we may prove that if, for some integers $k,l$ with $1 < k \leq n'$ and $1 \geq n'$, the $k$-th column of $G_{i,n}$ is linearly independent of the columns of $Q_i(A,C)$ then the $(k-1)$-th column of $G_{i+1,n}$ is linearly independent of the columns of $Q_i(A,C)$. From this it is clear that the $r$ linearly independent columns of $G_{i,n}$ which are also linearly independent of the columns of $Q_i(A,C)$ for every $i \geq n'$, can be chosen to be the first $r$ columns of $G_{i,n}$.

On the other hand we can prove that if the $k$-th column of $G_{i,n}$, with $r < k \leq n'$ and $l > 1$, depends linearly on the columns of $[Q_i(A,C), G_{i,r}]$ then the $(k+1)$-th column of $G_{i-1,n}$ depends linearly on the columns of $[Q_{i-1}(A,C), G_{i-1,r}]$.

Therefore we may conclude that (A2) is equivalent to

for every $i \geq n'$: rank $[Q_i(A,C), G_{i,r}] = \text{rank} [Q_i(A,C), G_{i,r+1}] = n'$.