Rational Representations of Behaviors:
Well-Posedness, Stability
and Stabilizability

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Abstract
This paper presents an analysis of representation and stability properties of dynamical systems whose signals are assumed to be square summable sequences. Systems are understood as families of trajectories with no more structure than linearity and shift-invariance. We depart from the usual input-output and operator theoretic setting and view relationships among system variables as a more general starting point for the study of dynamical systems. Parametrizations of two model classes are derived in terms of analytic functions which define kernel and image representations of dynamical systems. It is shown how state space models are derived from these representations. Uniqueness and minimality of these representations are completely characterized. Elementary properties like stability, stabilizability, well-posedness and interconnectability of dynamical systems are introduced and characterized in this set-theoretic framework.

Keywords: system representation, $l_2$-systems, well-posedness, system interconnections, stability.
1 Introduction

The purpose of this paper is to present a detailed study on representations and basic properties of the class of $\ell_2$ systems. We consider discrete time systems, where time is running over the non-negative integers, and we will be interested in signals which belong to the class of square summable sequences. The analysis of this class of systems has led to important applications in robust controller design [1, 6, 3, 7, 10], in sampled data systems [2], in model reduction and in system identification. For many contributions in $\mathcal{H}_\infty$ control theory, signal processing and system identification, the $\ell_2$ assumption on system trajectories is often justified by the physical nature of the problem when dissipativity, power and energy considerations play a natural role. Furthermore, the $\ell_2$ assumption is often implicitly made by considering stable or stabilized dynamical systems only. The motivation for a separate study of $\ell_2$ discrete time systems is threefold.

Firstly, we aim to set-up a theory for the representation of a class of discrete time dynamical systems in which system variables are not necessarily partitioned in inputs and outputs. Especially in the control community, dynamical systems are dominantly viewed as operators acting on inputs and producing output signals. As argued in [12, 13], for many applications in modeling, control and simulation, the traditional input-output framework is not a natural starting point. Indeed, the causality structure of systems is often assumed or imposed to facilitate modeling and simulation of complex processes, while more often than not the causality structure is a mathematical artifact which has no equivalent counterpart in the physical world. For example, it is common practice to construct simulation models of physical systems based on interconnections of components or devices which have a predefined flow chart of input-output information which may or may not correspond to the physical system. Also, for a general modeling problem it may often be unclear which variables classify as inputs and which as outputs. In this paper we treat system variables in a symmetric way, not distinguishing between inputs and outputs.

Secondly, in the recent work of Willems [12, 13, 15] polynomial representations are dominantly used to represent the behavior of dynamical systems. In this paper we consider more general classes of analytic functions to represent dynamical systems and we will specifically develop a theory of rational system representations. Clearly, every polynomial matrix can be considered as a rational one and for this reason such a generalization may seem of little interest at first sight. However, the fact that polynomials form a principal ideal domain and rational functions constitute a field, yields decisive advantages for rational representations of dynamical systems not only from an abstract mathematical point of view, but also for computational reasons. Furthermore, we will view the algebraic dual of a signal space as a distinguished space for representing sets of signals. For the space of all sequences its algebraic dual is isomorphic to the space of polynomials. This observation is at the basis of the representation results in the behavioral framework developed in [12, 13, 15]. Since the Hilbert space of square summable sequences is equal to its algebraic dual, the orthogonal complement of a set of square summable sequences is again a set of square summable sequences. This duality structure is consistently exploited in this paper and leads to a theory in which $\ell_2$ systems are naturally represented by the image or the kernel of a map. This yields an immediate generalization of polynomial system representations to a framework in which $\ell_2$ systems are represented in terms of analytic operators acting on suitable Hilbert spaces. Moreover, the specific structure of the Hilbert space of square summable sequences leads to a concise theory in which duality between time and frequency domain representations of dynamical systems are developed in parallel.

Our third motivation stems from recent contributions [3, 7, 10] in a line of research in which the ($\ell_2$)-graph of an input-output operator is viewed as the basic object for studying model uncertainty, stability, feedback control and robust stabilization of dynamical systems. Many analytic concepts
developed in this paper are similar to this line of thinking but have the main advantage that non-controllable systems, which are excluded in the graph theoretic approach, are naturally considered in the framework presented here. This has particular advantages in feedback control configurations where autonomous closed-loop systems (i.e. no inputs) can not be derived in a straightforward manner using the graph of an input-output operator as a starting point. Georgiou and Smith [3], Ober and Sefton [7, 10] studied representation issues and stability of feedback interconnections and we will clearly point out the intimate relationship between these approaches and the results presented here.

The paper basically consists of two parts. The first part (section 3 till section 8) concerns representation issues of the class of linear time invariant $\ell_2$ systems. The second part (section 9 till section 10) is motivated by applications in control and concerns elementary properties like stability, stabilizability, well-posedness and interconnectability of dynamical systems on a set theoretic level.

The main contributions of this paper can be summarized as follows. For the class of linear time invariant and complete $\ell_2$ systems we provide parametrizations in terms of kernel and image representations. It is shown that every such system admits a rational kernel and image representation which is defined both in the frequency as well as in the time domain. We address the issue of model equivalence and completely characterize non-uniqueness and minimality of kernel and image representations. Normalized representations are defined and it is shown how both kernel and image representations can be used to deduce $\ell_2$ state space models in a direct and natural way.

The second part of the paper is devoted to a set theoretic analysis of stability, stabilizability and interconnectability of dynamical systems. We give parametric and non-parametric characterizations of these important control theoretic notions. The latter results are compared to similar results which were recently obtained using graph theoretic methods. Throughout the paper no assumptions are made on input-output partitionings of signal variables.

The paper is organized as follows. Preliminary mathematical notation is introduced in section 2. Section 3 introduces the classes of dynamical systems which are analyzed in this paper and additional motivation for the study of left-shift invariant $\ell_2$ systems is given in this section. Basic results on the relations between left- and right shift invariance and completeness and finite dimensionality of factor spaces are presented in section 4. Section 5 introduces kernel representations of dynamical systems and characterizes non-uniqueness of these representations. Image representations of $\ell_2$ systems are introduced and analyzed in section 6 and the main representation results are presented in section 7. We proceed in section 8 with the derivation of state space models based on rational representations of systems. Section 9 formalizes the concepts of stabilizability and system interconnections and completely characterizes the property of instantaneous interconnectability of dynamical systems. We proceed in section 10 with a comparison of these results to recent contributions on stability and stabilizability of systems in a graph theoretic context. Conclusions are deferred to section 11.

2 Notation

Let $T \subseteq \mathbb{Z}$ be a set and let $(W, \| \cdot \|)$ be a normed vector space. We define the following objects:

- $\mathbb{Z}_+ := \{ t \in \mathbb{Z} | t \geq 0 \}$, $\mathbb{Z}_- := \{ t \in \mathbb{Z} | t < 0 \}$, and $\mathbb{D}$ denotes the unit circle,
- $\ell(T, W) := W^T = \{ w | w : T \to W \}$,
- $\ell_2(T, W) := \{ w \in \ell(T, W) | \sum_{t \in T} \| w(t) \|^2 < \infty \}$,
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- $\ell_2^+ := \ell_2(\mathbb{Z}_+, W)$, the set of all square summable trajectories $w : \mathbb{Z}_+ \to W$ with norm
  $$\| w \|_2 := \left( \sum_{t=0}^{\infty} \| w(t) \|^2 \right)^{1/2}.$$

- $\ell_2^- := \ell_2(\mathbb{Z}_-, W)$, the set of all square summable trajectories $w : \mathbb{Z}_- \to W$ with norm
  $$\| w \|_2 := \left( \sum_{t=-\infty}^{-1} \| w(t) \|^2 \right)^{1/2}.$$

- The concatenation at $t \in T$ of $w_1, w_2 \in \ell(T, W)$ is the trajectory
  $$(w_1 \wedge_t w_2)(t') := \begin{cases} w_1(t') & \text{for } t' \leq t \\ w_2(t') & \text{for } t' > t. \end{cases}$$

- $\mathcal{L}_2$ is the space of complex valued functions $f : \mathbb{D} \to W$ which are square integrable on the unit circle.

- $\mathcal{H}_2^+$ is the Hardy space of complex valued functions $f : \mathbb{D} \to W$ which are square integrable on the unit circle with analytic continuation outside the unit circle (including $\infty$).

- $\mathcal{H}_2^-$ is the orthogonal complement of $\mathcal{H}_2^+$ in $\mathcal{L}_2$. (Thus, basically $\mathcal{H}_2^-$ is the Hardy space of complex valued functions which are square integrable on the unit circle with analytic continuation inside the unit circle excluding the constant functions.)

- $\mathcal{H}_2^+, \mathcal{H}_2^-$ are the Hardy spaces of complex valued functions which are bounded on the unit circle with a bounded analytic continuation in $|z| < 1$ and $|z| > 1$, respectively.

- $\Pi_+ : \mathcal{L}_2 \to \mathcal{H}_2^+$ and $\Pi_- : \mathcal{L}_2 \to \mathcal{H}_2^-$ are the canonical projections $\Pi_+ w := w_+$ and $\Pi_- w = w_-$ where $w \in \mathcal{L}_2$ is decomposed as $w = w_+ + w_-$ with $w_+ \in \mathcal{H}_2^+$ and $w_- \in \mathcal{H}_2^-$. It is assumed that $\ell_2, \ell_2^+, \ell_2^-, \mathcal{H}_2^+$ and $\mathcal{H}_2^-$ are equipped with their natural inner product. The prefix $\mathcal{R}$ is used to denote rational elements of Hardy spaces, i.e., $\mathcal{R}\mathcal{H}_2^+, \mathcal{R}\mathcal{H}_2^-$, etc. We would like to note that since we work in discrete time we have $\mathcal{R}\mathcal{H}_2^- = \mathcal{R}\mathcal{H}_2^+$. The $z$-transform of a function $w \in \ell_2(\mathbb{Z}_+, W)$ is defined as
  $$\hat{w}(z) := \sum_{t=0}^{\infty} w(t)z^{-t}$$
  where $z \in \mathbb{C}$. We will omit the hat $\hat{\cdot}$ whenever it is clear from the context that signals are treated in the frequency domain.

3 Classes of $\ell_2$ systems

Following the work of Willems, dynamical systems or systems for short, are specified by families of maps $w : T \to W$ defined on a time set $T$ and taking values in a signal space $W$. A dynamical system is defined by a triple $\Sigma = (T, W, B)$ where the set $B \subseteq \ell(T, W)$ is referred to as the behavior of the system. In this paper we exclusively consider systems with time set $T = \mathbb{Z}_+$ and finite dimensional signal space $W = \mathbb{R}^q$ where $q$ is some positive integer. We will further focus on $\ell_2$ systems which are defined as follows.
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Definition 3.1 An $\ell_2$ system is a triple $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, B)$ whose behavior $B$ is a closed subset of $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$.

Of crucial interest will be the shift operators defined on $\ell(\mathbb{Z}_+, W)$. The left-shift operator$^1$ is a map $\sigma_L : \ell(\mathbb{Z}_+, W) \to \ell(\mathbb{Z}_+, W)$ which, given $w \in \ell(\mathbb{Z}_+, W)$, is defined by

$$ (\sigma_L w)(t) := w(t + 1) \quad (3.1) $$

The right-shift operator is a map $\sigma_R : \ell(\mathbb{Z}_+, W) \to \ell(\mathbb{Z}_+, W)$ defined by

$$ (\sigma_R w)(t) = \begin{cases} 0 & \text{for } t = 0 \\ w(t - 1) & \text{for } t \geq 1. \end{cases} \quad (3.2) $$

Note that the composition $\sigma_L \sigma_R$ is the identity map on $\ell(\mathbb{Z}_+, W)$. For $n > 0$, we denote $n$ compositions of $\sigma_L$ and $\sigma_R$ by $\sigma_L^n$ and $\sigma_R^n$, respectively.

Definition 3.2 A system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, B)$ is said to be left-shift invariant if $\sigma_L B \subseteq B$ and right-shift invariant if $\sigma_R B \subseteq B$.

Remark 3.3 We emphasize that for the analysis of dynamical systems with time set $T = \mathbb{Z}_+$ left-shift invariance is a more appealing property than right-shift invariance. Since by (3.2), trajectories in a right-shift invariant subset of $\ell(\mathbb{Z}_+, \mathbb{R}^q)$ can be preceded by an arbitrary number of zeros it is intuitively clear that in the context of systems defined by difference equations, right-shift invariant systems correspond to systems with "zero initial conditions". In a theory for right-shift invariant subspaces of $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$, autonomous behaviors (typically obtained by feedback interconnections) are necessarily trivial (See theorem 4.1 below). This has the consequence that for an important class of feedback interconnections of right-shift invariant dynamical systems the resulting closed-loop systems are not rich enough to further investigate performance or stability issues. In view of the practical importance of autonomous systems, transient phenomena, non-zero initial conditions, off-sets, etc. this makes the class of right-shift invariant $\ell_2$ systems less suitable for general modeling purposes. In this paper we therefore concentrate on left-shift invariant $\ell_2$ systems.

The system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, B)$ is said to be linear if $B$ is a linear subspace of $\ell(\mathbb{Z}_+, \mathbb{R}^q)$ and it is called complete if a trajectory $w$ belongs to $B$ whenever its restrictions $w|_{[t_0, t_1]}$ belong to $B|_{[t_0, t_1]}$ for all (finite) intervals $[t_0, t_1] \subset \mathbb{Z}_+$. The system $\Sigma$ is said to be autonomous if there exists an interval $[t_0, t_1] \subset \mathbb{Z}_+$ such that the mapping $\pi|_{[t_0, t_1]} : B \to B|_{[t_0, t_1]}$, defined by the restriction $\pi|_{[t_0, t_1]}(w) := w|_{[t_0, t_1]}$, is injective. $\Sigma$ is called controllable if for all $w', w'' \in B$ and all $t_0 \in \mathbb{Z}_+$ there exists $t_1 \geq t_0$ and $w_c : [t_0, t_1] \to \mathbb{R}^q$ such that the concatenation

$$ w := w' \wedge_{t_0} w_c \wedge_{t_1} \sigma_R^{(t_1-t_0)}w'' $$

belongs to $B$. That is, at any time $t_0 \in \mathbb{Z}_+$ the trajectory $w'$ can in finite time be 'steered' to $w''$ by means of a control trajectory $w_c$ in such a way that the concatenation $w$ is in $B$.

We introduce the following classes of dynamical systems$^2$.

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$^1$The terminology used here refers to left shifting of the signal with respect to the time axis.

$^2$Since the time set and the signal space are throughout assumed fixed, we will not always formally distinguish between properties of systems $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, B)$ and their behavior $B$. 

Definition 3.4 The model classes $\mathbb{B}$, $\mathbb{B}_2$ and $\mathbb{B}^\text{complete}_2$ are defined by

$$\mathbb{B} := \{ B \subseteq \ell(Z^+, \mathbb{R}^d) \mid B \text{ is linear, left-shift invariant, complete} \} \quad (3.3)$$

$$\mathbb{B}_2 := \{ B \subseteq \ell_2(Z^+, \mathbb{R}^d) \mid B \text{ is linear, left-shift invariant, closed} \} \quad (3.4)$$

$$\mathbb{B}^\text{complete}_2 := \{ B \subseteq \ell_2(Z^+, \mathbb{R}^d) \mid B \text{ is linear, left-shift invariant, complete} \} \quad (3.5)$$

Here, 'closed' in (3.4) is understood as closedness in the standard topology on $\ell_2^+$ and 'complete' in (3.5) is understood in the sense that $w \in B \in \mathbb{B}^\text{complete}_2$ whenever for $w \in \ell_2^+$ the restrictions $w_{[t_0,t_1]} \in B_{[t_0,t_1]}$ for all intervals $[t_0,t_1] \subset Z^+$.

Remark 3.5 The model class $\mathbb{B}$ has been extensively studied in [13, 14, 15] and has been shown to be parameterizable by means of polynomial (kernel) representations. In [4, 8] state space representations of $\ell_2$ systems with doubly infinite time sets are derived. Georgiou and Smith [3] proposed a theory for right-shift invariant $\ell_2$ systems by taking the $\ell_2$ graph of an input-output operator as the basic object of study. The role of the $\ell_2$ graph has been further investigated in [7, 10] in the context of stability, model uncertainty and robust stabilization.

In this paper we mainly concentrate on the model classes $\mathbb{B}_2$ and $\mathbb{B}^\text{complete}_2$. In [14, 15] the model class $\mathbb{B}$ has been topologically characterized as those linear, left-shift invariant subspaces of $\ell(Z^+, \mathbb{R}^d)$ which are closed in the topology of pointwise convergence. In other words a linear left-shift invariant subspace in $\ell(Z^+, \mathbb{R}^d)$ is complete if and only if the subspace is closed in the topology of pointwise convergence. The fact that behaviors in $\mathbb{B}_2$ are closed in the $\ell_2^+$ topology does not imply that the behavior is complete. However, for subsets of $\ell_2^+$ closedness in the topology of pointwise convergence (in the sense that $w_n \to w$ with $w_n \in B$ pointwise and $w \in \ell_2^+$ yields $w \in B$) implies closedness in the $\ell_2^+$ topology. Therefore we have that complete behaviors are closed in the $\ell_2^+$ topology. Using the above it is easy to show that $B \in \mathbb{B}$ generates an element $B_2$ in $\mathbb{B}^\text{complete}_2$ by the restriction

$$B_2 = B \cap \ell_2^+.$$  

However, this restriction is in general not injective. In other words, there are $B', B'' \in \mathbb{B}$ with $B' \neq B''$ for which

$$B' \cap \ell_2^+ = B'' \cap \ell_2^+.$$  

We will see that in this case the difference between $B'$ and $B''$ amounts to unstable, uncontrollable dynamics.

One of the main advantages to consider $\ell_2$ systems is that we can interchangeably consider their behavior in the time domain and in the frequency domain. Specifically, define for all $B \in \mathbb{B}_2$:

$$\mathcal{F} = \{ \hat{w} \in \mathcal{H}_2^+ \mid w \in B \}$$

Before considering representations of $\ell_2$ systems we give some results pertaining to shift invariant $\ell_2$ systems in the next section.

4 Left- and right-shift invariant systems

In the frequency domain the left and right shift operators are the mappings $\hat{\sigma}_L, \hat{\sigma}_R : \mathcal{H}_2^+ \to \mathcal{H}_2^+$ which, given $w \in \mathcal{H}_2^+$, are defined by

$$\hat{\sigma}_L w(z) := zw(z) - zw(\infty)$$

$$\hat{\sigma}_R w(z) := z^{-1}w(z).$$
With these definitions a subset \( B \subseteq \ell^+_2 \) is left-shift invariant [right-shift invariant] if and only if \( \hat{\sigma}_n B \subseteq \hat{\sigma}_n B \) [\( \hat{\sigma}_n B \subseteq B \)]. With respect to the standard inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H}^+_2 \) we define the orthogonal complement \( B^\perp \) of a subspace \( B \subseteq \mathcal{H}^+_2 \) by the set

\[
B^\perp := \{ w \in \mathcal{H}^+_2 \mid \langle w, v \rangle = 0 \text{ for all } v \in \hat{B} \}.
\]

Of course we can similarly define \( B_1^\perp \) which is the orthogonal complement of \( B \) in \( \ell^+_2 \) or equivalently the inverse z-transform of \( B^\perp \). The first result of this section provides some elementary properties of left- and right-shift invariant systems.

**Theorem 4.1** Let \( \Sigma = (\mathbb{Z}_+, W, B) \) be a linear \( \ell_2 \) system. Then

1. \( \Sigma \) is left-shift invariant and right-shift invariant only if \( \Sigma \) is memoryless, i.e. if \( w', w'' \in B \) and \( n \in \mathbb{Z}_+ \) then the concatenation \( w' \wedge_n w'' \in B \).
2. \( \Sigma \) is autonomous and right-shift invariant if and only if \( B = \{0\} \).
3. \( \Sigma \) is autonomous if and only if \( B \) is finite-dimensional.
4. \( \sigma_1 B \subseteq B \iff \sigma_1 B^\perp \subseteq B^\perp \)
5. \( \sigma_n B \subseteq B \iff \sigma_n B^\perp \subseteq B^\perp \)

**Proof.** 1. Let \( w', w'' \in B \) and \( n \in \mathbb{Z}_+ \). Since \( B \) is \( \sigma_n \) and \( \sigma_1 \) invariant, both \( \sigma_n^\perp \sigma_1^\perp w' \) and \( \sigma_n^\perp \sigma_1^\perp w'' \) belong to \( B \). Linearity of \( B \) implies that then also \( \sigma_n^\perp \sigma_1^\perp w'' (1 - \sigma_n^\perp \sigma_1^\perp) w' \in B \) which is precisely the concatenation of \( w' \) and \( w'' \) at time \( n \).

2. Let \( B \) be autonomous and right-shift invariant. Hence there exists \( t_1 > 0 \) such that each \( w \in B \) is uniquely determined by its restriction \( w|_{[0,t_1]} \). Let \( w \in B \) be given. Let \( t_0 > 0 \) and consider \( w_{t_0} = \sigma_n^\perp \sigma_1^\perp w \). Then \( w_{t_0} \in B \) and \( w_{t_0}(t) = 0 \) for \( t < t_0 \). Since \( w_{t_0} \in B \), \( w_{t_0} \) is uniquely determined by its restriction \( w_{t_0}|_{[0,t_1]} \). However, the latter vanishes for \( t > t_1 \) from which it follows that \( w_{t_0}(t) = 0 \) for all \( t \in \mathbb{Z}_+ \). This immediately yields that \( w = 0 \). The reverse implication is trivial.

3. (if) If \( B \) is finite-dimensional then there exists some finite basis \( w_1, \ldots, w_n \). Obviously there exists \( t_1 > 0 \) such that \( w_1|[0,t_1], \ldots, w_n|[0,t_1] \) are independent. But then it is clear that \( B \) is uniquely determined by \( w|[0,t_1] \) and hence \( \Sigma \) is autonomous.

(only if) If \( \Sigma \) is autonomous then there exists \( t_1 > 0 \) such that \( w \in B \) is uniquely determined by \( w|[0,t_1] \). But then it is obvious that

\[
\dim B \leq \dim B|[0,t_1] \leq (t_1 + 1) \dim W < \infty.
\]

4. Let \( \sigma_1 B \subseteq B \) and \( w \in B^\perp \). Then \( \langle w, v \rangle = 0 \) for all \( v \in B \). By shift invariance of \( B \) also \( \langle w, \sigma_n v \rangle = \langle \sigma_n^\perp w, v \rangle = \langle \sigma_n^\perp w, v \rangle = 0 \) for all \( v \in B \). Thus, \( \sigma_n^\perp w \in B^\perp \) which implies that \( \sigma_n^\perp B^\perp \subseteq B^\perp \). The reverse implication follows from a similar argument.

5. The proof of 5. is a straightforward modification of the proof of statement 4. \( \square \)

**Definition 4.2** Let \( \Sigma = (\mathbb{Z}_+, \mathbb{R}^2, B) \) be an \( \ell_2 \) system with \( B \in \mathbb{B}_2 \). The **equilibrium response** of \( \Sigma \) is the set \( B^* \) defined as

\[
B^* = \{ w \in B \mid \sigma_n^\perp w \in B, \forall t \in \mathbb{Z}_+ \}.
\]
The equilibrium response $B^*$ therefore consists of all time series $w \in B$ which can be preceded with an arbitrary number of zeros. It follows from Theorem 4.1 that $B^* = B$ only if the system defined by $B$ is memoryless. Furthermore, $B^*$ is a closed subset of $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$ and therefore it defines in itself a behavior of an $\ell_2$ system in the sense of Definition 3.1. Finally, $B^*$ is right shift invariant and is in fact the largest right-shift invariant subspace of $B$ in the sense that $B^*$ contains every right-shift invariant subspace $B' \subseteq B$.

The notion of state is defined in terms of $B$ and the equilibrium response $B^*$. We call two trajectories $w_1, w_2 \in B$ right-shift equivalent if $w_1 - w_2 \in B^*$. This obviously defines an equivalence relation on $B$ and we denote by $B$ (mod $B^*$) the factor space consisting of the set of all equivalence classes $w$ (mod $B^*$) with $w \in B$. Intuitively, $w$ (mod $B^*$) consists of all trajectories that depart from the same state and we will therefore view the equivalence class $w := w$ (mod $B^*$) as the initial state of the system when $w \in B$ is observed. The factor space $\mathcal{X} := B$ (mod $B^*$) is thus identified as the state space of $\Sigma$. The dimension of $\mathcal{X}$ is a measure of the complexity, i.e., the internal dynamical structure of the system.

**Definition 4.3** The complexity of a system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, B)$ with $B \in \mathbb{B}_2$ is $n(B) := \dim(\mathcal{X}) = \dim[B$ (mod $B^*$)]

Thus the complexity of an $\ell_2$ system is finite if and only if $\mathcal{X}$ is finite dimensional. In fact, the following result shows that finite dimensionality of $\mathcal{X}$ precisely characterizes the model set $\mathbb{B}_2^\text{complete}$. 

**Theorem 4.4** The following statements are equivalent.

1. $B \in \mathbb{B}_2^\text{complete}$
2. $B \in \mathbb{B}_2$ and $\mathcal{X} := B$ (mod $B^*$) is finite dimensional.

**Proof.** Using similar arguments as in [13] we can show that $B$ is complete if and only if there exists $t > 0$ such that $B$ is $t$-complete. Here $t$-complete means that $w \in B$ whenever for $w \in \ell_2^t$ the restrictions $w|_{[t_0, t_0+t]} \in B|_{[t_0, t_0+t]}$ for all $t_0 \in \mathbb{Z}_+$, in other words we only have to focus on intervals of length $t$. In order to show the implication $1. \Rightarrow 2.$ we assume that we have a $t$-complete behavior $B$. It is easy to check that this implies $B \in \mathbb{B}_2$. Remains to show that $\mathcal{X}$ is finite dimensional. It is easy to check that, since the behavior is $t$-complete, $B^*$ is given by those $w \in B$ for which

$$[\sigma^t_0 w]|_{[0, t]} \in B|_{[0, t]}$$

for all $t_0 \in \mathbb{Z}_+$. In other words whether given $w \in B$ we have $w \in B^*$ only depends on $w|_{[0, t]}$. Since $B|_{[0, t]}$ is finite-dimensional this implies that $\mathcal{X}$ is finite dimensional.

Next consider $B \in \mathbb{B}_2$ such that $\mathcal{X}$ is finite-dimensional. We will show that $B \in \mathbb{B}_2^\text{complete}$. Note that since $\mathcal{X}$ is finite-dimensional there exist $w_1, \ldots, w_n \in B$ for some $n > 0$ such that the equivalence classes of $w_1, \ldots, w_n$ span $\mathcal{X}$. Let $B_1$ be the finite dimensional linear subspace spanned by $w_1, \ldots, w_n$ and note that $B_1$ is a finite-dimensional linear subspace of $B$.

Let $k$ be such that

$$\{ w \in B, w|_{[t-k, t]} = 0, t > k \} \implies \{ \sigma^t_0 w \in B^* \}.$$

We claim that $k$ is well-defined (and hence finite). Indeed, if not, then there exists a sequence of signals $w_j \in B$ and $t_j > k_j \in \mathbb{Z}_+$ with $k_j \to \infty$ as $j \to \infty$ such that $w_j|_{[t_j-k_j, t_j]} = 0$ and $\sigma^{t_j}_j w_j \notin B^*$. It is easy to check that we can assume without loss of generality that $\sigma^{t_j}_j w_j \in B_1$ and $|| \sigma^{t_j}_j w_j ||_2 = 1$. But then there exists a subsequence $w_{j_s}$ of $w_j$ such that $\sigma^{t_j}_j w_{j_s}$ converges
to some non-zero element \( v \in \mathcal{B}_1 \). In that case, for any \( t > 0 \) \( \sigma_{t \mu}^{-1}w_j \) also converges to \( \sigma_{t \mu}^tv \). Since \( \mathcal{B} \) is closed we find \( \sigma_{t \mu}^tv \in \mathcal{B} \) for all \( t > 0 \). But then \( v \in \mathcal{B}^* \) which gives a contradiction since \( \mathcal{B}^* \cap \mathcal{B}_1 = \{0\} \).

The above implies that for \( t > k \), \( w_{[0,t]} \in \mathcal{B} \) has a unique continuation \( v \in \mathcal{B}_1 \) such that

\[
\begin{cases}
  w(t') & t' \leq t \\
  v(t' - t) & t' > t
\end{cases}
\]

is in \( \mathcal{B} \). After all two different continuations \( v_1 \) and \( v_2 \) would yield signals \( w_1, w_2 \in \mathcal{B} \) such that \( (w_1 - w_2)(t') = 0 \) for \( t' < k < t \) implying that \( v_1 - v_2 \) belongs to \( \mathcal{B}^* \). Since \( v_1 - v_2 \in \mathcal{B}_1 \) and \( \mathcal{B}_1 \cap \mathcal{B}^* = \{0\} \) we get \( v_1 = v_2 \).

Let \( w \in \ell_2^+ \) be such that \( w_{[0,t]} \in \mathcal{B} \) for all \( t > 0 \). Hence for all \( t \) there exists a \( v_t \in \mathcal{X} \) such that \( w_t \in \mathcal{B} \) where

\[
\begin{cases}
  w(t') & t' \leq t \\
  v_t(t' - t) & t' > t
\end{cases}
\]

Because \( v_t \in \mathcal{B}_1 \) is also a continuation of \( [\sigma_{t \mu}^{-k}w]_{[0,k]} \), this continuation is uniquely determined by \( [\sigma_{t \mu}^{-k}w]_{[0,k]} \). In particular, \( v_t \) is a linear function of \( w_{[t-k,t]} \). Since \( w \in \ell_2^+ \) we have \( w_{[t-k,t]} \to 0 \) but then also \( v_t \to 0 \). This immediately yields that \( w_t \to w \) and since \( \mathcal{B} \) is closed we have \( w \in \mathcal{B} \).  

Due to this characterization, the model class \( \mathbb{B}^{\text{complete}}_2 \) is referred to as the class of finite dimensional \( \ell_2 \) systems. Note that this does not imply that elements \( \mathcal{B} \in \mathbb{B}^{\text{complete}}_2 \) are finite dimensional subspaces of \( \ell_2^+ \). Finite dimensional sets \( \mathcal{B} \in \mathbb{B}^{\text{complete}}_2 \) were characterized in Theorem 4.1 as behaviors corresponding to autonomous systems.

## 5 Kernel representations

Suppose that \( \Theta \in \mathcal{H}_\infty^+ \). We can view \( \Theta \) as a mapping from \( \mathcal{H}_2^+ \) to \( \mathcal{L}_2 \) defined by the multiplication \((\Theta w)(z) := \Theta(z)w(z), \ z \in \mathbb{C} \). To each such \( \Theta \) we will associate a behavior \( \hat{\mathcal{B}} \) which consists of those functions \( w \in \mathcal{H}_2^+ \) for which \( \Theta w \) is an element of \( \mathcal{H}_2^+ \). Formally, we introduce the set

\[
\hat{\mathcal{B}}_{\text{str}}(\Theta) := \{ w \in \mathcal{H}_2^+ \mid \Theta w \in \mathcal{H}_2^+ \} = \{ w \in \mathcal{H}_2^+ \mid [\Pi_+(\Theta w)](z) = 0 \text{ for all } z \in \mathbb{C} \} = \ker \Pi_+ \Theta.
\]

Clearly, \( \hat{\mathcal{B}}_{\text{str}}(\Theta) \) is a linear and closed subset of \( \mathcal{H}_2^+ \). Moreover, it is easy to see that \( \hat{\mathcal{B}}_{\text{str}}(\Theta) \) is \( \hat{\sigma}_L \) invariant.

In the time domain the set \( \hat{\mathcal{B}}_{\text{str}}(\Theta) \) has an equivalent interpretation. Let \( \Theta_k \in \mathbb{R}^{r \times r}, k \in \mathbb{Z} \), be constant real matrices which uniquely define the Laurent series expansion

\[
\Theta(z) = \sum_{k=-\infty}^{\infty} \Theta_k z^{-k}
\]

where \( z \in \mathbb{C} \). Since \( \Theta \in \mathcal{H}_\infty^+ \) the coefficients \( \Theta_k \) are zero for \( k > 0 \) so that only non-negative powers of \( z \) appear in the expansion (5.2). Introduce the map \( \Theta(\hat{\sigma}_L) : \ell_2(\mathbb{Z}_+,\mathbb{R}^r) \to \ell_2(\mathbb{Z}_+,\mathbb{R}^r) \)
which, given \( w \in \ell_2^+ \), is defined by the convolution
\[
[\Theta(\sigma_w)w](t) := \sum_{k=-\infty}^{\infty} \Theta_k \sigma_w^{-k} w(t) = \sum_{k=0}^{\infty} \Theta_{t-k} w(k)
\]
where \( t \geq 0 \). This map is well defined and corresponds to the convolution of the sequences \( \{\Theta_k\}_{k \in \mathbb{Z}} \) and \( \{w(k)\}_{k \in \mathbb{Z}_+} \). For all \( \Theta \in \mathcal{H}_\infty^\circ \) the set
\[
\mathcal{B}_{\text{ker}}(\Theta) := \{ w \in \ell_2^+ \mid \Theta(\sigma_w)w = 0 \} = \ker \Theta(\sigma_w)
\]
defines a linear, left-shift invariant and closed subset of \( \ell_2^+ \), i.e., \( \mathcal{B}_{\text{ker}}(\Theta) \subseteq \ell_2^+ \). The notation \( \mathcal{B}_{\text{ker}}(\Theta) \) in (5.1) and \( \mathcal{B}_{\text{ker}}(\Theta) \) in (5.3) are indeed consistent as is shown in the following theorem.

**Theorem 5.1** For all \( \Theta \in \mathcal{H}_\infty^\circ \) there holds \( \hat{\mathcal{B}}_{\text{ker}}(\Theta) = \mathcal{Z}(\mathcal{B}_{\text{ker}}(\Theta)) \) where \( \mathcal{Z} : \ell_2^+ \rightarrow \mathcal{H}_2^+ \) denotes the z-transform \( \mathcal{Z}w := \hat{w} \).

**Proof.** Let \( \hat{w} \in \mathcal{H}_2^+ \) and \( \Theta \in \mathcal{H}_\infty^\circ \). Then there holds
\[
\Theta(z)\hat{w}(z) = \sum_{k=-\infty}^{\infty} \Theta_k z^{-k} \sum_{j=0}^{\infty} w(j)z^{-j} = \sum_{t=-\infty}^{\infty} v(t)z^{-t}
\]
where for all \( t \in \mathbb{Z} \), we have \( v(t) = \sum_{k=0}^{\infty} \Theta_{t-k} w(k) \). Therefore \( \hat{w} \in \hat{\mathcal{B}}_{\text{ker}}(\Theta) \) if and only if \( v(t) = 0 \) for all \( t < 0 \). On the other hand
\[
v(t) = \sum_{k=0}^{\infty} \Theta_{t-k} w(k) = \sum_{k=0}^{\infty} \Theta_{t-k} w(t+k) = (\Theta(\sigma_w)w)(t)
\]
since \( \Theta_{t-k} = 0 \) for \( t > k \) because \( \Theta \in \mathcal{H}_\infty^\circ \). Hence \( \hat{w} \in \hat{\mathcal{B}}_{\text{ker}}(\Theta) \) if and only if \( (\Theta(\sigma_w)w)(t) = 0 \) for all \( t < 0 \).

**Definition 5.2** A subset \( B \subseteq \ell_2^+ \) is said to have a kernel representation if there exist \( \Theta \in \mathcal{H}_\infty^\circ \) such that \( B = \mathcal{B}_{\text{ker}}(\Theta) \) [\( \hat{B} = \hat{\mathcal{B}}_{\text{ker}}(\Theta) \)].

**Remark 5.3** A kernel representation \( \Theta \) will either refer to a subset \( B \subseteq \ell_2^+ \) or to its frequency domain analogue \( \hat{B} \subseteq \mathcal{H}_2^+ \). If there is no confusion of interpretation we will not distinguish between the two objects represented by \( \Theta \).

**Example 5.4** To further motivate the implications of (5.1) and (5.3) consider the example of a rational function \( \Theta(z) = (z - \alpha)/(z - \beta) \) with \( |\beta| > 1 \) and \( \alpha \neq \beta \). It has expansion (5.2) with
\[
\Theta_k = \begin{cases} 0 & \text{for } k > 0 \\ \alpha/\beta & \text{for } k = 0 \\ (\alpha - \beta)\beta^{k-1} & \text{for } k < 0. \end{cases}
\]
The \( \ell_2^+ \) kernel of \( \Theta(\sigma_w) \) consists of exponentials of the form \( w_\lambda(t) := \lambda(\beta)^t \) with \( t > 0 \) and \( \lambda \in \mathbb{R} \). Consequently, \( \mathcal{B}_{\text{ker}}(\Theta) \) is an autonomous system. The \( \mathcal{H}_2^+ \) kernel of \( \Pi_+ \Theta \) equals \( \hat{\mathcal{B}}_{\text{ker}}(\Theta) = \{ \lambda(z - \beta)^{-1} \mid \lambda \in \mathbb{R}, \ z \in \mathbb{C} \} \). The kernel of \( \Theta \) (when viewed as a multiplicative operator on \( \mathcal{H}_2^+ \)) contains only the trivial function \( \hat{w} = 0 \) which for this reason is a less appealing representation for describing dynamical systems. Also, \( \Theta(\sigma_w) \) contains only \( w = 0 \) in its kernel when \( \ell_2(\mathbb{Z}, \mathbb{R}) \) is taken as its domain of definition.
We proceed this section with a complete characterization of subset inclusions and non-uniqueness of kernel representations. The main result is as follows.

**Theorem 5.5** For $i = 1, 2$, let $\Theta_i \in \mathcal{H}_\infty^-$ be a rational kernel representation of $B_i = B_{\text{mr}}(\Theta_i)$. Then

1. $B_1 \subseteq B_2$ if and only if there exists $U \in \mathcal{R}\mathcal{H}_\infty^-$ such that $\Theta_2 = U \Theta_1$.

2. If $\Theta_1$ and $\Theta_2$ have full row rank then $B_1 = B_2$ if and only if there exist a unit $^3 U \in \mathcal{R}\mathcal{H}_\infty^-$ such that $\Theta_2 = U \Theta_1$.

**Proof.** 1. (if) Suppose that $\Theta_2 = U \Theta_1$ for some $U \in \mathcal{R}\mathcal{H}_\infty^-$. For $w \in B_1$ we have that $\tilde{w} := \Theta_1 \tilde{w} \in \mathcal{H}_\infty^-$. Then $U \tilde{w} = U \Theta_1 \tilde{w} = \Theta_2 \tilde{w} \in \mathcal{H}_\infty^+$ which implies that $\Pi_+ \Theta_2 \tilde{w} = 0$. Hence $w \in B_2$ from which we conclude that $B_1 \subseteq B_2$.

(only if) We need to show that $B_1 \subseteq B_2 \Rightarrow \Theta_2 = U \Theta_1$ for some $U \in \mathcal{R}\mathcal{H}_\infty^-$. To see this, first observe that

$$
\mathcal{B}_{\text{un}}(\Theta_1) = \{ \tilde{w} \in \mathcal{H}_\infty^+ \mid \langle \Theta_1 \tilde{w}, \tilde{v} \rangle = 0 \text{ for all } \tilde{v} \in \mathcal{H}_\infty^+ \} = \\
= \{ \tilde{w} \in \mathcal{H}_\infty^+ \mid \langle \tilde{w}, \Theta_1^\top \tilde{v} \rangle = 0, \text{ for all } \tilde{v} \in \mathcal{H}_\infty^+ \} = \\
= (\text{im} \Theta_1^\top)^{\perp}
$$

(5.4)

where $\Theta_1^\top : \mathcal{H}_\infty^+ \rightarrow \mathcal{H}_\infty^+$ is the dual operator in $\mathcal{R}\mathcal{H}_\infty^+$ defined by $\Theta_1^\top(z) = \Theta_1^\top(z^{-1})$. Thus, $B_1 \subseteq B_2$ implies that $B_2^{\perp} \subseteq B_1^{\perp}$ which, by (5.4), implies that $\text{im} \Theta_2^{\perp} \subseteq \text{im} \Theta_1^{\perp}$ where the bar denotes closure in $\mathcal{H}_\infty^+$. For rational operators, the latter implies that $\text{im} \Theta_2^{\perp} \subseteq \text{im} \Theta_1^{\perp}$ since in that case the images are closed. Let $e_i$ be the $i$th unit vector then $\Theta_2 e_i \in \text{im} \Theta_2^{\perp}$ and hence there exists $v_i \in \mathcal{R}\mathcal{H}_\infty^+$ such that $\Theta_2 e_i = \Theta_1^\top v_i$. Hence, if $p$ denotes the number of rows of $\Theta_2$ then we have $\Theta_2^\top = \Theta_1^\top X$ with

$$
X = (v_1 \ldots v_p) \in \mathcal{R}\mathcal{H}_\infty^+ = \mathcal{R}\mathcal{H}_\infty^+
$$

Hence $U = X^\top$ satisfies $\Theta_2 = U \Theta_1$.

2. Part 1 implies that $B_1 = B_2$ if and only if $\Theta_2 = U \Theta_1$ and $\Theta_1 = U_2 \Theta_2$ with both $U_1$ and $U_2$ in $\mathcal{R}\mathcal{H}_\infty^-$. Moreover, if $U_1$ and $U_2$ satisfy these conditions then $\Theta_2 = U_1 U_2 \Theta_2$ and $\Theta_1 = U_2 U_1 \Theta_1$. Since $\Theta_1$ and $\Theta_2$ are both full row rank we find $U_1 = U_2^{-1}$ which completes the proof.

**Definition 5.6** A kernel representation $B_{\text{mr}}(\Theta)$ is called normalized if $\Theta$ is co-inner$^4$.

Using the fact that every rational operator $\Theta \in \mathcal{R}\mathcal{H}_\infty^-$ admits an inner-outer factorization, we obtain as an immediate consequence of Theorem 5.5 that every subset $B \in \mathcal{B}_\text{mr}^{\text{simp}}$ which admits a kernel representation, also admits a normalized kernel representation. We finally remark that rationality of $\Theta$ is not necessary to prove the sufficiency parts of Theorem 5.5.

$^3$A unit $U \in \mathcal{R}\mathcal{H}_\infty^-$ is a square matrix with entries in $\mathcal{H}_\infty^-$ whose inverse $U^{-1}$ exists and also belongs to $\mathcal{H}_\infty^-$. $^4$An operator $\Theta \in \mathcal{H}_\infty^-$ is called co-inner if $\Theta \Theta^\top = I$ where $\Theta^\top(z) := \Theta^\top(z^{-1})$. A function $\Psi \in \mathcal{H}_\infty^+$ is inner if $\Psi^\top \Psi = I$. 
6 Image representations

In this section we will be interested in representing elements $B$ of $\mathcal{B}_2$ as images of maps. The Beurling-Lax theorem (for details see [9]) is known to provide the existence of an image representation for shift invariant subspaces $\hat{B}$ of a Hilbert space $\mathcal{H}$.

**Theorem 6.1 (Beurling-Lax)** Let $\sigma$ be an isometry on a Hilbert space $\mathcal{H}$ such that

$$
\|\sigma^k w\| \to 0 \text{ for all } w \in \mathcal{H} \text{ as } k \to \infty.
$$

Then for any closed, $\sigma$-invariant linear subspace $\hat{B}$ of $\mathcal{H}$ there exists a bounded isometric linear operator $\Psi$ from some Hilbert space $\mathcal{W}$ to $\mathcal{H}$ such that $B = \text{im } \Psi$.

**Remark 6.2** The left-shift $\sigma_\ell$ does not satisfy the conditions of the Beurling-Lax theorem and hence Theorem 6.1 cannot be used to prove the existence of image representations of behaviors $B \in \mathcal{B}_2$. On the other hand, we do have that $\sigma = \sigma_\rho$ defines an isometry on $\mathcal{H}^+_{\rho}$ which satisfies (6.1) so that for any $\sigma_\rho$ invariant subspace $B \subset \mathcal{H}^+_{\rho}$ we infer from Theorem 6.1 the existence of an element $\Psi \in \mathcal{H}^+_{\rho}$ such that

$$
\hat{B} = \hat{B}_{\text{im}}(\Psi) := \text{im } \Psi.
$$

Here, $\Psi : \mathcal{H}^+_{\rho} \to \mathcal{H}^+_{\rho}$ is the multiplicative operator $(\Psi \psi)(z) := \psi(z) \psi(z)$ with $z \in \mathbb{C}$. Moreover, $\Psi$ can be chosen to be an inner (or norm-preserving) map from $\mathcal{H}^+_{\rho}$ to $\hat{B}$.

From Theorem 4.1 we infer that image representations of the form (6.2) are not applicable for elements of $\mathcal{B}_2$ except for some trivial cases where the system is memoryless. In order to represent left-shift invariant systems we introduce a different type of image representation. Let $\Psi_a, \Psi_c$ be elements of $\mathcal{H}^+_{\rho}$ and consider the set

$$
\hat{B}_{\text{im}}(\Psi_a, \Psi_c) = \Pi_+ \Psi_a \mathcal{H}^+_2 + \Pi_+ \Psi_c \mathcal{L}_2 = \left\{ \left( \Pi_+ (\Psi_a \Psi_c) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \mid s_1 \in \mathcal{H}^-_2 \text{ and } s_2 \in \mathcal{L}_2 \right) \right\}.
$$

The operators $\Psi_a$ and $\Psi_c$ constitute a decomposition of $\hat{B}$ in an autonomous and a controllable part. Precisely, $\hat{B} = \hat{B}_a + \hat{B}_c$ where $\hat{B}_c = \Pi_+ \Psi_c \mathcal{L}_2$ is the controllable part of $\hat{B}$ and $\hat{B}_a = \Pi_+ \Psi_a \mathcal{H}^-_2$ is a (non-unique) autonomous part of $\hat{B}$. Controllable and autonomous $\ell_2$ systems are characterized in terms of image representations as follows.

**Theorem 6.3** Let $\Sigma = (\mathbb{Z}_+, \mathbb{R}^\ell, B)$ be an $\ell_2$ system whose behavior $\hat{B}$ admits a rational image representation of the form (6.3). Then

1. $B$ is autonomous if and only if $\hat{B} = \{ \Pi_+ \Psi_a s \mid s \in \mathcal{H}^-_2 \}$.  
2. $B$ is controllable if and only if $\hat{B} = \{ \Pi_+ \Psi_c s \mid s \in \mathcal{L}_2 \}$.

**Proof.** We first prove that if $\Psi_c$ is non-zero then there exists for all $t > 0$ a $w \in B$ with $w \neq 0$ but $w|_{[0,t]} = 0$. This would clearly imply the system is not autonomous. Let $s_1 \in \mathcal{H}^+_2$ be such that $\Psi_c s_1 \neq 0$. Then $\hat{w} = \Psi_c (\sigma_\rho s_1) \in \hat{B}$ and since $\Psi_c \in \mathcal{H}^+_\infty$ we find $w|_{[0,t]} = 0$. On the other hand $\hat{w} = \sigma_\rho \Psi_c s_1 \neq 0$ since $\sigma_\rho$ is injective. Conversely, if $\Psi_c = 0$ then rationality of $\Psi_a$ implies that $\hat{B} = \Pi_+ \Psi_a \mathcal{H}^-_2$ is finite dimensional, which by Theorem 4.1 yields that $B$ is autonomous.
Next we will prove that $B$ is controllable if and only if $B_a \subset B_c$ where $B_a$ and $B_c$ are defined in the frequency domain by $B_a = \Pi_+ \Psi_a \mathcal{L}_2$ and $B_c = \Pi_+ \Psi_c \mathcal{H}^2_\omega$. It is easy to check that $B_a \subset B_c$ implies that $B$ is controllable. Remains to show the converse. For each $w \in B_a$ which is not in $B_c$ there exists $t > 0$ such that $w|_{[0,t]} \notin B_c|_{[0,t]}$. Since $B_a$ is finite dimensional we conclude there exists $T > 0$ such that for any $w \in B_a$ we have $w \in B_c$ if and only if $w|_{[0,T]} \in B_c|_{[0,T]}$. Let $w \in B_a$ be such that $w \notin B_c$ and suppose $B$ is controllable. Then given $T$ as before there exists $t_1 > T$ and $w_e : [T, t_1] \to \mathcal{W}$ such that $w_e \in B$ where $w_e$ is the concatenation $w_e = 0 \wedge T w_a \wedge t_1, \sigma^w_k w$. Since $w_e|_{[0,T]} \in B_c|_{[0,T]}$ we know by definition of $T$ that $w_e \in B_c$. This yields a contradiction since:

$$w_e|_{[t_1, t_1 + T]} = w|_{[0,T]} \notin B_c|_{[0,T]} = B_c|_{[t_1, t_1 + T]}$$

where we used time-invariance of the behavior.

In the time domain (6.3) can be interpreted as follows. Associate with $\Psi_a$ and $\Psi_c$ the operators $\Psi_a(\sigma_L) : \ell^+_2 \to \ell^+_2$ and $\Psi_c(\sigma_L) : \ell_2 \to \ell^+_2$ defined by the convolutions

$$[\Psi_a(\sigma_L)](t) := \sum_{k=-\infty}^{0} \Psi_a,t-k s(k); \quad [\Psi_c(\sigma_L)](t) := \sum_{k=-\infty}^{t} \Psi_c,t-k s(k)$$

where $t \geq 0$ and $\Psi_a,k$ and $\Psi_c,k$ are constant real matrices which are uniquely defined by the Laurent series expansions

$$\Psi_a(z) = \sum_{k=0}^{\infty} \Psi_a,k z^{-k}; \quad \Psi_c(z) = \sum_{k=0}^{\infty} \Psi_c,k z^{-k}$$

with $z \in \mathbb{C}$. We define

$$B_m(\Psi_a, \Psi_c) := \Psi_a(\sigma_L) \ell^+_2 + \Psi_c(\sigma_L) \ell_2 =$$

$$\{ w \in \ell^+_2 \mid \exists s_1 \in \ell^+_2, s_2 \in \ell_2 \text{ such that } w = \Psi_a(s_1) + \Psi_c(s_2) \}$$

which is a left-shift invariant linear subspace of $\ell^+_2$ and corresponds to the inverse $z$-transform of (6.3). The notation is therefore consistent.

**Definition 6.4** A subset $B \subseteq \ell^+_2 \quad (B \subseteq \mathcal{H}^2_\omega)$ is said to have an *image representation* if there exist $\Psi_a, \Psi_c \in \mathcal{H}_\infty^\omega$ such that $B = B_m(\Psi_a, \Psi_c)$ ($B = B_m(\Psi_a, \Psi_c)$).

We conclude this section with a complete characterization of non-uniqueness of image representations of the form (6.3). In words, the result states that two image representations define the same $\ell_2$ system if and only if they have a common square and all pass left factor. These common factors define a notion of normalized image representations which we define first.

**Definition 6.5** An image representation $B_m(\Psi_a, \Psi_c)$ is said to be *normalized* if $(\Psi_a, \Psi_c)$ is square and inner and if $\Psi_c$ has no finite or infinite zeros.

The interpretation of this definition is as follows. If $B_m(\Psi_a, \Psi_c)$ is normalized then the image of $\Pi_+ [\Psi_a, \Psi_c]$ when acting on $\mathcal{H}^2_\omega$ is a finite dimensional subset $B_a$ of $\mathcal{H}^2_\omega$ which is orthogonal to the equilibrium response $B^*$. $B_m(\Psi_a, \Psi_c)$. It is in this sense that a normalized image representation achieves a decomposition $B = B_a \oplus B^*$ where $B_a \perp B^*$. (For details we refer to the proofs of Theorem 6.6 and Theorem 7.2 below).
Theorem 6.6 Let $B \in \mathbb{B}_2$ be a behavior which admits a rational image representation. Then

1. $B$ admits a normalized image representation $B = B_m(\Psi_a, \Psi_c)$.

2. $B_m(\Psi_a, \Psi_c)$ is an image representation of $B$ if and only if there exists $R \in \mathcal{H}_\infty^+, T \in \mathcal{H}_\infty^+$ and $S \in \mathcal{L}_\infty$ such that $R$ and $T$ have a right-inverse in $\mathcal{L}_\infty$,

$$\Psi_a = \bar{\Psi}_a R + \bar{\Psi}_c S \quad (6.4)$$

$$\Psi_c = \bar{\Psi}_c T \quad (6.5)$$

and such that there are no stable pole-zero cancelations between $\bar{\Psi}_a$ and $R$ or, in other words, the number of stable poles of $\bar{\Psi}_a$ equals the number of stable poles of $\Psi_a R$.

Proof. 1. Let $B_m(\Psi_a, \Psi_c)$ be an arbitrary image representation of $B$. It is straightforward that we can factorize $\Psi_c$ as $\Psi_c = \tilde{\Psi}_c T$ where $\tilde{\Psi}_c$ is inner without any zeros and $T \in \mathcal{H}_\infty^+$ is square and of full rank. Then clearly $\Psi_c L_2 = \tilde{\Psi}_c L_2$. Let $\psi_{ce} \in \mathcal{H}_\infty^+$ be such that $(\psi_{ce} \tilde{\Psi}_c)$ is square and inner and such that $\psi_{ce}$ has no zeros. Next consider the set

$$B_a = \Pi_+ \psi_{ce} \psi_a \mathcal{H}_2^-. $$

By Theorem 4.1.3 this defines the behavior of an autonomous system. In general, $\psi_{ce} \psi_a \not\in \mathcal{H}_\infty^+$ but there exist $X_+ \text{ and } X_-$ such that $\psi_{ce} \psi_a = X_+ + X_-$ with $X_+ \in \mathcal{H}_\infty^+$ and $X_- \in \mathcal{H}_\infty^-$. We find $B_a = \Pi_+ X_+ \mathcal{H}_2^-$. Let $N, M$ be a right coprime factorization over $\mathcal{H}_\infty^+$ of $X_+$ with $N$ square and $N^-N = I$, i.e. $X_+ = N^{-1}M$. We claim that $B_a = \Pi_+ N^{-1} \mathcal{H}_2^-$. To see this, note that $X_+, N^{-1} \in \mathcal{H}_\infty^+$ and $M \in \mathcal{H}_\infty^-$. Since there are no stable pole-zero cancelations between $N$ and $M$ we find that the McMillan degree of $N$ and $X_+$ is equal. For $M \mathcal{H}_2^+ \subseteq \mathcal{H}_2^+$ we have

$$B_a = \Pi_+ X_+ \mathcal{H}_2^- = \Pi_+ N^{-1} M \mathcal{H}_2^- \subseteq \Pi_+ N^{-1} \mathcal{H}_2^-.$$ 

But the subspaces on the left and on the right have the same dimension since $N$ and $X_+$ have the same McMillan degree. Hence $B_a = \Pi_+ N^{-1} \mathcal{H}_2^-$. Define $\Psi_a = \psi_{ce} N^{-1}$. It remains to prove that

$$\tilde{B}_m(\psi_a, \psi_c) := \Pi_+ \psi_a \mathcal{H}_2^- + \Pi_+ \psi_c L_2 = \Pi_+ \psi_a \mathcal{H}_2^- + \Pi_+ \psi_c L_2 =: \tilde{B}_m(\psi_a, \psi_c). \quad (6.6)$$

To see this, first note that since $\psi_{ce} \psi_a = N^{-1} M + X_-$ we find that $\psi_a = \psi_{ce} (N^{-1} M + X_-) + \psi_c S$ for some $S \in \mathcal{L}_\infty$. Since $\psi_{ce}$ is a tall rational matrix without zeros, the number of stable poles of $\psi_{ce}$ and $\psi_{ce} \psi_{ce}$ are equal. This implies:

$$\Pi_+ \psi_{ce} \mathcal{H}_2^- = \Pi_+ \psi_{ce} \psi_{ce} \mathcal{H}_2^-$$

Similarly, we find:

$$\Pi_+ \psi_c \mathcal{H}_2^- = \Pi_+ \psi_c \psi_{ce} \mathcal{H}_2^-$$

But then:

$$\Pi_+ \psi_a \mathcal{H}_2^- = \Pi_+ \psi_{ce} \psi_{ce} \mathcal{H}_2^- = \Pi_+ (I - \psi_c \psi_{ce}) \mathcal{H}_2^- = \Pi_+ \psi_c \mathcal{H}_2^-.$$ 

After these preliminaries we return to proving the identity (6.6). We have:

$$\Pi_+ \psi_a \mathcal{H}_2^- + \Pi_+ \psi_c L_2 = \Pi_+ \psi_{ce} N^{-1} \mathcal{H}_2^- + \Pi_+ \psi_c L_2$$

$$= \Pi_+ \psi_{ce} N^{-1} M \mathcal{H}_2^- + \Pi_+ \psi_c L_2$$

$$= \Pi_+ \psi_a \mathcal{H}_2^- + \Pi_+ \psi_c L_2 + \Pi_+ (\psi_{ce} X_- + \psi_c S) \mathcal{H}_2^-$$

$$= \Pi_+ \psi_a \mathcal{H}_2^- + \Pi_+ \psi_c L_2.$$
where the second equality follows because $M \mathcal{H}_2^+ \subseteq \mathcal{H}_2^+$ and the rank of the two Hankel operators $\Pi_+ \Psi_{ee} N^{-1}$ and $\Pi_+ \Psi_{ee} N^{-1} M$ are equal because $N$ and $M$ are coprime. The last equality follows since
\[ \Pi_+ \Psi_{ee} \mathcal{H}_2^+ = \Pi_+ \Psi_{ee} \mathcal{H}_2^+ \subseteq \Pi_+ \Psi_{ee} \mathcal{L}_2 = \Pi_+ \Psi_{ee} \mathcal{L}_2. \]

2. The proof of part 1 implies that we can always find a normalized representation $B_{im}(\tilde{\Psi}_a, \tilde{\Psi}_c)$, and elements $R \in \mathcal{H}_\infty^-$, $T \in \mathcal{H}_\infty^+$ and $S \in \mathcal{L}_\infty$ such that $\tilde{\Psi}_a$ and $\tilde{\Psi}_c$ satisfy (6.4) and (6.5). It remains to be proven that for each image representation we can choose the same $\tilde{\Psi}_a$ and $\tilde{\Psi}_c$. Suppose that $\tilde{\Psi}_{a,1}, \tilde{\Psi}_{a,2}, \tilde{\Psi}_{c,1}, \tilde{\Psi}_{c,2} \in \mathcal{H}_\infty^+$ are such that $B_{im}(\tilde{\Psi}_{a,1}, \tilde{\Psi}_{c,1})$ and $B_{im}(\tilde{\Psi}_{a,2}, \tilde{\Psi}_{c,2})$ are two normalized image representations. To prove part 2 it is sufficient to prove that
\[ B_{im}(\tilde{\Psi}_{a,1}, \tilde{\Psi}_{c,1}) = B_{im}(\tilde{\Psi}_{a,2}, \tilde{\Psi}_{c,2}). \]
implies the existence of constant unitary matrices $U_a$ and $U_c$ such that $\tilde{\Psi}_{c,1} U_c = \tilde{\Psi}_{c,2}$ and $\tilde{\Psi}_{a,1} U_a = \tilde{\Psi}_{a,2}$. Note that we have:
\[ \Pi_+ \tilde{\Psi}_{c,1}^+ (\Pi_+ \tilde{\Psi}_{a,1} \mathcal{H}_2^- + \Pi_+ \tilde{\Psi}_{c,1} \mathcal{L}_2) = \mathcal{H}_2^+. \]
Hence we must have
\[ \Pi_+ \tilde{\Psi}_{c,1}^+ (\Pi_+ \tilde{\Psi}_{a,2} \mathcal{H}_2^- + \Pi_+ \tilde{\Psi}_{c,2} \mathcal{L}_2) = \Pi_+ \tilde{\Psi}_{c,1}^+ \tilde{\Psi}_{a,2} \mathcal{H}_2^- + \Pi_+ \tilde{\Psi}_{c,1}^+ \tilde{\Psi}_{c,2} \mathcal{L}_2 = \mathcal{H}_2^+. \]
and it is easy to see that this implies that $U_c := \tilde{\Psi}_{c,1}^+ \tilde{\Psi}_{c,2}$ is right-invertible as a rational matrix and $\tilde{\Psi}_{c,1} U_c = \tilde{\Psi}_{c,2}$. A dual argument shows that $U_c$ is actually invertible. Since $\tilde{\Psi}_{c,1}$ and $\tilde{\Psi}_{c,2}$ are stable and have no zeros this implies that $U_c$ and $U_c^{-1}$ are both stable. Moreover, since $\tilde{\Psi}_{c,1}$ and $\tilde{\Psi}_{c,2}$ are both inner, we find that $U_c$ is inner. Thus $U_c$ is inner and outer and hence it must be a constant unitary matrix.

Similarly we get
\[ \Pi_+ \tilde{\Psi}_{a,1}^+ (\Pi_+ \tilde{\Psi}_{a,1} \mathcal{H}_2^- + \Pi_+ \tilde{\Psi}_{c,1} \mathcal{L}_2) = 0. \]
Hence
\[ \Pi_+ \tilde{\Psi}_{a,1}^+ (\Pi_+ \tilde{\Psi}_{a,2} \mathcal{H}_2^- + \Pi_+ \tilde{\Psi}_{c,2} \mathcal{L}_2) = \Pi_+ \tilde{\Psi}_{a,1}^+ \tilde{\Psi}_{a,2} \mathcal{H}_2^- = 0 \]
where we used that $\tilde{\Psi}_{c,2} = \tilde{\Psi}_{c,1} U_c$. We conclude that $U_a := \tilde{\Psi}_{a,1}^+ \tilde{\Psi}_{a,2}$ is in $\mathcal{H}_\infty^-$ and $\tilde{\Psi}_{a,1} U_a = \tilde{\Psi}_{a,2}$. Along the same line we can prove that $U_a$ is invertible and $U_a^{-1} \in \mathcal{H}_\infty^+$. But since $\tilde{\Psi}_{a,1}$ and $\tilde{\Psi}_{a,2}$ are both inner also $U_a$ is inner. Therefore, $U_a$ is inner and outer and hence a constant unitary matrix as desired. $\square$

7 Rational representations of $\ell_2$ systems

The main result of this section claims that all $\ell_2$ systems whose behavior belongs to either $\mathbb{B}_2$ or $\mathbb{B}_2^{\text{complete}}$ allow for both kernel and image representations. We moreover characterize minimal kernel representations and minimal image representations and provide a complete characterization of the equilibrium response of these systems. The following theorem is the first main result of this section.

Theorem 7.1 The following statements are equivalent.
1. \( \mathcal{B} \in \mathcal{B}_2 \)

2. \( \mathcal{B} \) admits a kernel representation \( \mathcal{B} = \mathcal{B}_m(\Theta) \).

3. \( \mathcal{B} \) admits an image representation \( \mathcal{B} = \mathcal{B}_m(\Psi_u, \Psi_c) \).

**Proof.** (1 \( \iff \) 2). We know from the Beurling-Lax theorem that any \( \sigma_n \)-invariant subspace (i.e. \( \sigma_n \) invariant subspace) has an image representation. From theorem 4.1 we know that if \( \mathcal{B} \) is \( \sigma_n \)-invariant then \( \mathcal{B}^\perp \) is \( \sigma_n \)-invariant. Therefore, by the Beurling-Lax theorem, there exists \( \Psi \in \mathcal{H}^+ \) such that \( \mathcal{B}^\perp = \text{im} \Psi \). The proof of the theorem is then completed by the following sequences of equalities

\[
\mathcal{B} = [\text{im} \Psi]^L = \{ \bar{v} \in \mathcal{H}_2^+ \mid \langle \bar{w}, \bar{v} \rangle = 0 \text{ for all } \bar{w} \in \text{im} \Psi \} = \{ \bar{v} \in \mathcal{H}_2^+ \mid \langle \Psi \hat{x}, \bar{v} \rangle = 0 \text{ for all } \hat{x} \in \mathcal{H}_2^+ \} = \{ \bar{v} \in \mathcal{H}_2^+ \mid \langle \hat{x}, \Psi^* \bar{v} \rangle = 0 \text{ for all } \hat{x} \in L_2 \} = \{ \bar{v} \in \mathcal{H}_2^+ \mid \Pi_+ \Psi^* \bar{v} = 0 \} = \ker(\Pi_+ \Psi^*),
\]

where we used that \( \mathcal{B}^\perp \) is closed. Setting \( \Theta = \Psi^* \) yields the result.

(2 \( \iff \) 3) From the previous we know that \( \mathcal{B} = \mathcal{B}_m(\Theta_a) \) for some \( \Theta_a \in \mathcal{H}_\infty \). Moreover, by theorem 5.5 we can assume without loss of generality that \( \Theta_a \) is normalized. Construct \( \Theta_c \) such that:

\[
\Theta_a = \begin{pmatrix} \Theta_a^* \\ \Theta_c \end{pmatrix}
\]

is a square and co-inner matrix in \( \mathcal{H}_\infty^+ \). (If \( \Theta_a \) is square then \( \Theta_c \) is a trivial 0-dimensional matrix).

We first prove that:

\[
\left\{ \Pi_+ \left( \Theta_a^*, \Theta_c^* \right) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right| s_1 \in \mathcal{H}_2^-, s_2 \in L_2 \right\} \subset \mathcal{B}
\]

Let \( w \) be an element of the set on the left hand side, i.e. there exist \( s_1 \in \mathcal{H}_2^- \) and \( s_2 \in L_2 \) such that

\[ w = \Pi_+ \Theta_a^* s_1 + \Pi_+ \Theta_c^* s_2 \]

Since \( \mathcal{B} \) is closed and \( \mathcal{B}^\perp = \text{im} \Theta_a^* \) we find that \( w \in \mathcal{B} \) if and only if \( \langle w, \Theta_a^* v \rangle = 0 \) for all \( v \in \mathcal{H}_2^+ \).

This follows easily by observing that:

\[
\langle w, \Theta_a^* v \rangle = \langle \Theta_a^* s_1 + \Theta_c^* s_2, \Theta_a^* v \rangle = \langle \Theta_a^* s_1, \Theta_a^* v \rangle = \langle s_1, v \rangle = 0,
\]

where we used that \( \Theta_c \Theta_a^* = 0 \) and \( \Theta_a \Theta_a^* = I \). To prove the reverse inclusion it is sufficient to show that:

\[
\left\{ \Pi_+ \left( \Theta_a^*, \Theta_c^* \right) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right| s_1 \in \mathcal{H}_2^-, s_2 \in L_2 \right\} + \mathcal{B}^\perp = \mathcal{H}_2^+.
\]

The latter follows from our image representation of \( \mathcal{B}^\perp \) and the fact that \( \Pi_+ \left( \Theta_a^*, \Theta_c^* \right) \) is surjective as a map from \( L_2 \) to \( \mathcal{H}_2^+ \). \( \square \)

The proof of the above result provides a constructive algorithm to convert a kernel representation of a behavior \( \mathcal{B} \in \mathcal{B}_2 \) to an image representation of \( \mathcal{B} \). The next theorem characterizes kernel and image representations of the equilibrium response \( \mathcal{B}^* \) of \( \mathcal{B} \), as introduced in definition 4.2.
Theorem 7.2 Let \( B \in \mathbb{B}_2^{\text{complete}} \) and suppose that \( B = B_w(\Theta) = B_w(\Psi_\alpha, \Psi_c) \) define a kernel and normalized image representation of \( B \). The equilibrium response \( B^* \) of \( B \) is given by

\[
B^* = \ker \Theta = \text{im} \Psi_c
\]

(7.1)

where both \( \Theta \) and \( \Psi_c \) in (7.1) are viewed as multiplicative operators defined on \( \mathcal{H}_2^+ \).

Proof. To see that \( \ker \Theta \subseteq B^* \), let \( w \in \mathcal{H}_2^+ \) be such that \( \Theta w = 0 \). Then clearly \( \Pi_+ \Theta w = 0 \) so that \( w \in \tilde{B} \). Moreover, \( (\Theta \tilde{\sigma}_k^i w)(z) = z^{-i} \Theta(z)w(z) = 0 \) implies that \( \Pi_+ \Theta \tilde{\sigma}_k^i w = 0 \) for all \( t \in \mathbb{Z}_+ \). The latter equality yields that \( \tilde{\sigma}_k^i w \in \tilde{B} \) so that \( w \in B^* \).

To prove the converse inclusion let \( w \in B^* \) and define \( v(t) := \Theta(\sigma_t)w(t) = \sum_{k=0}^{\infty} \Theta_{1-k} w(k) \) where \( t \in \mathbb{Z} \). As \( w \in B^* \subseteq \tilde{B} \) it follows that \( v(t) = 0 \) for \( t \in \mathbb{Z}_+ \) and \( v|_{\mathbb{Z}_-} \in \ell_2^+ \). Moreover, since \( \tilde{\sigma}_k^i w \in \tilde{B} \) for all \( t \in \mathbb{Z}_+ \), it follows that \( v(t) := \Theta(\sigma_t)\tilde{\sigma}_k^i w(t) = v(t - \tau) = 0 \) for all \( t, \tau \in \mathbb{Z}_+ \). Hence, \( v(t) = 0 \) for all \( t \in \mathbb{Z} \) which yields that \( w \in \ker \Theta(\sigma_t) \). We conclude that \( B^* \subseteq \ker \Theta \) which together with the previous inclusion yields the first equality of (7.1).

To prove the second equality, let \( w \in \text{im} \Psi_c \). Thus there exists \( v \in \mathcal{H}_2^+ \) such that \( w = \Psi_c v \). By (6.3), \( w \in \tilde{B} \) and since \( \tilde{\sigma}_k^i v \in \mathcal{H}_2^+ \) for all \( t \in \mathbb{Z}_+ \) it follows that \( \tilde{\sigma}_0^i w = \Psi_c \tilde{\sigma}_0^i v \in \text{im} \Psi_c \) which also belongs to \( \tilde{B} \) for all \( t \in \mathbb{Z}_+ \). Hence \( \text{im} \Psi_c \subseteq B^* \).

Finally, we show that \( B^* \subseteq \text{im} \Psi_c \). Assume there exists \( w \in B^* \) with \( \not\in \text{im} \Psi_c \). We know that \( \tilde{B} \) (mod \( \text{im} \Psi_c \)) is finite dimensional (say of dimension \( n \)) and that \( \tilde{\sigma}_k^i w \in \tilde{B} \) for \( t = 1, \ldots, n + 1 \) are independent. Hence there exist \( \alpha_i \in \mathbb{R} \) \( (i = 1, \ldots, n + 1) \) which are not all zero such that

\[
\sum_{i=1}^{n+1} \alpha_i \tilde{\sigma}_k^i w = \Psi_c y
\]

for some \( y \in L_2 \). Let \( f(z) := \sum_{i=1}^{n+1} \alpha_i z^{-i} \) and note that \( w = \Psi_c f^{-1} y \). The proof is complete if we show that \( v := f^{-1} y \) belongs to \( \mathcal{H}_2^+ \). Indeed, since the image representation is normalized, \( \Psi_c \) admits a left inverse \( \Psi_c^{-1} \in \mathcal{H}_\infty^+ \) so that

\[
v = f^{-1} y = \Psi_c^{-1} w \in \mathcal{H}_2^+
\]

where we used that \( w \in \mathcal{H}_2^+ \). Hence \( w \in \Psi_c \mathcal{H}_2^+ \) which yields a contradiction. Conclude that \( B^* \subseteq \text{im} \Psi_c \) as desired. \( \square \)

Note that the equilibrium response \( B^* \) does not depend on \( \Psi_\alpha \). This is quite natural as \( \Psi_\alpha \) represents a subset of \( B \) which corresponds to system trajectories which do not depart from the zero equilibrium. Further, we remark that the normalization of the image representation is necessary to prove that \( B^* \subseteq \text{im} \Psi_c \). For non-normalized image representations this inclusion does not hold. The above result on the representation of the equilibrium response of a given behavior \( B \in \mathbb{B}_2 \) turns out to be extremely useful to characterize and to construct state space representations of systems with behavior \( B \in \mathbb{B}_2 \). This will be the topic of the next section. Using Theorem 4.4 and Theorem 7.2, a parameterization of the model set \( \mathbb{B}_2^{\text{complete}} \) is now given as follows.

Theorem 7.3 The following statements are equivalent.

1. \( B \in \mathbb{B}_2^{\text{complete}} \)

2. \( B \) admits a kernel representation \( B = B_w(\Theta) \) with \( \Theta \in \mathcal{R}\mathcal{H}_\infty \).
3. B admits an image representation $B = B_m(\Psi_a, \Psi_c)$ with $\Psi_a, \Psi_c \in \mathcal{R}\mathcal{H}_\infty^+$.

Proof. The existence of kernel and image representations has been shown in theorem 7.1. Remains to show that there exists rational image and kernel representations. By theorem 7.2 we know that $B^*$ has a kernel representation $B^* = \ker \Theta$ while $B = \ker \Pi_+ \Theta$. Without loss of generality we can assume that $\Theta$ is normalized. By theorem 4.4, $B \in \mathcal{B}^\text{complete}$ is equivalent to $B \in \mathcal{B}_2$ and $B \mod (B^*)$ finite dimensional. It therefore suffices to show that $B \mod (B^*)$ is finite dimensional if and only if $\Theta$ has finite rank. To see this, consider the map $R$ from $B$ into $\mathcal{H}_2^-$ defined by $x \rightarrow \Pi_- \Theta x$. Since $\Pi_+ \Theta B = 0$ we find that $B^* = \ker R$. Then it is immediate that $B \mod (B^*)$ is finite dimensional if and only if $\ker R$ is finite dimensional. There is an obvious extension of $R$ to a map $R_\epsilon$ from $\mathcal{H}_\infty^+$ to $\mathcal{H}_\infty^+$. Let $z \in \ker R$, i.e. there exists $y \in \mathcal{H}_\infty^+$ such that $z = \Pi_- \Theta y$. We can decompose $y = y_1 + y_2$ with $y_1 \in B$ and $y_2 \in B^\perp$. Since $B^\perp = \ker \Theta^\perp$ there exists $z_2 \in \mathcal{H}_\infty^+$ such that $y_2 = \Theta^\perp z_2$. But then

$$z = R_\epsilon y = \Pi_- \Theta (y_1 + y_2) = R y_1 + \Pi_- \Theta \Theta^\perp z_2 = R y_1 + \Pi_- z_2 = R y_1.$$

Therefore the image of $R$ is equal to the image of $R_\epsilon$. Moreover, $R_\epsilon$ is precisely the Hankel operator associated to $\Theta$ and hence the image of $R_\epsilon$ is finite-dimensional if and only if $\Theta$ is rational.

From the proof of theorem 7.1 it is immediate that the existence of a rational kernel representation implies the existence of a rational image representation. $\square$

8 State space representations

In this section we will show that state space representations of systems $B \in \mathcal{B}_2$ can be constructed directly from the kernel and image representations which have been introduced so far. Such a construction is non-trivial as it amounts to define the state of a system on the basis of a representation of the external behavior only. First of all, a state space needs to be defined, and second, the evolution of the state variable as a function of time needs to be specified. The construction of state space representations will be based on the subsets $B$ and $B^*$ and will exploit the difference between right- and left shift invariance.

8.1 The state space construction

In section 4 we defined the state space $X$ of a behavior $B \in \mathcal{B}_2$ as the factor space $B \mod (B^*)$. With this definition of state every $w \in B$ defines an element $x(0) := w \mod (B^*)$ in $X$ consisting of the equivalence class of all trajectories $w' \in B$ which are right-shift equivalent with $w$. Similarly, at an arbitrary time instant $t \in \mathbb{Z}_+$ the state $x(t)$ is determined by the equivalence class of all trajectories $w' \in B$ for which the left-shifts $\sigma_t^l w$ and $\sigma_t^l w'$ are right-shift equivalent. Thus, with $B^*_t$ denoting the equilibrium response of $\sigma_t^l B$ we have that

$$x(t) := \sigma_t^l w \mod (B^*_t)$$

defines the state of the system at time $t \in \mathbb{Z}_+$ and the factor space $X_t := \sigma_t^l B \mod (B^*_t)$ is identified with the state space of the system $\Sigma = (\mathbb{Z}_+, \mathbb{R}, \mathcal{B})$ at time $t \in \mathbb{Z}_+$. We see that in general the state space $X_t$, $t \in \mathbb{Z}_+$ of $\Sigma$ is a time varying object. However, the following lemma shows that for left-shift invariant systems $B^*_t$ is independent of $t \in \mathbb{Z}_+$. 

Lemma 8.1 Let \( B \) be left-shift invariant. Then the equilibrium response \( B_1^* \) of \( \sigma_1 B \) is independent of \( t \in \mathbb{Z}_+ \) and equals \( B^* \) as defined in definition 4.2.

Proof. We will show that \( B_0^* = B_1^* \). The result then follows from an induction argument. To see that \( B_0^* \subseteq B_1^* \) let \( w \in B_0^* \). Then, by definition, \( \sigma_1 w \in B \) for all \( t \in \mathbb{Z}_+ \). Define \( w' := \sigma_1 w \). Then \( w' \in B \) by assumption and since \( w = \sigma_1 w' \) it follows that \( w \in \sigma_1 B \). Moreover, as \( \sigma_n w = \sigma_1 \sigma_{n+1} w \) with (by assumption) \( \sigma_n w \in \sigma_1 B \) for all \( t \in \mathbb{Z}_+ \). Hence, \( w \in B_1^* \) and thus \( B_0^* \subseteq B_1^* \). To prove the converse inclusion, let \( w \in B_1^* \). Then, \( \sigma_n w \in \sigma_n B \subseteq B \) for all \( t \in \mathbb{Z}_+ \). Conclude that \( w \in B_0^* \) so that \( B_0^* = B_1^* \). \( \square \)

For left-shift invariant systems we have that \( \sigma_1 B \subseteq \sigma_1^{-1} B \) and together with Lemma 8.1 this implies that the sequence \( \{X_t\}_{t \in \mathbb{Z}_+} \) is monotonically non-increasing in the sense that \( X_{t+1} \leq X_t \) for all \( t > 0 \). In other words, the minimal dimension of the state space is time varying for systems with behavior \( B \in \mathbb{B}_2 \). We introduce the following types of state space systems.

Definition 8.2 Let \( \mathcal{X} \) be a normed space.

1. An output-nulling state space system is a quadruple \( \Sigma_{\text{ON}} := (\mathbb{Z}_+, \mathbb{R}^q, \mathcal{X}, B_{\text{ON}}) \) where

\[
B_{\text{ON}} := \left\{ (w, x) \in \ell_2^+ \mid \begin{array}{l}
x(t + 1) = Ax(t) + Bw(t) \\
0 = Cx(t) + Dw(t)
\end{array} \right\}. \tag{8.1}
\]

2. A driving-variable state space system is a quadruple \( \Sigma_{\text{DV}} := (\mathbb{Z}_+, \mathbb{R}^q, \mathcal{X}, B_{\text{DV}}) \) where

\[
B_{\text{DV}} := \left\{ (w, x) \in \ell_2^+ \mid \exists v \in \ell_2(\mathbb{Z}_+), w(t) = Cx(t) + Dw(t) \right\}. \tag{8.2}
\]

Here, \( x \in \ell_2(\mathbb{Z}_+, \mathcal{X}) \) is called the state, \( w \in \ell_2(\mathbb{Z}_+, \mathbb{R}^q) \) is the external variable and \( v \in \ell_2(\mathbb{Z}_+, \mathcal{V}) \) in (8.2) is called a driving variable. The operators \( A, B, C \) and \( D \) in (8.1) and (8.2) are bounded and compatible with the indicated partitioning.

A state space system \( \Sigma_{\text{ON}} [\Sigma_{\text{DV}}] \) is viewed as an output-nulling [driving variable] state space representation of an \( \ell_2 \) system \( \Sigma = (\mathbb{Z}_+, \mathbb{R}^q, B) \) if the projection of \( B_{\text{ON}} [B_{\text{DV}}] \) on the external variables \( w \) coincides with \( B \). In other words, \( B_{\text{ON}} [B_{\text{DV}}] \) represents \( B \) if

\[
B = \{ w \in \ell_2^+ \mid \exists x \in \ell_2^+ \text{ such that } (w, x) \in B_{\text{ON}} [B_{\text{DV}}] \}. 
\]

Note that we require the state trajectory to be an element of \( \ell_2^+ \).

8.2 The state evolution map

A state space representation of \( (\mathbb{Z}_+, \mathbb{R}^q, B) \) with \( B \in \mathbb{B}_2 \) is obtained as follows. Let \( B^* \subseteq B \) be as defined in (4.1) and let \( B^0 := \{ w \in B^* \mid w(0) = 0 \} \) be a subspace of \( B^* \). The factor space \( B^* \text{ (mod } B^0) \) is identified with a subspace \( \mathcal{V} \subset \mathbb{R}^q \) which consists of all initial values \( w(0) \) of trajectories \( w \in B^* \). Formally,

\[
\mathcal{V} := \{ w_0 \in \mathbb{R}^q \mid \exists w \in B^*; w(0) = w_0 \}. \tag{8.3}
\]
Since $\mathcal{B}^*$ consist of all trajectories $w \in \mathcal{B}$ which are initially at rest, the factor space $\mathcal{B}^* \mod \mathcal{B}^0$, and thus $\mathcal{V}$, has the interpretation of an 'input space' for $\mathcal{B}$.

Let $\pi^*$ and $\pi^*$ define the canonical projections from $\mathcal{B}$ to $\mathcal{B} (\mod \mathcal{B}^0)$ and $\mathcal{B} (\mod \mathcal{B}^*)$, respectively. We know that $\mathcal{B}^*$ and $\mathcal{B}^0$ are both closed subspaces of $\ell_2^+$ and hence these projections define bounded operators. Since $\sigma_l \mathcal{B}^0 \subseteq \mathcal{B}^*$ there exists a map

$$F : \mathcal{B} (\mod \mathcal{B}^0) \rightarrow \mathcal{B} (\mod \mathcal{B}^*)$$

such that $F \pi^0 = \pi^* \sigma_l$. In other words, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\sigma_l} & \mathcal{B} \\
\downarrow{\pi^0} & & \downarrow{\pi^*} \\
\mathcal{B} (\mod \mathcal{B}^0) & \xrightarrow{F} & \mathcal{B} (\mod \mathcal{B}^*)
\end{array}
\]

Since the left-shift operator defines a possibly non-surjective map $\sigma_l : \mathcal{B} \rightarrow \mathcal{B}$, also $F$ is possibly non-surjective. By Lemma 8.1 the factor spaces $\sigma_l^1 \mathcal{B} (\mod \mathcal{B}^0)$ and $\sigma_l^1 \mathcal{B} (\mod \mathcal{B}^*)$ allow for a natural embedding in $\mathcal{B} (\mod \mathcal{B}^0)$ and $\mathcal{B} (\mod \mathcal{B}^*)$, respectively. A similar construction can therefore be applied to define for each $t \in \mathbb{Z}_+$ a map $F_t : (\sigma_l^1 \mathcal{B}) (\mod \mathcal{B}^0) \rightarrow (\sigma_l^1 \mathcal{B}) (\mod \mathcal{B}^*)$ such that $F_t \pi^0_t = \pi^* \sigma_l$ with $\pi^0_t$ and $\pi^* t$ the canonical projections from $\sigma_l^1 \mathcal{B}$ to $(\sigma_l^1 \mathcal{B}) (\mod \mathcal{B}^0)$ and $(\sigma_l^1 \mathcal{B}) (\mod \mathcal{B}^*)$, respectively. Using left-shift invariance of the system we can actually show that $F_t = F | \sigma_l^1 \mathcal{B}^0$ for all $t \in \mathbb{Z}_+$.

If $\mathcal{B} \in \mathbb{H}_2^{\infty}$, then by theorem 4.4 both $\mathcal{B} (\mod \mathcal{B}^0)$ as well as $\mathcal{B} (\mod \mathcal{B}^*)$ are finite dimensional and the map $F$ yields a state space representation of $\mathcal{B}$ by choosing a basis for $\mathcal{B} (\mod \mathcal{B}^*)$ and a basis for $\mathcal{B}^* (\mod \mathcal{B}^0)$. In other words, we can construct matrices $A$ and $B$ such that

$$x(t+1) = F[\sigma_l^1 w \ (\mod \mathcal{B}^0)] = Ax(t) + Bu(t)$$

defines a matrix representation of a state evolution of $\mathcal{B}$ where $x(t) = \sigma_l^1 w \ (\mod \mathcal{B}^*)$ is the state and $v(t) = \Pi \Pi w(t)$ is a driving variable. Here, $w \in \mathcal{B}$, $t \in \mathbb{Z}_+$ and $\Pi$ is the projection from $\mathbb{R}^4$ onto $\mathcal{V}$.

### 8.3 The construction of state space representations

The construction of state space representations which we present here uses the basic properties of Hankel operators associated with kernel and image representations of $\ell_2$ systems. We first state a result in which the complexity $n(\mathcal{B})$ of a set $\mathcal{B} \in \mathbb{H}_2$ is compared with the complexity of its representation. The complexity of a kernel representation $\Theta$ of $\mathcal{B}$ is defined as its McMillan degree and denoted $\deg(\Theta)$. Similarly, $\deg(\Psi, \Psi)$ will denote the McMillan degree of $(\Psi, \Psi)$. Note that the McMillan degree of an operator in $\mathcal{R} \mathcal{H}_\infty^+$ or $\mathcal{R} \mathcal{H}_\infty^-$ is equal to the dimension of the image of a Hankel operator which has this representation as its symbol.

**Theorem 8.3** Let $\mathcal{B} \in \mathbb{H}_2^{\infty}$ have complexity $n(\mathcal{B})$ and suppose that

$$\mathcal{B} = \mathcal{B}_{\mathcal{B}^0}(\Theta) = \mathcal{B}_{\mathcal{B}^0}(\Psi, \Psi)$$

define kernel and image representations of $\mathcal{B}$. Then
1. \( n(B) = \dim(B \cap B^*) \).

2. \( n(B) \leq \deg(\Theta) \).

3. If \( \Theta \) is normalized then \( \hat{B} \cap \hat{B}^* = \Pi_+ \Theta^* \mathcal{H}_2^* \) and \( n(B) = \deg(\Theta) \).

4. \( n(B) \leq \deg(\Psi_a \Psi_e) \).

5. If \( (\Psi_a \Psi_e) \) is normalized then \( \hat{B} \cap \hat{B}^* = \Pi_+ (\Psi_a \Psi_e) \mathcal{H}_2^* \) and \( n(B) = \deg(\Psi_a \Psi_e) \).

Proof. 1. Obvious.

3. If \( B = B_m(\Theta) \) then, by theorem 7.2, \( \hat{B}^* = \ker \Theta \) where \( \Theta \) is viewed as a multiplicative map \( \Theta : \mathcal{H}_2^* \rightarrow \mathcal{L}_2 \). Since

\[
\hat{B}^* = \{ w \in \mathcal{H}_2^* \mid \langle \Theta w, v \rangle = 0 \text{ for all } v \in \mathcal{L}_2 \}
\]

\[
= \{ w \in \mathcal{H}_2^* \mid \langle w, \Theta^* v \rangle = 0 \text{ for all } v \in \mathcal{L}_2 \}
\]

\[
= \{ w \in \mathcal{H}_2^* \mid \langle w, \Pi_+ \Theta^* v \rangle = 0 \text{ for all } v \in \mathcal{L}_2 \}
\]

\[
= [\Pi_+ \Theta^* \mathcal{L}_2]^1,
\]

it follows that \( \hat{B}^* = \Pi_+ \Theta^* \mathcal{L}_2 \), where we used that the image of a rational operator is closed. Suppose that \( \Theta \) is normalized and let \( w \in B \cap \hat{B}^* \). The latter equality yields that there exists \( v \in \mathcal{L}_2 \) such that \( w = \Pi_+ \Theta^* v \). Decompose \( v = v_+ + v_- \) where \( v_+ \in \mathcal{H}_2^* \) and \( v_- \in \mathcal{H}_2^- \). Then \( w = \Theta^* v_+ + \Pi_+ \Theta^* v_- \) where we used that \( \Theta^* \in \mathcal{H}_2^* \). By (5.4), \( \Theta^* v_+ \in \hat{B}^* \). On the other hand, \( \Pi_+ \Theta^* v_- \in \hat{B} \) since \( \hat{B} = \Theta^* \mathcal{H}_2^* \) and for all \( v \in \mathcal{H}_2^* \) we have

\[
\langle \Pi_+ \Theta^* v_-, \Theta^* v \rangle = \langle \Theta^* v_-, \Theta^* v \rangle = \langle v_-, v \rangle = 0.
\] (8.5)

Since \( w \in \hat{B} \) it thus follows that \( \Theta^* v_+ = 0 \) and \( w = \Pi_+ \Theta^* v_- \). We conclude that \( \hat{B} \cap \hat{B}^* \subseteq \Pi_+ \Theta^* \mathcal{H}_2^- \). Conversely, let \( w = \Pi_+ \Theta^* v_- \) for some \( v_- \in \mathcal{H}_2^- \). Since \( \hat{B}^* = \Pi_+ \Theta^* \mathcal{L}_2 \) it is immediate that \( w \in \hat{B}^* \). Since (8.5) holds for all \( v \in \mathcal{H}_2^* \), it follows that \( w \in \hat{B} \) and hence \( \hat{B} \cap \hat{B}^* = \Pi_+ \Theta^* \mathcal{H}_2^- \) as desired. The claim that \( n(B) = \deg(\Theta) \) is a trivial consequence of 1. and the fact that the McMillan degree is equal to the dimension of \( \Pi_+ \Theta^* \mathcal{H}_2^- \).

2. If \( \Theta \) is not inner then we can find an outer-inner factorization of \( \Theta \), i.e. there exists \( \Theta_o, \Theta_i \) which are coprime such that \( \Theta_i \Theta_o, \Theta_o^{-1} \in \mathcal{H}_2^- \), \( \Theta_i \Theta_o = I \) and \( \Theta_i \Theta_i = \Theta \). Since

\[
\hat{B} \cap \hat{B}^* = \Pi_+ \Theta_o^* \mathcal{H}_2^-;
\]

it follows that \( n(B) = \deg(\Theta_i) \leq \deg(\Theta) \).

5. Suppose that \( (\Psi_a \Psi_e) \) is normalized and let \( w \in \hat{B} \cap \hat{B}^* \). Then there exist \( v_1 \in \mathcal{H}_2^* \) and \( v_2 \in \mathcal{L}_2 \) such that \( w = \Pi_+ \Psi_a v_1 + \Pi_+ \Psi_e v_2 \). Let \( v_2 = v_+ + v_- \) be a decomposition of \( v_2 \) with \( v_+ \in \mathcal{H}_2^* \) and \( v_- \in \mathcal{H}_2^- \). Observe that for all \( v \in \mathcal{H}_2^* \),

\[
\langle \Pi_+ \Psi_a v_1 + \Pi_+ \Psi_e v_-, \Psi_e v \rangle = \langle \Psi_a v_1 + \Psi_e v_-, \Psi_e v \rangle = \langle v_-, v \rangle = 0
\] (8.6)

where we used that \( \Psi_a^* \Psi_e = 0 \) and \( \Psi_e^* \Psi_e = I \). Together with theorem 7.2 this implies that \( \Pi_+ \Psi_a v_1 + \Pi_+ \Psi_e v_- \in \hat{B}^* \) and \( \Pi_+ \Psi_e v_+ = \Psi_e v_+ \in \hat{B}_* \). Since \( w \) also belongs to \( \hat{B}^* \) it follows that \( w = \Pi_+ \Psi_a v_1 + \Pi_+ \Psi_e v_- \). Conversely, if \( w = \Pi_+ \Psi_a v_1 + \Pi_+ \Psi_e v_- \) for some \( v_1, v_- \in \mathcal{H}_2^* \) then obviously \( w \in \hat{B} \), while (8.6) implies that \( \langle w, \hat{B}^* \rangle = 0 \), i.e., \( w \in \hat{B} \cap \hat{B}^* \).
4. If \( \mathcal{B} = \mathcal{B}_{in}(\Psi_a, \Psi_c) \) then from theorem 6.6 we know there exists a normalized representation \( \mathcal{B}_{in}(\tilde{\Psi}_a, \tilde{\Psi}_c) \) of \( \mathcal{B} \) such that
\[
\begin{align*}
\Psi_c &= \tilde{\Psi}_a R + \tilde{\Psi}_c S \\
\tilde{\Psi}_a &= \tilde{\Psi}_c T
\end{align*}
\]
for some \( R \in \mathcal{H}_-^\infty, \ T \in \mathcal{H}_+^\infty \) and \( S \in \mathcal{L}_\infty \) where \( R \) and \( T \) have a right-inverse in \( \mathcal{L}_\infty \). Using a similar argument as in the proof of statement 5, it follows that
\[
n(\mathcal{B}) = \text{deg}(\begin{bmatrix} \tilde{\Psi}_a & \tilde{\Psi}_c \end{bmatrix}) \leq \text{deg}(\begin{bmatrix} \Psi_a & \Psi_c \end{bmatrix})
\]
which completes the proof. \( \square \)

**Definition 8.4** A kernel representation \( \Theta \) of \( \mathcal{B} \in \mathbb{B}_2 \) is said to be **minimal** if \( \text{deg}(\Theta) = n(\mathcal{B}) \). Similarly, an image representation \( (\Psi_a, \Psi_c) \) of \( \mathcal{B} \in \mathbb{B}_2 \) is said to be **minimal** if \( \text{deg}(\Psi_a, \Psi_c) = n(\mathcal{B}) \).

The following corollary is an interesting and immediate consequence of Theorem 8.3.

**Corollary 8.5** Every normalized kernel representation and normalized image representation of an element in \( \mathbb{B}_2 \) is minimal.

We have the following construction of (possibly non-minimal) state space representations of \( \ell_2 \) systems. Let \( \mathcal{B} \in \mathbb{B}_2 \) be the behavior of a dynamical system and suppose that
\[
\mathcal{B} = \mathcal{B}_{ker}(\Theta) = \mathcal{B}_{in}(\Psi_a, \Psi_c)
\]
define kernel and image representations of \( \mathcal{B} \). Define
\[
\begin{align*}
\mathcal{X}_{ker} &= \Pi_- \Theta \mathcal{H}_{\mathcal{A}}^+ \\
\mathcal{X}_{im} &= \Pi_+(\Psi_a \quad \Psi_c) \mathcal{H}_{\mathcal{A}}^-
\end{align*}
\]
which we refer to as the **state space** associated with \( \mathcal{B}_{ker}(\Theta) \) and \( \mathcal{B}_{in}(\Psi_a, \Psi_c) \), respectively.

**Remark 8.6** Note that \( \mathcal{X}_{ker} \) is a subset of the infinite dimensional space \( \mathcal{H}_{\mathcal{A}}^+ \) and \( \mathcal{X}_{im} \) is a subset of the infinite dimensional space \( \mathcal{H}_{\mathcal{A}}^- \). Their dimensions, however, are **finite** if and only if \( \Theta \) and \( (\Psi_a \quad \Psi_c) \) are rational operators.

Let \( w \in \mathcal{B} \) and suppose that \((v_1, v_2) \in \ell_2^2 \times \ell_2 \) are such that \( \hat{w} = \Pi_+ \Psi_a \hat{v}_1 + \Pi_+ \Psi_c \hat{v}_2 \). The **state trajectories** associated with \( w \in \mathcal{B} \) are the maps \( x_{ker} : \mathbb{Z}^+ \to \mathcal{X}_{ker} \) and \( x_{im} : \mathbb{Z}^+ \to \mathcal{X}_{im} \) defined by
\[
\begin{align*}
x_{ker}(t) &= \Pi_- \Theta \hat{v}_1(t) \tag{8.7} \\
x_{im}(t) &= \Pi_+ \Psi_a \hat{v}_1(t) + \Pi_+ \Psi_c \hat{v}_2(t) \tag{8.8}
\end{align*}
\]
where \( \hat{v}_1 : \mathcal{L}_2 \to \mathcal{L}_2 \) is the \( \mathcal{L}_2 \)-extended left shift \( (\hat{v}_1 w)(z) := zw(z) \). To describe the time evolutions of \( x_{ker} \) and \( x_{im} \) we introduce the operator \( P : W \to \mathcal{H}_{\mathcal{A}}^+ \) defined by \( (Pw)(z) = w \) for all \( z \in \mathbb{C} \). Note that for \( w \in \ell_2 \) and \( t \in \mathbb{Z}^+ \),
\[
P[w(t)] = (1 - \hat{\sigma}_t \hat{\sigma}_0) \hat{\sigma}_t \Pi_+ \hat{w}
\]
where \( \hat{w} \) denotes the Laplace transform of \( w \). The following identity is easily derived from the definitions of the shift operators

\[
\Pi_- \hat{\sigma}_L = \hat{\sigma}_L \Pi_- + \hat{\sigma}_L (1 - \hat{\sigma}_L \hat{\sigma}_L) \Pi_+.
\]

Using this identity, the time evolutions of the state are described by

\[
x_{\text{sn}}(t+1) = \hat{\sigma}_L x_{\text{sn}}(t) - \Pi_- \hat{\sigma}_L \Theta P w(t) \\
x_{\text{im}}(t+1) = \hat{\sigma}_L x_{\text{im}}(t) + \Pi_+ \Psi_e \hat{\sigma}_L P v_2(t).
\]

(8.9) (8.10)

Observe that the time evolution of \( x_{\text{im}} \) is independent of \( W_a \), the autonomous part of \( B \). This is in accordance with the intuitive idea that the 'autonomous subset' \( \Pi_+ \Psi_e \mathcal{H}_2^- \) of \( B \) is only determined by the initial conditions \( x_{\text{im}}(0) \).

To construct operators \( A, B, C, D \) which define state space representations of \( B \), let \( \Pi_0 : \mathcal{H}_2^+ \rightarrow W \) denote the map \( \Pi_0[\hat{w}(z)] = \lim_{\|z\| \rightarrow \infty} \hat{w}(z) \). As \( \mathcal{H}_2^+ \) functions are analytic at infinity, this limit is well defined and \( \Pi_0 \) maps \( \hat{w} \in \mathcal{H}_2^+ \) to the coefficient \( w_0 \) of the Laurent series expansion of \( \hat{w} \). Finally let \( \Pi_X \) be the orthogonal projection on \( X_\infty \). The following result concerns the construction of output nulling and driving variable state space representations and is the main result of this section.

**Theorem 8.7** Let \( B \in \mathcal{B}_2 \) and suppose that \( B = B_{\text{on}}(\emptyset) = B_{\text{im}}(\Psi_a, \Psi_e) \) define kernel and image representations of \( B \).

1. Define

\[
A := \Pi_X \hat{\sigma}_z; \quad B := -\Pi_X \Pi_- \hat{\sigma}_L \Theta P \\
C := (I - \Pi_X) \hat{\sigma}_z; \quad D := -(I - \Pi_X) \Pi_- \hat{\sigma}_L \Theta P
\]

Then \( B_{\text{on}}(A, B, C, D) \) defines an output-nulling state space representation of \( B \).

2. Define

\[
A := \hat{\sigma}_z; \quad B := \Pi_+ \Psi_e \hat{\sigma}_L P \\
C := \Pi_0; \quad D := \Pi_0 \Psi_e P
\]

Then \( B_{\text{pr}}(A, B, C, D) \) defines a driving-variable state space representation of \( B \).

**Proof.** 1. We first show that \( w \in \mathcal{B} \) implies that \((w, x) \in B_{\text{on}}(A, B, C, D)\) where \( x \) is defined by \( x(t) = \Pi_- \Theta \hat{\sigma}_L^t \hat{w} \). The fact that \( x(t+1) = Ax(t) + Bw(t) \) is immediate from (8.9). To see that \( Cx(t) + Dw(t) = 0 \), we note that

\[
\hat{\sigma}_L \Pi_- \Theta \hat{\sigma}_L^t \hat{w} - \Pi_- \hat{\sigma}_L \Theta P w(t) = \Pi_- \Theta \hat{\sigma}_L^{t+1} \hat{w} \in X
\]

where we used that \( \Pi_+ \hat{\sigma}_L \Theta P w(t) = 0 \). It is then immediate that \( Cx(t) + Dw(t) = 0 \). Remains to be shown that \( x \in \ell_2^+ \). We note that since the behaviour is complete it is also \( k \)-complete for some \( k \) (see the proof of theorem 4.4). Therefore \( x(t) \) is uniquely determined by \( w(t), \ldots, w(t+k) \). Moreover, it is straightforward that the mapping \((w(t), \ldots, w(t+k)) \mapsto x(t) \) is linear and bounded. Because the system is left-shift invariant this map does moreover not depend on \( t \). Let \( M \) be the norm of this map. Then we find

\[
\|x(t)\|^2 \leq M^2 (\|w(t)\|^2 + \cdots + \|w(t+k)\|^2),
\]
which immediately yields that \( x \in \ell_2^+ \).

Conversely, suppose that \((w, x) \in \mathcal{B}_{\text{con}}(A, B, C, D)\). We will show that for all \( k \in \mathbb{Z}_+ \) there exists \( v_k \in \ell_2^+ \)

1. \( x(i) = \Pi_- \Theta \hat{\sigma}_i^+ \hat{u}_k \) for \( i = 0, \ldots, k \).
2. \( \hat{\sigma}_k^+ \Pi_+ \Theta \hat{v}_k \in \mathcal{H}_2^+ \)
3. \( v_k|_{[0, k-1]} = w|_{[0, k-1]} \).

To show this, we start with \( x(k + 1) \in \mathcal{X}_\text{ext} \). Then

\[
x(k + 1) = \hat{\sigma}_L x(k) - \Pi_- \hat{\sigma}_L \Theta P w(k) \in \mathcal{X}_\text{ext}
\]

which by definition of \( \mathcal{X}_\text{ext} \) yields the existence of \( \hat{v} \in \mathcal{H}_2^+ \) such that

\[
\hat{\sigma}_L x(k) = \Pi_- \hat{\sigma}_L \Theta P w(k) + \Pi_- \Theta \hat{v}.
\]

This implies

\[
x(k) = \Pi_- \Theta P w(k) + \Pi_- \Theta \hat{\sigma}_n \hat{v} \\
0 = (1 - \hat{\sigma}_n \hat{\sigma}_L)[\Pi_+ \Theta P w(k) + \Pi_+ \Theta \hat{\sigma}_n \hat{v}].
\]

In other words, if we define \( \hat{v}'_k \in \mathcal{H}_2^+ \) by

\[
\hat{v}'_k := P w(k) + \hat{\sigma}_n \hat{v}
\]

then we have for \( i = k \):

4. \( x(j) = \Pi_- \Theta \hat{\sigma}_L^j \hat{v}'_k \) for \( i \leq j \leq k \).
5. \( \hat{\sigma}_k^k+1 \Pi_+ \Theta \hat{v}'_k \in \mathcal{H}_2^+ \)
6. \( \hat{v}'_k|_{[0, k-1]} = w|_{[i, k]} \).

Suppose that the above holds for \( i = s \) then we will show it also holds for \( i = s - 1 \). Define \( \hat{v}'_{s-1} \in \mathcal{H}_2^+ \) by

\[
\hat{v}'_{s-1} = P w(s - 1) + \hat{\sigma}_n \hat{v}'_k.
\]

Using similar arguments as above and the fact that \( \hat{v}'_k = \hat{\sigma}_l \hat{v}'_{s-1} \) we find 4. and 6. immediately. Let \( \hat{y} = \Theta \hat{v}'_k \). Then statement 5. holds if and only if \( y(t) = 0 \) for \( 0 \leq t < k - i + 1 \). The latter follows by observing that, by constrution of \( \hat{v}'_k \)

\[
P[y(t)] = (1 - \hat{\sigma}_n \hat{\sigma}_L)[\Pi_+ \Theta \hat{v}'_{s+1}] = 0.
\]

Finally, set \( v_k = v'_0 \). Statements 4-5 then imply that \( v_k \) satisfies 1-3. It is easy to see from the above construction that we can make sure that \( v_k \) is a bounded sequence in \( \mathcal{L}_2 \). Since \( \ell_2^+ \) is weakly compact there exists a subsequence converging to some \( \hat{v} \in \ell_2^+ \). Since \( v_k \) converges to \( w \) pointwise we must have \( \hat{v} = w \). But then

\[
\Pi_- \hat{\sigma}_L^i \Pi_+ \Theta v_k = 0
\]

for all \( k > i \) implies that

\[
\Pi_- \hat{\sigma}_L^i \Pi_+ \Theta w = 0
\]
for all i. This yields that $\Pi_i \Theta w = 0$ and hence $w \in \mathcal{B}$.

2. Let $w \in \mathcal{B}$. Hence there exists $v_1 \in \mathcal{H}_2^*$ and $v_2 \in \mathcal{L}_2$ such that $\dot{w} = \Pi_+ (\Psi_1 v_1 + \Psi_2 v_2)$. Let $x = x_{im}$ be defined by the right hand side of (8.8). The fact that $x(t + 1) = Ax(t) + Bv_2(t)$ then follows from (8.10). We find

$$w(t) = \Pi_0 \sigma_1 \Pi_+ (\Psi_1 v_1 + \Psi_2 v_2)$$
$$= \Pi_0 \Pi_+ \sigma_1 (\Psi_1 v_1 + \Psi_2 v_2)$$
$$= \Pi_0 (\dot{x}(t) + \Psi_2 \Pi_+ \sigma_1 v_2)$$
$$= \Pi_0 (x(t) + \Psi_2 [Pv_2(t) + \sigma_1 \Pi_+ \sigma_1^t v_2])$$
$$= \Pi_0 x(t) + \Pi_0 \Psi_2 \dot{v}_2(t)$$

where we used in the last equality that $\Pi_0 \Psi_2 \sigma_1 \Pi_+ = 0$, since $\Psi_2 \in \mathcal{H}_2^*$. Hence $w(t) = Cx(t) + Dv_2(t)$. Remains to show that $x \in \ell^+_2$. To see this, we use a similar argument as for kernel representations. Since the system is $k$-complete for some $k$ we know that for all $t > k$ the state $x(t)$ is uniquely determined by $w(t - k), \ldots, w(t)$. This immediately yields that $x \in \ell^+_2$.

Conversely, suppose that $(w, x) \in \mathcal{B}_{ov}$. Let $v_2 \in \ell^+_2$ be the corresponding driving variable. Since $x(0) \in X_{im}$ we can write $x(0) = \Pi_+ \Psi_1 \dot{v}_1 + \Pi_+ \Psi_2 \Pi_+ \dot{v}_2$ for some $v_1, v_2 \in \ell^+_2$. It is then straightforward to show recursively that

$$x(t) = \Pi_+ \Psi_1 \sigma_1 \dot{v}_1 + \Pi_+ \Psi_2 \Pi_+ \sigma_1 \dot{v}_2 + \dot{v}_2(t)$$
$$w(t) = \Pi_0 x(t) + \Pi_0 \Psi_2 \dot{v}_2(t)$$

for all $t \in \mathbb{Z}_+$. It is then immediate that $w(t) = \Psi_1 (\sigma_1) v_1 + \Psi_2 (\sigma_1) (v_2 + v_2')$, i.e., $w \in \mathcal{B}$.

## 9 System interconnections and stabilizability

The analysis of system interconnections is the core of many problems in modeling, simulation and control. Before studying general interconnection structures we first concentrate on some conceptual issues related to the interconnection of two systems.

Consider two dynamical systems $\Sigma_i = (\mathbb{Z}_+, W_i \times \mathcal{W}_{int}, B_i), i = 1, 2$, which have a common non-empty subset $W_{int}$, the interconnection space, in their respective signal spaces.

**Definition 9.1** The interconnection of two systems $\Sigma_i = (\mathbb{Z}_+, W_i \times \mathcal{W}_{int}, B_i), i = 1, 2$, is the system

$$\Sigma_1 \cap \Sigma_2 := (\mathbb{Z}_+, W_1 \times W_2 \times \mathcal{W}_{int}, B_1 \cap B_2)$$

where

$$B_1 \cap B_2 := \{(w_1, w_2, w_{int}) \mid (w_i, w_{int}) \in B_i, i = 1, 2\}.$$  \hspace{1cm} (9.1)

Signals $w_{int}$ are called the interconnection variables. If both $W_1$ and $W_2$ are void then $\Sigma_1 \cap \Sigma_2$ is called a full interconnection of $\Sigma_1$ and $\Sigma_2$.

The concept of system interconnection therefore coincides with the intuitive idea of imposing an algebraic constraint on a set of distinguished variables of two models. Note that in a full interconnection $B_1 \cap B_2 = B_1 \cap B_2$. 


If interconnection variables $w_{int}$ are viewed as latent variables after the moment of interconnection, then the interconnection induces a system $\Sigma_1 \times \Sigma_2 = (\mathbb{Z}_+, W_1 \times W_2, B)$ with behavior

$$B = \{(w_1, w_2) | \exists w_{int} \text{ such that } (w_1, w_2, w_{int}) \in B_1 \cap B_2\}. \quad (9.2)$$

**Remark 9.2** Interconnection is strongly related to intersection. Indeed, if in Definition 9.1, $E_i$ is extended to the system $E_i := (\mathbb{Z}_+, W_1 \times W_2 \times W_{int}, B_i')$ with

$$B_i' = \{(w_1, w_2, w_{int}) | (w_i, w_{int}) \in B_i\},$$

for $i = 1, 2$, then

$$B_1 \cap B_2 = B_1' \cap B_2' \quad (9.3)$$

which shows that interconnection amounts to intersection of the extended system behaviors.

Well-posed interconnections are defined as follows.

**Definition 9.3** Let $\Sigma_i = (\mathbb{Z}_+, W_i \times W_{int}, B_i), \ i = 1, 2$, be given. The interconnection $\Sigma_{int} := \Sigma_1 \cap \Sigma_2$ is said to be well-posed if there exists $t_0 \in \mathbb{Z}_+$ such that

$$\{ (w_1, w_2, w'_{int}), (w_1, w_2, w''_{int}) \in B_1 \cap B_2, w'_{int}(t) = w''_{int}(t) \text{ for } t \leq t_0 \} \implies \{ w'_{int} = w''_{int} \}. \quad (9.4)$$

This means that once the external trajectories $(w_1, w_2)$ in an interconnected system are specified, the set of all interconnection variables $w_{int}$ which are compatible with $(w_1, w_2)$ in the sense that $(w_1, w_2, w_{int}) \in B_1 \cap B_2$ define an autonomous behavior.

The interconnection variables of two systems which are interconnected at time instant $t_0 \in \mathbb{Z}_+$ are forced to be jointly compatible with both systems for all $t \geq t_0$. If such an interconnection can be made irrespective of the past behavior of the interconnecting systems then we call these systems instantaneous interconnectable. This is formalized as follows.

**Definition 9.4** Let $\Sigma_i = (\mathbb{Z}_+, W_i \times W_{int}, B_i), \ i = 1, 2$, be two left-shift invariant dynamical systems. $\Sigma_1$ and $\Sigma_2$ are said to be instantaneous interconnectable if for all $t > 0$, $(w_1, w'_{int}) \in B_1[0, t]$ and $(w_2, w''_{int}) \in B_2[0, t]$ there exists $w_{1e}, w_{2e}, w'_{int,e}$ and $w''_{int,e}$ with

$$w_{1e}[0, t] = w_1, \quad w_{2e}[0, t] = w_2, \quad w'_{int,e}[0, t] = w'_{int} \quad \text{and} \quad w''_{int,e}[0, t] = w''_{int}$$

such that $(w_{1e}, w'_{int,e}) \in B_1$ and $(w_{2e}, w''_{int,e}) \in B_2$ with

$$w'_{int,e}[t, \infty) = w''_{int,e}[t, \infty). \quad (9.4)$$

This condition is quite natural since it states that any past for $B_1$ together with any past for $B_2$ can, after interconnection at time $t$, yield a common future\(^5\). Note that for $\ell_2$ systems the

\(^5\)The time instant $t$ in equation (9.4) can be interpreted as the interconnection time. Equation (9.4) in Definition 9.4 can be replaced by the weaker condition that $w'_{int,e}[t+L, \infty) = w''_{int,e}[t+L, \infty)$ with $L \geq 0$. This requires the systems $\Sigma_1$ and $\Sigma_2$ to be interconnectable with lag $L$, and has the interpretation that it takes at least $L$ time units before a common future is attainable from any pair of past trajectories of $\Sigma_1$ and $\Sigma_2$. We will not further pursue this weaker notion of interconnectability in this paper.
property of instantaneous interconnectable implies a common future of interconnection variables in $\ell_2$. This shows clear links to stability of the interconnection.

We proceed this section with the definition of stabilizability. Intuitively, in a stabilizable system any trajectory can at any time be concatenated with a future system trajectory that is asymptotically converging to zero. We formalize this as follows.

**Definition 9.5** A dynamical system $\Sigma = (\mathbb{Z}_+, W, B)$ is said to be **stabilizable** if for all $w' \in B$ and $t_0 > 0$ there exists $w'' \in B$ with $\lim_{t \to \infty} w''(t) = 0$ such that $w'|[0,t_0] = w''|[0,t_0]$.

Just like the notion of controllability, stabilizability is therefore a property of the *external* behavior of a dynamical system. The notion of stabilizability of dynamical systems has an elegant characterization in terms of $\ell_2$ systems.

**Theorem 9.6** Let $\Sigma = (\mathbb{Z}_+, W, B)$ be a system with behavior $B \subseteq \mathbb{B}$. Then $\Sigma$ is stabilizable if and only if

$$B = \overline{B \cap \ell_2^+}$$

(9.5)

where the closure is taken in the topology of pointwise convergence.

**Proof.** We first prove that the equality (9.5) implies that the system is stabilizable. Choose $w \in B$. Equality (9.5) implies that for all $t_0$ we have

$$B|[0,t_0] = \overline{(B \cap \ell_2^+)}|[0,t_0]$$

(9.6)

On the other hand these are finite dimensional spaces and therefore closed. Since $w|[0,t_0] \in B|[0,t_0]$ we find $w|[0,t_0] \in \overline{(B \cap \ell_2^+)}|[0,t_0]$. Hence there exists $v \in B \cap \ell_2^+$ such that $w|[0,t_0] = v|[0,t_0]$. Since this is possible for all $w \in B$ this implies by definition that $B$ is stabilizable.

Again choose an arbitrary element $w \in B$. We have to show that $B$ stabilizable guarantees that $w \in B \cap \ell_2^+$. $B$ stabilizable implies by definition that (9.6) is satisfied for all $t_0$. Hence for all $t$ there exists $v_t$ such that $w|[0,t] = v_t|[0,t]$ and $v_t \in B \cap \ell_2^+$. But then it is straightforward to check that $v_t \to w$ as $t \to \infty$ in the topology of pointwise convergence. Therefore $w \in B \cap \ell_2^+$. $\square$

The following result is of importance for applications in control and shows that the interconnection of stabilizable and instantaneous interconnectable systems results in a stabilizable system. Stated otherwise, Theorem 9.7 claims that the property of stabilizability is closed under the operation of system interconnection provided the systems are instantaneous interconnectable.

**Theorem 9.7** Let $\Sigma_1 = (\mathbb{Z}_+, W_1 \times W_{int}, B_1)$ and $\Sigma_2 = (\mathbb{Z}_+, W_2 \times W_{int}, B_2)$ be dynamical systems. Suppose that both $\Sigma_1$ and $\Sigma_2$ are stabilizable and $B_1 \cap \ell_2$ and $B_2 \cap \ell_2$ are instantaneous interconnectable. Then $\Sigma_1 \cap \Sigma_2$ is stabilizable.

**Proof.** Suppose the hypothesis holds. Choose $(w_1, w_2, w_{int}) \in B_1 \cap B_2$. By Theorem 9.6 it then suffices to prove that

$$(w_1, w_2, w_{int}) \in (B_1 \cap \ell_2) \cap (B_2 \cap \ell_2).$$

(9.7)

Stabilizability of $\Sigma_1, \Sigma_2$ implies, by Theorem 9.6, that for all $t > 0$:

$$(w_1, w_{int})|[0,t] \in (B_1 \cap \ell_2)|[0,t]$$

and

$$(w_2, w_{int})|[0,t] \in (B_2 \cap \ell_2)|[0,t].$$
Together with the fact that $B_1 \cap \ell_2$ and $B_2 \cap \ell_2$ are instantaneous interconnectable, this yields the existence of a triple $(v_1^t, v_2^t, v_{\text{int}}^t)$ such that for $i = 1, 2$, $(v_i^t, v_{\text{int}}^t) \in B_i \cap \ell_2$ and the restriction

$$(v_1^t, v_2^t, v_{\text{int}}^t)[0,t] = (w_1, w_2, w_{\text{int}})[0,t].$$

It is then clear that in the topology of pointwise convergence $(v_1^t, v_2^t, v_{\text{int}}^t) \to (w_1, w_2, w_{\text{int}})$ as $t \to \infty$. Hence, (9.7) holds, as desired.

The following result gives a complete characterization of the concept of instantaneous interconnectibility of systems. It is the main result of this section and it provides necessary and sufficient conditions for instantaneous interconnectibility both in terms of system complexities as well as in kernel and image representations.

**Theorem 9.8** Let $\Sigma_1 = (\mathbb{Z}_+, \mathbb{R}^n, B_1)$ and $\Sigma_2 = (\mathbb{Z}_+, \mathbb{R}^n, B_2)$ be controllable systems and suppose that

\[
B_1 = B_\text{int}(\Theta_1) = B_\text{int}(\Psi_{1e}, \Psi_{1s}) \\
B_2 = B_\text{int}(\Theta_2) = B_\text{int}(\Psi_{2e}, \Psi_{2s})
\]

define normalized kernel and normalized image representations of their behaviors. Consider a full interconnection of $B_1$ and $B_2$. Then the following statements are equivalent

1. The systems $\Sigma_1$ and $\Sigma_2$ are instantaneous interconnectable.
2. The inclusions

   \[
   B_1 \cap B_{1+}^\perp \subseteq B_1^+ + B_2^-
   \]

   \[
   B_2 \cap B_{2+}^\perp \subseteq B_1^+ + B_2^-
   \]

   hold simultaneously.
3. The inclusions

   \[
   \Pi_+ \Theta_1^\perp H_2^- \subseteq \Theta_1^\perp H_2^+ + \Theta_2^\perp H_2^+
   \]

   \[
   \Pi_+ \Theta_2^\perp H_2^- \subseteq \Theta_1^\perp H_2^+ + \Theta_2^\perp H_2^+
   \]

   hold simultaneously.
4. The inclusions

   \[
   \Pi_+ (\Psi_{1e}, \Psi_{1s}) H_2^- \subseteq (\Psi_{1e}, \Psi_{2s}) H_2^+
   \]

   \[
   \Pi_+ (\Psi_{2e}, \Psi_{2s}) H_2^- \subseteq (\Psi_{1e}, \Psi_{2s}) H_2^+
   \]

   hold simultaneously.

Moreover, if the interconnection is well posed, then each of these statements is equivalent to

\[
n(B_1 \cap B_2) = n(B_1) + n(B_2).
\]
Proof. (1 $\Rightarrow$ 2). Suppose that $\Sigma_1$ and $\Sigma_2$ are instantaneous interconnectable. By definition 9.4, this means that for all $t \geq 0$, $w_1 \in B_1$ and $w_2 \in B_2$, there exist $w \in B_1 \cap B_2$ such that the concatenations $w_1 \Lambda_t w$ and $w_2 \Lambda_t w$ belong to $B_1$ and $B_2$, respectively. Since $w$ is a "common continuation" of both $w_1$ and $w_2$ from time $t$ on, it follows that $\sigma_t^i w$ is right-shift equivalent with $\sigma_t^j w_1 \in \sigma_t^i B_1$ and with $\sigma_t^j w_2 \in \sigma_t^j B_2$. Stated otherwise,

$$w \in (\sigma_t^i w_1 + B_t^i) \cap (\sigma_t^j w_2 + B_t^j).$$

(9.12)

Taking $w_1 \in B_1 \cap B_t^i$, $w_2 = 0$ and $t = 0$ this implies that the intersection $(w_1 + B_t^i) \cap B_t^j \neq \emptyset$. Hence, there exists a trajectory $b_2 \in (w_1 + B_t^i) \cap B_t^j$ and $b_1 \in B_t^i$ such that $b_2 = w_1 - b_1$. As $b_2 \in B_t^j$ we obtain that $w_1 = b_1 + b_2$, i.e., $w_1 \in B_t^i + B_t^j$. Conclude that

$$B_1 \cap B_t^i \subseteq B_t^i + B_t^j$$

which is the first inclusion of (9.8). Similarly, taking $w_2 \in B_2 \cap B_t^j$, $w_1 = 0$ and $t = 0$ yields the second inclusion of (9.8).

(2 $\Rightarrow$ 1). Suppose (9.8) holds and let $t > 0$, $w_1 \in B_1$ and $w_2 \in B_2$. We need to show that there exist $w \in B_1 \cap B_2$ such that $w_1 \Lambda_t w \in B_1$ and $w_2 \Lambda_t w \in B_2$. Since any such $w$ is compatible with both $w_1|_{[0,t]}$ and $w_2|_{[0,t]}$ it suffices to construct a $w$ which satisfies (9.12). Let $\Pi_{B_1}^*$ and $\Pi_{B_2}^*$ denote the orthogonal projections of $H_2^+$ onto $B_1^*$ and $B_2^*$, respectively. Then $w_1$ and $w_2$ can be decomposed as $w_1 = w_1' + w_1''$ and $w_2 = w_2' + w_2''$ where

$$w_1' = \Pi_{B_1}^* w_1 \in B_1^*; \quad w_1'' = [I - \Pi_{B_1}^*] w_1 \in B_1 \cap B_t^i$$

$$w_2' = \Pi_{B_2}^* w_2 \in B_2^*; \quad w_2'' = [I - \Pi_{B_2}^*] w_2 \in B_2 \cap B_t^j.$$

By (9.8), we can further find elements $b_{11}, b_{21} \in B_t^i$ and $b_{22}, b_{12} \in B_t^j$ such that

$$w_1' = b_{11} + b_{21}; \quad w_2' = b_{22} + b_{12},$$

Now define

$$b_1^* = b_{21} - w_1' - b_{11}$$

$$b_2^* = b_{12} - w_2' - b_{22}$$

and observe that $b_1^* \in B_1^*$ and $b_2^* \in B_2^*$. Define $w := b_{12} + b_{21}$ and verify that

$$w = w_1 + b_1^* = w_1'' + b_{21} - b_{11} = b_{12} + b_{21} = w_2'' + b_{22} - b_{22} = w_2 + b_2^*$$

which implies that $w$ satisfies (9.12) for $t = 0$. By time invariance of $B_1$ and $B_2$ the trajectory $w_t := \sigma_t^i w$ satisfies (9.12) for all $t \geq 0$. This proves the implication.

(2 $\Leftrightarrow$ 3). This is an immediate consequence of Theorem 7.2 and statement 3 of Theorem 8.3.

(2 $\Leftrightarrow$ 4). This is an immediate consequence of Theorem 7.2 and statement 5 of Theorem 8.3.

To prove the last claim, let $B := B_1 \cap B_2$ and let $B^*$ denote the equilibrium response of $B$. It is easily seen that $B^* \subseteq B_1^* \cap B_2^*$ and, as $B_1^* \cap B_2^*$ is itself a right-shift invariant subspace, we have that $B^* = B_1^* \cap B_2^*$. Suppose that the interconnection is well posed. Then $B_1 \cap B_2$ is finite dimensional and by Theorem 4.1 $B^* = 0$. Hence, $n(B) = \dim(B)$, while for $i = 1, 2$, $n(B_i) = \dim(B_i \cap B_t^i)$. Suppose that $B_1 \cap B_2$ holds and let

$$f : (B_1 \cap B_t^i) \times (B_2 \cap B_t^j) \rightarrow B \cap B^*$$

(9.13)
be defined by \( f(w_1, w_2) := w \) where \( w = b_{12} + b_{21} \) and \( b_{12} \) and \( b_{21} \) are obtained by decomposing

\[
\begin{align*}
w_1 &= b_{11} + b_{12}; & b_{11} &\in B_1^*; \\
w_2 &= b_{21} + b_{22}; & b_{21} &\in B_1^*; \\
\end{align*}
\]

where existence of such a decomposition is implied by (9.8). Since \( B_t \cap B_{\infty}^* = 0 \), it is easily verified that \( f \) is well defined and it therefore suffices to prove that \( f \) is bijective. To see this, let \((w_1, w_2)\) and \((w'_1, w'_2)\) be such that \( f(w_1, w_2) = f(w'_1, w'_2) \). Then, using obvious notation, \( w_1 - w'_1 = b_{11} - b'_{11} + b_{12} - b'_{12} \in B_1^* \). For \( w_1 - w'_1 \) also belongs to \( B_1 \cap B_{\infty}^* \) it follows that \( w_1 = w'_1 \).

A similar argument yields that \( w_2 = w'_2 \) and we conclude that \( f \) is injective. To prove surjectivity of \( f \), let \( w \in B_1 \cap B_2 \) and define

\[
\begin{align*}
w_1 &= [I - \Pi_{B_1^*}]w; \\
w_2 &= [I - \Pi_{B_2^*}]w.
\end{align*}
\]

where \( \Pi_{B_1^*} \) is the orthogonal projection onto \( B_1^* \). Then, for \( i = 1, 2 \), \( w_i \in B_i \cap B_{\infty}^{*\perp} \) and by the construction of the proof \((2 \Rightarrow 1)\), we have that \( f(w_1, w_2) = w \). Hence \( f \) is surjective. Conclude that \( f \) is bijective which implies (9.11).

(\((9.11) \Rightarrow 2)\). Suppose (9.11) holds. Then \( \dim(B) = \dim(B_1 \cap B_{\infty}^{*\perp}) + \dim(B_2 \cap B_{\infty}^{*\perp}) \). Since

\[
B = B_1 \cap B_2 = (B_1 \cap B_{\infty}^{*\perp} + B_2^*) \cap (B_2 \cap B_{\infty}^{*\perp} + B_1^*)
\]

this implies that for each \( w_1 \in B_1 \cap B_{\infty}^{*\perp} \) and \( w_2 \in B_2 \cap B_{\infty}^{*\perp} \) there exists

\[
w \in (w_1 + B_1^*) \cap (w_2 + B_2^*).
\]

Taking \( w_2 = 0 \) this yields that \( w \) belongs to \( B_2^* \) and can be written as \( w = w_1 + b_1^* \) with \( b_1^* \in B_1^* \). Thus, \( w_1 = w - b_1^* \) belongs to \( B_1^* + B_2^* \) from which the first inclusion of (9.8) follows. A similar argument with \( w_1 = 0 \) yields the second inclusion of (9.8).

Theorem 9.8 characterizes the notion of instantaneous interconnectibility for the class of \( \ell_2 \) systems. The following result shows that under suitable stabilizability conditions of the interconnected systems the \( \ell_2 \) assumption on system trajectories can be made without loss of generality.

**Theorem 9.9** Suppose that both the systems \( \Sigma_1 = (\mathbb{Z}_+, W_1 \times W_{\text{int}}, B_1) \) and \( \Sigma_2 = (\mathbb{Z}_+, W_2 \times W_{\text{int}}, B_2) \) are stabilizable. Then the following statements are equivalent

1. \( \Sigma_1 \cap \Sigma_2 \) is stabilizable and \( B_1 \) and \( B_2 \) are instantaneous interconnectable.

2. \( B_1 \cap \ell_2 \) and \( B_2 \cap \ell_2 \) are instantaneous interconnectable.

**Proof.** \((2 \Rightarrow 1)\). Suppose 2. holds. Then stabilizability of \( \Sigma_1 \cap \Sigma_2 \) is an immediate consequence of Theorem 9.7. Moreover, since both \( \Sigma_1 \) and \( \Sigma_2 \) are stabilizable Theorem 9.6 yields that

\[
\begin{align*}
B_1 &= B_1 \cap \ell_2; \\
B_2 &= B_2 \cap \ell_2.
\end{align*}
\]

(\((1 \Rightarrow 2)\). Suppose that 2. does not hold. Then there exist \( t > 0 \) and pairs \( (w_i, w_{\text{int}, i}) \in [B_i \cap \ell_2]_{[0, t]} \) with \( i = 1, 2 \), for which no continuation \( (v_1, v_2, v_{\text{int}}) \in (B_1 \cap \ell_2) \cap (B_2 \cap \ell_2) \) exists with the property that \( (w_i', w_{\text{int}, i}') \in B_i \) where

\[
\begin{align*}
w_i'(t') &= \begin{cases} 
  w_i(t') & \text{for } t' \leq t, \\
  v_i(t' - t) & \text{for } t' > t
\end{cases}; & w_{\text{int}, i}'(t') &= \begin{cases} 
  w_{\text{int}, i}(t') & \text{for } t' \leq t, \\
  v_{\text{int}, i}(t' - t) & \text{for } t' > t
\end{cases}.
\end{align*}
\]
On the other hand, if \( B_1 \) and \( B_2 \) are instantaneous interconnectable then there exists \( (v_1, v_2, v_{int}) \in B_1 \cap B_2 \) satisfying (9.15). Given the assumptions on stabilizability of \( \Sigma_1, \Sigma_2 \) and \( \Sigma_1 \cap \Sigma_2 \) we infer from Theorem 9.6 that

\[
B_1 \cap B_2 = B_1 \cap \ell_2 \cap B_2 \cap \ell_2 = (B_1 \cap \ell_2) \cap (B_2 \cap \ell_2). \tag{9.16}
\]

Consequently, \( (v_1, v_2, v_{int}) \) belong to the right hand side of (9.16), which contradicts the assumption that \( (w_i, w_{int,i}) \) have no continuation in \( (B_1 \cap \ell_2) \cap (B_2 \cap \ell_2) \).

The next step is a definition of stability in this behavioural setting. In most classical definitions, (internal) stability of a closed loop system means that for all initial conditions and for zero exogenous inputs the state of the closed loop system converges to 0 as \( t \to \infty \). Since we do not distinguish between inputs and outputs in the framework presented so far, this definition can not be directly extended. However, for autonomous closed loop systems (no exogenous inputs) a definition of stability is most natural given as follows.

**Definition 9.10** If the interconnection of two dynamical system \( \Sigma_1 \) and \( \Sigma_2 \) is autonomous then the interconnection is called **stable** if \( \Sigma_1 \cap \Sigma_2 \) is stabilizable.

Using this notion of stability we obtain the following corollary from Theorem 9.9.

**Corollary 9.11** Suppose that the interconnection of two stabilizable systems \( \Sigma_1 = (\mathbb{Z}_+, W_1, B_1) \) and \( \Sigma_2 = (\mathbb{Z}_+, W_2, B_2) \) is autonomous. Then the following conditions are equivalent:

1. The interconnection is stable and \( B_1 \) and \( B_2 \) are instantaneous interconnectable.
2. \( B_1 \cap \ell_2 \) and \( B_2 \cap \ell_2 \) are instantaneous interconnectable.

A definition of stability is much harder for the class of non-autonomous closed loop systems. Suppose that a causality structure is defined for a closed loop system with input \( u \) and output \( y \) which is assumed to be invertible and stable in the classical sense. Invertibility of the system implies that we can reverse the role of inputs and outputs. The inverse system is then stable in the classical sense if and only if the original system is minimum-phase.

In the present setting we can define a concept of **strong stability** if for any input-output partitioning of the closed loop system the autonomous subsystem obtained by setting the exogenous inputs equal to zero, is stable. Similarly, a notion of **weak stability** can be formalized if there exists at least one input-output decomposition of the closed loop system such that the autonomous closed loop system we obtain by setting the exogenous inputs equal to zero, is stable. We do not further pursue these definitions here.

### 10 Stability of left- and right-shift invariant systems

Various authors [3, 7, 10, 5] studied the stability of closed loop systems in terms of the graph of the system and the controller. In these papers the study of closed-loop systems is restricted to autonomous systems only. Moreover, since the graph of a system is defined in terms of the input-output operator associated with the system, zero-initial conditions of the defining relationships among the system variables need to be assumed implicitly. For systems defined on the time set \( T = \mathbb{Z}_+ \) we have seen that these graphs correspond to right-shift invariant subspaces. With
reference to Theorem 4.1, this has the main disadvantage that controlled systems in such a setting are not rich enough to characterize and analyze closed loop autonomous behaviors.

We considered representations of left-shift invariant $\ell_2$ systems and derived characterizations of systems which do not necessarily have zero initial conditions and include in particular the class of autonomous systems. In this section we relate our definitions of stability to the conditions presented by Georgiou and Smith in [3].

Let $G \in \mathcal{RL}_\infty$ be a transfer function mapping inputs $u$ of dimension $m$ to outputs $y$ of dimension $p$. We associate with $G$ its graph defined by

$$
\mathcal{B} := \{ (y, u) \in \mathcal{H}_2^+ \times \mathcal{H}_2^+ \mid y = Gu \}.
$$

This clearly defines an $\ell_2$ system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{p+m}, \mathcal{B})$ and we note that its behavior is a right-shift invariant subspace of $\mathcal{H}_2^+$. Therefore, by theorem 6.1, there exists $\Psi \in \mathcal{H}_\infty^+$ such that the graph is represented as the image of $\Psi$ when viewed as a multiplicative map from $\mathcal{H}_2^+$ to $\mathcal{H}_2^+$, i.e.,

$$
\mathcal{B} = \text{im}\Psi.
$$

Given a system with transfer function $G$ we can construct the graph associated to the system from a right-coprime factorization of $G$. If $G = NM^{-1}$ with $N, M$ elements of $\mathcal{RH}_\infty$, then

$$
\mathcal{B} = \text{im}\Psi = \text{im}\left( \begin{pmatrix} N \\ M \end{pmatrix} \right).
$$

Note that since $N$ and $M$ are coprime the rational matrix $\Psi$ does not have any zeros. The interconnection of a system with transfer function $G_p$ and a controller with transfer matrix $G_c$ is generally considered to be stable [1, 3, 5] if the transfer function

$$
H(G_p, G_c) := \begin{pmatrix}
(I - G_p G_c)^{-1} & (I - G_p G_c)^{-1} G_p \\
G_c (I - G_p G_c)^{-1} & G_c (I - G_p G_c)^{-1} G_p
\end{pmatrix}
$$

belongs to $\mathcal{H}_\infty^+$. We have the following theorem from [3] relating the above stability condition to the graphs associated to the system and the controller.

**Theorem 10.1** Let a system with graph $\mathcal{B}_p = \text{im}\Psi_p$ and a controller with graph $\mathcal{B}_c = \text{im}\Psi_c$ be given. Assume that $\Psi_c$ and $\Psi_p$ have no zeros\(^\dagger\). Then the following statements are equivalent

1. The interconnection of the system and the controller is stable.
2. The transfer function

$$
(\Psi_c \quad \Psi_p)
$$

is a unit in $\mathcal{H}_\infty^+$.
3. The two conditions

$$
\mathcal{B}_p + \mathcal{B}_c = \mathcal{H}_2^+
$$

$$
\mathcal{B}_p \cap \mathcal{B}_c = \{0\}
$$

hold simultaneously.

\(^\dagger\)This is equivalent of saying that both $\mathcal{B}_p$ and $\mathcal{B}_c$ are controllable.
Just like a left-shift invariant subspace \( B \) of \( \mathcal{H}_+^2 \) induces a right-shift invariant subset \( B^* \subseteq B \), we can treat the graph of a transfer function as the equilibrium response of a left-shift invariant \( \ell_2 \) system. Moreover, it also uniquely generates a left-shift invariant subset of \( \mathbb{B} \) by using inverse \( z \)-transforms.

**Definition 10.2** Let \( \bar{B} \) be a right-shift invariant subspace of \( \mathcal{H}_+^2 \). The *smallest left-shift invariant \( \ell_2 \) extension* of \( \bar{B} \) is the subspace

\[
\mathcal{B} := \{ w \in \ell_2^+ \mid \sigma^t w \in \bar{B} \text{ for all } t \in \mathbb{Z}_+ \}. \tag{10.2}
\]

The *smallest left-shift invariant extension* of \( \bar{B} \) in \( \mathbb{B} \) is denoted by \( \mathcal{B}^{cl} \) and is defined as the closure in the topology of pointwise convergence of \( Z^{-1} \bar{B} \) where \( Z : \ell_2^+ \to \mathcal{H}_+^2 \) denotes the \( z \)-transform \( Zw := \bar{w} \).

Clearly, the smallest left-shift invariant extensions \( \mathcal{B} \) and \( \mathcal{B}^{cl} \) of \( \bar{B} \) are unique. Furthermore, it is easy to check that

\[
\{ \mathcal{B} = \text{im} \Psi \} \implies \{ \mathcal{B} = \Pi_+ \Psi L_2 \}.
\]

Hence, by theorem 6.3, \( \mathcal{B} \) is controllable and, by theorem 7.2, \( \tilde{B} \) is in turn the equilibrium response \( B^* \) of \( B \).

We have the following theorem relating the stability condition in terms of the right-shift invariant subspaces \( \bar{B}_p \) and \( \bar{B}_c \) given in theorem 10.1 to a condition on the left-shift invariant subspaces \( B_p \) and \( B_c \).

**Theorem 10.3** Let \( B_p \) and \( cB_c \) denote the smallest left-shift invariant \( \ell_2 \) extensions of the graphs \( \bar{B}_p \) and \( \bar{B}_c \), respectively. Then

1. \( B_p \cap B_c = \{0\} \) if and only if \( B_p \cap B_c \) is autonomous.

2. Suppose that \( B_p \cap B_c \) is autonomous. Then \( \bar{B}_p + \bar{B}_c = \mathcal{H}_+^2 \) if and only if \( B_p \) and \( B_c \) are instantaneous interconnectable.

3. Suppose that \( B_p \cap B_c \) is autonomous. Then the interconnection of \( \bar{B}_p \) and \( \bar{B}_c \) is stable if and only if the interconnection of \( B_p^{cl} \) and \( B_c^{cl} \) is stable in the sense of definition 9.10.

**Proof.** We have:

\[
B_p = \{ \Pi_+ \Psi_p w \mid w \in \mathcal{H}_+^2 \} + \{ \Pi_+ \Psi_c w \mid w \in \mathcal{H}_-^2 \}.
\]

Note that the first component is equal to \( \bar{B}_p \) and, since \( \Psi_p \) is rational, the second component is finite-dimensional. Therefore \( \bar{B}_p \cap \bar{B}_c \) is finite dimensional if and only if \( B_p \cap B_c \) is finite-dimensional. By theorem 4.1, \( \bar{B}_p \cap \bar{B}_c \) is finite dimensional if and only if \( B_p \cap B_c = \{0\} \) which proves the first statement.

To prove the second statement, note that by theorem 9.8 the system is instantaneous interconnectable if and only if for all \( w_c \in \mathcal{H}_-^2 \) and \( w_p \in \mathcal{H}_+^2 \) there exists \( v_c \in \mathcal{H}_+^2 \) and \( v_p \in \mathcal{H}_-^2 \) such that

\[
\Pi_+ \Psi_c v_c + \Pi_+ \Psi_p v_p = \Pi_+ \Psi_p w_p + \Pi_+ \Psi_p w_p
\]
or, equivalently

\[
(\Psi_c \quad \Psi_p) \begin{pmatrix} v_c \\ v_p \end{pmatrix} = \Pi_+ (\Psi_c \quad \Psi_p) \begin{pmatrix} w_c \\ w_p \end{pmatrix}
\]
$\tilde{B}_c + \tilde{B}_p = \mathcal{H}_2^+$ is equivalent to the requirement that $(\Psi_c, \Psi_p)$ is surjective as a map from $\mathcal{H}_2^+$ to $\mathcal{H}_2^+$. From the above it is obvious that $\tilde{B}_c + \tilde{B}_p = \mathcal{H}_2^+$ implies that the interconnection is instantaneous interconnectable.

Remains to prove that instantaneous interconnectable implies $\tilde{B}_c + \tilde{B}_p = \mathcal{H}_2^+$. Theorem 9.8 yields that

$$\Pi_+ (\Psi_c, \Psi_p) w \in (\Psi_c, \Psi_p) \mathcal{H}_2^+$$

(10.3)

for all $w \in \mathcal{H}_2^-$. Clearly we also have (10.3) for all $w \in \mathcal{H}_2^+$ since in this case the projection $\Pi_+$ can be removed. This implies (10.3) is satisfied for all $w \in \mathcal{L}_2$. On the other hand we know the interconnection is autonomous which implies that $(\Psi_c, \Psi_p)$ is square and has full normal rank. This implies that $(\Psi_c, \Psi_p)$ is surjective as a mapping from $\mathcal{L}_2$ to $\mathcal{L}_2$. Therefore the fact that (10.3) holds for all $w \in \mathcal{L}_2$ implies that $(\Psi_c, \Psi_p)$ is surjective as a map from $\mathcal{H}_2^+$ to $\mathcal{H}_2^+$ and the latter was equivalent to the requirement that $B_c$ and $B_p$ are instantaneous interconnectable.

The last statement is an immediate consequence of Theorem 9.11.

\[ \Box \]

11 Conclusions

In this paper we developed a theory for the representation of the class of linear left-shift invariant and complete $\ell_2$ systems. The study of this class of dynamical systems is motivated by various applications in digital control where the assumption of square summability of system trajectories is made to analyze the qualitative behavior of controlled systems. In particular, in $\mathcal{H}_\infty$ optimal control, applications in sampled data systems, $\mathcal{H}_2$ optimal control and for the study of robust stability of systems, the $\ell_2$ assumption is made implicitly. We put forward a set theoretic analysis of this class of discrete time systems which refrains from using the common input-output framework. Such an approach has particular conceptual and methodological advantages in control, as the problem of controller synthesis in this setting does not require the design of a map, but rather the design of a set whose intersection with a set of system trajectories yields the controlled system behavior. As such, there is no a priori need to distinguish between actuators and sensors.

We showed that the class of linear time invariant $\ell_2$ systems with two-sided time axis has essentially different properties than the class of linear time invariant $\ell_2$ systems where time is running over the non-negative integers. We proved the existence of rational kernel and image representations for the class of left-shift invariant and complete $\ell_2$ systems and characterized non-uniqueness of these representations. We put forward an analytic framework in which state space models can be derived from both kernel as well as image representations in a straightforward way. The important problem of minimality of state space representations has been addressed and we proved that normalized kernel and normalized image representations of $\ell_2$ systems give always rise to minimal state space representations of $\ell_2$ systems using the construction of Theorem 8.7.

We introduced well known concepts such as stability, stabilizability, well-posedness of system interconnections and interconnectability of dynamical systems in a set theoretic framework and characterized these notions in terms of kernel and image representations. We proved that a system is stabilizable if and only if it is equal to (the closure of) its $\ell_2$ behavior. We introduced the concept of instantaneous interconnectability of systems and showed that two systems are instantaneous interconnectable if and only if the complexity of their interconnection is equal to the sum of the complexities of the constituting systems. A comparison to the graph theoretical approach has been made and we proved that the interconnection of two systems results in a stable interconnection if
and only if the interconnecting systems are instantaneously interconnectable and yield a well-posed interconnection.

References


