The maximum number of states after projection

Schols, H.M.J.L.

Published: 01/01/1987

Citation for published version (APA):
THE MAXIMUM NUMBER OF STATES
AFTER PROJECTION

BY

HUUB M.J.L. SCHOLS

April 1987
This is a series of notes of the Computing Science Section of the Department of Mathematics and Computing Science of Eindhoven University of Technology. Since many of these notes are preliminary versions or may be published elsewhere, they have a limited distribution only and are not for review. Copies of these notes are available from the author or the editor.

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB EINDHOVEN
The Netherlands
All rights reserved
editor: F.A.J. van Neerven
The maximum number of states after projection

Huub M.J.L. Schols

Department of Mathematics and Computing Science
Eindhoven University of Technology
Eindhoven, the Netherlands

ABSTRACT

Projecting a minimal deterministic state graph onto an alphabet might yield a minimal deterministic state graph that contains more states than the original one, due to the introduction of nondeterminism, cf. [Kaldewaij0, example 5.4, p. 36]. In this paper we show that for every natural number \( N, N \geq 2 \), there exists an alphabet \( A \) and a minimal deterministic state graph \( S \), that contains exactly \( N \) states, such that projecting \( S \) onto \( A \) yields a minimal deterministic state graph that contains \( (3 \cdot 2^{(N-1)} - 1) \) states. It is easily shown that \( (3 \cdot 2^{(N-1)} - 1) \) is the upper limit. Robert Huis in ’t Veld was the first who showed us this upper limit.

0 Introduction

This problem arises from studying communicating processes, cf. [Hoare] and [Kaldewaij0]. Transitions in a state graph denote communication actions in which a process may involve. Projecting transitions away corresponds to hiding communication actions. These are referred to as internal moves or \( e \)-transitions. In the remainder of this paper an alphabet is a set of symbols that denote transitions. Furthermore, by referring to state graph we mean minimal deterministic state graph. \( x \xrightarrow{a} y \) denotes a transition labeled \( a \) from the state labeled \( x \) to the state labeled \( y \).

1 Projecting a state graph onto an alphabet

Consider a state graph \( S \) with \( N, N \geq 2 \), states labeled with natural numbers from 0 up to \( (N-1) \). Projecting \( S \) yields a state graph, say \( T \). Due to introduction of nondeterminism states of \( T \) correspond to subsets of states of \( S \), which is proven formally by Kaldewaij [Kaldewaij1]. Therefore, we label the states of \( T \) with subsets of \( \{k \mid 0 \leq k < N\} \). The
terminology introduced in this section is used throughout the remainder of this paper.

2 The upper limit

The number of subsets of \( \{ k \mid 0 \leq k < N \} \) equals \( 2^N \). As a consequence, \( T \) contains at most \( 2^N \) states. If only transitions of type \( x \xrightarrow{a} x \) are projected away, no nondeterminism is introduced. In this case, the number of states of \( T \) is at most \( N \), the number of states of \( S \). Now, assume that at least one transition of type \( x \xrightarrow{a} y \) is projected away, where \( x \neq y \). Consider a state in \( T \) labeled with a set \( P \) that contains \( x \). Due to the introduced nondeterminism, viz. internal move \( a \) may or may not have happened, \( P \) contains \( y \) as well. Therefore, subsets of \( \{ k \mid 0 \leq k < N \} \) that contain \( x \) but not \( y \) do not occur as the label of some state of \( T \). Since there are \( \frac{1}{2} \times 2^N \) subsets of \( \{ k \mid 0 \leq k < N \} \) that contain \( x \) but not \( y \), \( T \) contains at most \( (3 \times 2^{(N-2)}-1) \) states. Furthermore, the empty set does not occur as the label of some state of \( T \) neither. In short, \( T \) contains at most \( (3 \times 2^{(N-2)}-1) \) states. This proof owes to Robert Huis in 't Veld.

The set discussed above, which consists of \( (3 \times 2^{(N-2)}-1) \) elements, is denoted by \( L \). In short, \( L \) consists of all subsets \( l \) of \( \{ k \mid 0 \leq k < N \} \), such that \( i \) is non-empty and \( 0 \in l \) implies \( l \in l \).

3 The maximum

In section 3.0 we define state graphs \( S \) and \( T \). We show in section 3.1 that all \( (3 \times 2^{(N-2)}-1) \) states defined in section 2 be distinct. Finally, section 3.2 deals with the reachability of these states from the initial state.

Notice that we do not need mathematical induction to \( N \), the number of states.

3.0 Definitions

We consider \((N+2)\) distinct symbols: \( di \), for \( 0 \leq i < N \), \( e \), and \( g \). The alphabet \( A \) denotes the set \( \{ g \} \cup \{ di \mid 0 \leq i < N \} \). State graph \( S \) contains the following transitions:

- \( 0 \xrightarrow{a} 1 \)
- \( 0 \xrightarrow{a} 0 \)
- \( k \xrightarrow{a} (k + 1) \) for \( 1 \leq k < (N - 1) \), and
- \( k \xrightarrow{a} k \) for \( (0 \leq k < N) \land (0 \leq i < N) \land (i \neq k) \).

The state labeled \( 0 \) is the initial state of \( S \). State graph \( T \) is the projection of \( S \) onto \( A \). Therefore, the label of the initial state of \( T \) is \( \{ 0, 1 \} \). In the remainder of this section we consider state graph \( T \). Furthermore, by state \( P \), for \( P \) an element of \( L \), we mean the state labeled with \( P \).

Notice that for \( i \neq 1 \) transition \( di \) is possible from each state, say \( P \), such that \( P \) contains at least one element besides perhaps \( i \); \( di \) is possible from each state, say \( P \), such that \( P \) contains at least one element besides perhaps \( i \) and \( P \) does not contain \( 0 \). From such a state \( di \), \( 0 \leq i < N \), leads to the state \( P \setminus \{ i \} \), i.e. \( di \) "deletes" the element \( i \) from the set that labels the state.
the maximum number of states after projection

3.1 Distinctness
We consider two distinct subsets, say \( P \) and \( Q \), of \( L \). Without loss in generality we assume that \( x \) be such that \( x \in P \setminus Q \). Since \( Q \) is nonempty, cf. section 3.0, we choose \( y \) in \( Q \), such that \( y \neq x \) if \( 0 \in Q \). This is possible since \( 0 \in Q \) implies \( 1 \in Q \), cf. section 3.0. Let \( s \) be a (perhaps empty) sequence of the transitions \( d_i \), for which \( i \in Q \setminus \{ y \} \), i.e. \( s \) is a permutation of the elements of \( Q \setminus \{ y \} \). This sequence \( s \) leads from state \( Q \) to state \( \{ y \} \); moreover, \( s \) leads from state \( P \) to some state, say \( R \). Since \( x \in P \setminus Q \), \( x \neq y \) and \( x \in R \) hold. Transition \( dy \) is possible from state \( R \) as it is not from state \( \{ y \} \). As a consequence, the sequence \( (s ; dy) \), i.e. \( s \) followed by \( dy \), of transitions is possible from state \( P \), as it is not from state \( Q \). We conclude that states \( P \) and \( Q \) be distinct.

3.2 Reachability
Due to the nondeterminism that is introduced by the projection, the path \( \{ 0, 1 \} \xrightarrow{3(N-2)} \{ k | 0 \leq k < N \} \), i.e. the path from \( \{ 0, 1 \} \) via a sequence of \( (N-2) \) transitions \( g \) to \( \{ k | 0 \leq k < N \} \), exists. Furthermore, it is obvious that all \( (3 \times 2^{(N-2)} - 1) \) states mentioned in section 3.1 are reachable from state \( \{ k | 0 \leq k < N \} \) by *deleting* symbols from the latter, cf. section 3.0. Combining this with our first observation, we conclude that all \( (3 \times 2^{(N-2)} - 1) \) states mentioned in section 3.1 are reachable from the initial state \( \{ 0, 1 \} \).

4 Remarks
Projecting away two transitions, say \( h \xrightarrow{\ast} j \) and \( k \xrightarrow{\ast} l \), yields a state graph with less than \( (3 \times 2^{(N-2)} - 1) \) states, provided that \( h \neq j \), \( k \neq l \), and \( ((h \neq k) \lor (j \neq l)) \). This is due to the introduction of more nondeterminism. Since the proof hereof is easy and analogous to the proof of our upper limit, we do not present it here.

In a preliminary version of this paper we suggested to use an alphabet consisting of \( (3 \times 2^{(N-2)} + 1) \) symbols. By using such large an alphabet it is possible to go directly via a transition from state \( \{ k | 0 \leq k < N \} \) to any particular state. Of course, there exists a trade-off between the number of symbols of the alphabet and the (maximum) length of the path of transitions, that is needed to go from \( \{ k | 0 \leq k < N \} \) to an arbitrary state. The solution presented in section 3 arises from suggestions by Tom Verhoeff.

5 Acknowledgements
Acknowledgements are due to the members of the Eindhoven VLSI Club for discussing the problem and the solution presented above. In particular Robert Huis in 't Veld showed (the existence of) the upper limit, and Tom Verhoeff suggested to use an alphabet consisting of only \( (N+2) \) symbols.
References


Eindhoven, March 19, 1987
<table>
<thead>
<tr>
<th>No.</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>85/01</td>
<td>R.H. Mak</td>
<td>The formal specification and derivation of CMOS-circuits</td>
</tr>
<tr>
<td>85/02</td>
<td>W.M.C.J. van Overveld</td>
<td>On arithmetic operations with M-out-of-N-codes</td>
</tr>
<tr>
<td>85/03</td>
<td>W.J.M. Lemmens</td>
<td>Use of a computer for evaluation of flow films</td>
</tr>
<tr>
<td>85/04</td>
<td>T. Verhoeff, H.M.J.L. Schols</td>
<td>Delay insensitive directed trace structures satisfy the foam rubber wrapper postulate</td>
</tr>
<tr>
<td>86/01</td>
<td>R. Koymans</td>
<td>Specifying message passing and real-time systems</td>
</tr>
<tr>
<td>86/02</td>
<td>G.A. Bussing, K.M. van Hee, M. Voorhoeve</td>
<td>ELISA, A language for formal specifications of information systems</td>
</tr>
<tr>
<td>86/03</td>
<td>Rob Hoogerwoord</td>
<td>Some reflections on the implementation of trace structures</td>
</tr>
<tr>
<td>86/04</td>
<td>G.J. Houben, J. Paredsens, K.M. van Hee</td>
<td>The partition of an information system in several parallel systems</td>
</tr>
<tr>
<td>86/05</td>
<td>Jan L.G. Dietz, Kees M. van Hee</td>
<td>A framework for the conceptual modeling of discrete dynamic systems</td>
</tr>
<tr>
<td>86/06</td>
<td>Tom Verhoeff</td>
<td>Nondeterminism and divergence created by concealment in CSP</td>
</tr>
<tr>
<td>86/07</td>
<td>R. Gerth, L. Shira</td>
<td>On proving communication closedness of distributed layers</td>
</tr>
<tr>
<td>Year</td>
<td>Author(s)</td>
<td>Title</td>
</tr>
<tr>
<td>------</td>
<td>-------------------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>86/09</td>
<td>C. Huizing, R. Gerth, W.P. de Roever</td>
<td>Full abstraction of a real-time denotational semantics for an OCCAM-like language</td>
</tr>
<tr>
<td>86/10</td>
<td>J. Hooman</td>
<td>A compositional proof theory for real-time distributed message passing</td>
</tr>
<tr>
<td>86/11</td>
<td>W.P. de Roever</td>
<td>Questions to Robin Milner - A responder's commentary (IFIP86)</td>
</tr>
<tr>
<td>86/12</td>
<td>A. Boucher, R. Gerth</td>
<td>A timed failures model for extended communicating processes</td>
</tr>
<tr>
<td>86/14</td>
<td>R. Koymans</td>
<td>Specifying passing systems requires extending temporal logic</td>
</tr>
<tr>
<td>87/01</td>
<td>R. Gerth</td>
<td>On the existence of sound and complete axiomatizations of the monitor concept</td>
</tr>
<tr>
<td>87/02</td>
<td>Simon J. Klaver, Chris F.M. Verberne</td>
<td>Federatieve Databases</td>
</tr>
<tr>
<td>87/03</td>
<td>G.J. Houben, J. Paredaens</td>
<td>A formal approach to distributed information systems</td>
</tr>
<tr>
<td>87/04</td>
<td>T. Verhoeff</td>
<td>Delay-insensitive codes - An overview</td>
</tr>
</tbody>
</table>
87/05 R.Kuiper  Enforcing non-determinism via linear time temporal logic specification.

87/06 R.Koymans  Temporele logica specificatie van message passing en real-time systemen (in Dutch).

87/07 R.Koymans  Specifying message passing and real-time systems with real-time temporal logic.

87/08 H.M.J.L. Schols  The maximum number of states after projection.