The recurrence behaviour of random walks on locally compact groups

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on locally compact groups

by

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Eindhoven, May 1979

The Netherlands
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Introduction and summary

Let \((X, \Sigma)\) be a measurable space and let \(M^+\) be the class of nonnegative extended real valued measurable functions on \((X, \Sigma)\). A Markov process on \((X, \Sigma)\) is a mapping \(P\) of \(M^+\) into itself such that

\[
\sum_{n=1}^{\infty} \alpha_n P^n f_n = P \sum_{n=1}^{\infty} \alpha_n n^n f_n \quad (\alpha_n \geq 0, f_n \in M^+) ,
\]

\[
P_1 \leq 1.
\]

If we put \(P(x, A) = P_\lambda^{-1}(x)\) for every \(x \in X\) and every \(A \in \Sigma\), then \(P(\cdot, \cdot)\) is the (sub) transition probability describing the process on the state space \((X, \Sigma)\). Conversely, every (sub) transition probability on \((X, \Sigma)\) determines a Markov process on \((X, \Sigma)\).

Sometimes we want to consider Markov processes in a more global sense i.e. modulo a \(\sigma\)-finite measure \(\mu\) on the state space \((X, \Sigma)\). Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space and let \(M^+(\mu)\) be the space of equivalence classes of \(\mu\)-almost everywhere equal nonnegative extended real valued measurable functions on \((X, \Sigma)\). A Markov process on \((X, \Sigma, \mu)\) is a mapping \(P\) of \(M^+(\mu)\) into itself such that (1) and (2) hold \(\mu\)-a.e.

When \(P\) is a Markov process on a measurable space \((X, \Sigma)\) and \(\mu\) is a \(\sigma\)-finite measure on \((X, \Sigma)\), then in general \(P\) is not a Markov process on the measure space \((X, \Sigma, \mu)\). This is the case if and only if \(P\) is nonsingular with respect to \(\mu\) i.e. for every \(A \in \Sigma\) with \(m(A) = 0\) we have \(m(\{P^n_A > 0\}) = 0\). Conversely, if \(P\) is a Markov process on a measurable space \((X, \Sigma, \mu)\), then in general there does not exist a (sub) transition probability on the measurable space \((X, \Sigma)\) describing the process.

In this note we shall discuss the recurrence behaviour of a random walk on a locally compact abelian group \((G, \Sigma)\), where \(\Sigma\) denotes the \(\sigma\)-algebra of the Borel sets of \(G\).

Our main tool to investigate the recurrence behaviour of random walks will be the use of embedded processes. Let \(P\) be a Markov process on a measurable space \((X, \Sigma)\), then for every \(A \in \Sigma\) the embedded process \(P_A\) is the Markov process on \((X, \Sigma)\) given by

\[
P_A f = \sum_{n=0}^{\infty} (PI_A^n)^n P_I f \quad \text{for all } f \in M^+ .
\]
Here $A'$ stands for $X \setminus A$ and $I_A$ for multiplication by the indicator function $1_A$ of the set $A$. The expression $P_{AB}^{A}(x)$ is the probability that, starting in $x$, at its first visit to the set $A$ the process will enter the set $B$.

Section one gives preliminaries concerning Markov processes. In section two we shall show that a random walk is either conservative or dissipative and the random walk is conservative if and only if the random walk is recurrent. Section three gives criteria for a random walk to be Harris (cf. [5], ch. 3 theorem 4.4) and to be ergodic. In section four some recurrence criteria for random walks on $\mathbb{R}^n$ (cf. [2]) will be given.

1. Preliminaries concerning Markov processes

In this section we collect some results about Markov processes, which can be found in [7]. $P$ will be a Markov process on an arbitrary measurable space $(X, \Sigma)$. A $\sigma$-finite measure $\mu$ on $(X, \Sigma)$ is said to be invariant with respect to $P$ if

$$\int P f(x) \mu(dx) = \int f(x) \mu(dx) \quad \text{for all } f \in M^+. $$

**Proposition 1.1.** If $P$ is a Markov process on the measurable space $(X, \Sigma)$ and the $\sigma$-finite measure $\mu$ on $(X, \Sigma)$ is invariant with respect to $P$, then $P$ is nonsingular with respect to $\mu$ and therefore $P$ can also be considered as a Markov process on the measure space $(X, \Sigma, \mu)$.

**Proof.** Suppose $A \in \Sigma$ with $\mu(A) = 0$. Since $\mu$ is invariant we have

$$\int P_{IA}^{A}(x) \mu(dx) = \int 1_A(x) \mu(dx) = \mu(A) = 0 .$$

Hence $P_{IA}^{A} = 0$ $\mu$-a.e. and therefore $\mu(\{P_{IA}^{A} > 0\}) = 0$. $\square$

**Proposition 1.2.** If $P$ is a Markov process on the measurable space $(X, \Sigma)$, then for all sets $A \in \Sigma$ and all $f \in M^+$ we have

$$\sum_{n=1}^{\infty} P_{IA}^{n} f = \sum_{n=1}^{\infty} P_{IA}^{n} f .$$

**Proof.** Replace in [7] proposition 2.2 A by $A'$, B by A and f by $I_A f$. $\square$

Let $\mu$ be a $\sigma$-finite measure on $(X, \Sigma)$ such that the Markov process $P$ on $(X, \Sigma)$ is nonsingular with respect to $\mu$, then $P$ is also a Markov process on the measure space $(X, \Sigma, \mu)$. The decomposition theorem of E. Hopf states, that there
exists a mod $\mu$ unique decomposition of the state space $X$ in a conservative part $C$ and a dissipative part $D$ ([3], ch. II). In the following theorem we collect some important properties of the conservative and dissipative parts.

**Theorem 1.1.** Let $P$ be a Markov process on the $\sigma$-finite measure space $(X, \Sigma, \mu)$ and let $C$ and $D$ be the conservative and dissipative parts of $X$ with respect to $P$, then we have

i) For all $A \in \Sigma$ with $A \subset C$

$$P^\infty_A = 1 \text{ $\mu$-a.e. on } A$$

$$\sum_{n=0}^{\infty} P^n_A = \infty \text{ $\mu$-a.e. on } A .$$

ii) There exists a measurable partition of $D = D_1 \cup D_2 \cup \ldots$ and a sequence $q_1, q_2, \ldots$ such that

$$P_{D_i}^1 \leq q_i < 1 \text{ $\mu$-a.e. on } D_i$$

$$\sum_{n=0}^{\infty} P^n_{D_i} \leq \frac{1}{1-q_i} \text{ $\mu$-a.e. on } X \text{ for all } i .$$

For a proof of theorem 1.1 see [7], theorem 2.1 and 2.2.

**Theorem 1.2.** If $P$ is a conservative Markov process on a measure space $(X, \Sigma, \mu)$ and for a function $f \in M^+$ we have $Pf = f \text{ $\mu$-a.e.}$, then $Pf = f \text{ $\mu$-a.e.}$.

For a proof of theorem 1.2 see [3], ch. II theorem B.

Let $P$ be a Markov process on a measure space $(X, \Sigma, \mu)$. A set $R \in \Sigma$ is said to be invariant, if $P^1_R \geq 1_R \text{ $\mu$-a.e.}$ Intuitively, this means that it is impossible to leave the set $R$ under the action of the process. There is no unanimity in the definition of invariant set. The definition in [3] differs from our definition, but for conservative Markov processes they are equivalent because of theorem 1.2.

**Proposition 1.3.** If $P$ is a conservative Markov process on a measure space $(X, \Sigma, \mu)$, then for every $A \in \Sigma$ we have $P^1_A = 1_A^* \text{ $\mu$-a.e.}$, where $A^*$ is the mod $\mu$ smallest invariant set containing $A$.

For a proof of proposition 1.3 see [7], proposition 3.5.
Proposition 1.4. Let $P$ be a Markov process on the measure space $(X, \Sigma, \mu)$ and $\nu$ a finite invariant measure with $\nu << \mu$, then the set $\{\frac{d\nu}{d\mu} > 0\}$ belongs to the conservative part.

For a proof of proposition 1.4 see [3] Ch. IV theorem E.

If $P$ is a Markov process on a measure space $(X, \Sigma, \mu)$, then it is easily verified that for every $A \in \Sigma$ the embedded process $P_A$ is also a Markov process on $(X, \Sigma, \mu)$. Let $C(P)$ be the conservative part of $X$ with respect to $P$ and $C(P_A)$ the conservative part of $X$ with respect to $P_A$.

Proposition 1.5. Let $P$ be a Markov process on the measure space $(X, \Sigma, \mu)$, then for every $A \in \Sigma$ we have $C(P_A) = A \cap C(P)$.

For a proof of proposition 1.5 see [7], proposition 2.4.

Proposition 1.6. Let $P$ be a Markov process on the $\sigma$-finite measure space $(X, \Sigma, \mu)$ such that $\mu$ is invariant with respect to $P$. If $A \in \Sigma$ with $0 < \mu(A) < \infty$ and $P_A 1 = 1 \mu$-a.e. on $A$ then $A \subset C(P)$.

Proof. The restriction of $\mu$ to the set $A$ will be denoted by $\mu_A$. We shall prove that $\mu_A$ is invariant with respect to $P_A$. Using the invariance of $\mu$ with respect to $P$, it is a straightforward verification by induction, that for all $N \in \mathbb{N}$ and all bounded functions $f \in M^+$ we have

$$\int I_A f(x) \mu(dx) = \sum_{k=0}^{N} I_A (P_A)^k I_A f(x) \mu(dx) + \int (P_A)^{N+1} I_A f(x) \mu(dx).$$

Hence

$$\int I_A f(x) \mu(dx) \geq \int I_A P_A f(x) \mu(dx).$$

(1)

Define the measure $\nu$ on $(X, \Sigma)$ by

$$\nu(B) = \int I_A P_A 1_B(x) \mu(dx) \text{ for all } B \in \Sigma.$$ 

Since $\nu << \mu$ the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ exists and from (1) we conclude $\frac{d\nu}{d\mu} \leq 1 A \mu$-a.e.

Since $P_A 1 = 1 \mu$-a.e. on $A$ we have

$$0 \leq \int (I_A(x) - \frac{d\nu}{d\mu}(x)) \mu(dx) = \mu(A) - \int I_A P_A 1(x) \mu(dx) = 0.$$
Hence \( \frac{dv}{du} = 1 \) \( \mu \)-a.e.

For every \( f \in M^+ \) we now have

\[
\int P_A f(x) \mu_A(dx) = \int I_A P_A f(x) \mu(dx) = \int f(x) v(dx)
\]

\[
= \int f(x) \frac{dv}{du}(x) \mu(dx) = \int I_A f(x) \mu(dx) = \int f(x) \mu_A(dx).
\]

Hence \( \mu_A \) is invariant with respect to \( P_A \). From proposition 1.4 we conclude \( A \subset C(P_A) \) and by proposition 1.5 we have \( A \subset C(P) \).

2. Recurrence and conservatively for random walks

Throughout the sections two and three \( G \) will be a locally compact abelian group. We shall suppose that \( G \) is metrizable and \( d \) is a metric on \( G \) compatible with the topology. The group operation in \( G \) will be written additively. The \( \sigma \)-algebra of the Borel sets of \( G \) will be denoted by \( \Sigma \) and the Haar measure on \((G,\Sigma)\) by \( \lambda \).

A random walk on \((G,\Sigma)\) is determined by a probability measure \( \mu \) on \((G,\Sigma)\). Following Revuz (see [5], p. 27) the probability measure \( \mu \) is called the law of the random walk. A random walk with law \( \mu \) is a Markov process on \((G,\Sigma)\) with transition probability

\[
P(x, A) = \int 1_A (x+y) \mu(dy) \quad \text{for all } x \in G \text{ and all } A \in \Sigma.
\]

The corresponding Markov operator \( P \) on \( M^+ \) is

\[
Pf(x) = \int f(x+y) \mu(dy) \quad \text{for all } x \in G \text{ and all } f \in M^+.
\]

In the sequel we shall frequently use the translation operator \( T_a \) \((a \in G)\) on \( M^+ \). For every \( a \in G \) define

\[
T_a f(x) = f(x+a) \quad \text{for all } x \in G \text{ and } f \in M^+.
\]

Proposition 2.1. Let \( P \) be a random walk with law \( \mu \), then for every \( a \in G \) and every \( A \in \Sigma \) we have

i) \( P T_a = T_a P \),

ii) \( P A T_a = T_a A + a \).

Proof. For every \( f \in M^+ \) we have

\[
PT_a f(x) = \int T_a f(x+y) \mu(dy) = \int f(x+y+a) \mu(dy) = T_a Pf(x).
\]
Hence the operators $P$ and $T_a$ commute, which proves i). For every $B \in \Sigma$ we have

$$I_B T f(x) = 1_B(x) f(x + a) = 1_B(x + a) f(x + a) = T_a I_B f(x) .$$

Hence

$$I_B T_a = T_a I_B .$$

We conclude

$$P T_a = \sum_{n=0}^{\infty} (P_t A)^n P T A a = T_a \sum_{n=0}^{\infty} (P_t (A + a))^n P A + a = T_a A + a .$$

Proposition 2.2. The Haar measure $\lambda$ is invariant with respect to every random walk.

Proof. Let $P$ be a random walk with law $p$, then for every $f \in M^+$ we have using Fubini's theorem

$$\int P f(x) \lambda(dx) = \int \left( \int f(x + y) p(dy) \right) \lambda(dx) = \int \left( \int f(x + y) \lambda(dx) \right) p(dy) = \int f(x) \lambda(dx) .$$

Proposition 2.3. If $A, B \in \Sigma$ with $\lambda(A) > 0$ and $\lambda(B) > 0$, then there exists a set $C \in \Sigma$ with $\lambda(C) > 0$ such that $\lambda((A + p) \cap B) > 0$ for all $p \in C$.

Proof. Suppose $\lambda((A + p) \cap B) = 0$ $\lambda$-a.e. on $G$. Then we have

$$0 = \int \lambda((A + p) \cap B) \lambda(dp) = \int (\int 1_A(x - p) 1_B(x) \lambda(dx)) \lambda(dp) = \lambda(-A) \lambda(B) > 0 .$$

This contradiction proves the proposition.

Let $P$ be a random walk on $(G, \Sigma)$ with law $p$. Because of the proposition 1.1 and 2.2 the random walk $P$ is also a Markov process on the measure space $(G, \Sigma, \lambda)$.

Theorem 2.1. Let $P$ be a random walk on $(G, \Sigma)$ with law $p$. The random walk as Markov process on the measure space $(G, \Sigma, \lambda)$ is either conservative or dissipative.
Proof. Let $C$ and $D$ be the conservative and dissipative parts of $C$ with respect to $P$ and suppose $\lambda(C) > 0$ and $\lambda(D) > 0$. It follows from theorem 1.1 that there exist a Borel set $A \subset D$ and a number $s$ such that $\lambda(A) > 0$ and \[ \sum_{n=0}^{\infty} p^1_{nA} \leq s \text{-a.e.} \]

From proposition 2.3 we conclude, that there exists a point $p \in G$ such that $\lambda((A+p) \cap C) > 0$. Put $B = (A+p) \cap C$, then $B \subset C$, $\lambda(B) > 0$ and by proposition 2.1 \[ \sum_{n=0}^{\infty} p^1_{nB} \leq \sum_{n=0}^{\infty} p^1_{nA+p} = \sum_{n=0}^{\infty} p^T_{n-pA} = \sum_{n=0}^{\infty} p^1_{nA} \leq s \text{-a.e.} \]

Since $B \subset C$ we conclude from theorem 1.1 $\lambda(B) = 0$. This contradiction proves the theorem.

Remark: When we say a random walk is conservative (dissipative), the random walk is considered as a Markov process on the measure space $(G, \Sigma, \lambda)$.

Let $P$ be a random walk on $(G, \Sigma)$. A set $A \in \Sigma$ is said to be \textit{recurrent} if for every starting point $x \in A$ the random walk will return to the set $A$ with probability one i.e. $P^1_A(x) = 1$ for all $x \in A$. The random walk $P$ is said to be \textit{recurrent} if every open set is recurrent (cf. [1], def. 3.31). Let $\mu$ be a $\sigma$-finite measure on $(G, \Sigma)$, then a set $A \in \Sigma$ is said to be $\mu$-\textit{recurrent} if $P^1_A = 1 \mu$-a.e. on $A$. It follows from theorem 1.1 that the random walk is conservative if and only if every set $A \in \Sigma$ is $\lambda$-recurrent.

One might think, that a recurrent Markov process is conservative. This, however is not the case, there exists a recurrent Markov process on $(\mathbb{R}, \Sigma, \lambda)$, which is dissipative (see [6]). For random walks however recurrence and conservativity are equivalent.

**Theorem 2.2.** A random walk is conservative if and only if the random walk is recurrent.

**Proof.** First suppose the random walk is conservative. Let $A$ be an open set such that there exists a point $p \in A$ with $P^1_A(p) = s < 1$. For all $\varepsilon > 0$ and all $x \in G$ put $U_\varepsilon(x) = \{ y \in G \mid d(x,y) < \varepsilon \}$. Choose $\varepsilon > 0$ such that $U_{2\varepsilon}(p) \subset A$.

Because of proposition 2.1 we have for all $q \in U_\varepsilon(p)$
By theorem 1.1 we have \( U_\varepsilon (p) \) belongs to the dissipative part of \( G \) and hence \( \lambda (U_\varepsilon (p)) = 0. \) This contradiction proves that the random walk is recurrent.

Conversely, suppose the random walk is recurrent. Let \( A \) be an open set with \( 0 < \lambda (A) < \infty. \) From proposition 1.6 we conclude that \( A \) belongs to the conservative part of \( G. \) It follows from theorem 2.1, that the random walk is conservative.

**Corollary.** A random walk is conservative if and only if there exists a \( \lambda \)-recurrent set \( A \in \Sigma \) with \( 0 < \lambda (A) < \infty. \)

**Proof.** If the random walk is conservative then it follows from theorem 1.1 that every set \( A \in \Sigma \) is \( \lambda \)-recurrent. Conversely, suppose there exists a \( \lambda \)-recurrent set \( A \in \Sigma \) with \( 0 < \lambda (A) < \infty, \) then it is an immediate consequence of proposition 1.6 and theorem 2.1 that the random walk is conservative. \( \square \)

### 3. Harris recurrence and ergodicity

We start with some definitions and properties of probability measures on \((G, \Sigma)\).

If \( \mu \) and \( \nu \) are two probabilities on \((G, \Sigma)\), then the convolution \( \mu \ast \nu \) is a probability on \((G, \Sigma)\) defined by

\[
(\mu \ast \nu)(A) = \iint 1_A(x+y)\mu(dx)\nu(dy) \quad \text{for all } A \in \Sigma.
\]

The n-fold convolution of \( \mu \) will be denoted by \( \mu^{*n}. \)

**Proposition 3.1.** If \( \mu \) is a probability on \((G, \Sigma)\) with \( \mu << \lambda, \) then for all probabilities \( \nu \) we have \( \mu \ast \nu << \lambda. \)

**Proof.** Suppose \( A \in \Sigma \) with \( \lambda(A) = 0. \) Since \( \mu << \lambda \) and \( \lambda(A) = 0 \) we have \( \mu(A-y) = 0 \) for all \( y \in G. \) Hence

\[
(\mu \ast \nu)(A) = \int \left( \int 1_A(x+y)\mu(dx) \right)\nu(dy) = \int \mu(A-y)\nu(dy) = 0.
\]

We conclude \( \mu \ast \nu << \lambda. \) \( \square \)
Let \( \mu \) be a probability on \((G, \Sigma)\), then for \( n \geq 1 \) there exists the decomposition \( \mu^{*n} = q_n + r_n \) with \( q_n \ll \lambda \) and \( r_n \) singular with respect to \( \lambda \).

**Proposition 3.2.** If \( \mu \) is a probability on \((G, \Sigma)\), then \( r_{n+m} \leq r_n \ast r_m \) for all \( n \) and \( m \).

**Proof.**

\[
\mu^{(n+m)} = \mu^{*n} \ast \mu^{*m} = (q_n + r_n) \ast (q_m + r_m) = q_n \ast q_m + q_n \ast r_m + r_n \ast q_m + r_n \ast r_m.
\]

It follows from proposition 3.1 that \( r_{n+m} \leq r_n \ast r_m \).

Following Revuz ([5], p. 91) we say that a probability \( \mu \) is spread out, if there exists an integer \( p \) such that \( \mu^{*p} \) is nonsingular with respect to \( \lambda \) i.e. \( q_p \neq 0 \).

**Proposition 3.3.** If the probability \( \mu \) on \((G, \Sigma)\) is spread out, then \( r_n (G) \downarrow 0 \) if \( n \to \infty \).

**Proof.** By proposition 3.2 we have

\[
r_{n+1} (G) \leq (r_n \ast r)(G) = \int r_n (dx) r(dy) \leq \int r_n (dx) = r_n (G).
\]

Hence there exists a number \( g \geq 0 \) such that \( r_n (G) \downarrow g \) if \( n \to \infty \).

Since \( \mu \) is spread out there exists an integer \( p \) such that \( r_p (G) = s < 1 \). From proposition 3.1 we conclude for all \( k \)

\[
0 \leq g \leq r_{kp} (G) \leq r_{kp}^*(G) = \int r_p (dx_1) \ldots r_p (dx_k) = s^k.
\]

Hence \( g = 0 \).

In the sequel we shall use the following well known fact (see e.g. [4], ch. 3, § 6.1).

**Proposition 3.4.** Given are two functions \( f \) and \( g \) on \( G \) such that \( f \in L_1 (\lambda) \) and \( g \in L_\infty (\lambda) \). The function \( \psi (x) = \int f(x+y)g(y)\lambda(dy) \) is a continuous function on \( G \).

A random walk \( P \) on \((G, \Sigma)\) with law \( p \) is called **recurrent in the sense of Harris** (or shortly **Harris**), if for every starting point \( x \in G \) and every \( A \in \Sigma \) with \( \lambda (A) > 0 \) the random walk will enter the set \( A \) with probability one.
i.e. for every $A \in \Sigma$ with $\lambda(A) > 0$ we have $P^A_1(x) = 1$ for all $x \in G$ (cf. [5], ch. 3, def. 2.8).

The random walk is said to be spread out if the law $p$ is spread out.

**Theorem 3.1.** A random walk $P$ with law $p$ on a connected group $G$ is Harris if and only if the random walk is recurrent and spread-out.

**Proof.** First suppose the random walk is Harris.

It follows from theorem 1.1 that the random walk is conservative and by theorem 2.2 recurrent.

Suppose the random walk is not spread out. Then there exists a set $A \in \Sigma$ with $\lambda(A) = 0$ and $p^*A_n = 1$ for all $n$. Then we have

$$P A^* = \int_{\Sigma} A^* \lambda(dy) p(dy) = 0.$$  
Since $\lambda(A') > 0$ we conclude that the random walk is not Harris. This contradiction proves that the random walk is spread out.

Conversely, suppose the random walk is recurrent and spread out. Take $A \in \Sigma$ with $\lambda(A) > 0$.


Because of the $\sigma$-additivity of $P$ we have

$$P g = P \{ \lim_{n \to \infty} P^A n \} = \lim_{n \to \infty} P^A n = g.$$  
Hence $P g = g$.

We now conclude

$$g(x) = P^A n g(x) = \int g(x+y) p^*(dy)$$

$$= \int g(x+y) q_n(dy) + \int g(x+y) r_n(dy)$$

$$= \int g(x+y) \frac{d q_n}{d \lambda}(y) dy + \int g(x+y) r_n(dy).$$

Since $g \leq 1$ we have

$$0 \leq g(x) - \int g(x+y) \frac{d q_n}{d \lambda}(y) dy \leq r_n(G).$$

Since the random walk is spread out it follows from proposition 3.3 that
where the convergence is uniform. Using proposition 3.4 we get g is a continuous function.

Since $P_A P A 1 \preceq P 1$ and the random walk is conservative it follows from theorem 1.2 that $g = P 1 \lambda$-a.e. By proposition 1.3 we get that g takes values one or zero $\lambda$-a.e. Since g is continuous g takes values one or zero only. Since G is connected g is identically one or zero. By theorem 1.1 we have $g = 1 \lambda$-a.e. on A and therefore $g = 1$ everywhere on G.

We now have $1 = g \leq P A 1 \leq 1$. Hence $P A 1 = 1$ everywhere on G and therefore $P$ is Harris.

Remark: Some condition like the connectedness of G in theorem 3.1 is necessary, which can be illustrated by the following example. The identity on $\mathbb{Z}$ is a recurrent random walk, which is spread out but not Harris.

A random walk $P$ on $(G, \Sigma)$ is said to be ergodic if for every $A \in \Sigma$ with $\lambda(A) > 0$ the probability that the random walk will visit the set $A$ is positive for $\lambda$-almost all starting points i.e. $P A 1 > 0 \lambda$-a.e. for all $A \in \Sigma$ with $\lambda(A) > 0$. Usually ergodicity is defined by the fact, that every invariant set or its complement is empty mod the measure under consideration. It is easily verified that our definition is equivalent with the latter definition.

Theorem 3.2. A random walk $P$ on $(G, \Sigma)$ with law $p$ is ergodic if and only if the random walk will visit every non-empty open set $A$ with positive probability when started in the point zero i.e. $\sum_{n=1}^{\infty} p^{*n}(A) > 0$ for all nonempty open sets $A$.

Proof. First suppose there exists a nonempty open set $A$, such that $p^{*n}(A) = 0$ for all $n \geq 1$. Since

$$P A 1(0) \leq \sum_{n=1}^{\infty} p^{n} A 1(0) = \sum_{n=1}^{\infty} \int_A \int_{\lambda} p^{*n}(dy) \leq \sum_{n=1}^{\infty} p^{*n}(A) = 0 ,$$

we have $P A 1(0) = 0$.

Choose $\varepsilon > 0$ such that $n \leq \varepsilon \ A \cap (A + \eta)$ has a nonempty interior and let $B$ be a nonempty open subset of $n \leq \varepsilon \ A \cap (A + \eta)$.

For all $x \in U \varepsilon (0)$ we have by proposition 2.1
We conclude, that \( P \) is not ergodic, since \( \lambda(B) > 0 \) and \( P_B 1 = 0 \) on \( U_e(0) \).

Conversely, suppose for every nonempty open set \( A \) we have \( \sum_{n=1}^{\infty} P^n_B(A) > 0 \). Let \( A \in \Sigma \) with \( \lambda(A) > 0 \). In order to show that \( P \) is ergodic it suffices to show that for all \( F \in \Sigma \) with \( 0 < \lambda(F) < \infty \) there exists an integer \( n \) such that \( P^n_\lambda(x) = p^n(A-x) > 0 \) on a subset of \( F \) with positive Haar measure i.e.

\[
\int_1 F(x) p^n(A-x) \lambda(dx) > 0 .
\]

Using Fubini's theorem we get

\[
\int_1 F(x) p^n(A-x) \lambda(dx) = \int_1 F(x) \left( \int_1 A(x+y) p^n(dy) \right) \lambda(dx) = \int \left( \int_1 F(x) 1_A(x+y) \lambda(dx) \right) p^n(dy) .
\]

Since \( 1_F \in L_1(\lambda) \) and \( 1_A \in L_\infty(\lambda) \) it follows from proposition 3.4 that \( \phi(y) = \int_1 F(x) 1_A(x+y) \lambda(dx) \) is a continuous function. From \( \int \phi(y) \lambda(dy) = \lambda(F) \lambda(A) > 0 \) we conclude that \( \{ z \in G \mid \phi(z) > 0 \} \) is a non-empty open set. Hence \( \int_1 F(x) p^n(A-x) \lambda(dx) > 0 \) for some integer \( n \) and therefore \( P \) is ergodic.

If a random walk \( P \) is conservative and ergodic, then it follows from proposition 1.3 that for every \( A \in \Sigma \) with \( \lambda(A) > 0 \) we have \( P_A 1 = 1 \) \( \lambda \)-a.e. Hence Harris recurrency of a random walk is stronger than conservativity and ergodocity. We end this section with an example of a random walk, which is conservative and ergodic but not Harris.

Consider the random walk \( P \) on \( (\mathbb{R}, \Sigma, \lambda) \) with law \( p \), determined by

\[
p(\{n\}) = \frac{1}{1 + n} \quad \text{and} \quad p(\{-1\}) = \frac{\pi}{1 + \pi} .
\]

Since \( \pi \) is irrational the set \( A = \{ n\pi - m \mid n \in \mathbb{N}, m \in \mathbb{N} \} \) is dense in \( \mathbb{R} \) and it follows from theorem 3.2 that \( P \) is ergodic. Since \( \int xp(dx) = 0 \) it follows from theorem 4.5 that \( P \) is conservative. On the other hand, \( \lambda(A) = 0 \) and \( p^n(A) = 1 \) for all \( n \) and therefore the random walk is not spread out and by theorem 3.1 not Harris.
4. Recurrence criteria for random walks on $\mathbb{R}^n$

In this section we shall consider random walks on $(\mathbb{R}^n, \Sigma, \lambda)$ with $n \geq 1$, where $

\Sigma$ is the $\sigma$-algebra of the Borel sets of $\mathbb{R}^n$ and $\lambda$ the Lebesgue measure on $(\mathbb{R}^n, \Sigma)$. The theorems and propositions of this section are slight modifications of corresponding theorems in [8]. The techniques used in this section for random walks on $\mathbb{R}^n$ can also be used for random walks on locally compact groups. We start with some notations. For all $x$ and $y$ in $\mathbb{R}^n$ we write

$$ (x, y) = \sum_{i=1}^{n} x_i y_i $$

$$ |x| = \sqrt{(x, x)} $$

$$ d(x, y) = |x - y| $$

**Proposition 4.1.** Let $P$ be a dissipative random walk on $(\mathbb{R}^n, \Sigma, \lambda)$, then for every bounded set $A \in \Sigma$ we have $\sum_{n=0}^{\infty} P^n A$ is bounded on $\mathbb{R}^n$.

**Proof.** Put $U = \{x \mid |x| < 1\}$. Because of the corollary of theorem 2.2 there exists a point $p \in U$ such that $P^1_U(p) = s < 1$. Choose $\varepsilon > 0$ such that $U_{2\varepsilon}(p) \subset U$. For every $q \in U_{\varepsilon}(p)$ we have by proposition 2.1

$$ P^1_{U_{\varepsilon}(p)}(q) = P^1_{U_{\varepsilon}(p)} T_{p-q}^1(q) = T_{p-q}^1 U_{\varepsilon}(p) + p-q^1(q) = P^1_{U_{\varepsilon}(p)+p-q}(p) \leq P^1_{U_{\varepsilon}(p)}(p) \leq P^1_U(p) = s < 1. $$

Hence

$$ P^1_{U_{\varepsilon}(p)} \leq s. $$

We shall prove $P^n_{U_{\varepsilon}(p)} \leq s^{n-1}$ for $n \geq 1$. For $n = 1$ the assertion is true.

Suppose $P^n_{U_{\varepsilon}(p)} \leq s^{n-1}$ for some $n > 1$. Then

$$ P^{n+1}_{U_{\varepsilon}(p)} = P^1_{U_{\varepsilon}(p)} P^n_{U_{\varepsilon}(p)} \leq s P^n_{U_{\varepsilon}(p)} \leq s^n. $$

We conclude $P^n_{U_{\varepsilon}(p)} \leq s^{n-1}$ for $n \geq 1$. Hence

$$ \sum_{n=1}^{\infty} P^n_{U_{\varepsilon}(p)} \leq \frac{1}{1 - s} \text{ on } \mathbb{R}^n. $$

From proposition 1.2 we conclude $\sum_{n=1}^{\infty} P^n_{U_{\varepsilon}(p)} \leq \frac{1}{1 - s} \text{ on } \mathbb{R}^n$. By proposition 2.1 we have $\sum_{n=1}^{\infty} P^n_{U_{\varepsilon}(p)} \leq \frac{1}{1 - s} \text{ on } \mathbb{R}^n$ for all $a \in \mathbb{R}^n$.

Since $A$ can be covered by a finite number of translates of the hypersphere $U_{\varepsilon}(p)$ the proposition has been proved.
Proposition 4.2. Let $P$ be a conservative random walk on $(\mathbb{R}^n, \xi, \lambda)$, then for every open set $A$ we have $\sum_{n=0}^{\infty} P^n_A = \infty$ on $A$.

Proof. From theorem 2.2 we conclude $P_A 1 = 1$ on $A$. Suppose for some $n > 1$ we have $P^n_A 1 = 1$ on $A$. Then

$$P^{n+1}_A = P^n_A P_A 1 = P^n_A = 1$$

We conclude $P^n_A 1 = 1$ on $A$ for $n \geq 1$. Hence $\sum_{n=1}^{\infty} P^n_A = \infty$ on $A$. By proposition 1.2 we have $\sum_{n=1}^{\infty} P^n_A = \infty$ on $A$.

We now introduce a class $T$ of functions on $\mathbb{R}^n$. A function $f$ on $\mathbb{R}^n$ belongs to $T$ if the following conditions are satisfied.

1) $f$ is a nonnegative continuous even function on $\mathbb{R}^n$ such that $f(0) > 0$.

2) There exists a nonnegative nonincreasing function $\psi$ on the nonnegative real numbers such that

i) $f(x) \leq \psi(|x|)$ for all $x \in \mathbb{R}^n$,

ii) $\int_{\mathbb{R}^n} \psi(|x|)\lambda(dx) < \infty$.

Theorem 4.1. Let $P$ be a random walk on $(\mathbb{R}^n, \xi, \lambda)$. The random walk is conservative if and only if there exists a function $f \in T$ such that

$$\sum_{n=0}^{\infty} P^n f(0) = \infty.$$ 

Proof. Suppose the random walk is conservative. Let $A$ be an open set around zero such that $f \geq \gamma f(0) 1_A$. Then we have

$$\sum_{n=0}^{\infty} P^n f(0) \geq \gamma f(0) \sum_{n=0}^{\infty} P^n 1_A(0).$$

From proposition 4.2 we conclude $\sum_{n=0}^{\infty} P^n 1_A(0) = \infty$ and therefore $\sum_{n=0}^{\infty} P^n f(0) = \infty$. Conversely, suppose $\sum_{n=0}^{\infty} P^n f(0) = \infty$ for some $f \in T$. Let $\psi$ be a function associated with $f$ as described in the definition of $T$. Define $h(x) = \psi(|x|)$ for all $x \in \mathbb{R}^n$, then we have $\sum_{n=0}^{\infty} P^n h(0) = \infty$. Put

$$K = \{x \in \mathbb{R}^n \mid 0 \leq x_i < 1 \text{ for } i = 1, 2, \ldots, n\}.$$ 

There exists a countable partition of $\mathbb{R}^n \{K_1, K_2, \ldots\}$ such that $K_i$ is a translate of $K$ for all $i$. Define the function $k$ on $\mathbb{R}^n$. 

\[ k(x) = \sum_{i=1}^{\infty} \sup_{y \in K_i} h(y) 1_{K_i}(x) \quad \text{for all } x \in \mathbb{R}^n. \]

It is easily verified that \( k(x) \leq \psi(|x| - \sqrt{n}) \) for \( |x| \geq \sqrt{n} \) and therefore \( \int k(x) \lambda(dx) < \infty. \) Since \( h(x) \leq k(x) \) we have \( \sum_{n=0}^{\infty} P^n_k(0) = \infty. \) Suppose the random walk is dissipative, then we conclude from proposition 4.1 and 2.1 that there exists a number \( w \) such that

\[ \sum_{n=0}^{\infty} P^n_k(0) \leq w \quad \text{for all } i \in \mathbb{N}. \]

We now get

\[ \sum_{n=0}^{\infty} P^n_k(x) = \sum_{n=0}^{\infty} P^n \sum_{i=1}^{\infty} \sup_{y \in K_i} h(y) 1_{K_i}(x) = \sum_{i=1}^{\infty} \sup_{y \in K_i} h(y) \sum_{n=0}^{\infty} P^n_k(0) \]

\[ \leq w \sum_{n=0}^{\infty} \sup_{y \in K_i} h(y) = w \int k(x) \lambda(dx) < \infty. \]

Hence \( \sum_{n=0}^{\infty} P^n_k(0) < \infty. \) This contradiction proves, that the random walk is conservative.

**Proposition 4.3.** Let \( P \) be a random walk on \( (\mathbb{R}^n, \xi, \lambda) \) and define the function \( f \) on \( \mathbb{R}^n \) by

\[ f(x) = \prod_{i=1}^{n} \left( \frac{\sin^2 x_i}{x_i^2} \right)^n \quad \text{for all } x \in \mathbb{R}^n. \]

The random walk \( P \) is conservative if and only if

\[ \sum_{n=0}^{\infty} P^n f(0) = \infty. \]

**Proof.** It follows from theorem 4.1 that it suffices to show \( f \in T. \) Since for all \( x \in \mathbb{R} \) we have \( \sin^2 x + \frac{\sin^2 x}{x^2} \leq 2 \) we have

\[ \frac{\sin^2 x}{x^2} \leq \frac{2}{1+x^2} \quad \text{for all } x \in \mathbb{R}. \]

Hence

\[ f(x) = \prod_{i=1}^{n} \left( \frac{\sin^2 x_i}{x_i^2} \right) \leq 2^n \prod_{i=1}^{n} \left( \frac{1}{1+x_i^2} \right)^n \leq \frac{2^n}{(1+|x|^2)^n}. \]
The function $f$ satisfies the two conditions in the definition of the class $T$ with the function $\psi$ given by

$$\psi(r) = \frac{2^n}{(1 + r^2)^n} \quad \text{for } r \geq 0.$$ 

Let $p$ be a probability on $\mathbb{R}^n, \mathcal{E}$, then the characteristic function $\hat{p}$ of $p$ is defined by

$$\hat{p}(t) = \int e^{i(t,x)} p(dx) \quad \text{for all } t \in \mathbb{R}^n.$$ 

For every function $f \in L^1(\lambda)$ the Fourier transform $\hat{f}$ of $f$ is defined by

$$\hat{f}(t) = \int f(x) e^{-i(x,t)} \lambda(dx) \quad \text{for all } t \in \mathbb{R}^n.$$ 

**Proposition 4.4.** Let $P$ be a random walk on $\mathbb{R}^n$ with law $p$ and let $f \in T$ such that $\int |\hat{f}(t)| \lambda(dt) < \infty$ then

$$\sum_{n=0}^{\infty} p^n f(0) = \frac{1}{(2\pi)^n} \lim_{r \to 1} \int \hat{f}(t) \Re \frac{1}{1 - r \hat{p}(t)} dt.$$ 

**Proof.** Suppose $f \in T$ and $\int |\hat{f}(t)| \lambda(dt) < \infty$. We have

$$\hat{f}(t) = \int e^{-i(x,t)} p f(x) \lambda(dx) = \int e^{-i(x,t)} \left( \int f(x+y) p(dy) \right) \lambda(dx)$$

$$= \int e^{i(y,t)} \left( \int f(x+y) e^{-i(x+y,t)} \lambda(dx) \right) p(dy)$$

$$= \int e^{i(y,t)} \hat{f}(t) p(dy) = \hat{f}(t) \hat{p}(t).$$

Hence $\hat{p}^n f = \hat{f}(p^n) f$ for $n \geq 0$.

Choose $0 < r < 1$. Then we have

$$\sum_{n=0}^{\infty} r^n p^n f(t) = \int e^{-i(x,t)} r^n \sum_{n=0}^{\infty} p^n f(x) \lambda(dx)$$

$$= \sum_{n=0}^{\infty} \int e^{-i(x,t)} r^n p^n f(x) \lambda(dx)$$

(The series $\sum_{n=0}^{\infty} r^n p^n f(x)$ converges uniformly and is integrable.)

$$= \sum_{n=0}^{\infty} r^n f(t) (\hat{p}(t))^n = \hat{f}(t) \frac{1}{1 - r \hat{p}(t)} \left( |\hat{p}| \leq r < 1 \right).$$
Hence \( \sum_{n=0}^{\infty} r^n \hat{f} = \frac{\hat{f}}{1 - r \hat{1}} \).

Since \( f \) is a bounded continuous function, it follows from the dominated convergence theorem, that \( p^n f \) is continuous for all \( n \). Hence \( \sum_{n=0}^{\infty} r^n p^n f \) is a continuous function such that its Fourier transform is absolutely integrable. From a well known inversion theorem (see e.g. [4], p. 2) we conclude

\[
\sum_{n=0}^{\infty} r^n p^n f(x) = \frac{1}{(2\pi)^n} \int e^{i(x,t)} \hat{f}(t) \frac{1}{1 - r \hat{1}} \, dt \quad \text{for all } x \in \mathbb{R}^n.
\]

Since \( f \) is an even function the function \( \hat{f} \) is real valued and we get

\[
\sum_{n=0}^{\infty} p^n f(0) = \frac{1}{(2\pi)^n} \lim_{r \to 1} \int \hat{f}(t) \text{Re} \frac{1}{1 - r \hat{1}} \, dt.
\]

Proposition 4.5. Let \( p \) be a probability on \( \mathbb{R}^n, \mathcal{F} \) with characteristic function \( \hat{p} \). If for some \( t \in \mathbb{R}^n \) we have \( \hat{p}(t) = 1 \), then \( \hat{p} \) is periodic with period \( t \).

Proof. Suppose \( \hat{p}(t) = 1 \). Then we have

\[
\hat{p}(t) = \int \cos(x,t) p(dx) + i \int \sin(x,t) p(dx) = 1.
\]

Hence \( \cos(x,t) = 1 \) and \( \sin(x,t) = 0 \) for \( p \)-almost all \( x \in \mathbb{R}^n \) and therefore \( e^{i(x,t)} = 1 \) for \( p \)-almost all \( x \in \mathbb{R}^n \). Hence

\[
\hat{p}(x + t) = \int e^{i(y,t)} e^{i(y,x)} p(dy) = \int e^{i(y,x)} p(dy) = \hat{p}(x).
\]

Proposition 4.6. Let \( p \) be a probability on \( \mathbb{R}^n, \mathcal{F} \) with characteristic function \( \hat{p} \). If there exists a number \( w > 0 \) such that

\[
\lim_{r \to 1} \int_{|t| \leq w} \text{Re} \frac{1}{1 - r \hat{1}} \, dt \neq 0,
\]

then for all \( \alpha > 0 \) we have

\[
\lim_{r \to 1} \int_{|t| \leq \alpha} \text{Re} \frac{1}{1 - r \hat{1}} \, dt \neq 0.
\]

Proof. Suppose there exists a number \( w > 0 \) such that

\[
\lim_{r \to 1} \int_{|t| \leq w} \text{Re} \frac{1}{1 - r \hat{1}} \, dt \neq 0,
\]

and let \( \alpha > 0 \).
There exists a sequence $r_n \uparrow 1 \; (n \to \infty)$ such that

$$\lim_{n \to \infty} \int_{|t| \leq w} \text{Re} \left( \frac{1}{1 - r_n P(t)} \right) \, dt = \xi < \infty.$$ 

Put $A = \{ x \in \mathbb{R}^n \mid |x| \leq \alpha \}$ and $\tilde{p}(x) = 1$. Since $\tilde{p}$ is continuous the set $A$ is compact and there exists a finite number of points $a_1, a_2, \ldots, a_k$ in $A$ such that $A = \bigcup_{j=1}^k U_w(a_j)$. 

Put $B = \{ x \in \mathbb{R}^n \mid |x| \leq \alpha \}$ and $x \notin \bigcup_{j=1}^k U_w(a_j)$. Since $B$ is compact and $|\tilde{p}| < 1$ on $B$ we conclude from the dominated convergence theorem that

$$\lim_{r \uparrow 1} \int_{B} \text{Re} \left( \frac{1}{1 - r \tilde{p}(t)} \right) \, dt = s < \infty.$$ 

Using proposition 4.5 we conclude

$$\limsup_{r \uparrow 1} \int_{|t| \leq \alpha} \text{Re} \left( \frac{1}{1 - r \tilde{p}(t)} \right) \, dt \leq k \xi + s < \infty$$

and therefore

$$\lim_{r \uparrow 1} \int_{|t| \leq \alpha} \text{Re} \left( \frac{1}{1 - r \tilde{p}(t)} \right) \, dt \neq \infty. \quad \square$$

**Theorem 4.2.** Let $P$ be a random walk on $(\mathbb{R}^n, \Sigma, \lambda)$ with law $p$. The random walk is conservative if and only if there exists a number $\alpha > 0$ such that

$$\lim_{r \uparrow 1} \int_{|t| \leq \alpha} \text{Re} \left( \frac{1}{1 - r \tilde{p}(t)} \right) \, dt = \infty.$$ 

**Proof.** Let $f$ be the function as defined in proposition 4.3. 

Put $Q = \{ x \in \mathbb{R}^n \mid |x_i| \leq 1 \; \text{for} \; i = 1, 2, \ldots, n \}$ then the Fourier transform $\hat{f}_Q(t) = (2\pi)^n \prod_{i=1}^{n} \sin x_i \cdot x_i$. It follows from the convolution theorem for Fourier transforms that

$$\hat{f}_{2n}^* = (2\pi)^n \hat{f}$$

and from a well known inversion theorem ([4], p. 2) that

$$\hat{f}_{2n}(x) = \frac{4^{-n^2}}{(2\pi)^n} \int_{Q} f(t) e^{i(x,t)} \, dt.$$ 

Hence $\hat{f}(t) = 4^n (2\pi)^n \hat{f}_{2n}(-t)$ and therefore $\hat{f} \geq 0$ and has compact support. Hence there exists a number $w > 0$ such that $\hat{f}(t) = 0$ for all $|t| > w$ and we conclude from proposition 4.4.
First suppose there exists a number \( \alpha > 0 \) such that
\[
\lim_{r \to 1} \int_{|t| \leq \alpha} \Re \frac{1}{1 - rP(t)} \, dt = \infty,
\]
then it follows from proposition 4.6 that for all \( \beta > 0 \) we have
\[
\lim_{r \to 1} \int_{|t| \leq \beta} \Re \frac{1}{1 - rP(t)} \, dt = \infty.
\]
Since \( \tilde{f}(0) \neq 0 \) there exists a number \( v \) with \( 0 < v \leq w \) and \( \tilde{f}(t) \geq \frac{1}{2} \tilde{f}(0) \) for all \( |t| \leq v \). We then have
\[
\sum_{n=0}^{\infty} P^n f(0) \geq \frac{1}{2} \tilde{f}(0) \frac{1}{(2\pi)^n} \sum_{n=0}^{\infty} \int_{|t| \leq w} \Re \frac{1}{1 - rP(t)} \, dt = \infty.
\]
From proposition 4.3 we conclude the random walk is conservative. Conversely, suppose the random walk is conservative. Then by proposition 4.3 we have
\[
\sum_{n=0}^{\infty} P^n f(0) = \infty. \quad \text{Put } M = \max_{|t| \leq w} \tilde{f}(t), \quad \text{then we get}
\]
\[
\lim_{r \to 1} \int_{|t| \leq \alpha} \Re \frac{1}{1 - rP(t)} \, dt = \infty.
\]
Hence
\[
\int \Re \frac{1}{1 - P(t)} \, dt = \infty,
\]
for some \( \alpha > 0 \), then the random walk with law \( P \) is conservative.

**Proof.** From Fatou's lemma we conclude
\[
\int_{|t| \leq \alpha} \Re \frac{1}{1 - P(t)} \, dt \leq \liminf_{r \to 1} \int_{|t| \leq \alpha} \Re \frac{1}{1 - rP(t)} \, dt.
\]
Hence
\[
\int \Re \frac{1}{1 - P(t)} \, dt = \infty.
\]
\[
\lim_{r \to 1} \int_{|t| \leq \alpha} \Re \frac{1}{1 - r \hat{p}(t)} \, dt = \infty
\]

and therefore the random walk is conservative. \qed

A probability \( p \) on \((\mathbb{R}^n, \Sigma)\) is said to be reducible if there exists a linear subspace \( L \) of \( \mathbb{R}^n \) such that \( \dim L < n \) and \( p(L) = 1 \). A probability which is not reducible is called irreducible. A random walk with law \( p \) is called irreducible if its law \( p \) is irreducible.

Proposition 4.7. Let \( p \) be a probability on \((\mathbb{R}^n, \Sigma)\) and define for all \( \alpha \geq 0 \) the function \( \phi_\alpha \) on \( \mathbb{R}^n \) by

\[
\phi_\alpha(x) = \int_{|y| \leq \alpha} (x, y)^2 p(dy) \quad \text{for all } x \in \mathbb{R}^n.
\]

The probability \( p \) is irreducible if and only if there exists a number \( \beta > 0 \) such that \( \phi_\beta > 0 \) on \( \{x \mid |x| = 1\} \).

Proof. First suppose \( p \) is reducible, then there exists a linear subspace \( L \) of \( \mathbb{R}^n \) with \( \dim L < n \) and \( p(L) = 1 \). Let \( y \in \mathbb{R}^n \) with \( |y| = 1 \) and \( y \) orthogonal to \( L \). Then \( \phi_\alpha(y) = 0 \) for all \( \alpha > 0 \). Conversely, suppose \( p \) is irreducible.

For all \( n \in \mathbb{N} \) put \( W_n = \{x \in \mathbb{R}^n \mid |x| = 1 \text{ and } \phi_n(x) = 0\} \). If \( \cap_{n=1}^{\infty} W_n \neq \emptyset \), then there exists a point \( y \in \mathbb{R}^n \) with \( |y| = 1 \) and \( \int (z, y)^2 p(dy) = 0 \). It follows \( p(\{x \mid (x, y) = 0\}) = 1 \) and hence \( p \) is reducible. This contradiction proves \( \cap_{n=1}^{\infty} W_n = \emptyset \). Since \( \phi_n \) is continuous the set \( W_n \) is compact and hence there exists a number \( m \in \mathbb{N} \) such that \( \cap_{n=1}^{m} W_n = \emptyset \). Since the sequence \( (W_n) \) is nonincreasing we conclude \( W_m = \emptyset \). Hence \( \phi_m > 0 \) on \( \{x \mid |x| = 1\} \). \qed

Proposition 4.8. If \( p \) is an irreducible probability on \((\mathbb{R}^n, \Sigma)\) with characteristic function \( \hat{p} \), then there exist positive numbers \( \lambda \) and \( \mu \) with

\[
\Re(1 - \hat{p}(t)) \geq \lambda |t|^2 \quad \text{for } |t| \leq \mu.
\]

Proof. Since \( p \) is irreducible it follows from proposition 4.7 that there exist positive numbers \( \alpha \) and \( \sigma \) such that

\[
\int_{|x| \leq \alpha} (x, y)^2 p(dx) \geq \sigma \quad \text{for } |y| = 1.
\]

Since for all \( x \in \mathbb{R} \) with \( |x| \leq \frac{\pi}{2} \) we have \( |\sin x| \geq \frac{2|x|}{\pi} \) we also have
Isin \(x,t\) \(= I \) \(J(x,t)\) \(1T\) if \(|x| \leq \frac{\pi}{|t|}\).

Choose \(t \in \mathbb{R}^n\) such that \(0 < |t| \leq \frac{\pi}{\alpha}\). We now have

\[
\text{Re}(1 - \bar{p}(t)) = 2 \int_{|x| \leq \frac{\pi}{|t|}} \sin^2 \frac{1}{2}(x,t)p(dx) \geq 2 \int_{|x| \leq \alpha} \sin^2 \frac{1}{2}(x,t)p(dx)
\]

\[
\geq \frac{2}{\pi} \int_{|x| \leq \frac{\pi}{|t|}} (x,t)^2 p(dx) \geq \frac{2}{\pi} \int_{|x| \leq \alpha} (x,t)^2 p(dx)
\]

\[
= \frac{2|t|^2}{\pi} \int_{|x| \leq \alpha} (x,t)^2 p(dx) \geq \frac{2\sigma}{\pi}|t|^2.
\]

Hence

\[
\text{Re}(1 - \bar{p}(t)) \geq \frac{2\sigma}{\pi}|t|^2 \text{ for } |t| \leq \frac{\pi}{\alpha}.
\]

**Theorem 4.3.** If \(n \geq 3\), then every irreducible random walk on \((\mathbb{R}^n, \Sigma, \lambda)\) is dissipative.

**Proof.** Let \(P\) be a random walk with law \(p\). We have for \(0 < r < 1\)

\[
\text{Re} \left( \frac{1}{1 - r\bar{p}(t)} \right) = \frac{\text{Re}(1 - r\bar{p}(t))}{\left| 1 - r\bar{p}(t) \right|^2} \leq \frac{\text{Re}(1 - \bar{p}(t))}{\left( \text{Re}(1 - \bar{p}(t)) \right)^2}
\]

\[
= \frac{1}{\text{Re}(1 - \bar{p}(t))} \leq \frac{1}{\text{Re}(1 - \bar{p}(t))}.
\]

From proposition 4.8 we conclude, that there exist two positive numbers \(\lambda\) and \(\mu\) such that \(\text{Re}(1 - \bar{p}(t)) \geq \lambda |t|^2\) for all \(|t| \leq \mu\). Hence

\[
0 \leq \int_{|t| \leq \mu} \text{Re} \left( \frac{1}{1 - r\bar{p}(t)} \right) dt \leq \int_{|t| \leq \mu} \frac{1}{\text{Re}(1 - \bar{p}(t))} dt \leq \frac{1}{\lambda} \int_{|t| \leq \mu} \frac{dt}{|t|^2}.
\]

Since

\[
\int_{|t| \leq \mu} \frac{dt}{|t|^2} < \infty \text{ if } n \geq 3
\]

we conclude

\[
\lim_{r \uparrow 1} \int_{|t| \leq \mu} \text{Re} \left( \frac{1}{1 - r\bar{p}(t)} \right) dt \neq \infty
\]

and therefore by theorem 4.2 the random walk is dissipative.

Because of theorem 4.3 the situation with respect to conservativity of random walks on \(\mathbb{R}^n\) if \(n \geq 3\) is quite simple. Things are less simple for random walks on \(\mathbb{R}\) and \(\mathbb{R}^2\). We start with some technical results.
Proposition 4.9. Let $p$ be a probability on $(\mathbb{R}^n, \Sigma)$ with $\int |x|p(dx) < \infty$, then
\[
\lim_{t \to 0} \frac{\text{Re}(1 - \tilde{p}(t))}{|t|} = 0 .
\]

Proof. \[
\text{Re}(1 - \tilde{p}(t)) = 2 \int \sin^2 \theta(x, t) p(dx) \leq 2 |t| \int |x| |\frac{\sin^2 \theta(x, t)}{x(t)}| p(dx) .
\]
Since $|\frac{\sin^2 \theta(x, t)}{x(t)}|$ is bounded and tends to zero if $t \to 0$ we conclude from the dominated convergence theorem that
\[
\lim_{t \to 0} \frac{\text{Re}(1 - \tilde{p}(t))}{|t|} = 0 .
\]

Proposition 4.10. Let $p$ be a probability on $(\mathbb{R}^n, \Sigma)$ such that $\int |x|p(dx) < \infty$ and $\sigma^2 = \int |x|^2p(dx) < \infty$, then
i) $\text{Re}(1 - \tilde{p}(t)) \leq \frac{\sigma^2}{t^2}$,
ii) if $\int xp(dx) = 0$, then $|\text{Im} \tilde{p}(t)| \leq \frac{\sigma^2}{t^2}$.

Proof. \[
\text{Re}(1 - \tilde{p}(t)) = 2 \int \sin^2 \theta(x, t) p(dx) \leq \frac{\sigma^2}{t^2} \int |x|^2 p(dx) = \frac{\sigma^2}{t^2} |t|^2
\]
which proves i). $\sin(x, t) = (x, t) - \frac{1}{2} (x, t)^2 \sin(\theta(x, t))$ where $\theta$ depends on $(x, t)$ and $0 < \theta < 1$. Hence
\[
\text{Im} \tilde{p}(t) = \int \sin(x, t)p(dx) = \int (x, t)p(dx) - \frac{1}{2} \int (x, t)^2 \sin(\theta(x, t))p(dx).
\]
From $\int xp(dx) = 0$ we conclude $\int (x, t)p(dx) = (t, \int xp(dx)) = 0$ for all $t \in \mathbb{R}^n$. Hence $|\text{Im} \tilde{p}(t)| \leq \frac{\sigma^2}{|t|^2}$.

Proposition 4.11. Let $p$ be a probability on $(\mathbb{R}, \Sigma)$ such that $\int |x|p(dx) < \infty$. If we put $\mu = \int xp(dx)$ then
\[
\lim_{t \to 0} \frac{\text{Im} \tilde{p}(t)}{t} = \mu .
\]

Proof. $\text{Im} \tilde{p}(t) = \int \sin(x, t)p(dx) = t \int x \frac{\sin x}{x} p(dx)$.

From the dominated convergence theorem we conclude
\[
\lim_{t \to 0} \frac{\text{Im} \tilde{p}(t)}{t} = \mu .
\]

Proposition 4.12. Let $p$ be a probability on $(\mathbb{R}, \Sigma)$ such that $\int |x|p(dx) < \infty$ then
Proof. Define
\[ \phi(x) = \int_{0}^{1} \frac{1 - \cos xt}{t^2} \, dt \quad \text{for all } x \in \mathbb{R}. \]

Then
\[ \phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!(2n+1)} \int_{0}^{t} \, \text{d}t. \]

The uniform convergence of the series permits interchanging of summation and integration and we get
\[ \phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!(2n+1)}. \]

and therefore \( \phi''(x) = \frac{\sin x}{x} \). Since \( \phi'(0) = 0 \) and \( \int_{0}^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2} \) we have \( |\phi'(x)| \leq \frac{\pi}{2} x \) and \( \phi(x) \leq \frac{\pi}{2} x \) for all \( x \in \mathbb{R} \). Hence
\[ \int_{-1}^{1} \frac{\text{Re}(1 - \bar{Z}(t))}{t^2} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!(2n+1)} \int_{-1}^{1} \frac{\text{d}t}{t^2} \int_{-\infty}^{\infty} (1 - \cos xt)p(dx) \]
\[ = \int_{-\infty}^{\infty} \phi(x)p(dx) \leq \frac{\pi}{2} \int_{-\infty}^{\infty} |x|p(dx) < \infty. \quad \Box \]

Theorem 4.4. Let \( P \) be an irreducible random walk with law \( p \) on \( \mathbb{R} \) or \( \mathbb{R}^2 \) such that \( \int |x|p(dx) < \infty \) and \( \sigma^2 = \int |x|^2p(dx) < \infty \). If \( \mu = \int xp(dx) = 0 \), then the random walk is conservative.

Proof. From proposition 4.8 we conclude that there exist positive numbers \( \lambda \) and \( \rho \) such that
\[ \text{Re}(1 - \bar{Z}(t)) \geq \lambda |t|^2 \quad \text{if } |t| \leq \rho. \]

Since \( \mu = 0 \) we conclude from proposition 4.10
\[ (\text{Re}(1 - \bar{Z}(t)))^2 \leq \frac{\sigma^4}{4}|t|^4 \]
\[ (\text{Im} \bar{Z}(t))^2 \leq \frac{\sigma^4}{4}|t|^4. \]
Hence if $|t| \leq \rho$ we have

$$\text{Re} \frac{1}{1 - \tilde{P}(t)} = \frac{\text{Re}(1 - \tilde{p}(t))}{(\text{Re}(1 - \tilde{p}(t)))^2 + (\text{Im} \tilde{p}(t))^2} \geq \frac{2\lambda}{4|t|^2}. $$

Therefore

$$\int_{|t| \leq \rho} \text{Re} \frac{1}{1 - \tilde{P}(t)} \, dt \geq \frac{2\lambda}{4 \int_{|t| \leq \rho} \frac{dt}{|t|^2}} = \infty. $$

From the corollary of theorem 4.2 we conclude, that $P$ is conservative. \[ \square \]

**Theorem 4.5.** Let $P$ be a random walk on $\mathbb{R}$ with law $\mathbf{p}$ such that $\int |x|\mathbf{p}(dx) < \infty$. The random walk $P$ is conservative if and only if $\mu = \int x\mathbf{p}(dx) = 0$.

**Proof.** First suppose $\mu = 0$. Given any $\varepsilon > 0$ it follows from the propositions 4.9 and 4.11 that there exists a number $\alpha > 0$ such that $|\text{Im} \tilde{p}(t)| \leq \varepsilon |t|$ and $|\text{Re}(1 - \tilde{p}(t))| \leq \varepsilon |t|$ if $|t| \leq \alpha$. For $0 < r < 1$ we have

$$\text{Re} \frac{1}{1 - r\tilde{p}(t)} \geq \frac{1 - r}{(\text{Re}(1 - r\tilde{p}(t)))^2 + r^2(\text{Im} \tilde{p}(t))^2}. $$

It is easily verified, that for $0 < r < 1$ and $0 < \alpha < 1$ we have

$$(1 - ra)^2 \leq 2(1 - r)^2 + 2r^2(1 - \alpha)^2. $$

Therefore we can choose $\alpha$ small enough such that

$$(\text{Re}(1 - r\tilde{p}(t)))^2 \leq 2(1 - r)^2 + 2r^2(\text{Re}(1 - \tilde{p}(t)))^2 \quad \text{for} \quad |t| \leq \alpha. $$

We now have

$$\int_{-\alpha}^{\alpha} \text{Re} \frac{1}{1 - r\tilde{p}(t)} \, dt \geq \int_{-\alpha}^{\alpha} \frac{(1 - r)dt}{(\text{Re}(1 - r\tilde{p}(t)))^2 + (\text{Im} \tilde{p}(t))^2} \geq (1 - r) \int_{-\alpha}^{\alpha} \frac{dt}{2(1 - r)^2 + 3r^2 \varepsilon^2 t^2} \geq \frac{1}{3} \int_{-\alpha}^{\alpha} \frac{dt}{1 + \varepsilon^2 t^2}. $$

Hence

$$\lim_{r \uparrow 1} \int_{-\alpha}^{\alpha} \text{Re} \frac{1}{1 - r\tilde{p}(t)} \, dt \geq \frac{\pi}{3\varepsilon}$$

and therefore

$$\lim_{r \uparrow 1} \int_{-\alpha}^{\alpha} \text{Re} \frac{1}{1 - r\tilde{p}(t)} \, dt = \infty.$$
and by theorem 4.2 $P$ is conservative. Conversely, suppose $u \neq 0$

$$\text{Re} \frac{1}{1 - r \tilde{p}(t)} = \frac{\text{Re}(1 - \tilde{p}(t)) + (1 - r) \text{Re} \tilde{p}(t)}{(\text{Re}(1 - r \tilde{p}(t)))^2 + r^2 (\text{Im} \tilde{p}(t))^2}$$

$$\leq \frac{\text{Re}(1 - \tilde{p}(t))}{r^2 (\text{Im} \tilde{p}(t))^2} + \frac{1 - r}{|1 - r \tilde{p}(t)|^2}.$$

By proposition 4.11 we can choose $a > 0$ such that

$$|\text{Im} \tilde{p}(t)| \geq \left| \frac{\mu t}{2} \right| \text{ for } |t| \leq a.$$ 

Hence

$$\int_{-\alpha}^{\alpha} \frac{1}{1 - r \tilde{p}(t)} \, dt \leq \frac{4}{\mu} \int_{-\alpha}^{\alpha} \frac{\text{Re}(1 - \tilde{p}(t))}{t^2} \, dt + \int_{-\alpha}^{\alpha} \frac{1 - r}{|1 - r \tilde{p}(t)|^2} \, dt.$$

By proposition 4.12 the first integral on the right hand side is finite.

We are now going to estimate the second integral.

$$\int_{-\alpha}^{\alpha} \frac{1 - r}{|1 - r \tilde{p}(t)|^2} \, dt = \int_{|t| \leq 1 - r} \frac{1 - r}{|1 - r \tilde{p}(t)|^2} \, dt + \int_{1 - r \leq |t| \leq \alpha} \frac{1 - r}{|1 - r \tilde{p}(t)|^2} \, dt$$

$$\leq \int_{|t| \leq 1 - r} \frac{\text{dt}}{1 - r} + \int_{1 - r \leq |t| \leq \alpha} \frac{1 - r}{r^2 (\text{Im} \tilde{p}(t))^2} \, dt$$

$$\leq 2 + \frac{4(1 - r)}{r^2 \mu} \int_{1 - r \leq |t| \leq \alpha} \frac{\text{dt}}{t^2} = 2 + \frac{8(1 - r)}{r^2 \mu} \frac{1}{1 - r} \frac{1}{\alpha}.$$

Hence

$$\lim_{r \to 1} \int_{-\alpha}^{\alpha} \text{Re} \frac{1}{1 - r \tilde{p}(t)} \, dt \neq \infty$$

and therefore by theorem 4.2 the random walk is dissipative.

References


