Asymptotic treatment of the Elenbaas–Heller equation

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When the maximum temperatures within a high-pressure gas discharge arc are lower than the ionization temperature of the gas molecules by an order of magnitude, an asymptotic treatment of the temperature equation is possible. This is illustrated by means of the Elenbaas–Heller equation [e.g., M. F. Hoyaux, Arc Physics (Springer, Berlin, 1968), p. 36] for a nonradiating wall-stabilized arc. The asymptotics lead to a closed-form expression for the relationship between the arc current and the axis temperature. An expression for the heat loss per unit length is also given.

The Elenbaas–Heller equation for the temperature $t$

$$\frac{1}{r}\frac{d}{dr} r\lambda(t) \frac{dt}{dr} + \sigma(t)E^2 = 0$$

(1)

is one of the earliest attempts to investigate temperature distributions in a circular tube containing a high-pressure gas discharge arc. It was put forward by Elenbaas, and solutions are discussed in his book on the high-pressure mercury vapor discharge. Over the years this equation has been the subject of intensive study and all manner of approximate solution methods have been discussed. Examples are Refs. 3–7. A useful discussion of some of the earlier methods was given by Hoyaux. The basic difficulty presented by (1) is the strong nonlinear behavior of the thermal conductivity $\lambda$ and the electrical conductivity $\sigma$ as a function of the temperature $t$. The electric field $E$ is related to the total arc current $I$ as follows:

$$E = I / \left(2\pi \int_0^a \sigma(t) dr\right).$$

(2)

This, too, introduces a strong nonlinear element into the problem. Further, $r$ denotes the radial coordinate and $a$ is the radius of the inner tube wall.

Early attempts at solving (1) involved simplified assumptions concerning $\lambda(t)$ and $\sigma(t)$. Maecker, for instance, assumes $\sigma$, or rather a related transformed function, to be zero below a certain temperature and nonzero, but uniform above that temperature. Goldenberg distinguishes three temperature regimes, in each of which $\sigma$ is approximated by a linear rule. Whitman and Cohen follow an approach similar to that of Maecker, but instead of a uniform $\sigma$ they use a power law. With the advent of the computer the interest in the Elenbaas–Heller equation seemed to fade somewhat, as far more complex models could now be treated. Time dependence was added and multidimensional convection effects could now be modeled and solved successfully. However, a drawback of the numerical approach is that results have to be presented in graphical or tabulated form. It hardly ever produces rules in closed form. Quite often, the number of effects represented in the model is so large that it becomes difficult to present results systematically. As a result, one is forced to limit oneself to a few illustrative cases.

The purpose of this note is to show that Eq. (1) may yield an accurate explicit closed-form relationship between the arc current and the maximum arc temperature. We are able to derive such a rule when the arc temperatures remain far below the ionization temperature $t_i$ of the gas molecules, let us say by an order of magnitude. Since $t_i$ is on the order of many tens of thousands of degrees Kelvin, there are still many practical situations to which this solution applies.

The Elenbaas–Heller equation for a nonradiating arc can be written as follows:

$$\frac{1}{r} \frac{d}{dr} r\lambda(t) \frac{dt}{dr} + I^2 \sigma(t)/4\pi^2 \left(\int_0^a \sigma(t) dr\right)^2 = 0.$$  

(3)

This equation is an immediate consequence of (1) and (2). Boundary conditions are given on the axis of symmetry

$$\frac{dt}{dr} = 0 \quad \text{at} \quad r = 0,$$

(4)

and on the tube wall

$$t = t_w \quad \text{at} \quad r = a.$$  

(5)

For the electrical conductivity we shall use a temperature rule based on the Saha equation (see Elenbaas):

$$\sigma(t) = \gamma t^{1/4} \exp(-t/t_i),$$

(6)

where $t_i$ is the ionization temperature and $\gamma$ is a physical parameter which is inversely proportional to the gas pressure and the collision cross section of the gas molecules. For a mercury discharge of 5.32 atm, Zollweg gives tabulated values of $\sigma$ (see Zollweg's Table III) which can be modeled extremely accurately by (6) if we take $\gamma = 11.4$ mho/cm/K$^{1/4}$ and $t_i = 5.5820$ K (see Ref. 15).

Although the analysis which follows can be carried out in principle for any $\lambda(t)$, we shall model the thermal conductivity by the rule

$$\lambda(t) = \lambda_0 (t/t_i)^{3/4},$$

(7)

where $t_i$ is a reference temperature yet to be specified, and $\lambda_0$ is the corresponding thermal conductivity. Again, Zollweg's tabulated values show that (7) is a very good approximation for mercury arcs which are not too hot. Elenbaas himself used (7) in many of his papers.

An asymptotic analysis is possible after the introduction of dimensionless variables. It turns out that the analysis of the problem becomes much simpler if we invert the...
problem definition. Instead of asking what temperature rise will result from a given current \( I \), we shall look for the value of \( I \) which will result in a given temperature rise. Therefore, we prescribe the temperature on the axis, i.e.,

\[ t=t_o \quad \text{on} \quad r=0, \tag{8} \]

and we keep \( I \) as an unknown in the problem. The scaling rules are now as follows:

\[ r=aR, \quad t=t_o T. \tag{9} \]

Substituting this in Eq. (1), using Eqs. (6) and (7), we may derive the following dimensionless equation:

\[ \frac{1}{R} \frac{d}{dR} RT^{3/4} \frac{dT}{dR} + HT^{3/4} \exp \left[ T_i \left( 1 - \frac{1}{T} \right) \right] = 0, \tag{10} \]

and

\[ H=af(T_i), \tag{11} \]

with

\[ \alpha=\lambda T_i^{5/4} \gamma \lambda_i t_i^{1/4}, \tag{12} \]

\[ f(T_i) = e^{T_i T_i^{5/2}} \int_0^1 RT^{3/4} \exp \left[ T_i \left( 1 - \frac{1}{T} \right) \right] dR \tag{13} \]

Further

\[ T_i = t/t_o, \tag{14} \]

which is a given quantity, because of our inverse approach. Instead, the current parameter \( \alpha \) is unknown. It should be emphasized that in Eq. (12) \( \lambda_i \) is not the actual thermal conductivity at \( t = t_o \) but rather a fictitious one which follows by substitution of \( t = t_o \) in Eq. (7). Since \( \alpha \) is unknown, \( H \) is also an unknown parameter, the value of which becomes known as a result of the solution process.

The scaled boundary conditions are obtained from (2), (3), and (8):

\[ T=1, \quad \frac{dT}{dR}=0 \quad \text{at} \quad R=0, \tag{15} \]

\[ T=t_o t, = \beta \quad \text{at} \quad R=1. \tag{16} \]

Since \( t_o \) is much higher than the expected maximum arc temperatures, we have \( T_i \ll 1 \). This is the reason why the problem defined by (10), (15), and (16) can be treated by asymptotic methods. To facilitate the analysis we write

\[ T=\left( 1 - \frac{7}{4} \frac{Q}{T_i} \right)^{4/7}, \quad R=Z(HT_i)^{-1/2}. \tag{17} \]

After some manipulation, retaining leading-order terms in (10) only, we arrive at

\[ \frac{1}{Z} dZ Z \frac{dQ}{dZ} = e^{-Q}, \tag{18} \]

with

\[ Q=0 \]

and

\[ dQ/DZ=0 \quad \text{at} \quad Z=0, \tag{19} \]

which is solved by

\[ Q=2 \ln \left( 1 + \frac{1}{2}Z^2 \right). \tag{20} \]

The remaining boundary condition (16) is rewritten in terms of \( Z \) and \( Q \):

\[ Q=\frac{4}{7} T_i \left( 1 - \beta^{7/4} \right) \tag{21} \]

\[ Z=\left( HT_i \right)^{1/2}. \tag{22} \]

Substituting (21) in (20), and using the fact that \( HT_i \gg 1 \), we obtain

\[ H=8 T_i^{-1} \exp \left[ (2/7) T_i \left( 1 - \beta^{7/4} \right) \right]. \tag{23} \]

From (17) and (20) we may derive a first-order approximation for \( f(T_i) \) defined by Eq. (13):

\[ f(T_i) = 4 T_i^{5/4} \int_0^1 RT^{3/4} \exp \left[ T_i \left( 1 - \frac{1}{T} \right) \right] dR \tag{24} \]

which leads to the following relationship between \( \alpha \) and \( T_i \):

\[ \alpha \approx 2 T_i^{-7/2} \exp \left[ -T_i \left( \frac{9}{7} - \frac{2}{7} \beta^{7/4} \right) \right] \tag{25} \]

Using (7) and (12) we may derive from (24) an expression for the arc current as follows:

\[ I = 2^{3/2} \pi a \gamma^{1/2} \lambda_i^{1/2} t_i^{-1/8} t_i^{7/4} \exp \left[ -T_i \left( \frac{9}{7} - \frac{1}{7} \left( \frac{T_i}{t_o} \right)^{7/4} \right) \right], \tag{26} \]

giving the total arc current as a function of the axis temperature \( t \), and the inner tube wall temperature \( t_o \). Of course, \( t \ll t_o \). Another result concerns the heat loss per unit length of tube, which is equal to \( IE \). Using (2), (17), (20), and (25), we may express this as follows:

\[ I E = 8 \pi a \lambda_i^{3/4} t_i^{-1}. \tag{27} \]

Since we have assumed a rule such as that of Eq. (7) for the thermal conductivity and have neglected radiation losses in (1), we are limited to examples in which the maximum temperatures are not too high. Taking property values as tabulated in Table III of Zollweg (see also Ref. 15), we may obtain results as listed in Table I. As expected, the current \( I \) increases rapidly with the axis temperature. To first order the heat loss is independent of the

<table>
<thead>
<tr>
<th>( t_o ) (K)</th>
<th>heat loss ( (W/cm/\text{m}) )</th>
<th>( I_{t_o=1000 \text{K}} ) (A)</th>
<th>( I_{t_o=2000 \text{K}} ) (A)</th>
</tr>
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<tr>
<td>4000</td>
<td>3.7</td>
<td>0.11</td>
<td>0.16</td>
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<tr>
<td>4500</td>
<td>4.7</td>
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<td>1.94</td>
<td>2.30</td>
</tr>
<tr>
<td>6000</td>
<td>11.7</td>
<td>3.82</td>
<td>4.38</td>
</tr>
</tbody>
</table>
FIG. 1. Dimensionless representation of total arc current ($I$) vs axis temperature ($t_a$) for two different wall temperatures. ($t_w$) ionization temperature, (a) inner lamp radius, (γ) Saha parameter, see Eq. (6), (λ) fictitious thermal conductivity at $t_w$, see Eq. (7).

The current $I$, on the other hand, depends moderately upon $t_{wall}$. This dependence is strongest, in a relative sense, at low lamp powers. A dimensionless representation is given in Fig. 1. Again, it is clear that the dependence of the current on $t_{wall}$ is fairly weak. Further, it is interesting to note that $I$ is proportional to the lamp diameter and that the heat loss increases as the 11/4th power of the axis temperature [because of Eqs. (7) and (26)].

The asymptotic formalism put forward here can also be used for more complex cases which include radiative losses and thermal conductivities which also represent internal radiative heat transfer processes. Work is in progress to consider these more complex cases. These will allow us to consider arc temperatures higher than those put forward in this note.

1. W. Elenbaas, Physica 1, 673 (1934).