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ASYMPTOTICS IN NORMAL ORDER STATISTICS

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ASYMPTOTICS IN NORMAL ORDER STATISTICS

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ABSTRACT

In order statistics certain integrals involving the standard normal dis­
tribution play an important role. The asymptotic behaviour with respect to
a large parameter is studied.

1. INTRODUCTION

The expectation and variance of the maximum in a random sample of size n
from the standard normal distribution involve some of the integrals

(1) \( M_j(n) := \int_{-\infty}^{\infty} x^{j-1} \phi(x) (1 - \phi^{-1}(x)) \, dx \) \( (n \in \mathbb{N}, j \in \mathbb{N}) \); \( \)

where

(2) \( \phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \, ds \).

Integration by parts gives

(3) \( \mu_j(n) = j M_j(n) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^j e^{-x^2/2} \, dx \),

where

(4) \( \mu_j(n) := \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^j e^{-x^2/2} \phi^{-1}(x) \, dx \) \( (n \in \mathbb{N}, j \in \mathbb{N}) \).
The problem posed by two colleagues of the author is to determine the asymptotic behaviour for \( n \to \infty \) of \( \mu_j(n) \) for \( j = 1, 2, 3, 4 \). Moreover, they are interested especially in the asymptotic behaviour of

\[
(\mu_3 - \mu_1 \mu_2)(\mu_2 - \mu_1^2)^{-\frac{1}{2}} \quad \text{and} \quad (\mu_4 - \mu_2^2)^{\frac{1}{2}}.
\]

We remark that the differences

\[
M_j(n+1) - M_j(n) = \int_{-\infty}^{+\infty} x^{j-1}(1 - \phi(x)) \psi^n(x) dx
\]

are just the integrals occurring in the coefficients of the asymptotic formulas in a previous paper [1] (where \( f \) is defined by \( f(x) = \phi(x/\sqrt{2}) \)).

2. RESULTS

Let the asymptotic series of \((1 - \phi(x))(\phi'(x))^{-1}\) by denoted by \( A \), i.e.

\[
A := \sum_{\ell=0}^{\infty} (-1)^{\ell}(2\ell-1)!! \cdot x^{-2\ell-1} \quad (x \to \infty), \quad ((-1)!! = 1).
\]

Formal differentiations of \( A \) are denoted by \( A', A'' \), etc. Let \( x_1 = x_1(n) \) be defined by \( \phi(x_1) = 1 - \frac{1}{n} \). Then \( \mu_j(n) \) has an asymptotic expansion in powers of \( x_1^{-1} \), i.e.

\[
\mu_j(n) \approx \sum_{k=-j}^{\infty} C(j,k)x_1^{-k} \quad (n \to \infty),
\]

which can be computed as follows: \( x \) is considered to be a function of a variable \( z \) and \( \frac{dx}{dz} \approx \frac{1}{A} \). Higher derivatives can be computed by means of the chain rule. For instance, \( \frac{d^2 x}{dz^2} \approx \frac{1}{AA'} \). Then

\[
\mu_j(n) \approx \sum_{\ell=0}^{\infty} \left( \frac{d^\ell x}{dz^\ell} \right) \left( -2 \right)^\ell \frac{\Gamma(\ell+1)}{\ell!} \quad (n \to \infty),
\]

where the subscript 1 means that the value at \( x = x_1 \) has to be taken. \( x_1 \) has the following asymptotic series:

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where \( z = 2 \log \frac{n}{\sqrt{2\pi}} \) and the \( q_k \)'s are polynomials of degree \( k \). A few \( q_k \)'s are

\[
q_1(t) = -\frac{1}{2}t, \quad q_2(t) = -\frac{1}{8}t^2 + \frac{1}{4}t - 1,
\]

\[
q_3(t) = -\frac{1}{16}t^3 + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{2}.
\]

The coefficients \( C(j,k) \) have the property that \( C(j,-j+s) = 0 \) if \( s \) is odd and \( C(j,-j) = 1 \). A few more coefficients \( C(j,k) \) are:

\[
C(1,1) = -\Gamma'(1), \quad C(1,3) = \Gamma'(1) - \frac{1}{2}\Gamma''(1),
\]

\[
C(1,5) = 3\Gamma'(1) + 2\Gamma''(1) - \frac{1}{2}\Gamma'''(1);
\]

\[
C(2,0) = -2\Gamma'(1), \quad C(2,2) = 2\Gamma'(1), \quad C(2,4) = -6\Gamma'(1) + 2\Gamma''(1),
\]

\[
C(2,6) = 30\Gamma'(1) - 14\Gamma''(1) + \frac{8}{3}\Gamma'''(1);
\]

\[
C(3,-1) = -3\Gamma'(1), \quad C(3,1) = 3\Gamma'(1) + \frac{3}{2}\Gamma''(1),
\]

\[
C(3,3) = -9\Gamma'(1) + \frac{1}{2}\Gamma'''(1),
\]

\[
C(3,5) = 45\Gamma'(1) - \frac{21}{2}\Gamma''(1) - \frac{1}{2}\Gamma'''(1) + \frac{3}{8}\Gamma^{(4)}(1);
\]

\[
C(4,-2) = -4\Gamma'(1), \quad C(4,0) = 4\Gamma'(1) + 4\Gamma''(1),
\]

\[
C(4,2) = -12\Gamma'(1) - 4\Gamma''(1), \quad C(4,4) = 60\Gamma'(1) - \frac{8}{3}\Gamma'''(1).
\]

A routine computation shows that

\[
\frac{\nu_3 - \nu_1\nu_2}{(\nu_2 - \nu_1)^{\frac{3}{2}}} \approx d_0 + d_2 x_1^{-2} + d_4 x_1^{-4} + \ldots \quad (n + \infty)
\]

\[
\frac{\nu_4 - \nu_2}{(\nu_2)^{\frac{3}{2}}} \approx e_0 + e_2 x_1^{-2} + e_4 x_1^{-4} + \ldots \quad (n + \infty)
\]

where

\[
d_0 = e_0 = 2(\Gamma''(1) - (\Gamma'(1))^2)^{\frac{1}{4}}
\]
and
\[ d_2 = e_2 = -d_0. \]

3. PROOF OF THE RESULTS

We transform the integral in (4) by putting
\[ \phi(x) = 1 - \frac{s}{n}. \]

Then
\[ \frac{dx}{ds} = -\frac{\sqrt{2\pi}}{n} e^{x^2/2}, \]
whence
\[ \mu_j(n) = \int_{\infty}^{n} x^j (1 - \frac{s}{n})^{n-1} ds. \]

We observe that \( x = x(s) \) is monotonically decreasing on \([0, \infty)\), that \( x(s) \to \infty (s + 0) \), \( x(\frac{1}{2}n) = 0 \) and \( x(s) \to -\infty (s + n) \).

Now we shall prove that
\[ \mu_j(n) = \left(1 + \mathcal{O}\left(\frac{\log n}{n}\right) \right) \int_{0}^{\log n} x^j e^{-s} ds + \mathcal{O}(n^{-1}(\log n)^{1/2}) \quad (n \to \infty). \]

Since
\[ \left| \int_{\pi/2}^{n} x^j (1 - \frac{s}{n})^{n-1} ds \right| \leq \frac{n}{\sqrt{2\pi}} \int_{0}^{\infty} x^j e^{-x^2/2} \phi^{n-1}(x) dx \leq \frac{n^{2-n}}{\sqrt{2\pi}} \int_{0}^{\infty} x^j e^{-x^2/2} dx \]
we can write
\[ \mu_j(n) = \int_{0}^{\infty} x^j (1 - \frac{s}{n})^{n-1} ds + \mathcal{O}(n^{2-n}) \quad (n \to \infty). \]
Using

\[ \psi(x) \approx 1 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k=0}^{\infty} \frac{(-1)^k (2-k)!!}{k!} x^{-2k-1} \quad (x \to \infty) \]

we derive easily that at \( s = \log n \)

\[ x \sim \sqrt{2 \log n} \quad (n \to \infty). \]

Hence

\[ \int_{\log n}^{n/2} x^j (1 - \frac{s}{n})^{n-1} ds = O((\log n)^{j/2}) \int_{\log n}^{n/2} (1 - \frac{s}{n})^{n-1} ds = O(n^{-1} (\log n)^{j/2}) \quad (n \to \infty). \]

From (9) and (12) it follows that

\[ u_j(n) = \int_{0}^{\log n} x^j (1 - \frac{s}{n})^{n-1} ds + O(n^{-1} (\log n)^{j/2}) \quad (n \to \infty). \]

Furthermore

\[ \int_{0}^{\log n} x^j (1 - \frac{s}{n})^{n-1} ds = \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) \int_{0}^{\log n} x^j e^{-s} ds \quad (n \to \infty) \]

since

\[ (1 - \frac{s}{n})^{n-1} = e^{-s} \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) \quad (0 \leq s \leq \log n, n \to \infty). \]

Clearly (13) and (14) imply (8).

On the interval \( 0 \leq s \leq \log n \), corresponding with large values of \( x \), we can use (10) in order to solve \( x \) from (5) as a function of \( s \).

Introduction of

\[ z = 2 \log \frac{n}{\sqrt{2\pi}} - 2 \log s \]

transforms (5) into

\[ \psi(x) = 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{z}{2}}. \]
Clearly $z \rightarrow \infty$ if $n \rightarrow \infty$ and $0 < s < \log n$.

Using (10) and taking logarithms we get

$$ (18) \quad z \approx x^2 + \log(x^2) + \frac{2}{x^2} - \frac{5}{x^4} + \frac{74/3}{x^6} \ldots \quad (x \rightarrow \infty) $$

By asymptotic iteration we find

$$ (19) \quad x^2 \approx z - \log z + \sum_{k=1}^{\infty} z^{-k} p_k(\log z) \quad (z \rightarrow \infty) $$

where the $p_k$'s are polynomials of degree $k$. A few polynomials $p_k$ are

$$ (20) \quad p_1(t) = t - 2 $$
$$ p_2(t) = \frac{1}{2} t^2 - 3t + 7 $$
$$ p_3(t) = \frac{1}{3} t^3 - \frac{3}{2} t^2 + 17t - \frac{107}{3} . $$

The asymptotic expansions for $x^j$ have the form

$$ (21) \quad x^j \approx z^{j/2} \left(1 + \sum_{k=1}^{\infty} \frac{p_{jk}(\log z)z^{-k}}{k!} \right) \quad (z \rightarrow \infty) $$

where the $p_{jk}$ are polynomials of degree $k$.

The individual terms in the asymptotic expansions (21) have the following property: Let $f(z)$ be such a term occurring in the right side of (21).

Then $f(z)$ is of the form

$$ f(z) = (\log^m z)z^{-k+j} \quad \text{where} \quad m \leq k . $$

Let $z_1 := 2 \log \frac{n}{\sqrt{2\pi}}$. Clearly $z_1 \geq 2$ if $n \geq 7$. Let $n \geq 7$. Then the power-series about $z = z_1$

$$ (22) \quad f(z_1 + \varepsilon) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!} \varepsilon^{k} $$

is convergent for $|\varepsilon| < z_1$. Now it is easily seen that this power-series has the property that for all $N \in \mathbb{N}$, $N \geq |j-k|

$$ (23) \quad f(z_1 + \varepsilon) = \sum_{k=0}^{N} \frac{f^{(k)}(z_1)}{k!} \varepsilon^{k} + O((\log^m z_1)z_1^{-k+j-N-1} \varepsilon^{N+1}) $$

$$ (-\frac{1}{2} z_1 < \varepsilon < \infty, \quad z_1 \geq 2) .$$
Then it follows that upon substitution \( c = -2 \log s \) in (22) we get an asymptotic expansion for \( 0 < s < \log n, n \to \infty \), i.e. for all \( N \geq \frac{1}{2}j-k \)

\[
(24) \quad f(z - 2 \log s) = \sum_{k=0}^{N} \frac{f^{(k)}(z)}{k!} (-2 \log s)^k + \\
+ \mathcal{O}((\log z)^{k+j-N-1}(\log s)^{N+1}) \quad (0 < s < \log n, n \to \infty)
\]

since \( 2 \log \log n < \frac{1}{2}z \) for \( n \) sufficiently large. The hidden constant in the \( O \)-term is independent of \( n \).

Further, for every \( \ell \in \mathbb{N} \),

\[
(25) \quad \int_{\log n}^{\infty} \log^\ell s e^{-s} ds = \mathcal{O}(n^{-\ell} (\log n)^{\ell}) \quad (n \to \infty).
\]

Therefore we can proceed as follows: In the asymptotic expansion (21) of \( x_j \) we substitute \( z = z - 2 \log s \) and we expand formally into a power-series around \( z \). After multiplication with \( e^{-s} \) and integration over \( (0, \infty) \) we get an asymptotic expansion for \( \mu_j(n) \).

Denoting the asymptotic expansion (21) of \( x_j \) by \( X_j \) and writing \( X_j^{(k)} \) for its formal derivatives we have proved that

\[
(26) \quad \mu_j(n) \approx \sum_{k=0}^{n} X_j^{(k)}(z) (-2)^k \Gamma^{(k)}(1)(k!)^{-1} \quad (n \to \infty),
\]

where we have used that

\[
(27) \quad \int_{0}^{\infty} e^{-s} \log^k s ds = \Gamma^{(k)}(1).
\]

If we carry out the above program then, for instance, we find

\[
(28) \quad \mu_1 = z^{1/2} - \frac{1}{2} z^{-1/2} \log z + \gamma z^{-1/2} - \frac{1}{8} z^{-3/2} \log^2 z + \\
+ \frac{1}{2} (1+\gamma) z^{-3/2} \log z - \left(1 - \gamma - \frac{1}{2} \gamma^2 - \frac{1}{12} \pi^2\right) z^{-3/2} + \\
+ \mathcal{O}(z^{-5/2} \log^3 z) \quad (n \to \infty)
\]

where \( z = z \). We have used that \( \Gamma'(1) = \gamma \) (Euler's constant) and \( \Gamma''(1) = \gamma^2 + \frac{1}{6} \pi^2 \).
Of course we can also find asymptotic results for \( \mu_2, \mu_3 \) and \( \mu_4 \). We will not do so since there is a more convenient way to obtain asymptotic expansions for \( \mu_j(n) \). We shall show that \( \mu_j(n) \) has an asymptotic power-series expansion in powers of \( x_1^{-1} \), where \( x_1 \) is the value of \( x \) at \( z = z_1 := 2 \log \frac{n}{\sqrt{2\pi}} \), i.e. there are sequences \( (C(j,k))_{k=-j}^{\infty} \) of real numbers such that

\[
\mu_j(n) \approx \sum_{k=-j}^{\infty} C(j,k)x_1^{-k} \quad (n \to \infty).
\]

Considering \( x \) as a function of \( z \) defined by (17) we can write the integral in (8) as

\[
\log n \int_{0}^{\log n} x^j(z_1 - 2 \log s)e^{-s} ds.
\]

We shall prove that we can find the asymptotic expansion of (30) by termwise integration of the formal power series expansion of \( x^j(z_1 - 2 \log s) \) about \( z_1 \). We shall give the details of the proof for the case \( j = 1 \). The other cases \( j > 1 \) can be treated analogously. So for the moment being we suppose \( j = 1 \). Obviously we are done with the problem if we have proved that

\[
\forall_{k \in \mathbb{N}} \left( \frac{d^k x}{dz^k} \right)_1 \quad \text{has an asymptotic power series in } x_1^{-1}.
\]

\[
\forall_{k \in \mathbb{N}} \left( \frac{d^{k+1} x}{dz^{k+1}} + 1 \right)_1 = \sigma\left( \left( \frac{d^k x}{dz^k} \right)_1 \right) \quad (n \to \infty)
\]

\[
\forall_{N \in \mathbb{N}} \exists_{A > 0} \quad x(z_1 - 2 \log s) = x_1 + \sum_{k=1}^{N} \left( \frac{d^k x}{dz^k} \right)_1 \left( -2 \log s \right)^k +
\]

\[
+ \sigma\left( \left( \frac{d^{N+1} x}{dz^{N+1}} \right)_1 \left( \log s \right)^{N+1} \right) \quad (0 < s < \log n, n > A)
\]

where the subscript 1 in \( \left( \right)_1 \) means the value at \( z_1 \).
From (17) it follows that

\[ \frac{dx}{dz} = \frac{1}{4}a(x), \]

where

\[ a(x) = \frac{1 - \phi(x)}{\phi'(x)}. \]

From (10) we see that

\[ a(x) \approx \sum_{k=0}^{\infty} (-1)^k (2k-1)!! x^{-2k+1} \quad (x \to \infty). \]

From (35) we derive that

\[ \frac{da}{dx} = xa - 1. \]

Clearly (36) and (37) imply that all derivatives \( d^k a/dx^k \) have asymptotic powerseries in \( x^{-1} \) which can be obtained by formal differentiation of the asymptotic series in (36). By means of the chain rule we can compute from (34) all derivatives \( dx^k/dz^k \); clearly, \( dx^k/dz^k \) is a sum of products involving \( a(x) \) and its derivatives \( d^2 a/dx^2, k = 1, 2, \ldots, k-1 \). It follows that \( d^k x/dz^k \) has an asymptotic expansion in powers of \( x^{-1} \) for \( x \to \infty \).

Using that

\[ \frac{d^k a}{dx^k} \sim \frac{(-1)^k k!}{x^{k+1}} \quad (x \to \infty) \]

we easily derive that for \( k \geq 2 \)

\[ \frac{dx^k}{dz^k} \sim (-1)^{k+1} (2k-3)!! 2^{-k} x^{-2k+1} \quad (x \to \infty). \]

Thus we have proved (31) and (32).

Let \( N \in \mathbb{N} \). Let \( h \in \mathbb{R} \). Then there is a number \( \theta \in (0,1) \) such that

\[ x(z_1 + h) = x_1 + \frac{h}{1!} \left( \frac{dx}{dz} \right)_1 + \ldots + \frac{h^N}{N!} \left( \frac{d^N x}{dz^N} \right)_1 + R_N \]

where

\[ R_N = \frac{h^{N+1}}{(N+1)!} \left( \frac{d^{N+1} x}{dz^{N+1}} \right)_1 (z_1 + \theta h). \]
Restricting ourselves to $h > -\frac{1}{2}z_1$, we only have to prove that there is a number $A > 0$ and a number $K > 0$ such that for all $z > A$

\begin{equation}
\left| \frac{d^{N+1}}{dz^{N+1}} x(\eta) \right| < K \left| \frac{d^{N+1}}{dz^{N+1}} x(z) \right| \quad (\eta > \frac{1}{2}z).
\end{equation}

On account of (38) and the fact that $x(z) \sim z^\frac{1}{2} (z \to \infty)$, (41) is obviously true. Hence (33) holds, since $z_1 - 2 \log s > \frac{1}{2}z_1$ for $0 < s < \log n$ and $n$ sufficiently large.

Now using (33) in (30) for $j = 1$ we get an asymptotic series in powers of $x_1^{-1}$ for $\mu_1(n)$.

Analogously we can find asymptotic series for $\mu_j(n)$, $j = 2, 3, \ldots$. Only small adaptations are necessary; for instance in (33) we have to change $\forall n \in \mathbb{N}$ into $\forall n \in \mathbb{N}, n \geq j$.

REMARK. Especially for $(\mu_3 - \mu_1\mu_2)(\mu_2 - \mu_1^2)^{-1}$ and $(\mu_4 - \mu_2^2)^{\frac{1}{2}}$ the computations are most easily done if one postpones the replacement of $d^4x^1/dz^2$ by its asymptotic series as long as possible. For instance, in this way we get

$$
\mu_4 - \mu_2^2 = (8B - 4A^2)x'(x')^2 + (24C - 8AB)(x'x'' + x(x')^3) + o(x^4),
$$

where

$$
A = -2\Gamma'(1), \quad B = 2\Gamma''(1) \quad \text{and} \quad C = -\frac{2}{3}\Gamma'''(1),
$$

and $x'$, $x''$ denote first and second derivatives with respect to $z$. Now using the asymptotic series

$$
x' = \frac{1}{2x} - \frac{1}{2x^3} + \ldots, \quad x'' = -\frac{1}{4x^3} + \frac{1}{x^5} + \ldots
$$

we see that

$$
x^2x'x'' + x(x')^3 = o(x^{-4}) \quad \text{and} \quad x^2(x')^2 = \frac{1}{4} - \frac{1}{2x^2} + \ldots.
$$

We get

$$
\mu_4 - \mu_2^2 = d_0^2 - 2d_0^2 x^{-2} + o(x^{-4}),
$$

from which we can easily derive the result in section 2.
REFERENCE