THE $H_\infty$ CONTROL PROBLEM:
A STATE SPACE APPROACH

A.A. STOORVOGEL
The $H_\infty$ control problem: 
a state space approach

PROEFSCHRIFT

der verkrijging van de graad van doctor aan de Technische 
Universiteit Eindhoven, op gezag van de Rector Magnificus, 
prof. ir. M. Tels, voor een commissie aangewezen door het College van Dekanen in het openbaar te verdedigen op 
dinsdag 9 oktober 1990 te 16.00 uur.

door
ANTONIE ARIJ STOORVOGEL

geboren te Hengelo (O).
Dit proefschrift is goedgekeurd door de promotoren:

Prof.dr.ir. M.L.J. Hautus,
Prof.dr. R.F. Curtain.

Copromotor: Dr. H.L. Trentelman.
Acknowledgements

The research leading to this thesis has been carried out at the Mathematics department of the Eindhoven University of Technology from October 1, 1987 till August 1990. During these years several people contributed to my efforts to mature into a research scientist.

First of all, I would like to mention my promotor Malo Hautus and my copromotor Harry Trentelman. During my years in Eindhoven their doors were always open. Both of them gave me a lot of inspiration and assistance with my first research efforts. Each one of them gave help as a friend instead of simply being my boss. I am very grateful to both of them. I am also deeply in dept with my second promotor Ruth Curtain whose optimism and guidance in the last years is appreciated.

I would like to thank all the members of the reading committee, besides the already mentioned persons David Limebeer and Ton Backx, for reading my thesis and for the stimulating discussions and remarks regarding this thesis. Next, I would like to mention my roommates. At first I shared an office with Jacob Engwerda and later with Han Soethoudt. Both of them endured me as a roommate. Moreover they gave me the oppertunity of discussing problems of mutual interest which helped me to widen my horizon.

The other members of the system theory group: Kees Praagman, Marc Peters, Zhu Siquan, Luc Habets, Ton Geerts, Cees Withagen and Jan van Geldrop were always friendly and helpful. Kees Praagman and Zhu Siquan helped me on my first steps in the area of infinite dimensional systems theory and were always good colleagues. Marc Peters was my first masters student and is now a respected Ph.D. student and friend.

Also, outside the Eindhoven University of Technology a lot of people helped me and gave suggestions to improve my work. For a large part during the lectures of the graduate program of the Dutch Network on Systems and Control theory. I would like to thank Jan Willems. He was the driving force behind the graduate program and also in personal contact he is a man who is
Acknowledgements

extremely stimulating for everybody in his environment. Moreover, I would like to thank all the lecturers of the graduate program. Their lectures gave me a thorough foundation for my future research.

Before I became a Ph.D. student I did my masters work at the Philips Research Laboratories in Eindhoven. This was a very stimulating time and here I decided that I wanted to go on in research. This is mainly due to one man Guido Janssen, who has been extremely helpful and stimulating during this period. I would to thank him once again at this point.

Last but not least, I would like to thank my family. Both my parents as well as my brother were always understanding and helpful in the more personal problems of three years of very hard work.
Contents

1 Introduction
  1.1 Robustness analysis ........................................ 1
  1.2 The $H_\infty$ control problem ................................. 3
  1.3 Stabilization of uncertain systems .............................. 4
  1.4 The graph topology ........................................... 6
  1.5 The mixed-sensitivity problem .................................. 8
  1.6 Main items of this thesis ..................................... 10
    1.6.1 Singular systems ........................................ 11
    1.6.2 Differential game ........................................ 13
    1.6.3 Discrete time systems ..................................... 13

2 Notation and basic properties
  2.1 Introduction .................................................. 15
  2.2 Linear systems ................................................ 15
    2.2.1 Continuous time ........................................ 15
    2.2.2 Discrete time ............................................ 19
  2.3 Rational matrices ............................................. 20
  2.4 Geometric theory .............................................. 22
  2.5 $H_\infty$ theory ............................................... 27
    2.5.1 Continuous time ........................................ 27
    2.5.2 Discrete time ............................................ 33
  2.6 (Almost) Disturbance Decoupling Problems ..................... 36

3 The regular $H_\infty$ control problem: the full-information case 45
  3.1 Introduction .................................................. 45
  3.2 Problem formulation and main results .......................... 46
  3.3 Intuition for the formal proof ................................ 50
  3.4 Solvability of the Riccati equation .......................... 53
  3.5 Existence of a suitable controller ............................ 63
4 The general full-information $H_\infty$ control problem
4.1 Introduction ............................................................................................................. 67
4.2 Problem formulation and main results ...................................................................... 68
4.3 Solvability of the quadratic matrix inequality .......................................................... 72
4.4 Existence of state feedback laws .............................................................................. 73
4.5 A direct feedthrough matrix from disturbance to output ......................................... 78
4.6 Invariant zeros on the imaginary axis ...................................................................... 83
   4.6.1 Frequency domain loop shifting ................................................................. 83
   4.6.2 Cheap control ......................................................................................... 86
4.7 Conclusion ............................................................................................................ 88

5 The $H_\infty$ control problem with measurement feedback ........................................ 89
5.1 Introduction ............................................................................................................ 89
5.2 Problem formulation and main results .................................................................... 93
5.3 Reduction of the original problem to an almost disturbance decoupling problem ........................................................................................................... 98
5.4 The design of a suitable compensator .................................................................... 104
5.5 No assumptions on any direct feedthrough matrix .................................................. 107
   5.5.1 An extra direct feedthrough matrix from disturbance to output ..................... 107
   5.5.2 An extra direct feedthrough matrix from control to measurement ................. 111
5.6 Conclusion ............................................................................................................ 113

6 The singular zero-sum differential game with stability ........................................ 115
6.1 Introduction ............................................................................................................ 115
6.2 Problem formulation and main results ................................................................... 116
6.3 Existence of almost equilibria ............................................................................... 120
6.4 Necessary conditions for the existence of almost equilibria ................................... 125
6.5 The regular differential game ............................................................................... 127
6.6 Conclusions ......................................................................................................... 132

7 The discrete time $H_\infty$ control problem: the full-information case ....................... 133
7.1 Introduction .......................................................................................................... 133
7.2 Problem formulation and main results ................................................................... 134
7.3 Existence of a stabilizing solution of the Riccati equation ..................................... 139
7.4 Sufficient conditions for the existence of suboptimal controllers ......................... 150
7.5 Discrete algebraic Riccati equations ..................................................................... 152
7.6 Concluding remarks .............................................. 155

8 The discrete time $H_\infty$ control problem with measurement feedback .............................. 157
  8.1 Introduction ................................................ 157
  8.2 Problem formulation and main results ...................... 158
  8.3 A first system transformation .................................. 162
  8.4 The transformation into a disturbance decoupling problem with measurement feedback .......... 167
  8.5 Conclusions ................................................ 169

9 Conclusion ......................................................... 171
  9.1 Summary of results obtained .................................. 171
    9.1.1 Singular systems .......................................... 171
    9.1.2 Differential games ......................................... 173
    9.1.3 Discrete time systems ...................................... 174
  9.2 Application of our results to certain robustness problems ........................................ 175
    9.2.1 Additive perturbations ..................................... 175
    9.2.2 Multiplicative perturbations ................................ 178
    9.2.3 Perturbations in the realization of a system ................ 181
  9.3 Open problems ................................................ 183
    9.3.1 Invariant zeros ............................................ 183
    9.3.2 Practical applications ...................................... 185
    9.3.3 Mixed $H_\infty$/... problems ................................ 187
  9.4 Conclusion of this thesis ...................................... 188

A Preliminary basis transformations .................................................. 191
  A.1 A suitable choice of bases .................................... 191
  A.2 The quadratic matrix inequality .............................. 198

B Proofs concerning the system transformations ............................................ 205
  B.1 Proof of lemma 5.4 ............................................. 205
  B.2 Proof of lemma 5.5 ............................................. 208

Samenvatting .......................................................... 215

Bibliography .......................................................... 217

Index ................................................................. 227
Chapter 1

Introduction

In this thesis, we study the $H_\infty$ control problem. In this first chapter, we introduce a number of natural problems whose solutions depend on the results of $H_\infty$ control. This chapter will be more or less a justification of the extensive research effort in the area of $H_\infty$ control, and in particular, why we have written a thesis on this subject.

In section 1.1 we formulate and explain the importance of robustness and, in section 1.2, we sketch very briefly the basic $H_\infty$ control problem. The section on robustness outlines the importance of the problems of uncertain systems and the gap metric, which are explained in reasonable detail in sections 1.3 and 1.4, respectively. These problems are formulated with the same objective of improving the robustness although from a completely different point of view and have both a solution which is intimately connected to $H_\infty$ control. We shall also explain the mixed sensitivity problem in this chapter. This is in essence already formulated as an $H_\infty$ control problem and it is practically the only $H_\infty$ control problem which has already been applied in practice. To conclude this chapter we shall outline the $H_\infty$ control problems we shall treat in this thesis.

1.1 Robustness analysis

Control theory is concerned with the control of processes with inputs and outputs. We would like to know how we can achieve a desired goal we have for the output of our plant by choosing our inputs.
Example 1.1: Assume that we have a paper making machine. This machine has certain inputs: wood-pulp, water, pressure and steam. The wood-pulp is diluted with water. Then the fibres are separated from the water and a web is formed. Water is pressed out of the mixture and the paper is then dried on steam-heated cylinders (this is of course a very simplified view of the process). The product of the plant is the paper. More precisely we have two outputs: the thickness of the paper and the mass of fibres per unit area (expressing the quality of the paper). We would like both outputs to be equal to some desired value. That is, we have a process with a number of inputs and two outputs: the deviation from the desired values of the thickness and of the mass of fibres per unit area of the paper produced which we would like to make as small as possible.

The first step is to find a mathematical model describing the behaviour of our plant. The second step is to use mathematical tools to find suitable inputs for our plant based on measurements we make of all, or of a subset, of our outputs. However, we apply these inputs to our plant and not to our model. Since our model should be simple enough for the mathematical tools of step 2 (for instance in this thesis we require that the model be linear) the model will not describe the plant exactly. Because we do not know how sensitive our inputs are with respect to the differences between model and plant, the obtained behaviour might differ significantly from the mathematically predicted behaviour. Hence our inputs will in general not be suitable for our plant and the behaviour we obtain can be completely surprising.

Therefore it is extremely important that, when we search for a control law for our model, we keep in mind that our model is far from perfect. This leads to the so-called robustness analysis of our plant and suggested controllers. Robustness of a system says nothing more than that the stability of the system (or another goal we have for the system) will stand against perturbation (structured or unstructured, depending on the circumstances).

The classical approach to this problem from the 1960's was the Linear Quadratic Gaussian (LQG) approach. In that approach the uncertainty is modelled as a white noise Gaussian process added as extra inputs to the system. The major problem of this approach is that our uncertainty cannot always be modelled as white noise. While measurement error can be quite well described by a random process, this is not the case with parameter uncertainty. If we model $a = 0.9$ instead of $a = 1$, then the error is not random but deterministic. The only problem is that the deterministic error is
1.2 The $H_\infty$ control problem

unknown. Another problem of main importance with parameter uncertainty is that uncertainty in the transfer from inputs to outputs cannot be modelled as state or output disturbances, i.e. extra inputs. This is due to the fact that the size of the errors is relative to the size of the inputs and can hence only be modelled as an extra input in a non-linear framework.

In the last few years several approaches to robustness have been studied mainly for one goal: to obtain internal stability, where instead of trying to obtain this for one system, it is required to obtain internal stability for a class of systems simultaneously. It is then hoped that a controller which stabilizes all elements of this class of systems also stabilizes the plant itself.

In this chapter two approaches to this problem will be briefly discussed. Both approaches are in a linear, time-invariant setting and result in an $H_\infty$ control problem.

1.2 The $H_\infty$ control problem

We now state the $H_\infty$ control problem. Assume that we have a system $\Sigma$:

$$
\begin{array}{c}
z \\
y \\
\Sigma \\
w \\
u
\end{array}
$$

(1.1)

We assume $\Sigma$ to be a linear time-invariant system either in continuous time or in discrete time. We note that $\Sigma$ is a system with two kinds of inputs and two kinds of outputs. The input $w$ is an exogenous input representing the disturbance acting on the system. The output $z$ is an output of the system, whose dependence on the exogenous input $w$ we want to minimize. The output $y$ is a measurement we make on the system, which we shall use to choose our input $u$, which in turn is the tool we have to minimize the effect of $w$ on $z$. A constraint we impose is that this mapping from $y$ to $u$ should be such that the closed loop system is internally stable. This is quite natural since we do not want that the states become too large while we try to regulate our performance. The effect of $w$ on $z$ after closing the loop is measured in terms of the energy and the worst disturbance $w$. Our measure, which will turn out to be equal to the closed loop $H_\infty$ norm, is the supremum over all disturbances unequal to zero of the quotient of the energy flowing out of the system and the energy flowing into the system. A more precise definition is given in chapter 2.
Introduction

Note that this problem formulation in itself does not have any connection with robustness:

Example 1.2: Assume that we have the following system:

\[
\begin{align*}
\dot{x} &= -u + w, \\
\Sigma: \quad y &= x, \\
&z = u.
\end{align*}
\]

It can be checked that a feedback law which minimizes the effect of \( w \) on \( z \) in the above sense is given by

\[ u = \varepsilon x \]

where \( \varepsilon \) is a very small positive number. On the other hand it is easily seen that a small perturbation of the system parameters might yield a closed loop system which is unstable. Hence for this controller, the internal stability of the closed loop system is certainly not robust with respect to perturbations of the state matrix.

\[ \square \]

1.3 Stabilization of uncertain systems

As already mentioned, a method for tackling the problem of robustness is to treat the uncertainty as additional input(s) to the system. The LQG design method treats these inputs as white noise and we noted that parameter uncertainty is not suited to be treated as white noise. Also the idea of treating the error as extra inputs was not suitable because the size of the error might be relative to the size of the inputs. This yields an approach where parameter uncertainty is modelled as a disturbance system taking values in some range and modelled in a feedback setting (which allows us to incorporate the "relative" character of the error). We would like to know the effect with respect to stability of the "worst" disturbance in the prescribed parameter range. (We want guaranteed performance so even if the worst happens, then it should still be acceptable.) If this disturbance can not destabilize the system, then we are certain (under the assumption that the plant is exactly described by a system we obtain for some value of the parameters in the prescribed range) that the plant is stabilized by our control law.

It is easily verified that parameter uncertainty in a linear setting can very often be modelled as:
1.3 Stabilization of uncertain systems

\[
\begin{align*}
\Sigma_K & \quad \begin{cases}
    z = w \\
    y = u
\end{cases}
\end{align*}
\]

Here the system \( \Sigma_K \) represents the uncertainty and if the transfer matrix of \( \Sigma_K \) is zero, then we obtain our nominal model from \( u \) to \( y \). The system \( \Sigma_K \) might contain uncertainty of parameters or the ignored dynamics after model reduction. The goal is to find a feedback law which stabilizes the model for a large range of systems \( \Sigma_K \). In chapter 9 we shall give some examples of different kinds of uncertainties which can be modelled in the above sense. We shall also show how the results of this thesis applied to these problems look like. At this point, we only show by means of example that indeed a large class of parameter uncertainty can be considered as an interconnection of the form (1.2).

**Example 1.3**: Assume that we have a single-input, single-output system with two unknown parameters:

\[
\Sigma_n : \begin{cases}
    \dot{x} = -ax + bu, \\
y = x.
\end{cases}
\]

where \( a \) and \( b \) are parameters with values in the ranges \([a_0 - \epsilon, a_0 + \epsilon]\) and \([b_0 - \delta, b_0 + \delta]\) respectively. We can consider this system as an interconnection of the form (1.2) by choosing the system \( \Sigma \) to be equal to:

\[
\Sigma : \begin{cases}
    \dot{x} = -a_0x + b_0u + w, \\
y = x, \\
z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u.
\end{cases}
\]

and the system \( \Sigma_K \) to be the following static system:

\[
w = \begin{pmatrix} a - a_0 & b - b_0 \end{pmatrix} z.
\]

It is easily seen that by scaling we may assume that \( \epsilon = \delta = 1 \). \( \square \)
If we want to use some special structure of the system $\Sigma_K$ (e.g. that $\Sigma_K$ is static and not dynamic as in the above example), then we have to resort to the so-called $\mu$-synthesis of J. Doyle (see [Do2, Do3]) or to the theory of real stability radii (see [Hi]). $\mu$-synthesis has the disadvantage that this method is so general that, at this moment, no reasonably efficient algorithms are available to improve robustness via $\mu$-synthesis. The same is true for the related approach of working with real stability radii.

On the other hand, if we want to find a controller from $y$ to $u$ such that the closed loop system is stable for all stable systems $\Sigma_K$ with $H_\infty$ norm less than $\gamma$, then it has been shown (see [Hi]) that the problem is equivalent to the following problem: find a controller which is such that the closed loop system (if the transfer matrix of $\Sigma_K$ is zero) is internally stable and the $H_\infty$ norm from $w$ to $z$ is strictly less than $\gamma^{-1}$. This problem can be solved using the techniques given in this thesis.

1.4 The graph topology

When we treat parameter uncertainty as we did in the previous section then we need to know explicitly how this uncertainty is structured in our system. In many practical cases we do not have such information. It will turn out that for these cases a different approach is needed.

We could formulate the problem abstractly as follows: assume that some nominal plant is given and some controller stabilizing the plant. Does the same controller also stabilize systems which are "close" to our nominal plant?

The problem in this formulation is how to define the concept of distance between two systems. Since systems are in fact nothing else than input-output operators a natural distance concept would be the induced operator norm. In this view an unstable system must be seen as an unbounded operator and hence this distance concept cannot be used to define distances between unstable systems. On the other hand, it is very well possible that our plant is unstable and hence we need a concept of distance which is still valid in the case that one or both of the systems are unstable. The graph topology turns out to yield a useful distance concept for these cases.

When do we call two systems close to each other? Firstly, if we have a controller which internally stabilizes one system, then it should also be internally stabilizing for the other system. Secondly, if we apply the same stabilizing controller to each one of the systems, then the closed loop systems should be close to each other, measured in the induced operator norm. Note
that both closed loop systems are stable and hence this operator norm is always finite. The first requirement is natural because we consider robustness of internal stability. The second requirement is added because if we design our controller to satisfy other performance criteria besides stability, then we require that performance is preserved under small perturbations.

Next, we shall formalize the above intuitive reasoning. We define the graph topology for the set of linear time-invariant and finite-dimensional systems. A sequence of systems \( \{\Sigma_n\} \) is said to converge to \( \Sigma \) in the graph-topology if the following holds:

- For every internally stabilizing controller \( \Sigma_F \) applied to \( \Sigma \), there exists a natural number \( N \) such that for all \( n > N \), the controller \( \Sigma_F \) internally stabilizes \( \Sigma_n \).

- For every internally stabilizing controller \( \Sigma_F \) applied to \( \Sigma \) denote the closed loop operator by \( G_F \) and the closed loop operator obtained by applying \( \Sigma_F \) to \( \Sigma_n \) by \( G_{F,n} \). Then

\[
\|G_F - G_{F,n}\|_\infty \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Here \( \|\cdot\|_\infty \) denotes the induced operator norm which will be formally defined in section 2.5.

The graph topology has been introduced in [Vi]. In [Zh] it was shown that this topology is always equal to the gap topology (in [Zh] this is also shown for more general classes of infinite dimensional systems). The gap topology is yet another topology applicable to define convergence between possibly unstable plants. However, the gap topology is metrizable and hence we can now really discuss the distance between two plants. Note that the concept of coprime-factor perturbations as used in [Gl4, MF] again yields the same topology (see [Vi], in [GS] it was shown that the gap metric is related to so-called normalized coprime factor perturbations).

Now, the problem of finding a maximally robust controller in this setting is defined as follows: for each internally stabilizing controller of the nominal plant we search for the distance to the nearest system in the gap metric which is not stabilized by our controller. This is called the stability margin of the controller. Finally we search for the controller with the largest stability margin. The above problem has been reduced in [GS, Hab, Zh] to an \( H_\infty \) control problem.
1.5 The mixed-sensitivity problem

The mixed-sensitivity problem is a special kind of \( H_\infty \) control problem. In the mixed-sensitivity problem it is assumed that the system under consideration can be written as the following interconnection where \( \Sigma_F \) is the controller which has to satisfy certain prerequisites.

\[
\begin{array}{ccc}
\Sigma_W & \Sigma_V \\
\Sigma_F & \Sigma & \Sigma_W \\
r & u & y \\
+ & + & + \\
- & - & - \\
z_1 & z_2 & w \\
\end{array}
\]  

(1.3)

Many \( H_\infty \) control problems can be formulated in terms of an interconnection of the form (1.3). We shall show as an example how the tracking problem can be formulated in the setting described by the diagram (1.3). We first look at the following interconnection:

\[
\begin{array}{ccc}
\Sigma_F & \Sigma \\
r & u & y \\
+ & + & + \\
- & - & - \\
\end{array}
\]  

(1.4)

The problem is to regulate the output \( y \) of the system \( \Sigma \) to look like some given reference signal \( r \) by designing a precompensator \( \Sigma_F \) which has as its input the error signal, i.e. the input of the controller is the difference between the output \( y \) of \( \Sigma \) and the reference signal \( r \). To prevent undesirable surprises we require internal stability. We could formulate the problem as "minimizing" the transfer function from \( r \) to \( r - y \). As one might expect we shall minimize the \( H_\infty \) norm of this transfer function under the constraint of internal stability. Also the transfer matrix from \( r \) to \( u \) should be under consideration. In practice the process inputs will often be restricted by
1.5 The mixed-sensitivity problem

physical constraints. This yields a bound on the transfer matrix from \( r \) to \( u \). These transfer matrices from \( r \) to \( r - y \) and from \( r \) to \( u \) are given by:

\[
S := (I - G_F G)^{-1},
\]

\[
T := G_F (I - G_F G)^{-1},
\]

respectively where \( G \) and \( G_F \) denote the transfer matrices of \( \Sigma \) and \( \Sigma_F \). Here

\( S \) is called the sensitivity function and \( T \) is called the control sensitivity function. A small function \( S \) expresses good tracking properties while a small function \( T \) expresses small inputs \( u \). Note that there is a trade-off: making \( S \) smaller will in general make \( T \) larger. We add a signal \( w \) to the output \( y \) as in (1.3). Then the transfer matrix from \( w \) to \( y \) is equal to the sensitivity matrix \( S \) and the transfer matrix from \( w \) to \( u \) is equal to the control sensitivity matrix \( T \).

As noted in section 1.2 the \( H_{\infty} \) norm can be viewed as the maximum amount of energy coming out of the system, subject to inputs with unit energy. However, if we apply the Laplace transform, then we obtain a frequency domain characterization. For a single-input, single-output stable system the \( H_{\infty} \) norm is equal to the largest distance of a point on the Nyquist contour to the origin. Hence the \( H_{\infty} \) norm is a uniform bound over all frequencies on the transfer function. Although we assume the tracking signal to be, a priori, unknown, it might be that we know that our tracking signal will have a limited frequency spectrum. It is in general impossible to track signals of very high frequency reasonably well. On the other hand, in practice it is often sufficient to track signals of frequencies up to a certain bandwidth and we do not have to worry about tracking signals of very high frequency. In this situation straightforward application of \( H_{\infty} \) control might yield conservative results because it only investigates a uniform bound over all frequencies. Also for the error signal and the control input certain frequencies may be more important than others.

Thus in diagram (1.3), the systems \( \Sigma_{W_1}, \Sigma_{W_2} \) and \( \Sigma_V \) are weights which are chosen in such a way that we put more effort in regulating frequencies of interest instead of one uniform bound. For practical purposes the choice of these weights is extremely important. For single-input, single-output systems expressing performance criteria into requirements on the desired shape of the magnitude Bode plot is rather straightforward. This immediately translates into the appropriate choice for the weights. On the other hand for multi-input, multi-output systems it is in general very hard to translate practical performance criteria into an appropriate choice for the weights.
In this way, we obtain the interconnection (1.3). Note that the transfer matrix from the disturbance $\tilde{w}$ to $z_1$ and $z_2$ is

$$
\begin{pmatrix}
G_{W_1} & TG_V \\
G_{W_2} & SG_V
\end{pmatrix}
$$

where $G_{W_1}$, $G_{W_2}$ and $G_V$ are the transfer matrices of $\Sigma_{W_1}$, $\Sigma_{W_2}$ and $\Sigma_V$, respectively. Note that we can also use these weights to stress the relative importance of minimizing the sensitivity matrix $S$ with respect to the importance of minimizing the control sensitivity matrix $T$ by multiplying $G_{W_1}$ by a scalar.

We want to find a controller which minimizes the $H_{\infty}$ norm of the transfer matrix (1.5) and which yields internal stability. This problem can be solved using the techniques we present in this thesis.

1.6 Main items of this thesis

As already mentioned, this thesis will deal with several aspects of $H_{\infty}$ control. In recent years a large amount of papers has been published on this subject and in this section, we want to describe briefly the new contributions this thesis makes to the existing theory. We shall consider the time-domain approach to $H_{\infty}$ control, which has received a large impulse from the paper [Do4]. In this thesis, we investigate three aspects to this approach.

- Singular systems
- Differential games
- Discrete time systems

We first introduce some notation and give some preliminary results in chapter 2. In chapter 3, we give some results on the state feedback $H_{\infty}$ control problem of which the main results were already basically known in the literature. We have added this chapter for the sake of completeness and in order to have the results available for the rest of the thesis. In chapters 4 and 5 we extend the known $H_{\infty}$ theory to so-called singular systems. In chapter 6 we discuss the differential game and its relation to $H_{\infty}$ control. Then, in chapters 7 and 8 we investigate the $H_{\infty}$ control problem for discrete time systems. Finally chapter 9 contains a number of concluding remarks. Appendix A gives the details of the state decomposition on which the proofs.
1.6 Main items of this thesis

of our results for singular systems are based. Appendix B shows the proofs of two technical lemmas from chapter 5.

We shall discuss the three main subjects of this thesis in some detail in the next three subsections.

1.6.1 Singular systems

In the paper [Do4] linear finite-dimensional time-invariant systems were considered which satisfy two kinds of essential assumptions.

• The subsystem from the control input to the output should not have invariant zeros on the imaginary axis and its direct feedthrough matrix should be injective.

• The subsystem from the disturbance to the measurement should not have invariant zeros on the imaginary axis and its direct feedthrough matrix should be surjective.

Invariant zeros are defined in chapter 2. For the moment it suffices to think of invariant zeros as values of \( s \) where the transfer matrix loses rank.

Throughout this thesis we shall make the same assumption with respect to invariant zeros, by excluding invariant zeros on the imaginary axis for both subsystems. A discussion of the difficulty of invariant zeros on the imaginary axis is given in chapter 9.

We define singular systems (contrary to regular systems) to be systems which do not satisfy at least one of the two above assumptions on the direct feedthrough matrices. In chapters 4 and 5 we shall extend the results from [Do4] to the class of singular systems.

These singular systems are more difficult to analyse. In the case that the direct feedthrough matrix from the control input to the output is not injective then either the system has an invariant zero at infinity or the subsystem from the control input to the output is not injective.

• An invariant zero at infinity. This is as difficult as invariant zeros on the imaginary axis. The problems with invariant zeros on the imaginary axis are explained in chapter 9. Basically the method of handling an invariant zero is to choose a controller which creates a pole in the same point. Then the invariant zero is cancelled via pole-zero cancellation. Because of our requirement of internal stability this is only possible for invariant zeros in the open left half plane. For invariant zeros in the open right half plane it is clearly not possible. In the case
of an invariant zero on the imaginary axis we can achieve this cancellation *approximately* by creating a pole in the left half plane which is very close to the imaginary axis. A treatment like this for invariant zeros on the imaginary axis is given in [HSK]. At the moment we are only interested in invariant zeros at infinity. In general, the so-called central controller from [Do4] will be non-proper in this case. Hence we indeed have a pole in infinity which "cancels" our invariant zero at infinity. However, it will turn out that we can indeed approximate this controller by a proper controller. It will be shown that the problem of this approximation can be reduced to the problem of almost disturbance decoupling. In this problem one is looking for conditions under which we can find internally stabilizing controllers which make the $H_\infty$ norm arbitrarily small. Since this problem has been solved in [Tr, We2] we could use these results.

- The system from the control input to the output is not injective. This implies that there are several inputs which have the same effect on the output. Using a geometric approach this non-uniqueness can be filtered out. This is done explicitly in [SC2]. We do the same in this thesis but more implicitly because we handle invariant zeros at infinity at the same time.

On the other hand, the subsystem from the disturbance to the measurement might not be surjective. The problems related to this fact play a completely dual role and we shall tackle these problems by relating them to problems of a dual system.

In [Do4] necessary and sufficient conditions are given for regular systems under which the existence of an internally stabilizing controller which makes the $H_\infty$ norm less than some, a priori given, number $\gamma > 0$ is guaranteed. These conditions are in terms of two algebraic Riccati equations. For singular systems these Riccati equations will be replaced by quadratic matrix inequalities. This is completely analogous to Linear Quadratic (LQ) optimal control where for singular systems the role of the algebraic Riccati equation is replaced by a *linear* matrix inequality.

Finally, a few words about the interest in singular systems. First of all, for mathematicians removing annoying assumptions is always of interest. Moreover, singular systems do arise in natural control problems, for example the problem of treating parameter uncertainty via the method discussed in section 1.3 will often yield $H_\infty$ control problems for singular systems. This is worked out in more detail in chapter 9. Another example is the problem
1.6 Main items of this thesis

of Loop Transfer Recovery (see [Ni]) which also yields \( H_\infty \) control problems for singular systems.

Finally, a major reason for looking at singular systems is that quite often in applications direct feedthrough matrices appear which are nearly singular. The results of [Do4] may still be applied but for deriving numerically reliable algorithms it is useful to know exactly what will happen in the case that the direct feedthrough matrices are not injective or not surjective anymore.

1.6.2 Differential game

As we mentioned in the previous subsection, the conditions under which we can make the \( H_\infty \) norm less than some, a priori given, number, are either in terms of the solutions of two algebraic Riccati equations or in terms of the solutions of two quadratic matrix inequalities. The solutions of these equations or inequalities have no direct meaning in \( H_\infty \) control. On the other hand, it would be good for the overall picture if we could understand the role of these solutions better.

We shall try to achieve this in chapter 6 for a special case. First of all we assume that we have a continuous time system. Secondly we only investigate the special case of state-feedback. In chapters 3 and 4 it will be shown that in this case we have only one Riccati equation or one quadratic matrix inequality.

It turns out that the theory of differential games yields the desired understanding of the role of the solution of either equation or inequality. The quadratic form associated with the solution of our Riccati equation turns out to be a Nash equilibrium for a differential game with a special cost-criterion. The quadratic form associated with the solution of our quadratic matrix inequality turns out to be an almost Nash equilibrium for a differential game with the same cost-criterion.

It is shown that being able to make the \( H_\infty \) norm strictly less than our, a priori given, bound \( \gamma \) is a sufficient condition for the existence of an (almost) Nash equilibrium. On the other hand, being able to make the \( H_\infty \) norm less than or equal to \( \gamma \) is a necessary condition for the existence of an (almost) Nash equilibrium. Note that the cost-criterion explicitly depends on our bound \( \gamma \).

1.6.3 Discrete time systems

Early results for the \( H_\infty \) control problem were derived for the continuous time case. Also in the first chapters of this thesis we shall only concern
ourselves with continuous time systems. However, in practical applications one is often concerned with discrete time systems.

One major reason is that to control a continuous time system one often applies a digital computer on which we can only implement a discrete time controller. One possible approach is to derive a continuous time $H_\infty$ controller and then discretize the controller to be able to use your computer. This approach is followed in papers like [Ch, Ch2].

Discretizing the system first and then using $H_\infty$ control designed for discrete time systems might be a more useful approach. This comparison can however only be made after the discrete time $H_\infty$ control problem has been solved. We shall not make this comparison in this thesis but leave it as a subject for future research.

Also certain systems are in itself inherently discrete and certainly for these systems it is useful to have results available for $H_\infty$ control problems.

One approach to solve the discrete time $H_\infty$ control problem, is to apply a transformation in the frequency-domain which transforms discrete time systems to continuous time systems. The transformation we have in mind is for instance discussed in [Gen, appendix 1]. With this transformation discrete time $H_\infty$ functions are mapped isometrically onto continuous time $H_\infty$ functions. One can then use the results available for continuous time systems and afterwards apply the inverse transformation on the controller thus obtained.

This transformation is however not always attractive. It maps systems with a pole in 1 into non-proper systems. Also it clouds the understanding of specific features of discrete time $H_\infty$ control because of its complexity. If it is possible to derive results for discrete time systems, why not apply these results directly instead of performing this unnatural transformation. Another problem is that the state feedback $H_\infty$ control problem is transformed into a continuous time measurement feedback $H_\infty$ control problem where indeed one might have problems observing the state.

Therefore, in chapters 7 and 8 we shall derive results for the discrete time state feedback $H_\infty$ control problem and the discrete time measurement feedback $H_\infty$ control problem, respectively. We shall only consider discrete time analogues of regular systems, and hence it might not be too surprising that our conditions are formulated in terms of discrete time algebraic Riccati equations.
Chapter 2

Notation and basic properties

2.1 Introduction

In this chapter we introduce the notation and definitions we shall use throughout this thesis. Moreover we give a number of basic properties we shall need further on. Most of these properties will not be proven here but we shall give appropriate references. In this thesis we deal with both discrete time systems as well as continuous time systems. Therefore, two sections of this chapter are split up into a discrete time part and a continuous time part in order to emphasize the differences. While reading this thesis, one should always keep in mind that we want to minimize the output with respect to the worst case disturbance. This goal is all we want to achieve and we shall try to achieve it under several different circumstances.

Let $\mathcal{R}$ denote the real numbers, $\mathbb{C}$ denote the complex numbers and let $\mathcal{N}$ denote the non-negative integers. Let $\mathbb{C}^+ (\mathbb{C}^0, \mathbb{C}^-)$ denote the set of all $s \in \mathbb{C}$ such that $\text{Re } s > 0 (\text{Re } s = 0, \text{Re } s < 0)$. Finally by $\mathcal{D} (\delta \mathcal{D}, \mathcal{D}^+)$ we denote the set of all $s \in \mathbb{C}$ such that $|s| < 1 (|s| = 1, |s| > 1)$.

2.2 Linear systems

2.2.1 Continuous time

Except for chapters 7 and 8 we shall investigate systems with continuous time. These systems are described by a differential equation and two output equations.
16 Notation and basic properties

\[
\dot{x} = Ax + Bu + Ew,
\]
\[
y = C_1x + D_{11}u + D_{12}w,
\]
\[
z = C_2x + D_{21}u + D_{22}w.
\]  
(2.1)

We shall always assume that \(x, u, w, y\) and \(z\) take values in finite dimensional vector spaces: \(x(t) \in \mathcal{R}^n\), \(u(t) \in \mathcal{R}^m\), \(w(t) \in \mathcal{R}^l\), \(y(t) \in \mathcal{R}^q\) and \(z(t) \in \mathcal{R}^p\). The system parameters \(A, B, E, C_1, C_2, D_{11}, D_{12}, D_{21}\) and \(D_{22}\) are matrices of appropriate dimensions. We assume that the system is time-invariant, i.e. the system parameters are independent of time. Except when stated explicitly, we shall always assume that the initial state is zero, i.e. \(x(0) = 0\).

The input \(w\) is the disturbance working on the system, whose effect on one of the outputs we want to minimize. The input \(u\) is the control which we use to achieve this goal. The output \(y\) is the measurement on the basis of which we choose our input \(u\). The output \(z\) is the output we want to make small relative to the size of the disturbance \(w\). More precisely, we are searching for a feedback from \(y\) to \(u\), denoted by \(\Sigma_F\), such that the closed loop system \(\Sigma \times \Sigma_F\) mapping \(w\) to \(z\) has a small induced norm. If all information of the system is available for feedback, i.e. \(y = (x, w)\), or if the state is available for feedback, i.e. \(y = z\), then we shall often delete the second equation in (2.1).

In several chapters the indices of the \(C\)- and \(D\)-matrices are different from the indices used in system (2.1). This is done to simplify the notation in the respective chapters. When comparing results from different chapters the reader should be careful whether these differences arise or not.

When we apply an input \(u\) and a disturbance \(w\) with initial condition \(x(0) = \xi\) then we shall denote by \(x_{u,w,\xi}\) and \(z_{u,w,\xi}\) the state and the output of system (2.1), respectively. In the case that we have zero initial condition we shall write \(x_{u,w}\) and \(z_{u,w}\) instead of \(x_{u,w,0}\) and \(z_{u,w,0}\).

Quite often, we shall look at two special subsystems, one in which we restrict attention to the system from \(u\) to \(z\):

\[
\Sigma_{ci} : \begin{cases} 
\dot{x} = Ax + Bu, \\
 z = C_2x + D_{21}u,
\end{cases}
\]  
(2.2)

and one in which we restrict attention to the system from \(w\) to \(y\):

\[
\Sigma_{di} : \begin{cases} 
\dot{x} = Ax + Ew, \\
y = C_1x + D_{12}w.
\end{cases}
\]  
(2.3)

If we only have one input and one output, as in the above two systems, then we can associate with a system the quadruple of the four system parameters.
For the sake of simplicity we shall often write, for instance, the system 
\((A, B, C_2, D_{21})\) when, formally, we should write down the system equations 
(2.2).

The system equations will always be denoted by a \(\Sigma\) with some index to 
identify different systems. The input-output operator mapping the inputs 
to the outputs with zero initial state will always be denoted by \(G\) with, again, 
some index to identify various operators. Finally, the transfer matrix of a 
system which, for instance for the system \(\Sigma_{ci}\), is defined by

\[
G_{ci}(s) := C_2 (sI - A)^{-1} B + D_{21},
\]

will always be denoted by \(G\) with some index.

We shall investigate three kinds of feedbacks. In order of increasing gener-
ularity: static state feedback, static feedback and dynamic output feedback. 
For the first two we shall always implicitly assume that the measurement is 
\(y = x\) and \(y = (x, w)\), respectively.

A dynamic output feedback is a system of the form

\[
\Sigma_F : \begin{cases} 
\dot{p} = Kp + Ly, \\
u = Mp + Ny.
\end{cases}
\]  

(2.4)

If \(D_{11} \neq 0\), then the interconnection might be ill-posed, i.e. the equations 
(2.1) and (2.4) together might not have a unique solution for given \(w\). This 
is clearly undesirable and hence if \(D_{11} \neq 0\), then we only consider controllers 
which make the interconnection well posed, i.e. controllers of the form (2.4) 
such that the equations (2.1) and (2.4) together have a unique solution for 
given \(w\). It is easily checked that a controller of the form (2.4) makes the 
interconnection well posed if and only if \(I - D_{11}N\) is invertible.

If the interconnection is well posed, then it is called internally stable if, 
with \(w = 0\), for every initial state of the system and every initial state of 
the controller the state of the system and the state of the controller in the 
interconnection converge to zero as \(t \to \infty\). If the controller is given by 
(2.4), the system is given by (2.1) and if the interconnection is well posed, 
then this is equivalent to the requirement that the matrix

\[
\begin{pmatrix}
A + BN(I - D_{11}N)^{-1}C_1 & B(I - ND_{11})^{-1}M \\
L(I - D_{11}N)^{-1}C_1 & K + L(I - D_{11}N)^{-1}D_{11}M
\end{pmatrix}
\]  

(2.5)

be asymptotically stable, i.e. all its eigenvalues lie in the open left half com-
plex plane. For systems with discrete time, asymptotic stability will have a
different meaning (see the next subsection). Moreover, note that if $D_{11} = 0$, then the interconnection is always well posed. In that case the matrix (2.5) simplifies considerably and is equal to

$$
\begin{pmatrix}
A + BNC_1 & BM \\
LC_1 & K
\end{pmatrix}.
$$

(2.6)

For a finite dimensional system $(A, B, C, D)$ we shall call the matrix $A$ the *state matrix* of the system. (see [Pr]). Accordingly, the matrix (2.5) or (2.6) will be referred to as the *closed-loop state matrix*. We shall call $B$ the *input matrix*, $C$ the *output matrix* and $D$ the *direct feedthrough matrix*.

After applying the compensator $\Sigma_F$ described by the static feedback law $u = F_1 x + F_2 w$ the closed loop transfer matrix is given by

$$
G_F(s) := (C_2 + D_{21}F_1)(sI - A - BF_1)^{-1}(E + BF_2) + (D_{22} + D_{21}F_2).
$$

(2.7)

Just as for dynamic compensators, the closed loop system is called internally stable if, with $w = 0$, for all initial states the state converges to zero as $t \to \infty$. It is easily checked that the closed loop system is internally stable if and only if the matrix $A + BF_1$ is asymptotically stable. This can also be derived from the dynamic feedback case discussed above by noting that

$$
y = \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w
$$

and that the matrix (2.6) then becomes equal to $A + BF_1$. Here we use that $K$ is a $0 \times 0$ matrix and hence disappears and $N = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$. A compensator described by a static state feedback law $u = F x$ can be considered as a special case of a static feedback law and the above definitions will be used correspondingly.

We shall say that matrices $P$ and $Q$ satisfy dual properties if $P$ satisfies a certain property for the system (2.1) if and only if $Q^T$ satisfies the same property for the dual system defined by

$$
\Sigma^T : \begin{cases}
\dot{z} = A^T x + C_1^T u + C_2^T w, \\
y = B^T x + D_{11}^T u + D_{12}^T w, \\
z = E^T x + D_{21}^T u + D_{22}^T w.
\end{cases}
$$
2.3 Rational matrices

2.2.2 Discrete time

In chapters 7 and 8 we shall investigate systems with discrete time. These systems are described by a difference equation and two output equations.

\[
\Sigma: \begin{cases}
\sigma x = Ax + Bu + Ew, \\
y = C_1x + D_{11}u + D_{12}w, \\
z = C_2x + D_{21}u + D_{22}w.
\end{cases}
\tag{2.8}
\]

where \(\sigma\) denotes the shift-operator, which is defined by

\[(\sigma x)(k) := x(k + 1).\]

All assumptions and definitions for continuous time as given in the previous subsection have an analogous meaning in discrete time. However, one point has to be discussed explicitly.

A dynamic output feedback is a system of the form

\[
\Sigma_F: \begin{cases}
\sigma p = Kp + Ly, \\
u = Mp + Ny.
\end{cases}
\tag{2.9}
\]

As in the previous section we define the interconnection to be internally stable if the interconnection is well posed and if, with \(w = 0\), for every initial state of the system and for every initial state of the controller the state of the system and the state of the controller in the interconnection converge to zero as \(t \to \infty\). If the controller is given by (2.9) and the system is given by (2.8) this is equivalent to the requirement that the matrix (2.5) is asymptotically stable, i.e. all its eigenvalues lie in the open unit disc.

Note that for systems with continuous time asymptotic stability of a matrix has a different meaning. That is, we still have the same matrix as in the continuous time case only this time its eigenvalues should be in the open unit disc instead of the open left half plane.

Later on we shall also need a backwards difference equation of the form

\[\sigma^{-1}x = Ax + Bu.\]

A function \(x\) from \(\mathcal{N} \cup \{-1\}\) to \(\mathbb{R}^n\) is said to be a solution of this backwards difference equation if

\[x(k - 1) = Ax(k) + Bu(k),\]

for all \(k \in \mathcal{N}\).
2.3 Rational matrices

In this section we recall some basic notions on rational matrices which are either applied to system matrices (which will be defined on the next page) or transfer matrices. Let \( \mathcal{R}[s] \) denote the ring of polynomials with real coefficients. Let \( \mathcal{R}^{n \times m}[s] \) be the set of all \( n \times m \) matrices with coefficients in \( \mathcal{R}[s] \). An element of \( \mathcal{R}^{n \times m}[s] \) is called a polynomial matrix. Linear polynomial matrices of the form \( sE - F \) are sometimes called a matrix pencil. \( \mathcal{R}(s) \) denotes the field of rational functions with real coefficients, i.e. \( \mathcal{R}(s) \) is the quotient field of \( \mathcal{R}[s] \). Let \( \mathcal{R}^{n \times m}(s) \) be the set of all \( n \times m \) matrices with coefficients in \( \mathcal{R}(s) \). An element of \( \mathcal{R}^{n \times m}(s) \) is called a rational matrix. A rational matrix \( G \) is called proper if \( \lim_{s \to \infty} G(s) \) exists, and strictly proper if this limit is zero. Moreover for a proper rational matrix \( \lim_{s \to \infty} G(s) \) is called the direct feedthrough matrix. Note that the direct feedthrough matrix of a system is equal to the direct feedthrough matrix of its transfer matrix.

By \( \text{rank}_K \) we denote the rank of a matrix as a matrix with entries in the field \( K \). We shall often write only \( \text{rank} \) in the case that \( K = \mathbb{R} \) or \( K = \mathbb{C} \). (note that for a real matrix the rank over \( \mathbb{C} \) always equals its rank over \( \mathbb{R} \).) Moreover, we often use the term normal rank for \( \text{rank}_K \) in the case that \( K = \mathcal{R}(s) \).

We shall first discuss a number of properties of polynomial matrices. A square polynomial matrix is called unimodular if it is invertible over the ring of polynomial matrices. Two polynomial matrices \( P \) and \( Q \) are called unimodularly equivalent if there exist unimodular matrices \( U \) and \( V \) such that \( Q = UPV \). In this thesis, we denote the fact that \( P \) and \( Q \) are unimodularly equivalent by \( P \sim Q \). It is well known (see [Ga]) that for any \( P \in \mathcal{R}^{n \times m}[s] \) there exists \( \Psi \in \mathcal{R}^{n \times m}[s] \) of the form

\[
\Psi = \begin{pmatrix}
\psi_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & & \\
\vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \psi_r & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

such that \( P \sim \Psi \). Here \( \psi_i \) are monic polynomials with the property that \( \psi_i \) divides \( \psi_{i+1} \) for \( i = 1, \ldots, r - 1 \).
2.3 Rational matrices

The polynomial matrix $\Psi$ is called the Smith form of $P$ (see [Ga]). The polynomials $\psi_i$ are called the invariant factors of $P$. Their product $\psi = \psi_1 \psi_2 \cdots \psi_r$ is called the zero polynomial of $P$. The roots of $\psi$ are called the zeros of $P$. The integer $r$ is equal to the normal rank of $P$ as defined before.

If $s$ is a complex number, then $P(s)$ is an element of $C^{n \times m}$. It is easy to see that $\text{rank}_{\mathbb{R}(s)} P = \text{rank} P(s)$ for all $s \in \mathbb{C}$ if and only if $P$ is unimodularly equivalent to the constant $n \times m$ matrix

$$
\begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix},
$$

where $I_r$ is the $r \times r$ identity matrix.

Next we recall some important facts on the structure of a linear system $\Sigma_{ci} = (A, B, C, D)$. The system matrix of $\Sigma_{ci}$ is defined as the polynomial matrix

$$
P_{ci} := \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}.
$$

The invariant factors of $P_{ci}$ are called the transmission polynomials of $\Sigma_{ci}$. The transmission polynomials unequal to 1 are called the non-trivial transmission polynomials of $P_{ci}$. The zeros of $P_{ci}$ are called the invariant zeros of $\Sigma_{ci}$. Clearly, $s \in \mathbb{C}$ is an invariant zero of $\Sigma_{ci}$ if and only if

$$
\text{rank} P_{ci}(s) < \text{rank}_{\mathbb{R}(s)} P_{ci}.
$$

It is easy to see that if $F \in \mathbb{R}^{m \times n}$ and if $P_{ci,F}$ is the system matrix of $\Sigma_{ci,F} := (A + BF, B, C + DF, D)$, then $P_{ci} \sim P_{ci,F}$. In particular, this implies that the transmission polynomials of $\Sigma_{ci}$ and $\Sigma_{ci,F}$ coincide and, a fortiori, that the invariant zeros of $\Sigma_{ci}$ and $\Sigma_{ci,F}$ coincide.

We can also associate with the system $\Sigma_{ci}$ the controllability pencil

$$
L(s) := \begin{bmatrix} sI - A & -B \end{bmatrix},
$$

and the observability pencil

$$
M(s) := \begin{bmatrix} sI - A \\ C \end{bmatrix}.
$$

A special case which is often investigated in $H_\infty$ control theory is the case that $D$ is injective and $C^T D = 0$. In that case, the invariant zeros of $\Sigma_{ci}$
are the zeros of $M$. That is, the invariant zeros of $\Sigma_{ci}$ are the unobservable eigenvalues of $(C, A)$.

A system $\Sigma_{ci} = (A, B, C, D)$ is called left (right) invertible if the transfer matrix of $\Sigma_{ci}$ is left (right) invertible as a rational matrix. This is equivalent to the requirement that the system matrix of $\Sigma_{ci}$ be left (right) invertible as a rational matrix. It can be shown that a system is left invertible if and only if the input-output operator associated with this system is injective. However, it is not true in general that right invertibility implies surjectivity of the input-output operator. We define the class of functions $C_0^\infty$ as the class of infinitely often differentiable functions on $[0, \infty)$ such that all derivatives are equal to zero in zero. Then the continuous time system $\Sigma_{ci}$ is right invertible if and only if the input-output operator as a map from $C_0^\infty$ to $C_0^\infty$ is surjective. A discrete time system is right invertible if all sequences $f$ with $f(0) = f(1) = \ldots = f(n) = 0$ are contained in the image of the input-output operator. Here $n$ is the dimension of the state space.

2.4 Geometric theory

In this section we recall some basic notions from the geometric approach to linear system theory. We shall use the geometric approach only for continuous time systems and one should note that although we could use the geometric approach for discrete time systems too, its usefulness for our problems is mainly restricted to continuous time systems.

We first introduce the important concepts of controlled invariance and conditioned invariance (see [SH, Wo]).

**Definition 2.1**: A subspace $V$ of $\mathcal{R}^n$ is said to be conditioned invariant (also called $(C, A)$-invariant) if there exists a linear mapping $G$ such that

$$(A + GC)V \subseteq V.$$ 

A subspace $V$ of $\mathcal{R}^n$ is said to be controlled invariant (also called $(A, B)$-invariant) if there exists a linear mapping $F$ such that

$$(A + BF)V \subseteq V.$$ 

Note that the alternative terminology of $(C, A)$-invariant and $(A, B)$-invariant is such that the only difference in the terminology for these two concepts
is the order of the matrices. One has to remember that these two properties are in fact associated to the system \((A, B, C, 0)\) where \((A, B)\)-invariance means that there exists a state feedback \(F\) which makes the subspace \(V\) invariant under the closed loop state matrix \(A + BF\) while \((C, A)\)-invariance means that there exists an output injection \(G\) which makes this subspace invariant under the state matrix \(A + GC\). Finally we would like to remark that these properties are dual to each other. \(V\) is \((C, A)\)-invariant if and only if \(V^\perp\) is \((A^t, C^t)\)-invariant.

The following well-known lemma is a convenient tool for checking whether these properties hold for a certain subspace (see [SH, Wo]).

Lemma 2.2: A subspace \(V\) of \(\mathbb{R}^n\) is \((C, A)\)-invariant if and only if
\[
A (V \cap \text{ker}C) \subseteq V.
\]
A subspace \(V\) of \(\mathbb{R}^n\) is \((A, B)\)-invariant if and only if
\[
AV \subseteq V + \text{im}B.
\]

We now define a number of particular linear subspaces of the state space, among which the strongly controllable subspace. The latter will play a key role throughout this thesis.

Definition 2.3: Consider the system
\[
\Sigma_{ci} : \begin{cases} 
  \dot{x} = Ax + Bu, \\
  z = Cx + Du.
\end{cases}
\]
We define the strongly controllable subspace \(T(\Sigma_{ci})\) as the smallest subspace \(T\) of \(\mathbb{R}^n\) for which there exists a linear mapping \(G\) such that:
\[
(A + GC)T \subseteq T, \quad \text{(2.11)}
\]
\[
\text{Im} (B + GD) \subseteq T. \quad \text{(2.12)}
\]
We also define the detectable strongly controllable subspace \(T_g(\Sigma_{ci})\) as the smallest subspace \(T\) of \(\mathbb{R}^n\) for which there exists a linear mapping \(G\) such that (2.11) and (2.12) are satisfied and moreover \(A + GC\mid \mathbb{R}^n/T\) is asymptotically stable.

A system is called strongly controllable if its strongly controllable subspace is equal to the whole state space.
We also define the dual versions of these subspaces:

**Definition 2.4:** Consider the system (2.10). We define the weakly unobservable subspace \( \mathcal{V}(\Sigma_{ci}) \) as the largest subspace \( \mathcal{V} \) of \( \mathbb{R}^n \) for which there exists a mapping \( F \) such that:

\[
(A + BF)\mathcal{V} \subseteq \mathcal{V},
\]

\[
(C + DF)\mathcal{V} = \{0\}.
\]

We also define the stabilizable weakly unobservable subspace \( \mathcal{V}_g(\Sigma_{ci}) \) as the largest subspace \( \mathcal{V} \) for which there exists a mapping \( F \) such that (2.13) and (2.14) are satisfied and moreover \( A + BF | \mathcal{V} \) is asymptotically stable.

A system is called strongly observable if its weakly unobservable subspace is equal to \( \{0\} \).

We can give intuitive interpretations of these subspaces. \( \mathcal{V}(\Sigma_{ci}) \) is the subspace of all \( x_0 \in \mathbb{R}^n \) such that for the system (2.10) with initial condition \( x(0) = x_0 \) there exists an input function \( u \) on \([0, \infty)\) such that the output function \( y \) of the system is identical zero on \([0, \infty)\). \( \mathcal{V}_g(\Sigma_{ci}) \) has the same interpretation but with the extra constraint on the input \( u \) that the resulting state trajectory \( x(t) \) converges to 0 as \( t \rightarrow \infty \).

On the other hand \( \mathcal{T}(\Sigma_{ci}) \) consists of all \( x_0 \in \mathbb{R}^n \) such that for the system (2.10) with initial condition \( x(0) = x_0 \) and for all \( \varepsilon > 0 \) there exists \( T > 0 \) and an input function \( u \) such that the resulting state satisfies \( x(T) = 0 \) while the \( L_1 \)-norm of the output \( y \) is less than \( \varepsilon \), i.e. we can steer the initial state \( x_0 \) to 0 in finite time and, at the same time, we can make the \( L_1 \)-norm of the output \( y \) arbitrarily small. An interpretation of \( \mathcal{T}_g(\Sigma_{ci}) \) can be given in terms of observers but is not very intuitive. Therefore, we shall not explain this interpretation in this thesis.

Note that \( \mathcal{V}(\Sigma_{ci}) \) and \( \mathcal{T}(\Sigma_{ci}) \) are dual subspaces, i.e. \( \mathcal{V}(\Sigma_{ci})^\perp = \mathcal{T}(\Sigma_{ci})^\perp \). Also \( \mathcal{V}_g(\Sigma_{ci}) \) and \( \mathcal{T}_g(\Sigma_{ci}) \) are dual subspaces. The following lemma gives explicit recursive algorithms to calculate these subspaces.

In this lemma we need the concept of *modal subspace* of a matrix \( A \). Let some region \( C_d \) of the complex plane be given which is symmetric with respect to the real axis. The subspace \( \mathcal{X} \) is called the modal subspace of \( A \) with respect to \( C_d \) if \( \mathcal{X} \) is the largest \( A \)-invariant subspace of \( \mathbb{R}^n \) such that if we restrict the mapping \( A \) to \( \mathcal{X} \) then its spectrum is contained in \( C_d \).
Lemma 2.5: The strongly controllable subspace $T(\Sigma_{ci})$ is the limit of the sequence of subspaces $\{T_i(\Sigma_{ci})\}$ generated by the recursive algorithm:

\begin{align*}
T_0(\Sigma_{ci}) & := \{0\}, \\
T_{i+1}(\Sigma_{ci}) & := \{x \in \mathbb{R}^n \mid \exists \bar{x} \in T_i(\Sigma_{ci}), u \in \mathbb{R}^m \text{ such that } \\
& \hspace{1cm} x = A\bar{x} + Bu \text{ and } C\bar{x} + Du = 0\} \quad (2.15)
\end{align*}

$T_i(\Sigma_{ci}) \ (i = 0, 1, \ldots)$ is a non-decreasing sequence of subspaces which attains its limit in a finite number of steps. In the same way $V(\Sigma_{ci})$ equals the limit of the sequence of subspaces $\{V_i(\Sigma_{ci})\}$ generated by:

\begin{align*}
V_0(\Sigma_{ci}) & := \mathbb{R}^n, \\
V_{i+1}(\Sigma_{ci}) & := \{x \in \mathbb{R}^n \mid \exists \bar{u} \in \mathbb{R}^m, \text{ such that } \\
& \hspace{1cm} Ax + B\bar{u} \in V_i(\Sigma_{ci}) \text{ and } Cx + D\bar{u} = 0\} \quad (2.16)
\end{align*}

$V_i(\Sigma_{ci}) \ (i = 0, 1, \ldots)$ is a non-increasing sequence of subspaces which also attains its limit in a finite number of steps. Moreover, if $G$ is a mapping such that (2.11) and (2.12) are satisfied for $T = T(\Sigma_{ci})$ and if $F$ is a mapping such that (2.13) and (2.14) are satisfied for $V = V(\Sigma_{ci})$, then we have the following two equalities:

\begin{align*}
T_g(\Sigma_{ci}) & = [T(\Sigma_{ci}) + \chi_h(A + GC)] \cap < T(\Sigma_{ci}) + C^{-1} \text{im} \ D \mid A + GC > \quad (2.17) \\
V_g(\Sigma_{ci}) & = V(\Sigma_{ci}) \cap \chi_g(A + BF) + < A + BF \mid V(\Sigma_{ci}) \cap B \text{ker} D > \quad (2.18)
\end{align*}

Here $\chi_h(A + GC)$ denotes the modal subspace of the matrix $A + GC$ with respect to the closed right half complex plane and $\chi_g(A + BF)$ denotes the modal subspace of the matrix $A + BF$ with respect to the open left half complex plane. Moreover, $< A + BF \mid V(\Sigma_{ci}) \cap B \text{ker} D >$ denotes the smallest $A + BF$ invariant subspace containing $V(\Sigma_{ci}) \cap B \text{ker} D$ and finally, $< T(\Sigma_{ci}) + C^{-1} \text{im} \ D \mid A + GC >$ denotes the largest $A + GC$ invariant subspace contained in $T(\Sigma_{ci}) + C^{-1} \text{im} \ D$.

\begin{proof}
This is all well known except for possibly (2.17) and (2.18) in the case that the $D$-matrix is unequal to zero. This can be proven by first showing that there exists a $G$ satisfying (2.11) and (2.12) for which (2.17) holds and after that, showing that the equality is independent of our particular choice of $G$ satisfying (2.11) and (2.12). The same can be done for (2.18). Details are left to the reader.
\end{proof}
We shall give some properties of the strongly controllable subspace at this point which will come in handy in the sequel (see [Ha, SH]). The following lemma can be easily checked using the properties (2.11) and (2.12) of the strongly controllable subspace.

**Lemma 2.6**: For all $F \in \mathbb{R}^{m \times n}$, the strongly controllable subspace $T(\Sigma_{ci})$ is $(C + DF, A + BF)$-invariant.

**Proof**: Let $T$ be the smallest $(C + DF_0, A + BF_0)$-invariant subspace containing $B \ker D$. We know that $T(\Sigma_{ci})$ is $(C + DF_0, A + BF_0)$-invariant by lemma 2.6. Moreover, in definition 2.3 we have $T_1(\Sigma_{ci}) = B \ker D$. Since the $T_i(\Sigma_{ci})$ are non-decreasing this implies that $T(\Sigma_{ci}) \supseteq B \ker D$. Therefore we have $T \supseteq T(\Sigma_{ci})$.

Conversely we know:

\[ \exists G_1 : \text{im}(C + DF_0) \rightarrow \mathbb{R}^n \quad [(A + BF_0) + G_1(C + DF_0)]T \subseteq T, \]

\[ \exists G_2 : \text{im} D \rightarrow \mathbb{R}^n \quad \text{im}(B + G_0D) = B \ker D \subseteq T. \]

Since $D^\tau(C + DF_0) = 0$ the above two mappings $G_1$ and $G_2$ can be combined to one linear mapping $G$ such that

\[ G|_{\text{im}(C + DF_0)} = G_1, \]

\[ G|_{\text{im} D} = G_2, \]

and hence we have found a $G$ such that $(A + GC)T \subseteq T$ and $\text{im}(B + GD) \subseteq T$. Thus we find $T \supseteq T(\Sigma_{ci})$ and hence $T = T(\Sigma_{ci})$. 

We also have the following result available. (see [Ha])
Lemma 2.8: Assume that we have the system (2.10) with \((C \ D)\) surjective. The system is strongly controllable if and only if the system matrix

\[
\begin{pmatrix}
sI - A & -B \\ 
C & D
\end{pmatrix}
\]  
(2.19)

has full row rank for all \(s \in \mathbb{C}\).

Note that this last lemma immediately implies that a strongly controllable system with \((C \ D)\) surjective is controllable by applying the Popov-Belevitch-Hautus criterium (this result is true in general but it only follows from the above lemma if \((C \ D)\) is surjective). A major reason why strongly controllable systems are interesting is the following corollary which can be derived after some effort from lemma 2.8:

Corollary 2.9: Assume that we have the system (2.10) with \((C \ D)\) surjective. Denote the transfer matrix of \(\Sigma_{ci}\) by \(G_{ci}\). If the system is strongly controllable, then \(G_{ci}\) has a polynomial right inverse.

Note that this implies that the mapping from \(u\) to \(z\) is surjective as a function from \(\mathcal{L}_{\infty}^0\) to \(\mathcal{L}_{\infty}^0\). Hence we can achieve any desired output in \(\mathcal{L}_{\infty}^0\). The problem we shall encounter is that we cannot realize a polynomial part of a transfer matrix by a dynamic feedback of the form (2.4). It will turn out that it is possible to approximate the polynomial part of the desired controller arbitrarily well by a controller which is of the form (2.4).

2.5 \(H_{\infty}\) theory

2.5.1 Continuous time

We define the function space \(H_{\infty}^1\) as the set of all functions \(f\) on the open right half plane which are analytic and which satisfy

\[
\|f\|_{\infty} := \sup_{s \in \mathbb{C}^+} |f(s)| < \infty.
\]  
(2.20)
This function space is a Banach space with respect to the norm $\|\cdot\|_\infty$. It can easily be seen that the set of all proper rational functions with no poles in the closed right half plane is contained in $H^1_\infty$. Note that one can define such a Banach space of analytic functions on any simply connected region which is not equal to the whole complex plane. This is a direct consequence of Riemann’s mapping theorem (see [Ru]). By $H_\infty$ we denote all matrices with coefficients in $H^1_\infty$. Note that the transfer matrix of an internally stable system with continuous time will be in $H_\infty$ (Hence it is not surprising that for discrete time systems we should replace the right half plane by the complement of the closed unit disc). On the space $H_\infty$ we define the following norm:

$$\|G\|_\infty := \sup_{s \in \mathbb{C}^+} \sigma_1[G(s)].$$

Here $\sigma_1[M]$ denotes the largest singular value of the matrix $M$. Note that it is not a norm in the strict sense since $H_\infty$ is not a vector space (we cannot add matrices with different dimensions). However, any subset of $H_\infty$ of matrices with the same dimensions is a well-defined vector space on which $\|\cdot\|_\infty$ is a norm which makes this space into a Banach space. This norm will be referred to as the $H_\infty$ norm.

We define the space $L_\infty$ as the space of essentially bounded measurable matrix valued functions on the imaginary axis. On this space we define the following norm

$$\|F\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_1[F(i\omega)] < \infty. \quad (2.21)$$

With respect to this norm, subsets of all matrix valued functions in $L_\infty$ with the same dimension form a Banach space. For general matrix valued functions in $H_\infty$ we can identify a boundary function to the analytic function which is, a priori, only defined on the open left half plane. This boundary function is in $L_\infty$ and its $L_\infty$-norm is equal to the $H_\infty$ norm of the original function (see [Yo]). Note that rational matrices in $H_\infty$ do not have poles on the imaginary axis and hence this boundary function is simply the rational matrix itself evaluated on the imaginary axis. Since, in the above sense, we can embed $H_\infty$ isometrically into $L_\infty$ we shall use the same notation for the $H_\infty$-norm and the $L_\infty$-norm in this thesis.

Define $L^2_\mathbb{R}$ as the set of all Lebesgue measurable functions $f$ from $\mathbb{R}$ to $\mathbb{R}^n$ for which

$$\|f\|_2 := \left(\int_0^\infty \|f(t)\|^2 \, dt\right)^{1/2} < \infty,$$
where $\|\cdot\|$ denotes the Euclidian norm. With respect to the norm $\|\cdot\|_2$ the function space $L^2_2$ is a Banach space. It is even a Hilbert space since the $L_2$-norm is induced by the following inner product

$$< f, g >_2 := \int_0^\infty < f(t), g(t) > \, dt,$$

where $< \cdot, \cdot >$ denotes the standard Euclidian inner product. By $L_2$ we denote the set of all $f$ for which there exists an $n$ such that $f \in L^2_2$.

In this thesis we shall frequently use a time-domain characterization of the $H_\infty$ norm. Let $\Sigma \times \Sigma_F$ be the closed loop system when we apply a controller $\Sigma_F$ of the form (2.4) to the system $\Sigma$ given by (2.1). If the closed loop system is internally stable, then the closed loop transfer matrix $G_F$ is in $H_\infty$. Denote by $G_{cl}$ the closed loop operator mapping $w$ to $z$. The $H_\infty$ norm is equal to the $L_2$-induced operator norm of the closed loop operator, i.e.

$$\|G_F\|_\infty = \|G_F\|_\infty := \sup_w \left\{ \frac{\|G_{cl}w\|_2}{\|w\|_2} \mid w \in L^2_2, \ w \neq 0 \right\}. \quad (2.22)$$

Because of the above equality we shall often refer to the $L_2$-induced operator norm of the closed loop operator $G_F$ as the $H_\infty$ norm of $G_F$. By the $H_\infty$ norm of a stable system we denote the $H_\infty$ norm of the corresponding transfer matrix.

A system $(A, B, C, D)$ is called inner if the system is internally stable and the input-output operator $G$ is unitary, i.e. $G$ maps $L^2_T$ into itself and $G$ is such that for all $f \in L^2_T$ we have

$$\|Gf\|_2 = \|f\|_2.$$

Often inner is defined as a property of the transfer matrix but in our setting this is a more natural definition. It can be shown that $G$ is unitary if and only if the transfer matrix of the system, denoted by $G$, is stable and satisfies:

$$G^T(-z)G(z) = G(z)G^T(-z) = I, \quad (2.23)$$

for all $z \not\in \sigma(A)$ where $\sigma$ denotes the spectrum of $A$. We call a transfer matrix satisfying (2.23) unitary. Note that a stable transfer matrix is unitary if and only if $G(s)$ is a unitary matrix for all $s$ on the imaginary axis. However $G(s)$ is not a unitary matrix for all $s$ in the closed right half plane which one might expect at first glance.

We now formulate a result from [Gl].
Lemma 2.10: Assume that we have a system $\Sigma_{ci}$ described by (2.10) where $u$ and $z$ take values in the same vector space $\mathbb{R}^k$ and where $A$ is asymptotically stable. The system $\Sigma_{ci}$ is inner if there exists a matrix $X$ satisfying:

(i) $A^TX + XA + C^TC = 0,$
(ii) $D^TC + B^TX = 0,$
(iii) $D^TD = I.$

Remarks:

(i) If $(A, B)$ is controllable the reverse of the above implication is also true. However, in general, the reverse does not hold. A simple counter example is given by $\Sigma_{ci} := (-1, 0, 1, 1)$ which is inner but for which (ii) does not hold for any choice of $X$.

(ii) Note that if $A$ is asymptotically stable, then if a matrix $X$ satisfies part (i) of lemma 2.10 it is equal to the observability gramian of $(C, A)$. We know, for instance, that $X > 0$ if and only if $(C, A)$ is observable. In general we only have $X \geq 0$.

Inner systems are often used in $H_\infty$ control. Next, we shall give a lemma which is one of the main reasons why this class of systems is interesting. However, we first give a preliminary lemma needed to prove the lemma in which we are mainly interested.

Lemma 2.11: Let $K \in \mathcal{R}^{n \times n}(s)$ be such that

- $K \in \mathcal{L}_\infty$ and $\|K\|_\infty < 1,$
- $(I - K)^{-1} \in H_\infty.$

Then we have $K \in H_\infty.$

Proof: We know that $\|K(s)\| < 1$ for all $s$ on the imaginary axis. Therefore $\det(I - \alpha K(s)) \neq 0$ for all $s \in C^0$ and for all $\alpha \in [0, 1]$. The Nyquist contour of $\det I - \alpha K$, which is defined as the contour $\{\det(I - \alpha K(s) | s \in C^0\}$, therefore has the same winding number around zero for all $\alpha \in [0, 1]$. Since
the winding number for $\alpha = 0$ is zero we find that the winding number for $\alpha = 1$ is also zero. However plus or minus the winding number is equal to the number of poles minus the number of zeros of $(I - K)^{-1}$ in the right half plane. (depending on the direction one follows on the contour, see [Ru]) Since $I - K$ has no zeros in the right half plane (its inverse is in $H_\infty$) it does not have poles in the right half plane either and therefore its inverse $I - K$ is stable.

We now give the result which shows the importance of inner systems to $H_\infty$ control. The original result was obtained in [Re]. Our adapted version however stems from [Do4]:

Lemma 2.12 : Suppose that two systems $\Sigma_1$ and $\Sigma_2$, both described by some state space representation, are interconnected in the following way:

\begin{equation}
\begin{array}{c}
\overset{z}{\Sigma_1} \\
y \\
\underset{w}{\Sigma_2} \\
\underset{u}{\sigma_1}
\end{array}
\end{equation}

(2.24)

Assume that $\Sigma_1$ is inner and that its transfer matrix $G$ has the following decomposition:

\begin{equation}
G(w) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix}
\end{equation}

(2.25)

which is compatible with the sizes of $w$, $u$, $z$ and $y$, such that $G_{21}^{-1} \in H_\infty$ and $G_{22}$ is strictly proper.

Under the above assumptions the following two statements are equivalent:

(i) The closed loop system (2.24) is internally stable and its closed loop transfer matrix has $H_\infty$ norm less than 1.

(ii) The system $\Sigma_2$ is internally stable and its transfer matrix has $H_\infty$ norm less than 1.
Proof: Denote the transfer matrix of $\Sigma_2$ by $G_2$ and denote the closed loop transfer matrix of the interconnection (2.24) by $G_{cl}$. By $\mathcal{G}$ with some index we denote the input-output operator associated with the transfer matrix with the same index.

(i) $\Rightarrow$ (ii): Note that if the closed loop system (2.24) is internally stable, then $\Sigma_2$ is stabilizable and detectable. We have

$$G_{cl} = G_{11} + G_{12}G_2(I - G_{22}G_2)^{-1} G_{21}.$$  

Note that $I - G_{22}G_2$ is invertible as a rational matrix since $G_{22}$ is strictly proper. Assume that $G_2$ has a pole on the imaginary axis or has $L_\infty$ norm larger than or equal to one. Then, there exists an $s_0$ on the imaginary axis such that $s_0$ is not a pole of $G_2$ and $s_0$ is such that there exists an $0 \neq y \in C^q$ for which $\|G_2(s_0)y\| \geq \|y\|$ (in the case that we have to choose $s_0 = \infty$ we simply replace all transfer matrices evaluated in $s_0$ by their respective direct feedthrough matrices). Define

$$u = G_2(s_0)y,$$
$$w = G_{21}^{-1}(s_0)(I - G_{22}(s_0)G_2(s_0))y,$$
$$z = G_{11}(s_0)w + G_{12}(s_0)u.$$ 

Note that

$$\begin{pmatrix} z \\ y \end{pmatrix} = G(s_0) \begin{pmatrix} w \\ u \end{pmatrix}. $$

Hence, since $G(s_0)$ is a unitary matrix, we have $\|z\|^2 + \|y\|^2 = \|w\|^2 + \|u\|^2$. We already know that $\|y\| \leq \|u\|$ which implies $\|z\| \geq \|w\|$. However, $z = G_{cl}(s_0)w$ and since $\|G_{cl}\|_\infty < 1$ this yields a contradiction. Hence $G_2$ is in $L_\infty$ and has $L_\infty$ norm strictly less than 1. The closed loop system is internally stable which implies that $(I - G_{22}G_2)^{-1}$ and $G_2(I - G_{22}G_2)^{-1}$ are in $H_\infty$. Moreover $G_{22}G_2$ is in $L_\infty$ and has $L_\infty$ norm strictly less than 1. By applying lemma 2.11 this implies that $I - G_{22}G_2$ is in $H_\infty$. Using that $G_2(I - G_{22}G_2)^{-1}$ is in $H_\infty$ we find that $G_2$ is in $H_\infty$ and hence stable. Combined with the detectability and stabilizability of the realization of $\Sigma_2$ this yields the desired result.

(ii) $\Rightarrow$ (i) First note that, since both $\Sigma_1$ and $\Sigma_2$ are internally stable, the interconnection is internally stable if and only if $I - G_{22}G_2$ has an inverse in $H_\infty$. We have $\|G_2\|_\infty < 1$ and $G_{22}$, as a submatrix of a unitary matrix, satisfies $\|G_{22}\|_\infty \leq 1$. Hence, using a small gain argument, it can be shown that $I - G_{22}G_2$ has an inverse in $H_\infty$. 


Remains to prove that \( G_{cl} \) has \( H_\infty \) norm less than 1. Since \( G_{21} \) is invertible over \( H_\infty \) there exists \( \mu > 0 \) such that
\[
\mu \| G_{21}^{-1} (I - G_{22} G_2) \|_\infty < 1.
\]

Let \( w \in \mathcal{L}_2^\infty \) be given. We have
\[
y = (I - G_{22} G_2)^{-1} G_{21} w
\]
and thus we get \( \mu \| w \|_2 \leq \| y \|_2 \). Let \( u = G_2 y \). Since \( G \) is unitary and \( \| G_2 \|_\infty < 1 \) we find
\[
\| z \|_2^2 = \| w \|_2^2 + \| u \|_2^2 - \| y \|_2^2 \\
\leq \| w \|_2^2 + (\| G_2 \|_\infty^2 - 1) \| y \|_2^2 \\
\leq \| w \|_2^2 + \mu^2 (\| G_2 \|_\infty^2 - 1) \| w \|_2^2 \\
= (1 + \mu^2 (\| G_2 \|_\infty^2 - 1)) \| w \|_2^2.
\]
Because \( 1 + \mu^2 (\| G_2 \|_\infty^2 - 1) < 1 \) we find that \( \| G_{cl} \|_\infty < 1. \)

### 2.5.2 Discrete time

In chapters 7 and 8 we shall discuss systems with discrete time. In this subsection we shall repeat the definitions and results from the previous subsection but adapted to the discrete time case.

For the discrete time case we define the function space \( H_\infty^1 \) as the set of all functions \( f \) which are analytic outside the closed unit disc and which satisfy
\[
\| f \|_\infty := \sup_{z \in \mathbb{D}^+} |f(z)| < \infty.
\]

This function space is a Banach space with respect to the norm \( \| . \|_\infty \). It can be easily seen that the set of rational functions with all poles inside the open unit disc is contained in \( H_\infty^1 \). By \( H_\infty \) we denote all matrices with coefficients in \( H_\infty^1 \). On this space we define the following norm:
\[
\| G \|_\infty := \sup_{z \in \mathbb{D}^+} \sigma_1 [G(z)].
\]

Note that any subset of \( H_\infty \) of matrices with the same dimensions is a Banach space with respect to this norm. This norm will be referred to as the \( H_\infty \) norm. As in the continuous time case for rational matrices \( G \) we have the following equality:
\[ \|G\|_\infty = \sup_{z \in \mathbb{D}} \sigma_1[G(z)]. \]

Note that we use the same notation as in the continuous time case although in discrete time the role of the right half plane is replaced by the complement of the closed unit disc. Since it will always be clear from the context whether we have discrete time or continuous time this will cause no confusion.

Define \( \ell_2^n \) as the set of all Lebesgue measurable functions \( f \) from \( \mathcal{N} \) to \( \mathbb{R}^n \) for which

\[ \|f\|_2 := \left( \sum_{k=0}^{\infty} \|f(k)\|^2 \right)^{1/2} < \infty. \]

With respect to the norm \( \|\cdot\|_2 \) it can be shown that \( \ell_2^n \) is a Banach space. It is even a Hilbert space since this \( \ell_2 \)-norm is induced by the following inner product:

\[ <f,g>_2 := \sum_{k=0}^{\infty} <f(k),g(k)>, \]

where \( <\cdot,\cdot> \) denotes the standard Euclidian inner product. By \( \ell_2 \) we denote the set of all \( f \) for which there exists an \( n \) such that \( f \in \ell_2^n \).

As in the continuous time case we shall use a time-domain characterization of the \( H_\infty \) norm. Let \( \Sigma \times \Sigma_F \) be the closed loop system when we apply a controller \( \Sigma_F \) to the system \( \Sigma \). If the closed loop system is internally stable (as a discrete time system!), then the closed loop transfer matrix \( G_F \) is in \( H_\infty \). Denote the closed loop operator mapping \( w \) to \( z \) with zero initial conditions by \( Q_F \). The \( H_\infty \) norm is equal to the \( \ell_2 \)-induced operator norm of the closed loop operator, i.e.

\[ \|G_F\|_\infty = \|G_F\|_\infty := \sup_{w} \left\{ \frac{\|G_Fw\|_2}{\|w\|_2} \mid w \in \ell_2, \ w \neq 0 \right\}. \quad (2.26) \]

Because of the above equality, we often refer to the \( \ell_2 \)-induced operator norm of \( G_F \) as the \( H_\infty \) norm of \( G_F \) (like the continuous time case).

As in the continuous time case a system \( (A,B,C,D) \) is called inner if it is internally stable and the input-output operator is a unitary operator from \( \ell_2^k \) into itself. The input-output operator is unitary if and only if the transfer matrix of the system, denoted by \( G \), satisfies:

\[ G^T(z^{-1})G(z) = G(z)G^T(z^{-1}) = I. \quad (2.27) \]
We now formulate a result from [St6] which itself is a generalization of a result from [Gu]. A proof can be given by simply writing out (2.27).

Lemma 2.13 : Assume that we have a system

\[ \Sigma_{ci} : \begin{cases} \sigma x = Ax + Bu, \\ z = Cx + Du, \end{cases} \]

where \( A \) is asymptotically stable and where \( u \) and \( z \) both take values in the same vector space \( \mathcal{R}^m \). The system \( \Sigma_{ci} \) is inner if there exists a matrix \( X \) satisfying:

1. \( X = A^T X A + C^T C \),
2. \( D^T C + B^T X A = 0 \),
3. \( D^T D + B^T X B = I \).

Remark : As in the continuous time case it can be shown that the reverse implication holds if \( (A, B) \) is controllable. A counter example for the general case is this time given by \( \Sigma_{ci} := (0.5, 0, 1, 1) \).

We shall now rephrase lemma 2.12 for discrete time systems. We shall not prove this result since the proof of lemma 2.12 needs only minor adjustments to yield a proof of our discrete time version.

Lemma 2.14 : Suppose that two systems \( \Sigma_1 \) and \( \Sigma_2 \), both described by some state space representation, are interconnected in the following way:

[Diagram of interconnected systems]

Assume that \( \Sigma_1 \) is inner. Moreover, assume that if we decompose the transfer matrix \( L \) of \( \Sigma_1 \):

\[ \begin{cases} \bar{z} = \bar{A} \bar{x} + \bar{B} \bar{u}, \\ \bar{y} = \bar{C} \bar{x} + \bar{D} \bar{u}, \end{cases} \]
Notation and basic properties

\[ L \begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \] (2.29)

compatible with the sizes of \( w, u, z \) and \( y \), then we have \( L_{21}^{-1} \in H_\infty \) and \( L_{22} \) is strictly proper.

The following two statements are equivalent:

(i) The closed loop system (2.28) is internally stable and its closed loop transfer matrix has \( H_\infty \) norm less than 1.

(ii) The system \( \Sigma_2 \) is internally stable and its transfer matrix has \( H_\infty \) norm less than 1.

\[ \square \]

2.6 (Almost) Disturbance Decoupling Problems

In this section we shall discuss the problems of (almost) disturbance decoupling for continuous time systems. The disturbance decoupling problems for discrete time systems are defined in a similar way. We discuss both the case of state feedback as well as the case of dynamic measurement feedback. In this thesis our central objective is minimizing the \( H_\infty \) norm of the closed loop system over all internally stabilizing controllers. This section discusses the special case that this infimum is 0 (almost disturbance decoupling) and under which conditions the infimum is attained (disturbance decoupling). Several people contributed to this area. For the problem of disturbance decoupling excellent references are [SH, Wo]. For the problem of almost disturbance decoupling prime references are [Tr, We2, Wi2, Wi3].

We start by defining the several problems and after that we state several known solutions to the respective problems and some extensions. These extensions are mainly concerned with the inclusion of direct feedthrough matrices which were in general excluded in the literature. We shall mainly present results we need and not the most general results available.

Definition 2.15 : Consider the system

\[ \Sigma : \begin{cases} \dot{z} = Ax + Bu + Ew, \\ z = Cx + Du. \end{cases} \] (2.30)
We say that the Disturbance Decoupling Problem with internal Stability (denoted by DDPS) is solvable if there exists a state feedback $u = Fx$ for $\Sigma$ such that the closed loop system $\Sigma_{cl}$ is internally stable, i.e. $A + BF$ is asymptotically stable, and such that $\Sigma_{cl}$ has a transfer matrix which is equal to 0.

We say that the Almost Disturbance Decoupling Problem with internal Stability (ADDPS) is solvable if for all $\epsilon > 0$ there exists a state feedback $u = Fx$ for $\Sigma$ such that the closed loop system $\Sigma_{cl}$ is internally stable and $\Sigma_{cl}$ has $H_\infty$ norm less than $\epsilon$.

We also define related problems for the case of dynamic measurement feedback:

**Definition 2.16**: Consider the system (2.1). We say that the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS) is solvable if there exists a controller $\Sigma_F$ of the form (2.4) such that the closed loop system $\Sigma \times \Sigma_F$ is well posed, internally stable and has a transfer matrix which is equal to 0.

We say that the Almost Disturbance Decoupling Problem with Measurement feedback and internal Stability (ADDPMS) is solvable if there exists for all $\epsilon > 0$ a controller $\Sigma_F$ of the form (2.4) such that the closed loop system $\Sigma \times \Sigma_F$ is well posed, internally stable and has $H_\infty$ norm less than $\epsilon$.

It was shown in [Tr, Wo, Wi2] that necessary and sufficient conditions for the solvability of DDPS and ADDPS can be stated in terms of the strongly controllable subspace, $T(\Sigma_{cl})$, and the stabilizable weakly unobservable subspace, $V_g(\Sigma_{cl})$, associated with the system $\Sigma_{cl} = (A, B, C, D)$. (In [Tr, Wi2] the subspace $T(\Sigma_{cl})$ is denoted by $R_\Sigma^+(ker C)$ while $D$ is assumed to be equal to 0). The exact result (extended to include direct feedthrough matrices) is as follows:

**Theorem 2.17**: Consider the system (2.30) with zero initial condition. Let $T(\Sigma_{cl})$ and $V_g(\Sigma_{cl})$ denote the strongly controllable subspace and the stabilizable weakly unobservable subspace, respectively, associated with the system $\Sigma_{cl} = (A, B, C, D)$.

The disturbance decoupling problem is solvable if and only if
The almost disturbance decoupling problem with internal stability is solvable if

\[ \mathcal{V}_g(\Sigma_{ci}) + T(\Sigma_{ci}) = \mathbb{R}^n. \]

\[ \text{Im}E \subseteq \mathcal{V}_g(\Sigma_{ci}). \]

\[ \text{Proof:} \] We shall only prove the result on the almost disturbance decoupling problem with internal stability, as this is the result we shall explicitly use in this thesis.

By definition 2.4 we know there exists a mapping \( \tilde{F} \) such that

\[ (A + BF)\mathcal{V}_g(\Sigma_{ci}) = \mathcal{V}_g(\Sigma_{ci}) \]

\[ (C_2 + D_2F)\mathcal{V}_g(\Sigma_{ci}) = \{0\} \]

and such that \( A + BF | \mathcal{V}_g(\Sigma_{ci}) \) is asymptotically stable. Let \( \Pi \) denote the canonical projection \( \mathbb{R}^n \rightarrow \mathbb{R}^n / \mathcal{V}_g(\Sigma) \). By (2.31) and (2.32) there exists linear mappings \( \tilde{A}, \tilde{B}, \tilde{C} \) such that

\[ \tilde{A}\Pi = \Pi(A + BF), \]

\[ \tilde{B} = \Pi B, \]

\[ \tilde{C}\Pi = C_2 + D_2\tilde{F}. \]

We then define the system:

\[ \Sigma_{fs}: \begin{cases} \dot{p} = \tilde{A}p + \tilde{B}u, \\ z = \tilde{C}p + D_2u. \end{cases} \]

It can be easily shown by induction using algorithm 2.5 that for \( i = 0, 1, \ldots \) we have \( T_i(\Sigma_{fs}) = \Pi T_i(\Sigma_{ci}) \). Hence we have:

\[ T(\Sigma_{fs}) = \Pi T(\Sigma_{ci}) = \Pi \{ T(\Sigma_{ci}) + \mathcal{V}_g(\Sigma_{ci}) \} = \Pi \mathbb{R}^n = \mathbb{R}^n / \mathcal{V}_g(\Sigma_{ci}). \]

This implies that the system (2.36) is strongly controllable.

Let \( F_0 \) and \( M \) be such that

\[ (\tilde{C} + D_2F_0)^T D_2 = 0, \]

\[ \ker D_2 = \text{im} M. \]

Using lemma 2.7, it is straightforward to check that
2.6 (Almost) disturbance decoupling problems

\[ T(\Sigma_{fs}) = T(\bar{A} + \bar{B}F_{0}, \bar{B}M, \bar{C} + D_{2}F_{0}, 0). \]

Hence by [Tr, Theorem 3.36] we know that for all \( \varepsilon > 0 \) there exists an \( \bar{F} \) such that:

\[ \| (\bar{C} + D_{2}F_{0}) e((\bar{A} + BF_{0}) + BM\bar{F})^{t} \|_{1} < \varepsilon \]

and such that \( \bar{A} + \bar{B}F_{0} + \bar{B}M\bar{F} \) is asymptotically stable. Here \( \| \cdot \|_{1} \) denotes the \( \mathcal{L}_{1} \) norm which is defined by

\[ \| M \|_{1} := \int_{0}^{\infty} \| M(t) \| \, dt. \]

Define \( F := \bar{F} + (F_{0} + M\bar{F}) \Pi \) then

\[ A + BF \mid V_{g}(\Sigma_{ci}) = A + BM\bar{F} \mid V_{g}(\Sigma_{ci}), \]

\[ \Pi(A + BF) = (\bar{A} + \bar{B}F_{0} + BM\bar{F}) \Pi. \]

It can be easily shown that this implies that \( A + BF \) is asymptotically stable. Moreover we have:

\[ (C_{2} + D_{2}F)e^{(A+BF)t} = (\bar{C} + D_{2}F_{0}) e((\bar{A} + BF_{0}) + BM\bar{F})^{t}\Pi \quad (2.39) \]

for all \( t \geq 0 \). Using (2.39) we find for all \( s \in i\mathbb{R} \) (Use that \( |e^{st}| = 1 \)):

\[ \| (C_{2} + D_{2}F)(sI - (A + BF))^{-1} \| \]

\[ = \| \int_{0}^{\infty} (C_{2} + D_{2}F) e^{((A+BF)-sI)t} \, dt \| \]

\[ \leq \int_{0}^{\infty} \| (C_{2} + D_{2}F) e^{((A+BF)-sI)t} \| \, dt \]

\[ = \| (C_{2} + D_{2}F) e^{(A+BF)t}\|_{1} \]

\[ = \| (\bar{C} + D_{2}F_{0}) e((\bar{A} + BF_{0}) + BM\bar{F})^{t}\Pi\|_{1} \]

\[ \leq \varepsilon. \]

This implies that the closed loop system has \( H_{\infty} \) norm less than \( \varepsilon \). Therefore \( F \) satisfies all the requirements of the almost disturbance decoupling problem with internal stability for this specific but arbitrarily chosen \( \varepsilon \).

As an immediate consequence of the above we obtain the following fact: if \( \Sigma_{ci} = (A, B, C, D) \) is strongly controllable, then for all \( \varepsilon > 0 \) there exists a static state feedback \( u = Fx \) such that the closed loop system has \( H_{\infty} \) norm less than \( \varepsilon \) and such that \( A + BF \) is asymptotically stable.

In chapter 6 we shall need a different version of theorem 2.17 which includes initial states:
Lemma 2.18: Assume that the system \((A, B, C, 0)\) is strongly controllable. Then for all bounded sets \(V \subset \mathbb{R}^n\) and all \(\varepsilon > 0\) there exists \(F \in \mathbb{R}^{m \times n}\) such that

(i) \(A + BF\) is asymptotically stable.

(ii) For all \(w \in \mathcal{L}_2^1\) and all \(\xi \in V\) we have \(\|z\|_2 \leq \varepsilon (\|w\|_2 + 1)\), where \(z\) is given by

\[
\dot{x} = (A + BF)x + Ew, \quad x(0) = \xi \in V,
\]

\[
z = Cx.
\]

Proof: Let \(M_1\) be such that for all \(\xi \in V\) we have \(\|\xi\| < M_1\). In [Tr, theorem 3.36] it was shown that if the system \((A, B, C, 0)\) is strongly controllable, then for all \(\varepsilon > 0\) there exists a \(F\) such that \(A + BF\) is asymptotically stable and

\[
\|Ce^{(A+BF)t}\|_2 \leq \frac{\varepsilon}{M_1} \quad \text{and} \quad \|Ce^{(A+BF)t}\|_1 \leq \frac{\varepsilon}{\|E\| + 1}.
\]

Since the \(L_1\)-norm of the impulse response is an upper bound for the \(H_\infty\) norm of the transfer matrix as shown in the proof of theorem 2.17 we find

\[
\|C(sI - A - BF)^{-1}E\|_\infty < \varepsilon.
\]

Using the above it can be shown straightforwardly that \(F\) satisfies (ii) for this choice of \(\varepsilon\) and \(V\).

We shall now discuss the (almost) disturbance decoupling problems with measurement feedback and internal stability. For the disturbance decoupling problem with measurement feedback and internal stability we have the following result from [SH] available (again extended to include direct feedthrough matrices):

\[
\text{Theorem 2.19: Let the following system } \Sigma \text{ be given:}
\]

\[
\Sigma : \begin{cases}
\dot{x} = Ax + Bu + Ew, \\
y = C_1x + D_1w, \\
z = C_2x + D_2u.
\end{cases}
\]

\[(2.40)\]
The following conditions are equivalent

(i) We have

\[ T_g(\Sigma_{di}) \subseteq V_g(\Sigma_{ci}) \]

where \( \Sigma_{ci} = (A, B, C_2, D_2) \) and \( \Sigma_{di} = (A, E, C_1, D_1) \).

(ii) The disturbance decoupling problem with measurement feedback and internal stability is solvable, i.e. there exists a controller of the form (2.4) for \( \Sigma \) such that the closed loop system is internally stable and has a transfer matrix which is equal to 0.

For ADDPMS we shall only present sufficient conditions for solvability. These conditions are all we need in this thesis. General necessary and sufficient conditions for solvability are, as far as we know, not available. However, combining the previous theorems it is not hard to guess how these conditions should look like.

**Theorem 2.20:** Let the system (2.40) be given. Assume that \( \Sigma \) satisfies the following two rank conditions:

\[
\text{rank } \begin{pmatrix} sI - A & -B \\ C_2 & D_2 \end{pmatrix} = n + \text{rank } \begin{pmatrix} C_2 & D_2 \end{pmatrix} \quad \forall s \in C^0 \cup C^+, \quad (2.41)
\]

and

\[
\text{rank } \begin{pmatrix} sI - A & -E \\ C_1 & D_1 \end{pmatrix} = n + \text{rank } \begin{pmatrix} E \\ D_1 \end{pmatrix} \quad \forall s \in C^0 \cup C^+. \quad (2.42)
\]

Under the above assumptions the almost disturbance decoupling problem with measurement feedback and internal stability is solvable, i.e. for all \( \varepsilon > 0 \) there exists a time-invariant, finite-dimensional compensator \( \Sigma_F \) with McMillan degree \( n - \text{rank } \begin{pmatrix} C_1 & D_1 \end{pmatrix} + \text{rank } D_1 \) of the form (2.4) such that the closed loop system is internally stable and has \( H_\infty \) norm less than \( \varepsilon \).

Before we can prove this result we have to do some preparatory work. A system \( (A, B, C_2, D_2) \) satisfying (2.41) has all its invariant zeros in \( C^- \) and is right invertible. The property that a system has all its invariant zeros
Notation and basic properties

in the open left half complex plane is called minimum-phase. In the same way a system\((A, E, C_1, D_1)\) satisfying (2.42) is left invertible and minimum-phase. We can express the rank conditions (2.41) and (2.42) in terms of the subspaces introduced in section 2.4 (see [Fr, SH3]):

**Lemma 2.21:** The rank condition (2.41) is satisfied if and only if
\[ V_g(\Sigma_{ci}) + T(\Sigma_{ci}) = \mathcal{R}^n, \]
where\(\Sigma_{ci} = (A, B, C_2, D_2)\). The rank condition (2.42) is satisfied if and only if
\[ V(\Sigma_{di}) \cap T_g(\Sigma_{di}) = \{0\}, \]
where\(\Sigma_{di} = (A, E, C_1, D_1)\).

**Proof of theorem 2.20:** Let \(\varepsilon > 0\). We first choose a mapping \(F\) such that:
\[ \| (C_2 + D_2 F)(sI - A - BF)^{-1}\|_\infty < \frac{\varepsilon}{3\|E\| + 1}, \]  \quad (2.43)
and such that \(A + BF\) is asymptotically stable. This can be done according to lemmas 2.17 and 2.21. Next choose a mapping \(G\) such that:
\[ \| (sI - A - GC_1)^{-1} (E + GD_1) \|_\infty < \min\left\{ \frac{\varepsilon}{3\|D_2 F\| + 1}, \frac{\|E\|}{\|BF\| + 1} \right\}, \]  \quad (2.44)
and such that \(A + GC_1\) is asymptotically stable. The dual version of lemma 2.17 guarantees the existence of such a \(G\). We apply the following feedback compensator to the system (2.40):
\[ \Sigma_{F,G} : \begin{cases} \dot{p} = Ap + Bu + G(C_1 p - y), \\ u = F p. \end{cases} \]  \quad (2.45)

The closed loop system is given by (where \(e := x - p\)):
\[ \Sigma_{cl} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BF & -BF \\ 0 & A + GC_1 \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} E \\ E + GD_1 \end{pmatrix} w, \\ z = \begin{pmatrix} C_2 + D_2 F & -D_2 F \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}. \end{cases} \]
2.6 \textit{(Almost) disturbance decoupling problems}

It is clear that this is an internally stabilizing feedback. The transfer matrix from \( w \) to \( z \) can be shown to be equal to:

\[
(C_2 + D_2 F)(sI - A - BF)^{-1} E \\
-(C_2 + D_2 F)(sI - A - BF)^{-1} BF(sI - A - GC_1)^{-1}(E + GD_1) \\
-D_2 F(sI - A - GC_1)^{-1}(E + GD_1).
\]

Using (2.43) and (2.44) it can be easily shown that this closed loop transfer matrix has \( H_\infty \) norm less than \( \varepsilon \).

The way to prove that we can obtain a controller of McMillan degree \( n - \text{rank} \ (C_1 \ D_1) + \text{rank} \ D_1 \) instead of \( n \) is the standard idea of Luenberger observers (see [Lu, Lu2, Wo2]). The main step is using the fact that from our measurements we know \( \text{rank} \ (C_1 \ D_1) - \text{rank} \ D_1 \) states directly and therefore we do not need to build an observer for these states.

We conclude this section by noting that we can extend our results to the more general system (2.1) by some standard loop shifting arguments as discussed in sections 4.5 and 5.5. However, we should be careful. As discussed in example 4.4, we should make a careful distinction between proper and non-proper controllers.
Chapter 3

The regular $H_\infty$ control problem: the full-information case

3.1 Introduction

Both in this chapter and in the next one, we discuss the $H_\infty$ control problem where all states are available for feedback. We also discuss the alternative where all states and the disturbance are available for feedback.

The state feedback $H_\infty$ problem was the first to be solved using time domain techniques. The main ideas stem from the stabilization of uncertain systems where one of the problems is to find a controller which maximizes the complex stability radius of some system with structured uncertainty (see [Kh5, Pe, Pe2, Pe4, RK2, ZK]). This is closely related to the $H_\infty$ control problem since this problem can be reduced to the problem of finding a controller which minimizes the $H_\infty$ norm of some related system. Later these ideas were applied directly to $H_\infty$ control theory (see [Kh2, Kh3, Pe3, Pe5, ZK2]).

The main result of these papers is the following: there exists an internally stabilizing state feedback which makes the $H_\infty$ norm less than some a priori given bound $\gamma > 0$ if and only if there exists a positive definite matrix $P$ and a positive constant $\varepsilon$ such that $P$ is a stabilizing solution of an algebraic Riccati equation parametrized by $\varepsilon$. A main drawback of this result is that the Riccati equation is parametrized. However under certain assumptions it turned out that this parameter could be removed (see [Do4, St, Ta]). The assumptions are twofold. Firstly the direct feedthrough matrix from control input to output should be injective. Secondly a certain subsystem should have no invariant zeros on the imaginary axis. In this chapter we shall give necessary and sufficient conditions for the existence of a suitable controller
under these assumptions. In the next chapter we shall then remove the first assumption.

We shall generalize the results in [Do4, St, Ta] since we shall not assume that the direct feedthrough matrix from control input to output is zero. Moreover we do not need any extra assumptions besides the two assumptions mentioned above. Our result was given without proof in [Gl3].

It will be shown that there exists a "suitable" controller (i.e. an internally stabilizing controller such that the closed loop system has $H_\infty$ norm less than some a priori given bound $\gamma > 0$) if and only if the following condition is satisfied: there exists a positive semi-definite stabilizing solution of a certain algebraic Riccati equation and moreover, a given matrix, explicitly specified in terms of the system parameters, is positive definite. This Riccati equation has an indefinite quadratic term. Riccati equations of this type first appeared in the theory of differential games (see [Ban, Ma, We]). Our proof will have a strong relation to differential games. Differential games are discussed in more detail in chapter 6 of this thesis.

We shall show in the present chapter that, if there exists an arbitrary "suitable" compensator, then there exists a "suitable" static feedback. In the next chapter we shall show that this property is not necessarily true if we remove our assumption on the direct feedthrough matrix. We shall give an explicit formula for one "suitable" static feedback. It will be shown that if we only allow for static state feedback, then we have to make an extra assumption to guarantee existence of a suitable controller.

The main ideas for this chapter stem from [PM, Ta].

The outline of this chapter is the following. In section 3.2 we shall give the problem formulation and our main results. In section 3.3 we give an intuitive proof of the necessity part. This is done because the formal proof is rather technical so it appeared to be a good idea first to explain how we obtained the necessary intuition for the formal proof. In section 3.4 we shall then prove necessity in a formal way and in section 3.5 we shall prove sufficiency.

### 3.2 Problem formulation and main results

We consider the linear, time-invariant, finite-dimensional system:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ z = Cx + D_1 u + D_2 w, \end{cases}$$

(3.1)
3.2 Problem formulation and main results

where for each $t$ we have that $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the control input, $w(t) \in \mathbb{R}^l$ the disturbance and $z(t) \in \mathbb{R}^q$ the output to be controlled. $A, B, E, C, D_1$ and $D_2$ are matrices of appropriate dimensions. We want to minimize the effect of the disturbance $w$ on the output $z$ by finding an appropriate control input $u$. More precisely, we seek a compensator $\Sigma_F$ described by a static feedback law $u = F_1 x + F_2 w$ such that after applying this feedback law to the system (3.1), the resulting closed loop system $\Sigma \times \Sigma_F$ is internally stable and its transfer matrix, denoted by $G_F$, has minimal $H_\infty$ norm.

Although minimizing the norm is always our ultimate goal, in this chapter as well as in the rest of this thesis, we shall only derive necessary and sufficient conditions under which we can find an internally stabilizing compensator which makes the resulting $H_\infty$ norm of the closed loop system strictly less than some a priori given bound $\gamma$. In principle one can then obtain the infimum of the closed loop $H_\infty$ norm over all internally stabilizing compensators via a search procedure (a simple binary search procedure is straightforward, for more advanced, quadratically convergent, algorithms see [SC, SC2]).

We are now in the position to formulate our main result.

Theorem 3.1 : Consider the system (3.1) and let $\gamma > 0$. Assume that the system $(A, B, C, D_1)$ has no invariant zeros on the imaginary axis and $D_1$ is injective. Then the following three statements are equivalent:

(i) There exists a static feedback law $u = F_1 x + F_2 w$ such that after applying this compensator to the system (3.1) the resulting closed loop system is internally stable and the closed loop operator $G_F$ has $H_\infty$ norm less than $\gamma$, i.e. $\|G_F\|_\infty < \gamma$.

(ii) $(A, B)$ is stabilizable and there exists a $\delta < \gamma$ such that for all $w \in \mathcal{L}_2^l$ there exists an $u \in \mathcal{L}_2^m$ such that $x_{u,w} \in \mathcal{L}_2^p$ and $\|z_{u,w}\|_2 \leq \delta \|w\|_2$.

(iii) We have

$$D_2^T \left( I - D_1 (D_1^T D_1)^{-1} D_1^T \right) D_2 < \gamma^2 I.$$  \hspace{1cm} (3.2)
Moreover, there exists a positive semi-definite solution $P$ of the algebraic Riccati equation

$$0 = A^TP + PA + C^TC$$

$$- \begin{pmatrix} B^TP + D_1^TC \\ E^TP + D_2^TC \end{pmatrix}^T \begin{pmatrix} D_1^TD_1 & D_1^TD_2 \\ D_2^TD_1 & D_2^TD_2 - \gamma^2I \end{pmatrix}^{-1} \begin{pmatrix} B^TP + D_1^TC \\ E^TP + D_2^TC \end{pmatrix}$$

such that $A_{cl}$ is asymptotically stable where:

$$A_{cl} := A - \begin{pmatrix} B & E \end{pmatrix} \begin{pmatrix} D_1^TD_1 & D_1^TD_2 \\ D_2^TD_1 & D_2^TD_2 - \gamma^2I \end{pmatrix}^{-1} \begin{pmatrix} B^TP + D_1^TC \\ E^TP + D_2^TC \end{pmatrix}.$$ 

If $P$ satisfies the conditions in part (iii), then a controller satisfying the conditions in part (i) is given by:

$$F_1 := - (D_1^TD_1)^{-1} (D_1^TC + B^TP), \quad (3.3)$$

$$F_2 := - (D_1^TD_1)^{-1} D_1^TD_2. \quad (3.4)$$

Remarks:

(i) Note that our assumption that $D_1$ is injective together with (3.2) guarantees the existence of the inverse in the algebraic Riccati equation and in the definition of $A_{cl}$.

(ii) The implication (i) $\Rightarrow$ (ii) is trivial. Hence we only need to prove (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The first step will be done in the section 3.4 and in the section thereafter we shall complete the proof.

(iii) The case that either one of the assumptions made in the theorem above is not satisfied will be discussed in the next chapter. The extension to the case that $D_1$ is not necessarily injective is in the author's opinion completely satisfactory. However, an elegant extension of the above result to the case that invariant zeros on the imaginary axis are allowed is still an open problem.
3.2 Problem formulation and main results

In a large part of the literature one is searching for an internally stabilizing static state feedback which makes the $H_\infty$ norm less than $\gamma$, rather than a feedback that is also allowed to depend on $w$. The next theorem gives necessary and sufficient conditions for the existence of such a state feedback.

**Theorem 3.2**: Consider the system (3.1). Let $\gamma > 0$. Assume that the system $(A, B, C, D_1)$ has no invariant zero on the imaginary axis and $D_1$ is injective. Then the following statements are equivalent:

(i) There exists a static state feedback law $u = Fx$ such that after applying this compensator to the system (3.1) the resulting closed loop system is internally stable and the closed loop operator $G_F$ has $H_\infty$ norm less than $\gamma$, i.e. $\|G_F\|_\infty < \gamma$.

(ii) We have

$$D_2^TD_2 < \gamma^2 I.$$ 

Moreover, there exists a positive semi-definite solution $P$ of the algebraic Riccati equation

$$0 = A^TP + PA + C^TC - (B^TP + D_1^TC)^T \begin{pmatrix} D_1^TD_1 & D_1^TD_2 \\ D_2^TD_1 & D_2^TD_2 - \gamma^2 I \end{pmatrix}^{-1} (B^TP + D_1^TC)$$ 

such that $A_{\text{cl}}$ is asymptotically stable where:

$$A_{\text{cl}} := A - (B \ E) \begin{pmatrix} D_1^TD_1 & D_1^TD_2 \\ D_2^TD_1 & D_2^TD_2 - \gamma^2 I \end{pmatrix}^{-1} (B^TP + D_1^TC).$$

If $P$ satisfies the conditions in part (ii), then a controller satisfying the conditions in part (i) is defined by:

$$F := - \left( D_1^T \left(I - \gamma^{-2} D_2 D_2^T\right)^{-1} D_1 \right)^{-1} \times$$

$$\left[ D_1^TC + B^TP + D_1^TD_2 \left(\gamma^2 I - D_2^TD_2\right)^{-1} (D_2^TC + E^TP) \right]$$  \hspace{1cm} (3.5) \hspace{1cm} \square

**Remarks:**
(i) Note that the only difference between part (iii) of theorem 3.1 and part (ii) of this theorem is that we already have $D_2^T D_2 < I$ instead of having the possibility to obtain this condition after a preliminary disturbance feedforward. The condition $D_2^T D_2 < I$ is clearly necessary for the existence of a "suitable" state feedback, since if we are only allowed to apply state feedback then we cannot change $D_2$. The surprising part is that it is also sufficient: if $D_2^T D_2 < I$ and if there exists a "suitable" static feedback, then there also exists a "suitable" static state feedback.

(ii) If $D_2^T D_1 = 0$, then the compensator suggested is the same as the compensator suggested in theorem 3.1. Hence only if $D_2^T D_1 \neq 0$ we can possibly do better by allowing disturbance feedforward.

3.3 Intuition for the formal proof

Since the approach proving the result of this chapter is crucial for this thesis and is used again in chapter 7 for discrete-time systems, we first explain and prove the results intuitively. In the next section we give a formal proof. Clearly making the $H_\infty$ norm as small as possible is directly related to the following "sup-inf" problem:

$$\inf_{f} \sup_{w \neq 0} \left\{ \frac{\|z_{u,w}\|^2}{\|w\|^2} \right\} \quad w \in L_2^p, \ f : L_2^p \to L_2^p \text{ causal and } u = f(w)$$

for the initial condition $x(0) = 0$. We know from more classical results on "inf-sup" problems that in the case that the criterium function is not quadratic then it is nearly impossible to solve this problem explicitly. Therefore we note that if the number defined by (3.6) is smaller than some bound $\gamma > 0$ then clearly

$$\inf_{f} \sup_{w} \left\{ \|z_{u,w}\|^2 - \gamma^2 \|w\|^2 \right\} \quad w \in L_2^p, \ f \text{ causal and } u = f(w) \leq 0$$

for initial condition $x(0) = 0$. However, for $H_\infty$ control we have an extra constraint since we require that our controller is internally stabilizing. Internal stability is a property which is related to all initial conditions of the system. Thus, in order to be able to build the side constraint into our "inf-sup" problem, we investigate (3.7) for an arbitrary initial condition $x(0) = \xi$. We define
3.3 Intuition for the formal proof

\[ C(u, w, \xi) := \|z_{u,w,\xi}\|_2^2 - \gamma^2 \|w\|_2^2 \]

and we investigate

\[ \bar{C}^*(\xi) := \inf_{f} \sup_{w} \{ C(u, w, \xi) \mid f \text{ causal, } u = f(w), w \in L^1 \}
\text{ such that } x_{u,w,\xi} \in L^n_2 \}

for arbitrary initial state \( x(0) = \xi \). A related problem is the following:

\[ C^*(\xi) := \sup_{w} \inf_{u} \{ C(u, w, \xi) \mid u \in L^m_2, w \in L^l_2 \text{ such that } x_{u,w,\xi} \in L^n_2 \}
\]

for arbitrary initial state \( x(0) = \xi \). It can be easily shown that, without the extra assumption that \( f \) should be causal, both problems are equivalent. The surprising fact is that even with the extra assumption these problems are equal, i.e.

\[ C^*(\xi) = \bar{C}^*(\xi) \quad \forall \xi \in \mathcal{R}^n. \quad (3.8) \]

The main difference with our original problem formulation is that we do not require causality, i.e. \( u(t) \) may depend on the future of \( w \). This was excluded in our original formulation because we only look at controllers of the form (2.4) which are automatically causal. The surprising fact is that our formal proof will show that we cannot do better by allowing for a non-causal dependence on \( w \). From now on we only investigate \( C^*(\xi) \). It can be shown that \( C^*(\xi) < \infty \) for all \( \xi \in \mathcal{R}^n \) if the number defined by (3.6) is strictly less than \( \gamma \).

Next by using a method from [Mo] it can be shown that there exists a matrix \( P \) such that \( C^*(\xi) = \xi^t P \xi \) for all \( \xi \in \mathcal{R}^n \). (the same method can be used to show that for a large class of optimization problems with a quadratic cost criterion the optimal cost is also quadratic.)

We denote by \( x_{u,w,\xi,t}(\tau) \) and \( z_{u,w,\xi,t}(\tau) \) respectively the state and the output of our system at time \( \tau > t \) if we apply inputs \( u \) and \( w \) and \( \pi(t) = \xi \). We define:

\[ C(u, w, \xi, t) := \int_{t}^{\infty} \|z_{u,w,\xi,t}(\tau)\|^2 - \gamma^2 \|w(\tau)\|^2 \, d\tau. \]

Our system is time-invariant and therefore we find

\[ C^*(\xi, t) := \sup_{w} \inf_{u} \{ C(u, w, \xi, t) \mid u \in L^m_2, w \in L^l_2 \text{ such that } x_{u,w,\xi,t} \in L^n_2 \}
\]

\[ = \xi^t P \xi. \]
Now we make the, illegal, move to assume a priori that \( u \) is indeed a causal function of \( w \) (which, at this point, we justify by the equality (3.8)). Then the last "sup-inf" problem can be rewritten and we find

\[
0 = \sup_{u \in L^2_t} \inf_{w \in L^2_t} \sup_{u \in L^2_t} \inf_{w \in L^2_t} \int_0^\infty \|z_{u,w}(\tau)\|^2 - \gamma^2 \|w(\tau)\|^2 \, d\tau - \xi^T P \xi
\]

\[
= \sup_{u \in L^2_t} \inf_{w \in L^2_t} \int_0^t \|z_{u,w}(\tau)\|^2 - \gamma^2 \|w(\tau)\|^2 d\tau + x^T(t)Px(t) - \xi^T P \xi
\]

where \( u \in L^2_t \) and \( w \in L^2_t \) should be such that \( z_{u,w} \in L^2_t \). We differentiate this expression with respect to \( t \) and take the derivative at \( t = 0 \), acting as if this expression were differentiable and as if we were allowed to interchange the supremum and infimum with the differentiation operator. We then obtain

\[
0 = \sup_{u(0)} \inf_{w(0)} \|z(0)\|^2 - \gamma^2 \|w(0)\|^2 + \frac{d}{dt} x^T(t)Px(t) \bigg|_{t=0}
\]

We can now investigate this latter expression and note that we thus obtained a static "sup-inf" problem which we can solve explicitly. Writing this out in terms of the system parameters of our system \( \Sigma \) as defined by (3.1) then yields:

\[
0 = \sup_{u(0)} \inf_{w(0)} \begin{pmatrix} \xi \\ u(0) \\ w(0) \end{pmatrix}^T \begin{pmatrix} A^T P + PA + C^T C & B^T P + C^T D_1 & E^T P + C^T D_2 \\ PB + D_1^T C & D_1^T D_1 & D_1^T D_2 \\ PE + D_2^T C & D_2^T D_1 & D_2^T D_2 - \gamma^2 I \end{pmatrix} \begin{pmatrix} \xi \\ u(0) \\ w(0) \end{pmatrix}
\]

Next we define

\[
W := D_2^T D_2 - D_2^T D_1 (D_1^T D_1)^{-1} D_1^T D_2 - \gamma^2 I,
\]

\[
q := u + (D_1^T D_1)^{-1} (B^T P + D_1^T C) \xi + (D_1^T D_1)^{-1} D_1^T D_2 w(0),
\]

\[
p := w + W^{-1} [E^T P + D_2^T C - D_2^T D_1 (D_1^T D_1)^{-1} (B^T P + D_1^T C)] \xi,
\]

where we assume that \( W \) is invertible. Finally, we denote by \( R(P) \) the right hand side of the algebraic Riccati equation in theorem 3.1. By using Schur complements we can then rephrase our previous "sup-inf" problem in the form

\[
0 = \sup_{p} \inf_{q} \begin{pmatrix} \xi \\ q \\ p \end{pmatrix}^T \begin{pmatrix} R(P) & 0 & 0 \\ 0 & D_1^T D_1 & 0 \\ 0 & 0 & W \end{pmatrix} \begin{pmatrix} \xi \\ q \\ p \end{pmatrix}
\]
The above equality should be true for all initial conditions $\xi \in \mathcal{R}^n$. This implies $R(P) = 0$ and $W < 0$. These are two of the three conditions given in part (iii) of theorem 3.1. The optimal $p$ and $q$ are zero, i.e.

$$w^*(0) = -W^{-1} \left[ E^T P + D_2^T C - D_2^T D_1 (D_1^T D_1)^{-1} (B^T P + D_1^T C) \right] \xi,$$

$$u^*(0) = -(D_1^T D_1)^{-1} (B^T P + D_1^T C) \xi - (D_1^T D_1)^{-1} D_1^T D_2 w^*(0).$$

However, we can show in the same way that these equalities are satisfied by the optimal $u^*$ and $w^*$ for all $t$, i.e.

$$w^*(t) = -W^{-1} \left[ E^T P + D_2^T C - D_2^T D_1 (D_1^T D_1)^{-1} (B^T P + D_1^T C) \right] x^*(t),$$

$$u^*(t) = -(D_1^T D_1)^{-1} (B^T P + D_1^T C) x^*(t) - (D_1^T D_1)^{-1} D_1^T D_2 w^*(t),$$

which implies that

$$\dot{x}^* = A_d x^*.$$

Since $x^* \in \mathcal{L}_2^n$ for all initial conditions $\xi \in \mathcal{R}^n$ this implies that $A_d$ is asymptotically stable. This is the last condition in part (iii) of theorem 3.1. We shall formalize the above reasoning in the next section to yield a rigorous and reasonably elegant proof.

3.4 Solvability of the Riccati equation

Throughout this section we assume that there exists a $\delta < \gamma$ such that condition (ii) of theorem 3.1 is satisfied. We show that this implies that there exists a matrix $P$ satisfying condition (iii) of theorem 3.1. For the time being we assume that

$$D_1^T \begin{bmatrix} C & D_2 \end{bmatrix} = 0 \tag{3.9}$$

and $\gamma = 1$. We derive the more general statement at the end of this section. We first define the following function

$$C(u, w, \xi) := \|x_{u,w,\xi}\|_2^2 - \|w\|_2^2.$$

In order to prove the existence of the desired $P$ we shall investigate the following "sup-inf" problem:

$$C^*(\xi) := \sup_{u} \inf_{w} \left\{ C(u, w, \xi) \mid u \in \mathcal{L}_2^n, w \in \mathcal{L}_2^l \text{ such that } x_{u,w,\xi} \in \mathcal{L}_2^n \right\}.$$
The regular full-information case

for arbitrary initial state \( \xi \). Since condition (ii) of theorem 3.1 holds, it will turn out that the "sup-inf" is finite for all initial states. Moreover we shall show that there exists a \( P \geq 0 \) such that \( C^*(\xi) = \xi^T P \xi \). It will be proven that this matrix \( P \) exactly satisfies condition (iii) of theorem 3.1.

For given \( w \in \mathcal{L}_2^1 \) and \( \xi \in \mathbb{R}^n \), we shall first infimize the function \( C(u, w, \xi) \) over all \( u \in \mathcal{L}_2^0 \) for which \( x_{u,w,\xi} \in \mathcal{L}_2^0 \). After that we shall maximize over \( w \in \mathcal{L}_2^1 \).

As a tool we shall use Pontryagin's maximum principle. This only gives necessary conditions for optimality. However in [LM, Section 5.2] a sufficient condition for optimality is derived over a finite horizon. The stability requirement \( x_{u,w,\xi} \in \mathcal{L}_2^0 \) allows us to adapt the proof to the infinite horizon case. We start by constructing a solution to the Hamilton-Jacobi-Bellman boundary value problem associated with this optimization problem.

Let \( L \) be the positive semi-definite solution of the following algebraic Riccati equation:

\[
A^T L + LA + C^T C - LB (D_1^T D_1)^{-1} B^T L = 0
\]

for which

\[
A_L := A - B (D_1^T D_1)^{-1} B^T L
\]

is asymptotically stable. The existence and uniqueness of such \( L \) is guaranteed under the assumptions that \( (A, B, C, D_1) \) has no invariant zeros on the imaginary axis, \( D_1 \) is injective and \( (A, B) \) is stabilizable (see [Wi4]). Let \( w \in \mathcal{L}_2^1 \) be given. We define

\[
r(t) := - \int_t^\infty e^{A_L^T (\tau - t)} (C^T D_2 + LE) w(\tau) d\tau
\]

\((t \in [0, \infty))\). Note that \( r \) is well defined since \( A_L \) is asymptotically stable. Next we define \( x_+ \) and \( \eta \) by the equations:

\[
\dot{x}_+ = A_L x_+ + B (D_1^T D_1)^{-1} B^T r + E w
\]

\[
\eta = -L x_+ + r
\]

where \( x_+(0) = \xi \). It can be easily checked that \( r, x_+ \) and \( \eta \) are all \( \mathcal{L}_2 \) functions. Moreover we have

\[
\lim_{t \to \infty} r(t) = \lim_{t \to \infty} x_+(t) = \lim_{t \to \infty} \eta(t) = 0.
\]

After some calculations, we find the following lemma:
Lemma 3.3: Let $\xi \in \mathbb{R}^n$ and $w \in L^1_2$ be given. The function $\eta$ as defined by (3.13) satisfies:

$$\dot{\eta} = -A^T \eta + C^T C x_+ + C^T D_2 w.$$  \hspace{1cm} (3.15)

In the statement of Pontryagin's Maximum Principle this equation is the so-called "adjoint equation" and $\eta$ is called the "adjoint state variable". We have constructed a solution to this equation and we shall show that this $\eta$ indeed yields a minimizing $u$. The proof is adapted from [LM, Theorem 5.5]:

Lemma 3.4: Let the system (3.1) be given. Moreover let $w$ and $x(0) = \xi$ be fixed. Then

$$u_+ := (D_1^T D_1)^{-1} B^T \eta$$

$$= \arg \inf_u \{ C(u, w, \xi) \mid u \in L^m_2 \text{ such that } x_{u, w, \xi} \in L^2_2 \}.$$  \hspace{1cm} \Box

Proof: Since $w$ is fixed it is sufficient to minimize

$$\|x_{u, w, \xi}\|_2^2 = C(u, w, \xi) + \|w\|_2^2.$$  

It can be easily checked that $x_+ = x_{u_+, w, \xi}$. Let $u \in L^m_2$ be an arbitrary control input such that $x_{u, w, \xi} \in L^2_2$. Since $x_{u, w, \xi} \in L^2_2$ and $x_{u, w, \xi} \in L^2_2$ we have

$$\lim_{t \to \infty} x_{u, w, \xi}(t) = 0.$$  \hspace{1cm} (3.16)
For all \( t \in \mathcal{R}^+ \) we find
\[
\|z_{u,w,\xi}(t)\|^2 - 2 \frac{d}{dt} \eta^T(t)x(t) = \|D_2w(t)\|^2 + \|Cx(t)\|^2 - 2x^T(t)C^TCx_+(t)
+ [D_1^TD_1u(t) - 2B^T\eta(t)]^T u(t) - 2\eta^T(t)Ew(t)
\]  
(3.17)
and
\[
\|z_{u+,w,\xi}\|^2 - 2 \frac{d}{dt} \eta^T(t)x_+(t) = \|D_2w(t)\|^2 - \|Cx_+(t)\|^2
+ [D_1^TD_1u_+(t) - 2B^T\eta(t)]^T u_+(t) - 2\eta^T(t)Ew(t).
\]  
(3.18)
If we integrate (3.17) and (3.18) from zero to infinity, subtract from each other and use (3.14), (3.16) we find (note that \( x(0) = x_+(0) = \xi \))
\[
\|z_{u+,w,\xi}\|^2 - \|z_{u,w,\xi}\|^2 = -\int_0^\infty \|C[x(t) - x_+(t)]\|^2 dt + 
\int_0^\infty [D_1^TD_1u_+(t) - 2B^T\eta(t)]^T u_+(t) - [D_1^TD_1u(t) - 2B^T\eta(t)]^T u(t) dt
\]
Using the definition of \( u_+ \) we also find for each fixed \( t \in \mathcal{R}^+ \) that
\[
[D_1^TD_1u_+(t) - 2B^T\eta(t)]^T u_+(t) = \inf_v [D_1^TD_1v - 2B^T\eta(t)]^T v.
\]  
(3.19)
Combining the last two equations then yields
\[
\|z_{u+,w,\xi}\|^2 \leq \|z_{u,w,\xi}\|^2.
\]  
(3.20)
which is exactly what we had to prove. Since \( D_1 \) is injective it is straightforward to show the minimizing \( v \) in (3.19) is unique and hence the minimizing \( u \) is unique. 

We are now going to maximize over \( w \in \mathcal{L}_2^1 \). This will then yield \( C^*(\xi) \). Define \( \mathcal{F}(\xi, w) := (x_+, u_+, \eta) \) and \( \mathcal{G}(\xi, w) := z_{u+,w,\xi} \). It is clear from the previous lemma that \( \mathcal{F} \) and \( \mathcal{G} \) are bounded linear operators (linear in \( \xi, w \) not in \( \xi \) and \( w \) separately). Define
\[
C(\xi, w) := \|\mathcal{G}(\xi, w)\|^2 - \|w\|^2,
\]  
(3.21)
\[
\|w\|_C := (-\mathcal{C}(0, w))^{1/2}.
\]  
(3.22)
It can easily be shown, using condition (ii) of theorem 3.1 with \( \gamma = 1 \), that \( \|\cdot\|_C \) defines a norm on \( \mathcal{L}_2^1 \). Moreover we find
3.4 Solvability of the Riccati equation

\[ ||w||_2 \geq ||w||_C \geq \rho ||w||_2 \]  \hspace{1cm} (3.23)

where \( \rho > 0 \) is such that \( \rho^2 = 1 - \delta^2 \) and \( \delta \) is such that condition (ii) of theorem 3.1 with \( \gamma = 1 \) is satisfied. Hence \( ||.||_C \) and \( ||.||_2 \) are equivalent norms.

Note that lemma 3.4 still holds if condition (ii) of theorem 3.1 does not hold. However the result that \( ||.||_C \) is a norm and that even \( ||.||_C \) and \( ||.||_2 \) are equivalent norms is the essential property which is implied by condition (ii) of theorem 3.1 and which is the key to our derivation.

We have

\[ C^*(\xi) = \sup_{w \in L^1} C(\xi, w). \]  \hspace{1cm} (3.24)

We can derive the following properties of \( C^* \):

Lemma 3.5 :

(i) For all \( \xi \in \mathbb{R}^n \) we have

\[ 0 \leq \xi^T L \xi \leq C^*(\xi) \leq \frac{\xi^T L \xi}{1 - \rho^2}, \]  \hspace{1cm} (3.25)

where \( L \) as defined by (3.10) and the stability of (3.11) and where \( \rho \) is such that (3.23) is satisfied.

(ii) For all \( \xi \in \mathbb{R}^n \) there exists an unique \( w_\ast \in L^1 \) such that \( C^*(\xi) = C(\xi, w_\ast) \).

Proof : Part (i): It is well known that \( L \), as the stabilizing solution of the algebraic Riccati equation (3.10), yields the optimal cost of the linear quadratic problem with internal stability (see [Wi4]). Hence \( ||G(\xi, 0)||_2^2 = C(\xi, 0) = \xi^T L \xi \). Therefore we have \( 0 \leq \xi^T L \xi \leq C^*(\xi) \). Moreover

\[ C(\xi, w) = ||G(\xi, w)||_2^2 - ||w||_2^2 \]

\[ \leq (||G(\xi, 0)||_2 + ||G(0, w)||_2)^2 - ||w||_2^2 \]

\[ \leq \left( \sqrt{\xi^T L \xi + \delta ||w||_2} \right)^2 - ||w||_2^2 \]

\[ \leq \frac{\xi^T L \xi}{1 - \rho^2}. \]

Part (ii) can be proven in the same way as in [Ta]. First it is shown that \( ||.||_C \) satisfies:
The regular full-information case

\[ \|w_\alpha - w_\beta\|^2 = 2C(\xi, w_\alpha) + 2C(\xi, w_\beta) - 4C(\xi, 1/2(w_\alpha + w_\beta)) \quad (3.26) \]

for arbitrary \( \xi \in \mathbb{R}^n \). Then it can be shown that a maximizing sequence of \( C(\xi, w) \) is a Cauchy sequence with respect to the \( \| \cdot \|_C \)-norm and hence, since \( \| \cdot \|_C \) and \( \| \cdot \|_2 \) are equivalent norms, there exists a maximizing \( L_2 \) function \( w_* \). Using (3.26) it is straightforward to show uniqueness.

Define \( H : \mathbb{R}^n \rightarrow L_2^T \) by \( H : w = w_* \). In order to derive an explicit expression for \( w_* \) we first have to do some preliminary work (note that, by assumption, \( D_1^T D_2 = 0 \):

Lemma 3.6: If condition (ii) of theorem 3.1 is satisfied, then we have

\[ D_2^T D_2 < I. \quad (3.27) \]

Proof: Assume that \( D_2^T D_2 \geq I \). Then there exists a vector \( w_0 \neq 0 \) such that \( \| D_2 w_0 \| \geq \| w_0 \| \). We construct a sequence of disturbances parametrized by \( \varepsilon \):

\[ w_\varepsilon(t) = \begin{cases} w_0 & \text{if } t < \varepsilon \\ 0 & \text{otherwise} \end{cases} \]

We shall show that:

\[ \lim_{\varepsilon \to 0} \inf_u \left\{ \frac{\| x_{u, w_\varepsilon} \|_2}{\| w_\varepsilon \|_2} \left| u \in L_2^n \text{ such that } x_{u, w_\varepsilon} \in L_2^n \right. \right\} \geq 1. \quad (3.28) \]

This clearly is a contradiction with condition (ii) of theorem 3.1.

We denote the \( L_2 \) norm over the interval \([0, \varepsilon]\) by \( \| \cdot \|_{2, \varepsilon} \). Because of our assumption (3.9) we find

\[ \| x_{u, w_\varepsilon} \|^2 \geq \| C x_{u, w_\varepsilon} + D_2 w_\varepsilon \|^2 + \| D_1 u \|^2 \]

\[ \geq \left( \| D_2 w_\varepsilon \|_{2, \varepsilon} - \| C x_{u, w_\varepsilon} \|_{2, \varepsilon} \right)^2 + \| D_1 u \|^2 \]

\[ \geq \left( \| w_{\varepsilon} \|^2 - 2\| D_2 w_\varepsilon \|_{2} \| C x_{u, w_\varepsilon} \|_{2, \varepsilon} + \| D_1 u \|^2 \right. \]

Since \( D_1 \) is injective there exists a \( \mu > 0 \) such that \( \| D_1 u \|_2 \geq \mu \| u \|_2 \) for all \( u \in L_2^n \). We also have
3.4 Solvability of the Riccati equation

\[ \|Cx_{u,w}\|_{2,\varepsilon} \leq \|Cx_{u,0}\|_{2,\varepsilon} + \|Cx_{0,w}\|_{2,\varepsilon}. \]

Using Cauchy-Schwarz it can be easily checked that there exist \( M_1, M_2 > 0 \) such that

\[ \|Cx_{0,w}\|_{2,\varepsilon} \leq M_1 \varepsilon^{3/2} \|w_0\|, \]
\[ \|Cx_{u,0}\|_{2,\varepsilon} \leq M_2 \varepsilon^{3/2} \|u\|_2. \]

Combining the above equations we find that

\[ \|z_{u,w}\|_2^2 \geq \|w_\varepsilon\|_2^2 - 2 \varepsilon^{3/2} \|D_2 w_\varepsilon\|_2 (M_1 \|w_0\| + M_2 \|u\|_2) + \mu^2 \|u\|_2^2. \]

Infinimizing the expression on the right hand side over \( \|u\|_2 \) for each \( \varepsilon \) and letting \( \varepsilon \downarrow 0 \) then yields the desired expression (3.28). \( \blacksquare \)

**Lemma 3.7:** Let \( \xi \in \mathbb{R}^n \) be given. \( w_\ast = \mathcal{H}_\xi \) is the unique \( L_2 \)-function \( w \) satisfying

\[ w = (I - D_2^T D_2)^{-1} (D_2^T C x - E^T \eta), \quad (3.29) \]

where \((x, u, \eta) = \mathcal{F}(\xi, w)\).

**Proof:** Define \((x_\ast, u_\ast, \eta_\ast) = \mathcal{F}(\xi, w_\ast)\). Moreover, we define

\[ w_0 := -E^T \eta_\ast + D_2^T D_2 w_\ast + D_2^T C x_\ast \]

and \((x_0, u_0, \eta_0) := \mathcal{F}(\xi, w_0)\). We find

\[ \|z_{u_0, w_0, \xi}(t)\|^2 - \|w_0(t)\|^2 - 2 \frac{d}{dt} \eta_\ast^T(t) x_0(t) = \]
\[ \|w_0(t)\|^2 - \|z_{u_\ast, w_\ast, \xi}(t)\|^2 + \|z_{u_\ast, w_\ast, \xi}(t) - z_{u_0, w_0, \xi}(t)\|^2. \]

We also find:

\[ \|z_{u_\ast, w_\ast, \xi}(t)\|^2 - \|w_\ast(t)\|^2 - 2 \frac{d}{dt} \eta_\ast^T(t) x_\ast(t) = \]
\[ \|w_\ast(t)\|^2 - \|z_{u_\ast, w_\ast, \xi}(t)\|^2 - \|w_\ast(t) - w_0(t)\|^2. \quad (3.30) \]

Integrating the last two equations from zero to infinity and subtracting from each other gives us
The regular full-information case

\[ C(\xi, w_\ast) = C(\xi, w_0) - \|w_0 - w_\ast\|^2 - \|z_{w_0, w_0, \xi} - z_{w_\ast, w_\ast, \xi}\|^2. \]

Since \( w_\ast \) maximizes \( C(\xi, w) \) over all \( w \), this implies that \( w_0 = w_\ast \). Thus we find that \( w_\ast \) satisfies (3.29).

That \( w_\ast \) is the only solution of the equation (3.29) can be shown in a similar way. Assume that, apart from \( w_\ast \), also \( w_1 \) satisfies (3.29). Let \((x_1, u_1, \eta_1) = F(\xi, w_1)\). From (3.30) we find

\[ \|z_{u_1, w_1, \xi}(t)\|^2 - \|w_1(t)\|^2 - 2 \frac{d}{dt} \eta_1^T(t)x_1(t) = \|w_\ast(t)\|^2 - \|z_{u_\ast, w_\ast, \xi}(t)\|^2. \]

(3.31)

We also find:

\[ \|z_{u_1, w_1, \xi}(t)\|^2 - \|w_1(t)\|^2 - 2 \frac{d}{dt} \eta_1^T(t)x_1(t) = \|z_{u_1, w_1, \xi}(t)\|^2 - \|w_1(t)\|^2 + 2w_1^T(t)w_1(t) - 2z_{u_\ast, w_\ast, \xi}(t)z_{u_1, w_1, \xi}(t). \]

Hence, if we integrate the last two equations from 0 to \( \infty \) and subtract from each other we get

\[ C(\xi, w_\ast) = C(\xi, w_1) - \|w_\ast - w_1\|^2. \]

(3.32)

Since \( w_\ast \) was maximizing we find \( \|w_\ast - w_1\|_C = 0 \) and hence \( w_\ast = w_1 \). \( \Box \)

Next, we show that \( C^*(\xi) = \xi^T P \xi \) for some matrix \( P \). In order to do that we first show that \( \eta_\ast \) is a linear function of \( x_\ast \):

**Lemma 3.8**: There exists a constant matrix \( P \) such that

\[ \eta_\ast = -Px_\ast. \]

(3.33)

**Proof**: We shall first look at time 0. From (3.13) it is straightforward that \( \eta_\ast(0) \) depends linearly on \((\xi, x_\ast)\). By lemma 3.7 we know \( w_\ast \) is uniquely defined by equation (3.29). If we investigate initial condition \( \xi = \alpha \xi_1 + \beta \xi_2 \) and if we define \( w_p = \alpha \mathcal{H} \xi_1 + \beta \mathcal{H} \xi_2 \) then it is easily checked that \( w_p, \xi \) satisfy equation (3.29) and hence \( w_p = \mathcal{H} \xi \). In this way we have shown that
3.4 Solvability of the Riccati equation

\( \mathcal{H} : \xi \rightarrow w_* \) is linear. Since \( w_* \) depends linearly on \( \xi \), this implies that \( \eta_*(0) \) depends linearly on \( \xi \) and hence there exists a matrix \( P \) such that \( \eta_*(0) = -P\xi \).

We shall now look at time \( t \). The sup-inf problem starting at time \( t \) with initial state \( x(t) \) can now be solved. Due to time invariance we see that \( w_* \) restricted to \([t, \infty)\) satisfies (3.29) and hence, for this problem, the optimal \( x \) and \( \eta \) are \( x_* \) and \( \eta_* \). But since \( t \) is the initial time for this optimization problem, which is exactly equal to the original optimization problem, we find equation (3.33) at time \( t \) with the same matrix \( P \) as at time 0. Since \( t \) was arbitrary this completes the proof.


**Lemma 3.9**: The matrix \( P \) defined by lemma 3.8 satisfies

\[
C^*(\xi) = \xi^T P \xi. \tag{3.34}
\]

**Proof**: We integrate equation (3.31) from zero to infinity. By (3.14) we find

\[
C(\xi, w_*) + 2\eta_*^T(0)\xi = -C(\xi, w_*).
\]

Since \( C(\xi, w_*) = C^*(\xi) \) and \( \eta_*(0) = -P\xi \) we find (3.34).

Using the above it will be shown that this matrix \( P \) satisfies condition (iii) of theorem 3.1.

**Lemma 3.10**: Assume that the system \((A, B, C, D_1)\) has no invariant zeros on the imaginary axis. Moreover assume that \( D_1 \) is injective. Finally assume that

\[
D_1^T [C \quad D_2] = 0
\]

and \( \gamma = 1 \). If the statement in part (ii) of theorem 3.1 is satisfied, then there exists a symmetric matrix \( P \) satisfying part (iii) of theorem 3.1.
The regular full-information case

Proof: By lemma 3.8 we have \( \eta_\ast = -P_\ast x_\ast \). Using this we find that:

\[
\begin{align*}
  w_\ast &= (I - D_2^T D_2)^{-1} (E^T P + D_2^T C) x_\ast, \\
  u_\ast &= -(D_1^T D_1)^{-1} B^T P x_\ast.
\end{align*}
\]

(3.35)

Thus we get

\[
\dot{x}_\ast = A_{cl} x_\ast
\]

where \( A_{cl} \) as defined in theorem 3.1. Since \( x_\ast \in \mathbb{L}_2^\infty \) for every initial state \( \xi \) we know that \( A_{cl} \) is asymptotically stable. Next we show that \( P \) satisfies the algebraic Riccati equation as given in theorem 3.1. From (3.15) and (3.33) combined with (3.35) we find

\[
-P A_{cl} = A^T P + C^T C + C^T D_2 (I - D_2^T D_2)^{-1} (E^T P + D_2^T C).
\]

Using the definition of \( A_{cl} \) this equation turns out to be equivalent to the algebraic Riccati equation. Next we show that \( P \) is symmetric. Note that both \( P \) and \( P^T \) satisfy the algebraic Riccati equation. Using this we find that

\[
A_{cl}^T (P - P^T) + (P - P^T) A_{cl} = 0.
\]

Since \( A_{cl} \) is asymptotically stable this implies that \( P = P^T \). \( P \) can be shown to be positive semi-definite by combining lemma 3.5 and (3.34).

From lemma 3.10 we can derive the implication \( (ii) \Rightarrow (iii) \) in theorem 3.1 without the assumption \( \gamma = 1 \) and without the assumption \( D_1^T [C \quad D_2] = 0 \):

Corollary 3.11: Assume that \( (A, B, C, D_1) \) has no invariant zeros on the imaginary axis. Moreover assume that \( D_1 \) is injective. If part (ii) of theorem 3.1 is satisfied, then there exists a symmetric matrix \( P \geq 0 \) satisfying part (iii) of theorem 3.1.

Proof: First we scale the \( \gamma \) to 1, i.e. we set \( E_{new} = E/\gamma \) and \( D_{2,new} = D_2/\gamma \). The rest of the system parameters does not change. Then we apply a preliminary feedback \( u = \tilde{F}_1 x + \tilde{F}_2 w + v \) such that \( D_1^T (C + D_1 \tilde{F}_1) = 0 \) and \( D_1^T (D_{2,new} + D_1 \tilde{F}_2) = 0 \). Denote the new \( A, C, D_{2,new} \) and \( E_{new} \) by \( \tilde{A}, \tilde{C}, \tilde{D}_2 \) and \( \tilde{E} \). For this new system part (ii) of theorem 3.1 is satisfied for \( \gamma = 1 \).
3.5 Existence of a suitable controller

We also know that by applying a preliminary state feedback the invariant zeros of a system do not change. Therefore our new system does not have invariant zeros on the imaginary axis. Hence, since for this new system we also know that $D_1^T [\bar{C} \quad \bar{D}_2] = 0$, we may apply lemma 3.10. Thus we find conditions in terms of the new parameters. Rewriting in terms of the original parameters gives the desired conditions in part (iii) of theorem 3.1.

We conclude this section by proving the implication (i) $\Rightarrow$ (ii) of theorem 3.2.

Lemma 3.12 : Let $\gamma > 0$ be given. Assume that $(A, B, C, D_1)$ has no invariant zeros on the imaginary axis. Moreover assume that $D_1$ is injective. If part (i) of theorem 3.2 is satisfied, then there exists a symmetric matrix $P \geq 0$ satisfying part (ii) of theorem 3.2.

Proof : Since the conditions in part (i) of theorem 3.2 imply that the conditions in part (ii) of theorem 3.1 are satisfied we know by corollary 3.11 that there exists a $P$ satisfying the conditions in part (iii) of theorem 3.1. Therefore it only remains to be shown that $D_2^T D_2 < \gamma^2 I$. There exists a state feedback $\Sigma_F$ which makes the $H_\infty$ norm less than $\gamma$. For all $\omega \in \mathcal{R}$, the largest singular value of the closed loop transfer matrix $G_F(i\omega)$ is less than $\|G_F\|_\infty$ which in its turn is less than $\gamma$. Hence, by letting $\omega \rightarrow \infty$ we find that the largest singular value of $D_2$ is less than $\gamma$. Therefore all eigenvalues of $I - D_2^T D_2$ are less than 0 so $D_2^T D_2 < \gamma^2 I$.

3.5 Existence of a suitable controller

In this section we shall show that if there exists a matrix $P$ satisfying the conditions in part (iii) of theorem 3.1, then the feedback as suggested by theorem 3.1 satisfies condition (i). We also show the implication (ii) $\Rightarrow$ (i) in theorem 3.2. To this end we shall assume throughout this section that there exists a matrix $P$ satisfying the conditions in part (iii) of theorem 3.1 (which are the same as the conditions on $P$ in part (ii) of theorem 3.2.) Moreover we set $\gamma = 1$. The implications will only be proven under the assumption $\gamma = 1$. Just as in the proof of corollary 3.11 the general case can be reduced to the special case $\gamma = 1$ by scaling.

We define the following system:
The regular full-information case

\( \Sigma_U : \begin{cases} 
\dot{x}_U = A_U x_U + B_U u_U + E_U w, \\
y_U = C_{1,U} x_U + D_{12,U} w, \\
z_U = C_{2,U} x_U + D_{21,U} u_U + D_{22,U} w, 
\end{cases} \) (3.36)

where

\[
A_U := A - B (D_1^T D_1)^{-1} (D_1^T C + B^T P), \\
B_U := B (D_1^T D_1)^{-1/2}, \\
E_U := E - B (D_1^T D_1)^{-1} D_1^T D_2, \\
V := I - D_2^T (I - D_1 (D_1^T D_1)^{-1} D_1^T) D_2, \tag{3.37}
\]

\[
C_{1,U} := V^{-1/2} \left[ D_2^T D_1 (D_1^T D_1)^{-1} (B^T P + D_1^T C) - D_2^T C - E^T P \right], \\
C_{2,U} := C - D_1 (D_1^T D_1)^{-1} (D_1^T C + B^T P), \\
D_{12,U} := V^{1/2}, \\
D_{21,U} := D_1 (D_1^T D_1)^{-1/2}, \\
D_{22,U} := D_2 - D_1 (D_1^T D_1)^{-1} D_1^T D_2.
\]

Note that \( V \) is invertible by (3.2).

**Lemma 3.13**: The system \( \Sigma_U \) as defined by (3.36) is inner. Denote the transfer matrix of \( \Sigma_U \) by \( G_U \). We decompose \( G_U \):

\[
G_U \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} G_{11,U} & G_{12,U} \\ G_{21,U} & G_{22,U} \end{pmatrix} \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix}
\]

compatible with the sizes of \( w, u_U, z_U \) and \( y_U \). Then \( G_{21,U} \) is invertible as a rational matrix and its inverse is in \( H_\infty \). Moreover \( G_{22,U} \) is strictly proper.

**Proof**: Since \( P \) satisfies the conditions of part (iii) of theorem 3.1 it can be easily checked that \( X = P \) satisfies the equations (i) and (ii) of lemma.
3.5 Existence of a suitable controller

2.10 for the system $\Sigma_U$ with inputs $w, u_U$ and outputs $z_U, y_U$. Indeed, the algebraic Riccati equation in theorem 3.1 is equivalent to the equation in (i) of lemma 2.10. Also, by simply writing out the equations in the original system parameters of system (3.1) we find conditions (ii) and (iii) of lemma 2.10.

Moreover, we know that $P \geq 0$ and the equation in (i) is equivalent to

$$PA_U + A_U^T P + \begin{pmatrix} C_{1,u}^T & C_{2,u}^T \end{pmatrix} \begin{pmatrix} C_{1,u} & C_{2,u} \end{pmatrix} = 0.$$ 

Note that $A_{cl} = A_U - E_U D_{12,u}^{-1} C_{1,u}$. Therefore, since $A_{cl}$ is asymptotically stable we find that $(C_{1,u}, A_U)$ is detectable. Using standard Lyapunov theory it can then be shown that $A_U$ is asymptotically stable. (A more precise proof can be based on the proof of corollary A.8.) By applying lemma 2.10 we thus find that $\Sigma_U$ is inner.

To show that $G_{21,u}^{-1}$ is an $H_\infty$ function we can write down a realization for $G_{21,u}^{-1}$ and use the expression for $A_{cl}$ as given above.

**Lemma 3.14**: Assume that there exists a matrix $P$ satisfying the conditions in part (iii) of theorem 3.1. In that case the compensator $\Sigma_F$ described by the feedback law $u = F_1 x + F_2 w$, where $F_1, F_2$ are given by (3.9) and (3.4), satisfies condition (i) of theorem 3.1.

**Proof**: First note that for this particular $F$ the transfer matrix $G_F$ as given by (2.7) is equal to $G_{11,u}$ and, moreover, $A + BF_1$ is equal to $A_U$. This implies that the compensator $\Sigma_F$ is internally stabilizing. Because $G_{21,u}$ is invertible over $H_\infty$ we can derive the following inequalities:

$$\|G_{11,u}\|_\infty^2 + \|G_{21,u}^{-2}\|_\infty^2 \leq \|\begin{pmatrix} G_{11,u} & G_{21,u} \end{pmatrix}\|_\infty^2 \leq 1,$$  

(3.38)

which shows that we have $\|G_F\|_\infty < 1$. The second inequality in (3.38) follows because a submatrix of an inner matrix has $H_\infty$ norm less than or equal to 1.

Note that theorem 3.1 for $\gamma = 1$ is simply a combination of corollary 3.11 and lemma 3.14. The general result can then be obtained by scaling. For theorem
The regular full-information case

3.2 the implication (i) ⇒ (ii) is stated in lemma 3.12. The implication (ii) ⇒ (i) is a direct consequence of the following lemma.

**Lemma 3.15**: Assume that there exists a $P$ satisfying the conditions in (ii) of theorem 3.2. In that case the feedback $u = Fx$ where $F$ is given by (3.5) satisfies condition (i) of theorem 3.1. □

**Proof**: Since $P$ satisfies the conditions in part (ii) of theorem 3.2 it certainly satisfies the conditions in part (iii) of theorem 3.1. We again define $\Sigma_U$ by (3.36). We apply the compensator $\Sigma_F$ described by the static output feedback law $u_v = F y_u$ to the system $\Sigma_U$ where $F$ is defined by

$$F := (D_1^T D_1)^{-1/2} D_1^T D_2 V^{-1/2} (3.39)$$

and $V$ as defined by (3.37). We have

$$\bar{F}^T \bar{F} = I - V^{-1/2} (I - D_2^T D_2) V^{-1/2} < I. \quad (3.40)$$

Therefore the feedback operator defined by $\Sigma_F$ has $H_\infty$ norm less than 1. Moreover, by lemma 3.13, $\Sigma_U$ is inner. We now apply lemma 2.12 with $\Sigma_1$ equal to $\Sigma_U$ and $\Sigma_2$ equal to $\Sigma_F$. We find that the interconnection $\Sigma_U \times \Sigma_F$ is internally stable and has $H_\infty$ norm less than 1. It is easily checked that the systems $\Sigma \times \Sigma_F$ and $\Sigma_U \times \Sigma_F$ are equal, where $\Sigma_F$ is defined by the static state feedback law $u = Fx$ and $F$ is defined by (3.5). Hence the closed loop system $\Sigma \times \Sigma_F$ we obtain by applying the state feedback compensator $\Sigma_F$ to the original system $\Sigma$ is internally stable and has $H_\infty$ norm less than 1. □
Chapter 4

The general full-information $H_{\infty}$ control problem

4.1 Introduction

As we already mentioned in section 3.1 the full-information $H_{\infty}$ control problem was discussed in many papers. We showed that if we make two assumptions on the system parameters, then we could derive elegant necessary and sufficient conditions under which there exists an internally stabilizing controller which makes the $H_{\infty}$ norm of the closed loop system less than some, a priori given, bound. Moreover, under these assumptions, it was possible to derive an explicit formula for one "suitable" controller. These assumptions were

- The direct feedthrough matrix from the control input to the output is injective,
- The subsystem from the control input to the output has no invariant zeros on the imaginary axis.

In this chapter we shall discuss the general full-information $H_{\infty}$ problem where these conditions are not necessarily satisfied. We shall remove the first assumption in an, in our opinion, elegant way. The second assumption is removed in [SC3] but in this reference the first assumption is still made. At this moment no satisfactory method is available which removes both assumptions at the same time. Some methods to remove the second assumption will be briefly discussed in section 4.6.

In [Kh2, Kh3, Pe3, Pe5, ZK2] conditions were derived for the existence of suitable controllers without these assumptions. However, these conditions
are in terms of a family of algebraic Riccati equations parametrized by a positive constant $\varepsilon$. In the previous chapter we derived conditions which were expressed in terms of one single Riccati equation. The problem is that this Riccati equation no longer exists when the direct feedthrough matrix from control input to output is not injective. In this chapter we shall discuss the method from [St2]. We shall express our conditions in terms of a quadratic matrix inequality and a number of rank conditions. In the case that the direct feedthrough matrix is injective it will be shown that quadratic matrix inequality together with one rank condition reduce to the algebraic Riccati equation from the previous chapter. This quadratic matrix inequality is reminiscent of the dissipation inequality appearing in linear quadratic optimal control (see [Gee, SH2, Tr, Wi4]). The existence of solutions satisfying the rank conditions can be checked by solving a reduced order Riccati equation. However, in contrast with the previous chapter, we are not able to derive an explicit formula for one controller, if one exists, which is internally stabilizing and which makes the $H_\infty$ norm of the closed loop system less than some, a priori given, bound $\gamma$.

We shall first assume that the direct feedthrough matrix from disturbance to output is zero and we shall show at the end of this chapter how the general result can be obtained. This is done to prevent the complexity of the formulas from clouding the comprehension of the reader.

The outline of this chapter is as follows. In section 4.2 we briefly recall the problem to be studied and give a statement of our main result. We shall also show how the results of this chapter reduce to the results from chapter 3 in the case that the direct feedthrough matrix from control input to output is injective. In section 4.3 we shall prove necessity and in section 4.4 we shall show sufficiency. These sections will strongly depend on the decompositions as introduced in appendix A. In section 4.5 we shall give a method to solve the case that the direct feedthrough matrix from disturbance to output is unequal to zero. Finally, in section 4.6, we shall give methods for handling invariant zeros on the imaginary axis. None of these methods for handling invariant zeros on the imaginary axis is completely satisfactory, but combining the results of this chapter with the results from [SC3] will probably yield elegant conditions. However, this still has to be investigated.

4.2 Problem formulation and main results

We consider the finite-dimensional, linear, time-invariant system
4.2 Problem formulation and main results

\[ \dot{x} = Ax + Bu + Ew, \]
\[ z = Cx + D_1u, \]

(4.1)

where, for each \( t \), \( x(t) \in \mathcal{R}^n \) is the state, \( u(t) \in \mathcal{R}^m \) is the control input, \( w(t) \in \mathcal{R}^l \) is the disturbance and \( z(t) \in \mathcal{R}^q \) is the output to be controlled. \( A, B, E, C \) and \( D_1 \) are matrices of appropriate dimensions. As in the previous chapter we would like to minimize the effect of the disturbance \( w \) on the output \( z \) by finding an appropriate control input \( u \). More precisely, we seek a compensator \( \Sigma_F \) described by a static state feedback law \( u = Fx \) such that after applying this feedback law in the system (4.1), the resulting closed loop system \( \Sigma \times \Sigma_F \) is internally stable and its transfer matrix, denoted by \( G_F \), has \( H_\infty \) norm strictly less than some, a priori given, bound \( \gamma \). In principle one can then obtain, via a search procedure, the infimum over all internally stabilizing compensators of the \( H_\infty \) norm of the closed loop operator (see e.g. [SC, SC2]).

For any real number \( \gamma > 0 \) and matrix \( P \in \mathcal{R}^{n \times n} \) we define a matrix \( F_\gamma(P) \in \mathcal{R}^{(n+m) \times (n+m)} \) by

\[ F_\gamma(P) := \begin{pmatrix} PA + A^T P + \gamma^{-2} PEE^T P + C^T C & PB + C^T D_1 \\
B^T P + D_1^T C & D_1^T D_1 \end{pmatrix}. \]

(4.2)

If \( \gamma = 1 \) we shall simply write \( F(P) \) instead of \( F_1(P) \). Clearly if \( P \) is symmetric, then \( F_\gamma(P) \) is symmetric as well. If \( F_\gamma(P) \geq 0 \), then we shall say that \( P \) is a solution of the quadratic matrix inequality at \( \gamma \).

In addition to (4.2), for any \( \gamma > 0 \) and \( P \in \mathcal{R}^{n \times n} \), we define a \( n \times (n+m) \) matrix pencil \( L_\gamma(P,s) \) by

\[ L_\gamma(P,s) := \begin{pmatrix} sI - A - \gamma^{-2} E E^T P & -B \\
0 & D_1 \end{pmatrix}. \]

(4.3)

Again if \( \gamma = 1 \) we shall write \( L(P,s) \) instead of \( L_1(P,s) \). We note that \( L_\gamma(P,s) \) is the controllability pencil associated with the system

\[ \dot{x} = (A + \gamma^{-2} E E^T P) x + Bu. \]

\( G_{ci} \) will denote the transfer matrix of the system \( \Sigma_{ci} := (A, B, C, D_1) \) which is a subsystem of \( \Sigma \) as described by (4.1). We are now in the position to formulate the main result of this chapter:
Theorem 4.1: Consider the system (4.1). Let $\gamma > 0$. Assume that the system $(A, B, C, D_1)$ has no invariant zeros on the imaginary axis. Then the following three statements are equivalent:

(i) There exists a static feedback law $u = Fx$ such that after applying this compensator to the system (4.1) the resulting closed loop system is internally stable and the closed loop operator $G_F$ has $H_\infty$ norm less than $\gamma$, i.e. $\|G_F\|_\infty < \gamma$.

(ii) $(A, B)$ is stabilizable and for the system (4.1) there exists a $\delta < \gamma$ such that for all $w \in L_2^1$ there exists an $u \in L_\infty^* \delta$ such that $x_{u,w} \in L_2^1$ and $\|x_{u,w}\|_2 \leq \delta\|w\|_2$.

(iii) There exists a real symmetric solution $P \geq 0$ to the quadratic matrix inequality $F_\gamma(P) \geq 0$ such that

$$\text{rank } F_\gamma(P) = \text{rank}_{R(s)} G_{ci}$$

and

$$\text{rank } \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + \text{rank}_{R(s)} G_{ci} \quad \forall s \in C^0 \cup C^+.$$  

Remarks:

(i) The implication (i) $\Rightarrow$ (ii) is trivial. Hence we only need to prove (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The first step will be done in the next section and in the section after that we shall complete the proof. We shall only prove this result for $\gamma = 1$, the general result can be easily obtained by scaling.

(ii) The case that the assumption concerning the invariant zeros on the imaginary axis is not satisfied will be discussed in section 4.6. The extension to the more general system (3.1), where $D_2$ is arbitrary, will be discussed in section 4.5.

Before embarking on a proof of this theorem we would like to point out how the results from the previous chapter for $D_2 = 0$ and $D_1$ injective can be obtained from our theorem as a special case. First note that in this case we have
4.3 Solvability of the quadratic matrix inequality

\[ \text{rank}_{G_{ci}}(s) = \text{rank } D_1 = m. \]

Define

\[ R_\gamma(P) := PA + A^T P + \gamma^{-2}PEE^T P + C^T C \]
\[ - (PB + C^T D_1)(D_1^T D_1)^{-1}(B^T P + D_1^T C). \]

Furthermore, define a real \((n + m) \times (n + m)\) matrix by

\[ S(P) := \begin{pmatrix} I & -(PB + C^T D_1)(D_1^T D_1)^{-1} \\ 0 & I \end{pmatrix}. \]

Then we clearly have

\[ S(P)F_\gamma(P)S(P)^T = \begin{pmatrix} R_\gamma(P) & 0 \\ 0 & D_1^T D_1 \end{pmatrix}. \]

From this we can see that the conditions \(F_\gamma(P) \geq 0\) and \(\text{rank}_{G_{ci}}(s)F_\gamma(P) = m\) are equivalent to the single condition \(R_\gamma(P) = 0\). We now analyse the second rank condition appearing in our theorem. It is easily verified that for all \(s \in C\) we have

\[ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B(D_1^T D_1)^{-1} \\ -(PB + C^T D_1)(D_1^T D_1)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} \]

\[ = \begin{pmatrix} sI - A - \gamma^{-2}EE^T P + B(D_1^T D_1)^{-1}(B^T P + D_1^T C) & 0 \\ R_\gamma(P) & 0 \\ B^T P + D_1^T C & D_1^T D_1 \end{pmatrix}. \]

Consequently, if \(R_\gamma(P) = 0\), then for each \(s \in C\) the condition

\[ \text{rank} \left( \begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} \right) = n + \text{rank}_{G_{ci}}. \tag{4.6} \]

is equivalent to

\[ \text{rank} \left( sI - A - \gamma^{-2}EE^T P + B(D_1^T D_1)^{-1}(B^T P + D_1^T C) \right) = n \]

and hence, since (4.6) holds for all \(s \in C^0 \cup C^+\), we find that the matrix

\[ A + \gamma^{-2}EE^T P - B(D_1^T D_1)^{-1}(B^T P + D_1^T C) \]

is asymptotically stable. Theorem 3.1 for the case that \(D_2 = 0\) is then obtained immediately.
4.3 Solvability of the quadratic matrix inequality

In this section we shall establish a proof of the implication (ii) \( \Rightarrow \) (iii) in theorem 4.1: assuming that the condition in part (ii) is satisfied we will show that there exists a solution of the quadratic matrix inequality satisfying the two rank conditions. As announced just after theorem 4.1 we shall only prove this result for \( \gamma = 1 \). Therefore throughout this section we assume that \( \gamma = 1 \).

Consider our system (4.1). For the special case that \( D_1 \) is injective, we already have theorem 3.1 available. Our proof will use this result. This time we do not make assumptions on the matrix \( D_1 \). Choose bases in the state space, the input space and the output space as in appendix A (where \( D \) is replaced by \( D_1 \)) and apply the preliminary feedback \( u = F_0 x + v \) where \( F_0 \) is given by (A.1). We obtain the system

\[
\dot{x} = (A + BF_0)x + Bv + Ew,
\]
\[
z = (C + D_1F_0)x + D_1v
\]

(4.7)

After this transformation, in terms of our decomposition, we have the equations (A.5)-(A.7). The idea we want to pursue is the following. We know condition (ii) of theorem 4.1 is still satisfied after the preliminary feedback, i.e. there exists a \( \delta < 1 \) such that for all \( w \in L^2_2 \) there exists a \( v \in L^2 \) such that

\[
\| x_{v,w} \|_2 \leq \delta \| w \|_2,
\]
\[
x_{v,w} \in L^2_2.
\]

(4.8)

Then, for a given \( w \), let \( v \) be such that these conditions are satisfied. Define \( v_1 \) as the first component of \( v \) and take \( z_3 \) as the third component of \( x_{v,w} \). Interpret \( (v_1, z_3) \) as an input for the subsystem \( \hat{\Sigma} \) defined by the equations (A.5) and (A.7). It then follows from (4.8) that

\[
\| z_1 \|^2_2 + \| z_2 \|^2_2 \leq \delta^2 \| w \|^2_2,
\]
\[
x_1 \in L^2_2.
\]

Moreover note that the fictitious input \( z_3 \in L_2 \). Since \( w \in L_2 \) was arbitrary we note that \( \hat{\Sigma} \) satisfies condition (ii) of theorem 4.1 with "inputs" \( (v_1, z_3) \), disturbance \( w \) and output \( (z_1, z_2) \). The crucial observation is now that the direct feedthrough matrix of \( \hat{\Sigma} \) is injective (see lemma A.2). Thus we can apply theorem 3.1 to the system \( \hat{\Sigma} \). Before doing this we should make sure that \( (A, (B_{11}, A_{13})) \) is stabilizable and that \( \Sigma_{ci} \) given by (A.12) has
4.4 Existence of state feedback laws

no invariant zeros on the imaginary axis. It is easily seen that if \((A, B)\) is stabilizable, then also \((A, (B_{i1}, A_{13}))\) stabilizable. Furthermore if \(\Sigma_{\alpha} = (A, B, C, D_1)\) has no invariant zeros on the imaginary axis, then the same holds for \(\hat{\Sigma}_{\alpha}\) (see lemma A.3). Consequently, we may apply theorem 3.1 to \(\hat{\Sigma}\) and we obtain:

**Corollary 4.2:** Consider the system (4.1). Assume that \((A, B, C, D_1)\) has no invariant zeros on the imaginary axis. Moreover, assume that part (ii) of theorem 4.1 is satisfied. Then there exists a real symmetric solution \(P_1 \geq 0\) to the algebraic Riccati equation \(R(P_1) = 0\), where \(R\) is defined by (A.17), such that \(Z(P_1)\) is asymptotically stable, where \(Z\) is defined by (A.18).

The basic idea pursued in section A.2 is that there is a one-one relation between solutions of the algebraic Riccati equation \(R(P_1) = 0\) and solutions of the quadratic matrix inequality \(F(P) \geq 0\) which satisfy the first rank condition (4.4). This is formalized in theorem A.6. The implication (ii) \(\Rightarrow\) (iii) is then obtained directly by combining corollary 4.2 with theorem A.6.

4.4 Existence of state feedback laws

In this section we give a proof of the implication (iii) \(\Rightarrow\) (i) in theorem 4.1. We shall first explain the idea of the proof. Again, we consider the parameters of our control system with respect to the bases of appendix A and after applying the preliminary feedback \(u = F_0 x + v\) where \(F_0\) is defined by (A.1). In this way we obtain the system \(\hat{\Sigma}\) defined by (4.7).

Also, as in the previous section, we consider our control system as the interconnection of the subsystem \(\Sigma\) given by the equations (A.5),(A.7) and the subsystem \(\Sigma_0\) given by (A.6). Suppose that the quadratic matrix inequality has a positive semi-definite solution at \(\gamma = 1\) such that the rank conditions (4.4) and (4.5) hold. Then according to theorem A.6 the algebraic Riccati equation associated with the subsystem \(\Sigma\) (with "inputs" \(v_1\) and \(x_3\)) has a positive semi-definite solution \(P_1\) such that the corresponding matrix \(Z(P_1)\), defined by (A.18), is asymptotically stable. Thus by applying theorem 3.1 to the subsystem \(\hat{\Sigma}\) we find that the "feedback law"

\[
\begin{align*}
v_1 &= -\left(\bar{D}^T\bar{D}\right)^{-1}B_{11}^T P_1 x_1, \\
x_3 &= -(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{21}) x_1,
\end{align*}
\]
The general full-information $H_\infty$ control problem

applied to the system $\tilde{\Sigma}$ yields an internally stable system and the closed loop transfer matrix has $H_\infty$ norm smaller than 1. Now what we shall do is the following: we shall construct a state feedback law for the original system (4.1) in such a way that in the subsystem $\tilde{\Sigma}$ the equality (4.10) holds approximately and the equality (4.9) holds exactly. The closed loop transfer matrix of the original system will then be approximately equal to that of the subsystem $\tilde{\Sigma}$ and will therefore also have $H_\infty$ norm smaller than 1.

In our proof an important role will be played by a result in the context of the problem of almost disturbance decoupling as studied in [Tr] and [Wi2]. The results we need are recapitulated in section 2.6. We shall now formulate and prove the converse of corollary 4.2:

**Theorem 4.3:** Consider the system (4.1). Assume that $(A, B, C, D_1)$ has no invariant zeros on the imaginary axis. Assume that there exists a real symmetric solution $P_1 \geq 0$ to the algebraic Riccati equation $R(P_1) = 0$, where $R$ is defined by (A.17), such that $Z(P_1)$, defined by (A.18), is asymptotically stable. Then there exists $F \in R^{m \times n}$ such that $A + BF$ is asymptotically stable and such that after applying the state feedback law $u = Fx$ to $\Sigma$ we have $\|G_F\|_\infty < 1$.

**Proof:** Clearly it is sufficient to prove the existence of such state feedback law $v = Fx$ for the system (4.7). Let this system be decomposed according to (A.5)-(A.7). Choose

$$v_1 = -\left(\hat{D}^T \hat{D}\right)^{-1} B_{11}^T P_1 x_1$$

and introduce a new state variable $q_3$ by

$$q_3 := x_3 + (C_{22}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{21}) x_1$$

Then the equations (A.5)-(A.7) can be rewritten as

$$x_1 = \ddot{A}_{11} x_1 + A_{13} q_3 + E_1 w,$$

$$\begin{pmatrix} \dot{z}_2 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \ddot{A}_{33} \end{pmatrix} \begin{pmatrix} z_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} \ddot{A}_{21} \\ \ddot{A}_{31} \end{pmatrix} x_1 + \begin{pmatrix} E_2 \\ \ddot{E}_3 \end{pmatrix} w,$$
4.4 Existence of state feedback laws

\[
\begin{pmatrix}
  z_1 \\
  z_2
\end{pmatrix} = \begin{pmatrix}
  \tilde{c}_1 \\
  \tilde{c}_2
\end{pmatrix} z_1 + \begin{pmatrix}
  0 \\
  C_{23}
\end{pmatrix} q_3. 
\]  

(4.13)

Here we used the following definitions:

\[
\tilde{A}_{11} := A_{11} - A_{13}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{23}) \tilde{A}_{13},
\]

\[
\tilde{A}_{21} := A_{21} - A_{23}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{23}) \tilde{A}_{13},
\]

\[
\tilde{A}_{31} := A_{31} - A_{33}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{23}) \tilde{A}_{13},
\]

\[
\tilde{A}_{33} := A_{33} + (C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{23}) A_{13},
\]

\[
\tilde{C}_1 := -\tilde{D}(\tilde{D}^T \tilde{D})^{-1} B_{11} P_1,
\]

\[
\tilde{C}_2 := C_{21} - C_{23}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{23}) A_{13},
\]

\[
\tilde{E}_3 := E_3 + (C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{23}) E_1.
\]

According to theorem 3.1, if in the subsystem formed by the equations (4.11) and (4.13) we have \( q_3 = 0 \), then its transfer matrix from \( w \) to \( z \) has \( H_\infty \) norm smaller than 1. Moreover, by corollary (A.8), we know that \( \tilde{A}_{11} \) is asymptotically stable. Hence, there exist \( M > 0 \) and \( \rho > 0 \) such that for all \( w \) and \( q_3 \) in \( \mathbb{L}_2 \) we have

\[
\|z\|_2 < (1 - \rho)\|w\|_2 + M\|q_3\|_2. 
\]  

(4.14)

Also, since \( \tilde{A}_{11} \) is asymptotically stable, there exist \( M_1, M_2 \) > 0 such that for all \( w \) and \( q_3 \) in \( \mathbb{L}_2 \) we have

\[
\|z_1\|_2 < M_1\|w\|_2 + M_2\|q_3\|_2.
\]  

(4.15)

We claim that the following system is strongly controllable:

\[
\begin{bmatrix}
  (A_{22} & A_{23}) \\
  (A_{32} & \tilde{A}_{33})
\end{bmatrix}, \begin{pmatrix}
  B_{22} \\
  B_{32}
\end{pmatrix}, \begin{pmatrix}
  0 \\
  I
\end{pmatrix}, 0
\]  

(4.16)

This can be seen by the following transformation:

\[
\begin{pmatrix}
  I & 0 & 0 \\
  0 & I & A_{33} - \tilde{A}_{33}
\end{pmatrix} \begin{pmatrix}
  sI - A_{22} & -A_{23} & -B_{22} \\
  -A_{32} & sI - A_{33} & -B_{32}
\end{pmatrix}
\]
The general full-information $H_\infty$ control problem

\[
\begin{pmatrix}
    sI - A_{22} & -A_{23} & -B_{22} \\
    -A_{32} & sI - \bar{A}_{33} & -B_{32} \\
    0 & I & 0
\end{pmatrix}.
\]

Since the first matrix on the left is unimodular and the second matrix has full row rank for all $s \in \mathcal{C}$ (see lemma A.2), the matrix on the right has full row rank for all $s \in \mathcal{C}$. Hence the system (4.16) is strongly controllable by lemma 2.8.

Consider now the almost disturbance decoupling problem for the system (4.12) with output $q_3$ and "disturbance" $(x_1, w)$. According to lemma 2.17 the strong controllability of (4.16) implies that there exists a feedback law

\[
v_2 = F_1 \begin{pmatrix} x_2 \\ q_3 \end{pmatrix}
\]

such that in (4.12) we have

\[
\|q_3\|_2 < \frac{\rho}{2} (M + M_1 M + \rho M_2)^{-1} \{\|w\|_2 + \|x_1\|_2\} \tag{4.17}
\]

for all $w$ and $x_1$ in $\mathcal{L}_2$, and such that the matrix

\[
\bar{A} := \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \bar{A}_{33} \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} F_1
\]

is asymptotically stable. Combining (4.14), (4.15) and (4.17) gives us

\[
\|z\|_2 < \left(1 - \frac{\rho}{2}\right) \|w\|_2
\]

for all $w$ in $\mathcal{L}_2$. Summarizing, we have shown that if in our original system (4.7) we apply the state feedback law

\[
v_1 = -\left(\hat{D}^T \hat{D}\right)^{-1} B_{11}^T P_1 x_1
\]

\[
v_2 = F_1 \begin{pmatrix} x_2 \\ x_3 + (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{21}) x_1 \end{pmatrix}, \tag{4.18}
\]

then for all $w \in \mathcal{L}_2$ we have $\|z\|_2 < \delta\|w\|_2$ with $\delta < 1$. Thus, the $H_\infty$ norm of the resulting closed loop transfer matrix is smaller than 1.

It remains to be shown that the closed loop system is internally stable. We know that
4.4 Existence of state feedback laws

The closed loop A-matrix resulting from the feedback (4.18) is given by

\[
\begin{pmatrix}
\tilde{A}_{11} & 0 & A_{13} \\
\tilde{A}_{21} & \tilde{A} & \\
\tilde{A}_{31} & \tilde{A}
\end{pmatrix}
\]

Assume that \((x^T, y^T, z^T)^T\) is an eigenvalue of \(A_{cl}\) with eigenvalue \(\lambda > 0\). It can be seen that

\[
x = (\lambda I - \tilde{A}_{11})^{-1} A_{13} z.
\]

(4.21)

\[
z = \begin{pmatrix} 0 & I \end{pmatrix} (\lambda I - \tilde{A})^{-1} \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix} x.
\]

(4.22)

(Note that the inverses exist due to the fact that \(\tilde{A}_{11}\) and \(\tilde{A}\) are stable matrices). Combining (4.19) and (4.21) we find \(\|z\| \leq M_2\|z\|\) and combining (4.20) and (4.22) yields \(\|x\| \geq 2M_2\|z\|\). Hence \(x = z = 0\). This would however imply that \((y^T, 0)^T\) is an unstable eigenvector of \(\tilde{A}\). Since \(\tilde{A}\) is asymptotically stable this yields a contradiction. This proves that the closed loop system is internally stable.

Again using the one-one relation between solutions of the Riccati equation and solutions of the quadratic matrix inequality as given in theorem A.6, the implication (iii) \(\Rightarrow\) (i) in theorem 4.1 is now obtained by combining theorem A.6 and theorem 4.3.

Remark : In the regular case (i.e. \(D_1\) injective) it is quite easy to give an explicit expression for a suitable state feedback law. This was done in theorem 3.1. In the singular case (i.e. \(D_1\) is not injective) a suitable state feedback law is given by \(u = F_0 x + v\). Here, \(F_0\) is given by (A.1) and \(v = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}\) is given by (4.18). The matrix \(P_1\) is obtained by solving
the quadratic matrix inequality or, equivalently, by solving the reduced order Riccati equation $R(P_1) = 0$ where $R$ is defined by (A.17). The matrix $F_1$ is a "state feedback" for the strongly controllable auxiliary system (4.12). This state feedback achieves almost disturbance decoupling between the "disturbance" $(x_1, w)$ and the "output" $q_3$. The required accuracy of decoupling is expressed by (4.17). A conceptual algorithm to construct $F_1$ is given in section 5.4. However this algorithm is not numerically reliable.

4.5 A direct feedthrough matrix from disturbance to output

In this section we shall show how to handle the state feedback case for the more general system

$$
\begin{align*}
\Sigma : \left\{ \begin{array}{ll}
\dot{x} &= Ax + Bu + Ew, \\
z &= Cx + D_1u + D_2w.
\end{array} \right.
\end{align*}
$$

(4.23)

(i.e. we allow for $D_2 \neq 0$). We shall use a technique from [G15] to reduce this problem to the problem studied in section 4.2. Throughout this section we shall assume that $\gamma = 1$. The more general result can be easily derived by scaling. There is one notable difference between the regular case and the singular case if the extra direct feedthrough matrix is present. In the regular case conditions (i) and (ii) of theorem 3.1 are equivalent. This is not true in the singular case. An easy counterexample is the following:

Example 4.4 : Let the following system be given:

$$
\begin{align*}
\Sigma : \left\{ \begin{array}{ll}
\dot{x} &= u, \\
z &= x + 100w.
\end{array} \right.
\end{align*}
$$

After applying an arbitrary static feedback of the form $u = f_1x + f_2w$ the closed loop transfer matrix will be given by

$$
g_f(s) = \frac{f_2}{s - f_1} + 100.
$$

Clearly $\|g_f\|_\infty \geq 100$. However, if we set $u = -100\dot{w}$, then $z = 0$ for all $w$ which are differentiable and satisfy $w(0) = 0$. (If $w$ is not differentiable or if $w(0) \neq 0$, then we can approximate $w$ arbitrarily well by a differentiable function $w_1$ which satisfies $w_1(0) = 0$) This implies that condition (ii) of theorem 3.1 is satisfied but on the other hand condition (i) of theorem 3.1 is certainly not satisfied. This yields the desired contradiction. □
In general the controllers we would need in the case that condition (ii) is satisfied while condition (i) is not satisfied would be non-proper, i.e. they would include differentiations of the disturbance. Since this is very undesirable we shall focus all our attention to controllers of the form (2.4). We shall restrict ourselves to the case of state feedback, i.e. \( y = x \). The result for the full-information case, i.e. \( y = (x, w) \), can be obtained in a similar way.

It is easily seen that when there exists a dynamic controller of the form (2.4) where \( y = x \) which makes the \( H_\infty \) norm less than 1 then \( \|D_2\| < 1 \). We therefore assume that \( \|D_2\| < 1 \). We define the following matrices

\[
A_D := A + E(I - D_2^T D_2)^{-1} D_2^T C, \\
B_D := B + E(I - D_2^T D_2)^{-1} D_2^T D_1, \\
E_D := -E(I - D_2^T D_2)^{-1/2}, \\
C_D := (I - D_2 D_2^T)^{-1/2} C, \\
D_{1,D} := (I - D_2 D_2^T)^{-1/2} D_1.
\]

Using these matrices we define the following system:

\[
\Sigma_D : \begin{cases}
\dot{x}_D = A_D x_D + B_D u_D + E_D w_D, \\
z_D = C_D x_D + D_{1,D} u_D.
\end{cases}
\]

(4.24)

We have the following lemma connecting \( \Sigma_D \) and \( \Sigma \):

Lemma 4.5 : Let \( \Sigma_F \) be a dynamic controller of the form (2.4). Consider the following two systems:

\[
\begin{array}{c}
\Sigma \\
\Sigma_F
\end{array} \quad \begin{array}{c}
\Sigma_D \\
\Sigma_F
\end{array}
\]

The system on the left is the interconnection of \( \Sigma \) described by (4.23) and \( \Sigma_F \) described by (2.4). The system on the right is the interconnection of \( \Sigma_D \) described by (4.24) and the same feedback compensator \( \Sigma_F \). The following two conditions are equivalent

\[
\begin{array}{c}
\Sigma \\
\Sigma_F
\end{array} \quad \begin{array}{c}
\Sigma_D \\
\Sigma_F
\end{array}
\]

(4.25)
(i) The system on the left is internally stable and the closed loop transfer matrix from $w$ to $z$ has $H_\infty$ norm less than $1$.

(ii) The system on the right is internally stable and the closed loop transfer matrix from $w_D$ to $z_D$ has $H_\infty$ norm less than $1$. $\square$

Proof: We define the following static system

$$\Sigma_\Theta: \begin{pmatrix} z_\Theta \\ y_\Theta \end{pmatrix} = \begin{pmatrix} D_2 & (I - D_2D_2^T)^{1/2} \\ -(I - D_2^T D_2)^{1/2} & D_2^T \end{pmatrix} \begin{pmatrix} w_\Theta \\ u_\Theta \end{pmatrix}. $$

It is straightforward to check that $\Sigma_\Theta$ is inner. We investigate the following two interconnections:

Then it is easily shown that these systems have identical realizations. Next, we investigate the following interconnections:

Because for the system on the left and the system on the right in (4.26) we have identical realizations it is immediate that in (4.27) the system on the
left is internally stable and has $H_\infty$ norm less than 1 if and only if the system on the right is internally stable and has $H_\infty$ norm less than 1.

We would now like to apply lemma 2.12 to the interconnection on the right where $\Sigma_1 = \Sigma_\varnothing$ and $\Sigma_2$ is the dashed system. As required, $\Sigma_\varnothing$ is inner and the 2,1 block of its transfer matrix is invertible in $H_\infty$. However, the 2,2 block of its transfer matrix, $G_{22}$, is not strictly proper. On the other hand, if one checks the proof of lemma 2.12 it is easily seen that the requirement that $G_{22}$ is strictly proper is only needed to show that $I - G_{22}G_2$ is invertible as a rational matrix, where $G_2$ is the transfer matrix of the dashed system. It can be easily checked that $I - G_{22}G_2$ evaluated at infinity is equal to $I - D_2^T D_2$. Hence, since $I - D_2^T D_2$ is invertible, the rational matrix $I - G_{22}G_2$ is invertible. This implies that the result of lemma 2.12 is still valid. Therefore in (4.27) the system on the right is internally stable and has $H_\infty$ norm less than 1 if and only if the dashed system is internally stable and has $H_\infty$ norm less than 1.

Combining, we find that in (4.27) the system on the left is internally stable and has $H_\infty$ norm less than 1 if and only if the dashed system is internally stable and has $H_\infty$ norm less than 1. By noting that the system on the left in (4.27) is equal to the system on the left in (4.25) and the dashed system in (4.27) is equal to the system on the right in (4.25) this completes the proof.

We can also derive the following result:

**Lemma 4.6**: Assume that the system $\Sigma$, described by (4.23), is such that $(A, B, C, D_1)$ has no invariant zeros on the imaginary axis. Then the system $\Sigma_D$, described by (4.24), is such that $(A_D, B_D, C_D, D_{1,D})$ has no invariant zeros on the imaginary axis.

**Proof**: We have the following relationship between the system matrices of the systems $(A, B, C, D_1)$ and $(A_D, B_D, C_D, D_{1,D})$ respectively:

$$
\begin{pmatrix}
I & E D_1^T (I - D_2 D_1^T)^{-1/2}
\end{pmatrix}
\begin{pmatrix}
sI - A_D & -B_D \\
C & D_{1,D}
\end{pmatrix}
= 
\begin{pmatrix}
sI - A & -B \\
C & D_1
\end{pmatrix}.
$$
The general full-information $H_\infty$ control problem

Since the first matrix on the left is unimodular, the above equality immediately gives the desired result.

We shall now state and prove the results for the more general system (4.23). The following theorem generalizes theorem 3.2.

Theorem 4.7: Let the system $\Sigma$ described by 4.23 be given. Assume that $(A, B, C, D_1)$ has no invariant zeros on the imaginary axis. Then the following three statements are equivalent:

(i) There exists a compensator $\Sigma_F$ described by the static state feedback law $u = Fx$ such that the closed loop system $\Sigma \times \Sigma_F$ is internally stable and has $H_\infty$ norm less than 1.

(ii) There exists a compensator $\Sigma_F$ of the form (2.4) with $y=x$ such that the closed loop system $\Sigma \times \Sigma_F$ is internally stable and has $H_\infty$ norm less than 1.

(iii) We have $\|D_2\| < 1$ and $\Sigma_D$ defined by (4.24) satisfies condition (iii) of theorem 4.1 with $\gamma = 1$.

Proof: The implication (i) $\Rightarrow$ (ii) is trivial

(ii) $\Rightarrow$ (iii): Note that if there exists a dynamic state feedback which makes the $H_\infty$ norm less than 1, then $\|D_2\| < 1$. By lemma 4.5 if there exists an internally stabilizing controller $\Sigma_F$ which makes the $H_\infty$ norm less than 1, then the same controller is internally stabilizing for the system $\Sigma_D$ defined by (4.24) and this controller also makes the $H_\infty$ norm of the closed loop system $\Sigma_D \times \Sigma_F$ less than 1. By lemma 4.6 $\Sigma_D$ satisfies the assumptions of theorem 4.1 and we already know $\Sigma_D$ satisfies condition (ii) of theorem 4.1. Therefore condition (iii) of theorem 4.1 is satisfied for the system $\Sigma_D$.

(iii) $\Rightarrow$ (i): By lemma 4.6 the system $\Sigma_D$ satisfies the assumptions of theorem 4.1 and moreover condition (iii) of theorem 4.1 is satisfied. Therefore, the system $\Sigma_D$ satisfies condition (i) of theorem 4.1. This implies that for the system $\Sigma_D$ there exists an internally stabilizing static state feedback which makes the closed loop $H_\infty$ norm less than 1. Finally by lemma 4.5 this state feedback satisfies condition (i) of theorem 4.7 for the system $\Sigma$.  

\[\square\]
4.6 Invariant zeros on the imaginary axis

In this section we shall discuss two methods which can be used to solve the $H_{\infty}$ problem with zeros on the imaginary axis. A method we shall not discuss and which is given in [SC3] will probably lead to much more elegant conditions but these conditions are too complex to explain here in detail. (Some discussions on this alternative method are given in subsection 9.3.1.)

4.6.1 Frequency domain loop shifting

The basic method (see [Kh2, Li3, Sa2]) here is based on applying a transformation in the frequency domain. This method is applicable for the full-information feedback case. On the other hand, for the state feedback case we have, after transformation, a problem with measurement feedback which we can only solve using techniques which will be given in the next chapter. Therefore we assume that $y = (x, w)$.

For all $\epsilon > 0$ we define the following transformation:

$$ G(s) \rightarrow \tilde{G}(s) := G \left( \frac{\epsilon + s}{1 + \epsilon s} \right) $$

and instead of minimizing the $H_{\infty}$ norm of a system described by some transfer matrix $G$ we minimize the $H_{\infty}$ norm of a system described by the transfer matrix $\tilde{G}$. If $I - \epsilon A$ is invertible a state space realization of $\tilde{G}$ is given by:

$$ \hat{\Sigma}(\epsilon) : \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}u + \tilde{E}w, \\
y = \tilde{C}_1x + \tilde{D}_{11}u + \tilde{D}_{12}w, \\
z = \tilde{C}_2x + \tilde{D}_{21}u + \tilde{D}_{22}w, \end{cases} $$

(4.28)

where

$$ \tilde{A} := (A - \epsilon I)(I - \epsilon A)^{-1}, $$
$$ \tilde{B} := (1 - \epsilon^2)(I - \epsilon A)^{-1}B, $$
$$ \tilde{E} := (1 - \epsilon^2)(I - \epsilon A)^{-1}E, $$
$$ \tilde{C}_1 := \begin{pmatrix} (I - \epsilon A)^{-1} \\ 0 \end{pmatrix}, $$
$$ \tilde{C}_2 := C(I - \epsilon A)^{-1}, $$
$$ \tilde{D}_{11} := \begin{pmatrix} \epsilon(I - \epsilon A)^{-1}B \\ 0 \end{pmatrix}, $$
The general full-information $H_{\infty}$ control problem

$\tilde{D}_{12} := \left( \varepsilon (I - \varepsilon A)^{-1} E \right) I$

$\tilde{D}_{21} := D_1 + \varepsilon C (I - \varepsilon A)^{-1} B,$

$\tilde{D}_{22} := D_2 + \varepsilon C (I - \varepsilon A)^{-1} E.$

Assume that there exists a compensator $\tilde{\Sigma}_F$ described by $u = \tilde{F}_1 x + \tilde{F}_2 w$ for $\tilde{\Sigma}(\varepsilon)$ such that the closed loop system is internally stable, the closed loop transfer matrix $\tilde{G}_{cl}$ has $H_{\infty}$ norm less than 1 and, moreover, the matrix $I - \varepsilon BF_1$ is invertible (the latter we can always achieve by an arbitrarily small perturbation). Then the feedback for the system $\tilde{\Sigma}(\varepsilon)$ with $\bar{y}$ instead of $y = (x, w)$ defined by $u = F\bar{y}$ where

$$F := \begin{pmatrix} \tilde{F}_1 (I + \varepsilon B \tilde{F}_1) & \tilde{F}_2 \end{pmatrix} \begin{pmatrix} I - \varepsilon A & -\varepsilon (E - B \tilde{F}_2) \\ 0 & I \end{pmatrix}$$

yields the same closed loop system and is hence internally stabilizing and the closed loop transfer matrix is $\tilde{G}_{cl}$.

If we apply the feedback $\Sigma_F$ described by $u = F y$ to our original system, then the closed loop system $\Sigma_F \times \Sigma$, with transfer matrix $G_{cl}$, is related to the closed loop system $\hat{\Sigma}(\varepsilon) \times \hat{\Sigma}_F$, with transfer matrix $\hat{G}_{cl}$, via the above transformation, i.e.

$$\tilde{G}_{cl}(s) = G_{cl} \left( \begin{pmatrix} \varepsilon + s \\ 1 + \varepsilon s \end{pmatrix} \right).$$

Using this it can be shown that the state matrix of the closed loop system $\Sigma \times \Sigma_F$ has all its eigenvalues inside a circle which is symmetric with respect to the real axis and lies in the left half plane between $-\varepsilon$ and $-1/\varepsilon$. Hence the closed loop system is certainly internally stable. Denote the set of all $s \in \mathcal{C}$ outside that circle by $\bar{D}$. We have

$$1 > \|\tilde{G}_{F}\|_{\infty} = \sup_{s \in \mathcal{C}^+} \|\tilde{G}_{F}(s)\| = \sup_{s \in \bar{D}} \|G_{F}(s)\| \geq \sup_{s \in \mathcal{C}^+} \|G_{F}(s)\| = \|G_{F}\|_{\infty}.$$  

Hence $\Sigma_F$ makes the $H_{\infty}$ norm of the closed loop transfer matrix $G_{F}$ strictly less than 1.

On the other hand if for the system $\Sigma$ there exists a stabilizing static compensator $\Sigma_F$ with $y = (x, w)$ which makes the $H_{\infty}$ norm of the closed loop system strictly less than 1 then it can be shown that there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_1$ the transformed system $\tilde{\Sigma}(\varepsilon)$ with the same compensator $\tilde{\Sigma}_F$ but with measurement $\bar{y}$ is internally stable and the $H_{\infty}$ norm of the closed loop system is strictly less than 1.
Now for the system $\Sigma$ described by (4.23), for all but finitely many $\epsilon > 0$ the system $\Sigma(\epsilon)$, described by (4.28) is such that $(\tilde{A}, \tilde{B}, \tilde{C}_2, \tilde{D}_{21})$ has no invariant zeros on the imaginary axis. Therefore to check if for $\Sigma(\epsilon)$ there exists an internally stabilizing feedback which makes the $H_\infty$ norm of the closed loop system less than 1 we can use the results of the previous section.

We can formalize the above intuition in the following theorem. We shall not formally prove this theorem.

**Theorem 4.8**: Let a system $\Sigma$ be given described by (4.23). For all $\epsilon > 0$ such that $I - \epsilon A$ is invertible we define $\Sigma(\epsilon)$ by (4.28). The following two statements are equivalent:

(i) For the system $\Sigma$ there exists a static feedback $\Sigma_F$ described by $u = F_1 x + F_2 w$ which is internally stabilizing and which makes the $H_\infty$ norm of the closed loop system less than 1.

(ii) There exists $\epsilon_1 > 0$ such that for all $0 < \epsilon < \epsilon_1$ the matrix $I - \epsilon A$ is invertible and for the system $\Sigma(\epsilon)$ the subsystem $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the imaginary axis. Moreover, $\epsilon_1$ can be chosen such that for all $0 < \epsilon < \epsilon_1$ there exists a feedback $\Sigma_F$ for $\Sigma(\epsilon)$ of the form $u = \tilde{F}_1 x + \tilde{F}_2 w$ which is internally stabilizing and makes the $H_\infty$ norm of the closed loop system less than 1. 

**Remarks**

(i) Note that we can check part (ii) for some $\epsilon > 0$ using the methods from the previous section. The problem is that we do not know how small we must choose $\epsilon$. If for some $\epsilon > 0$ part (ii) is not satisfied, then either part (i) is not true or part (ii) is true for some smaller $\epsilon$. Therefore we are never sure.

(ii) A controller satisfying part (ii) such that $I - \epsilon B \tilde{F}_1$ is invertible can be transformed into a controller $u = F y$ which satisfies part (i) where $F$ is defined by (4.29). This condition is a well-posedness condition we need since the system $\Sigma(\epsilon)$ with measurement $\tilde{y}$ has $\tilde{D}_{11} \neq 0$.

(iii) The advantage of this method is that controllers for $\Sigma$ found in this way will have all poles outside $\tilde{D}$ and hence inside the left half plane. Since poles close to the imaginary axis with high imaginary part are in
they can never be eigenvalues of the closed loop state matrix. Since this kind of ill-damped, high-frequency poles is highly undesirable from a practical point of view this is a nice property.

(iv) We think that this transformation should make it clear to the reader that transformations in the frequency domain are not well-suited for the state feedback case or for the full-information case. Often these problems are transformed into systems with measurement feedback where we might really have trouble observing the state. This is the reason why most of the approaches we shall discuss in section 5.1 to the $H_{\infty}$ control problem with measurement feedback are often not very suitable to treat the special cases of state feedback and full-information separately as we do in this thesis.

4.6.2 Cheap control

In this subsection we shall briefly describe how the method used in [Kh3, Kh2, Pe3, Pe5, ZK2] still gives necessary and sufficient conditions for the existence of internally stabilizing controllers which make the $H_{\infty}$ norm of the closed loop system less than 1 even in the case that we have invariant zeros on the imaginary axis. This subsection is worked out in more detail in [St].

We assume that we have a system $\Sigma$ of the form (4.23). For each $\epsilon > 0$ we define the following system:

$$\tilde{\Sigma}(\epsilon) : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ z = \tilde{C}x + \tilde{D}_1u + \tilde{D}_2w, \end{cases} \quad (4.30)$$

where

$$\tilde{C} = \begin{pmatrix} C \\ \epsilon I \\ 0 \end{pmatrix}, \quad \tilde{D}_1 = \begin{pmatrix} D_1 \\ 0 \\ \epsilon I \end{pmatrix}, \quad \tilde{D}_2 = \begin{pmatrix} D_2 \\ 0 \end{pmatrix}. $$

The structure of the perturbations on the matrices is such that it is easily shown that any controller $\Sigma_F$ of the form (2.4) with $y = x$ has the property that $\Sigma_F$ is internally stabilizing when applied to $\Sigma$ if and only if the same controller $\Sigma_F$ is internally stabilizing when applied to $\tilde{\Sigma}(\epsilon)$. (The same is true for $y = (x, w).$)

Let $\Sigma_F$ be internally stabilizing when applied to $\Sigma$. Denote the closed loop operator by $G_F$. Moreover denote the closed loop operator when the controller is applied to $\tilde{\Sigma}(\epsilon)$ by $\tilde{G}_F(\epsilon)$. Then we have
\[ \|G_F\|_\infty \leq \|\tilde{G}_F(\varepsilon_1)\|_\infty \leq \|\tilde{G}_F(\varepsilon_2)\|_\infty \]

for all \(0 \leq \varepsilon_1 \leq \varepsilon_2\). Hence if we have an internally stabilizing controller which makes the \(H_\infty\) norm of the closed loop system less than 1 when applied to the system \(\tilde{G}_F(\varepsilon)\), then the same controller is internally stabilizing and makes the \(H_\infty\) norm of the closed loop system less than 1 when applied to the original system. Thus we can obtain the following result:

**Theorem 4.9**: Let a system \(\Sigma\) be given by (4.23). For all \(\varepsilon > 0\) define \(\tilde{\Sigma}(\varepsilon)\) by (4.28). The following two statements are equivalent:

(i) For the system \(\tilde{\Sigma}\) there exists a feedback \(\Sigma_F\) of the form (2.4) with \(y = x\) which is internally stabilizing and which makes the \(H_\infty\) norm of the closed loop system less than 1.

(ii) There exists \(\varepsilon_1 > 0\) such that for all \(0 < \varepsilon < \varepsilon_1\) there exists a feedback \(\Sigma_F\) of the form (2.4) for \(\tilde{\Sigma}(\varepsilon)\) with \(y = x\) which is internally stabilizing and which makes the \(H_\infty\) norm of the closed loop system less than 1.

Any controller satisfying part (ii) for some \(\varepsilon > 0\) also satisfies part (i). \(\Box\)

On the other hand for the system \(\tilde{\Sigma}(\varepsilon)\) the subsystem \((A, B, \tilde{C}, \tilde{D}_1)\) does not have any invariant zeros and hence certainly no invariant zeros on the imaginary axis. Moreover, \(\tilde{D}_1\) is injective. Hence we may apply the results of chapter 3 to the system \(\tilde{\Sigma}(\varepsilon)\) to obtain necessary and sufficient conditions for the existence of internally stabilizing controllers which make the \(H_\infty\) norm of the closed loop system less than 1.

If we compare the method of the previous subsection with the method of the current subsection, then we see that the method of this subsection is much easier. First of all because we do not have to apply transformations on the controller. Secondly because for all \(\varepsilon > 0\) the system (4.30) is well defined and satisfies the assumptions of chapter 3. In contrast the system (4.28) is not always well defined and for certain values of \(\varepsilon > 0\) there may be invariant zeros on the imaginary axis. Moreover, in general it will only satisfy the conditions of chapter 4 and at this point the reader is probably aware of the fact that the necessary and sufficient conditions of chapter 4 are more difficult to check than the necessary and sufficient conditions of chapter 3.
Both methods have the disadvantage that the conditions cannot actually be checked since both condition (ii) of theorem 4.8 as well as condition (ii) of theorem 4.9 have to be checked for infinitely many $\varepsilon > 0$.

One of the main reasons why nevertheless the method of the previous subsection is used is the following argument which was also mentioned in the previous subsection. If one uses the method of the previous subsection, then all closed loop poles will be placed inside a circle in the open left half plane and therefore the closed loop system will not have ill-damped high-frequency poles. Hence the method has some advantages from an engineering point of view.

4.7 Conclusion

In this chapter we have completed the results on the full-information $H_\infty$ control problem. It turns out that as long as we do not have invariant zeros on the imaginary axis, we find nice checkable necessary and sufficient conditions under which there exists an internally stabilizing controller which makes the $H_\infty$ norm of the closed loop system less than some, a priori given, bound $\gamma$ or not. In the case that we have a regular problem we have one Riccati equation and, if one exists, an explicit formula for one such controller. On the other hand for the singular problem we have a quadratic matrix inequality and a couple of rank conditions. Using a state space decomposition, the latter conditions can be shown to be equivalent to a reduced order Riccati equation (see appendix A). A problem is that we do not have an explicit formula for the desired controller and there may be numerical problems when trying to find such a controller.

In the case that we have invariant zeros on the imaginary axis the methods of section 4.6 can be used. However, these methods are not really satisfactory and one would have to investigate the method in [SC3] better in order to obtain nicer conditions for the general case. (the conditions we have in mind are suggested in subsection 9.3.1.)
Chapter 5

The $H_{\infty}$ control problem with measurement feedback

5.1 Introduction

In the previous two chapters we investigated two special cases of the $H_{\infty}$ control problem with measurement feedback, namely $y = x$ and $y = (x, w)$. In this chapter we shall investigate the more general system (2.1).

Around 1984 practically all the work on $H_{\infty}$ control theory with measurement feedback was done with a mixture of time domain and frequency domain techniques (see [Do3, Fr2, Gl, Gl5]). The main drawback of these methods was that it yielded high order controllers. In [Li, Li2] it was shown that the order of the controller could be reduced considerably: it was proved for the one block $H_{\infty}$ control problem (in (2.1) $D_{12}$ and $D_{21}$ are both square matrices) and for the two block $H_{\infty}$ control problem (in (2.1) either $D_{12}$ or $D_{21}$ is a square matrix) there always exist "suitable" controllers with order no greater than the order of the plant.

During the last few years the $H_{\infty}$ problem with measurement feedback was investigated via several new methods:

- **the interpolation approach.** In fact, the interpolation approach has already quite a history and several authors have worked on this problem (see e.g. [Gr, Kh, Li3, Za2]). However, it can only treat the special case that we have a one-block $H_{\infty}$ control problem. One should note that the classical interpolation techniques were used for discrete time systems. At first the continuous time case was treated via a frequency domain transformation to the discrete time case (a linear fractional transformation as given in [Gen, appendix 1]).
the time-domain approach. (see [Do4, Kh3, Kh4, Pe, Ta]) This method was the first to suggest the use of Riccati equations in $H_\infty$ control, which was an important breakthrough. The detailed formulations of the results obtained via this method will be given in this chapter.

the polynomial approach. (see [Bo, Kw, Kw2]) This method starts with a polynomial left (or right) coprime factorization of the transfer matrix of the system. Then it is shown that a controller which minimizes the $H_\infty$ norm of the closed loop system is a so-called equalizing solution of a certain minimization problem. Conditions for obtaining an internally stabilizing controller which minimizes the $H_\infty$ norm are then given in terms of diophantine equations. The current research in this area is to investigate the relation between these diophantine equations and the conditions found in other methods.

the $J$-spectral factorization approach. (see e.g. [Gr2, HSK, Ki]) This method is strongly based on the classical frequency domain approach. The conditions for the existence of an internally stabilizing controller which makes the $H_\infty$ norm of the closed loop system less than some, a priori given, bound are given in terms of a rational matrix which should have a so-called $J$-spectral factorization. Whether such a factorization exists can be checked via the solvability of certain algebraic Riccati equations.

All of these methods show that if suitable controllers exist, then a suitable controller can be found of the same complexity as the original plant. A second feature that all of these methods have in common is a number of basic assumptions they all have to make. The assumptions are twofold. Firstly two direct feedthrough matrices should be injective and surjective, respectively. Secondly, two given subsystems should have no invariant zeros on the imaginary axis. These assumptions exclude, for instance, the special cases of state feedback and full-information feedback.

A special feature of the time domain approach we are using throughout this thesis is that it first solves the full-information feedback case as we did in the previous chapters and then uses these results to obtain the general result.

Under the basic assumptions mentioned above and using the time domain approach, the necessary and sufficient conditions for the existence of internally stabilizing controllers which make the $H_\infty$ norm of the closed loop system less than some, a priori given, bound $\gamma$ are the following: two given
Riccati equations should have positive definite solutions and the product of these two matrices should have spectral radius less than $\gamma^2$. These two Riccati equations are not coupled and in fact one of them is the same as the Riccati equation from chapter 3. The other Riccati equation is dual to the first one and is related to the problem of state estimation. The coupling condition that the product of the solutions of these Riccati equations should have spectral radius less than $\gamma^2$ is very hard to explain intuitively. It is a kind of test whether state estimation and state feedback combined in some way yield the desired result: an internally stabilizing feedback which makes the $H_\infty$ norm less than $\gamma$.

In the literature two methods have been proposed to solve the $H_\infty$ problem without assumptions on the direct feedthrough matrices and without assumptions on the invariant zeros. For the special case of full-information feedback, these methods are discussed in section 4.6. Using the first method discussed in subsection 4.6.1, which applies a transformation in the complex plane combined with the results in [Do4, Kh3, Pe, Ta], we still have to make assumptions: two given subsystems should be left and right invertible respectively. The second method combined with the methods of the latter papers is able to tackle the most general case. These methods, however, have the drawback that the conditions are in terms of Riccati equations which are parametrized by some parameter $\epsilon > 0$.

In this chapter we shall present a method which, independently of the latter papers, solves the measurement feedback case using the results of our previous chapter. In contrast to [Do4, Kh3, Pe, Ta] we shall impose no assumptions on the direct feedthrough matrices. However, we still have to exclude invariant zeros on the imaginary axis.

Our method will not have the above mentioned drawback of a parametrized Riccati equation. Also our results reduce to the known results in [Do4, Ta] in the case that these singularities of the direct feedthrough matrices do not occur.

Another advantage of our (weaker) assumptions will be that the special cases of state feedback and full-information feedback fall within the framework of the general problem formulation.

The necessary and sufficient conditions under which there exists an internally stabilizing dynamic compensator which makes the $H_\infty$ norm strictly less than some a priori given bound $\gamma$ are formulated differently than in recent publications [Do4, Ta]. As mentioned above, in these papers the conditions are formulated in terms of two given Riccati equations. However, in the case that there are singularities of the direct feedthrough matrices
these Riccati equations do not exist. To replace the role of these Riccati equations we have two quadratic matrix inequalities. The solution of each of these quadratic matrix inequalities has to satisfy two rank conditions. Moreover, we have a condition which couples these two matrix inequalities. The spectral radius of the product of the two solutions of these matrix inequalities should be smaller than $\gamma^2$. In the regular case the quadratic matrix inequality together with the corresponding first rank condition reduces to a Riccati equation and the second rank condition guarantees that it is a stabilizing solution of the Riccati equation. As for the regular case, the first matrix inequality with the two corresponding rank conditions exactly form the necessary and sufficient conditions for the existence of a "suitable" state feedback as derived in chapter 4. The second quadratic matrix inequality with the remaining two rank conditions are, again as in the regular case, dual to the first matrix inequality and the first two rank conditions.

Our proof will use ideas given in [Do4] to solve the regular $H_{\infty}$ problem with measurement feedback but is independent of the results in [Do4] and is entirely self-contained. The results of this chapter already appeared in [St4].

The outline of this chapter is as follows: In section 5.2 we formulate the problem and present the main result for the case when two given direct feedthrough matrices are zero. Moreover we show that in the regular case and the state feedback case this result reduces to the known results in [Do4] and chapter 4, respectively. In section 5.3 it is shown that the conditions for the existence of a suitable compensator as given in our main theorem are necessary. It is also shown that the problem of finding such a compensator is equivalent to finding such a compensator for a certain transformed system, i.e. it is shown that any compensator which internally stabilizes this new system and makes the $H_{\infty}$ norm of the closed loop system less than $\gamma$ has the same properties when applied to the original system and vice versa. This new system has some desirable properties and using the results from section 2.6 it is shown that for this new system we can even make the $H_{\infty}$ norm of the closed loop system arbitrarily small. In section 5.4 a method for finding the desired compensator is discussed. In section 5.5 we shall briefly discuss how to extend the results to the case that all feedthrough matrices are allowed to be unequal to zero. We conclude in section 5.6 with some final remarks. The proofs of section 5.3 depend upon the basis transformations of appendix A and are given in appendix B since they are rather technical and detract from the main ideas of the proof.
5.2 Problem formulation and main results

We consider the linear, time-invariant, finite-dimensional system:

\[
\begin{align*}
\Sigma : & \quad \dot{x} = Ax + Bu + Ew, \\
& \quad y = C_1x + D_1w, \\
& \quad z = C_2x + D_2u,
\end{align*}
\]

(5.1)

where for all \( t \) we have that \( x(t) \in \mathcal{R}^n \) is the state, \( u(t) \in \mathcal{R}^m \) is the control input, \( w(t) \in \mathcal{R}^l \) is the unknown disturbance, \( y(t) \in \mathcal{R}^p \) is the measured output and \( z(t) \in \mathcal{R}^q \) is the unknown output to be controlled. \( A, B, E, C_1, C_2, D_1, \) and \( D_2 \) are matrices of appropriate dimensions. Note that compared with system (2.1) we assume that two direct feedthrough matrices are zero. In section 5.5 it will be shown how this assumption can be removed. As in the previous chapters we would like to minimize the effect of the disturbance \( w \) on the output \( z \) by finding an appropriate control input \( u \). This time however the measured output \( y \) is not necessarily \((x, w)\) or \( x \) but is a more general linear function of state and disturbance. In general the controller has less information and hence the necessary and sufficient conditions for the existence of internally stabilizing controllers which make the \( H_\infty \) norm of the closed loop system less than some given bound will be stronger in this chapter. It will turn out that we need an extra quadratic matrix inequality which tests how well we can observe the state and the disturbance. More precisely, we seek a dynamic compensator \( \Sigma_F \) described by (2.4) such that after applying the feedback \( \Sigma_F \) to the system (5.1), the resulting closed loop system is internally stable and has \( H_\infty \) norm strictly less than some a priori given bound \( \gamma \). We shall derive necessary and sufficient conditions under which such a compensator exists.

A central role in our study of the above problem will be played by the quadratic matrix inequality. For \( \gamma > 0 \) and \( P \in \mathcal{R}^{n \times n} \) we again consider the following matrix:

\[
F_\gamma(P) := \begin{pmatrix}
A^TP + PA + C_2^TC_2 + \gamma^{-2}PEE^TP & PB + C_2^TD_2 \\
B^TP + D_2^TC_2 & D_2^TD_2
\end{pmatrix}.
\]

Recall that if \( F_\gamma(P) \geq 0 \), we say that \( P \) is a solution of the quadratic matrix inequality at \( \gamma \). Note that this is the same quadratic matrix inequality as the one we used in the previous chapter (see (4.2), where \( D_1 \) is replaced by \( D_2 \)).
We also define a dual version of this quadratic matrix inequality. For any \( \gamma > 0 \) and matrix \( Q \in \mathcal{R}^{n \times n} \) we define the following matrix:

\[
G_\gamma(Q) := \begin{pmatrix}
AQ + QA^T + EE^T + \gamma^{-2}QC_2^TC_2Q & QC_1^T + ED_1^T \\
C_1Q + D_1E^T & D_1D_1^T
\end{pmatrix}.
\]

If \( G_\gamma(Q) \geq 0 \), we say that \( Q \) is a solution of the dual quadratic matrix inequality at \( \gamma \). In addition to these two matrices, we define two matrices pencils, whose roles are again duals:

\[
L_\gamma(P, s) := \begin{pmatrix}
sI - A - \gamma^{-2}EE^TP & -B
\end{pmatrix},
\]

\[
M_\gamma(Q, s) := \begin{pmatrix}
sI - A - \gamma^{-2}QC_2^TC_2 \\
-C_1
\end{pmatrix}.
\]

We note that \( L_\gamma(P, s) \) is the controllability pencil associated with the system:

\[
\dot{x} = (A + \gamma^{-2}EE^TP) x + Bu.
\]

Note that \( L_\gamma \) is the same controllability pencil as the one we used in the previous chapter (see (4.3)). On the other hand \( M_\gamma(Q, s) \) is the observability pencil associated with the system:

\[
\begin{cases}
\dot{x} = (A + \gamma^{-2}QC_2^TC_2) x, \\
y = -C_1x.
\end{cases}
\]

We define the following two transfer matrices which again play dual roles:

\[
G_{ci}(s) := C_2(sI - A)^{-1}B + D_2,
\]

\[
G_{di}(s) := C_1(sI - A)^{-1}E + D_1,
\]

where \( G_{ci} \) is the same transfer matrix as the one used in the previous chapter with \( D_1 \) replaced by \( D_2 \). Let \( \rho(M) \) denote the spectral radius of the matrix \( M \). We are now in a position to formulate the main result of this chapter.

**Theorem 5.1:** Consider the system (5.1). Assume that both the system \( (A, B, C_2, D_2) \) as well as the system \( (A, E, C_1, D_1) \) have no invariant zeros on the imaginary axis. Then the following two statements are equivalent:
5.2 Problem formulation and main results

(i) For the system (5.1) there exists a time-invariant, finite-dimensional dynamic compensator $\Sigma_F$ of the form (2.4) and with McMillan degree $n - \text{rank}(C_1 D_1) + \text{rank} D_1$ such that the resulting closed loop system, with transfer matrix $G_F$, is internally stable and has $H_\infty$ norm less than $\gamma$, i.e. $\|G_F\|_\infty < \gamma$.

(ii) There exist positive semi-definite solutions $P, Q$ of the quadratic matrix inequalities $F_\gamma(P) \geq 0$ and $G_\gamma(Q) \geq 0$ satisfying $\rho(PQ) < \gamma^2$, such that the following rank conditions are satisfied

(a) $\text{rank } F_\gamma(P) = \text{rank}_{\mathcal{R}(s)} G_{ei}$,

(b) $\text{rank } G_\gamma(Q) = \text{rank}_{\mathcal{R}(s)} G_{di}$,

(c) $\text{rank } \left( \begin{array}{c} L_\gamma(P, s) \\ F_\gamma(P) \end{array} \right) = n + \text{rank}_{\mathcal{R}(s)} G_{ei} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$,

(d) $\text{rank } \left( \begin{array}{c} M_\gamma(Q, s) \\ G_\gamma(Q) \end{array} \right) = n + \text{rank}_{\mathcal{R}(s)} G_{di} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$. 

Remarks:

(i) Note that the conditions on $P$ in part (iii) of theorem 4.1 are exactly the same as the conditions on $P$ in part (ii) of the above theorem. Hence the conditions on $P$ are related to the full-information $H_\infty$ control problem. The conditions on $Q$ are exactly dual to the conditions on $P$. It can be shown that the existence of $Q$ is related to the question how well we are able to estimate the state $x$ on the basis of our observations $y$. For instance if $y = x$ it will be shown that $Q = 0$. The test whether we are able to estimate and control simultaneously with the desired effect is expressed in the coupling condition $\rho(PQ) < \gamma^2$.

(ii) The construction of a dynamic compensator satisfying (i) can be done using a method that will be described in section 5.4. It turns out that it is always possible to find a compensator of the same dynamic order as the original plant. If $D_1$ is surjective and $D_2$ is injective an explicit formula for one controller satisfying part (i) can be given (see [Gl3]). That is, if $P$ and $Q$ exist satisfying the conditions of part (ii) of theorem 5.1, then a controller satisfying part (i) is given by:

$$\Sigma_F : \begin{cases} \dot{p} = K_{P,Q} p + L_{P,Q} y, \\ u = M_{P,Q} p, \end{cases}$$
where
\[
M_{p,q} := -(D_2^T D_2)^{-1} (D_2^T C_2 + B^T P),
\]
\[
L_{p,q} := [E D_1^T + (I - \gamma^{-2} Q P)^{-1} Q (C_1^T + \gamma^{-2} P E D_1^T)] (D_1 D_1)^{-1},
\]
\[
K_{p,q} := A + E E^T P + B M_{p,q} - L_{p,q} (C_1 + \gamma^{-2} D_1 E^T P).
\]

It is even possible for the regular \( H_\infty \) control problem to parametrize all suitable controllers. On the other hand, for the singular \( H_\infty \) control this is very hard. This is related to the fact that this parametrization is a kind of ball around a certain central controller given above. One can prove that for the singular \( H_\infty \) control problem this central controller will in general be non-proper. Since we require that all controllers should be proper this yields a problem which still has to be solved.

(iii) By corollary A.7 we know that a solution \( P \) of the quadratic matrix inequality \( F_\gamma(P) \geq 0 \) satisfying (a) and (c) is unique. By dualizing corollary A.7 it can also be shown that a solution \( Q \) of the dual quadratic matrix inequality \( G_\gamma(Q) \geq 0 \) satisfying (b) and (d) is unique. The existence of \( P \) and \( Q \) can be checked via state space transformations and investigating reduced order Riccati equations.

(iv) We shall prove this theorem only for the case \( \gamma = 1 \). The general result can then be easily obtained by scaling.

Before we shall prove this result we shall look more closely to the result for two special cases:

**State feedback:** \( C_1 = I, \ D_1 = 0. \)

In this case we have \( y = x \), i.e. we know the state of the system. The first matrix inequality \( F_\gamma(P) \geq 0 \) together with rank conditions (a) and (b) do not depend on \( C_1 \) or \( D_1 \) so we can’t expect a simplification there. However, \( G_\gamma(Q) \) does get a special form:

\[
G_\gamma(Q) = \begin{pmatrix} AQ + QA^T + EE^T + \gamma^{-2} QC_2^T C_2 Q & Q \\ Q & 0 \end{pmatrix}.
\]

Using this special form it can be easily seen that \( G_\gamma(Q) \geq 0 \) if and only if \( Q = 0 \). For the rank conditions it is interesting to investigate the normal rank of \( G_{di} \). We have:

\[
\text{rank}_{\mathcal{R}(s)} G_{di} = \text{rank}_{\mathcal{R}(s)} (sI - A)^{-1} E = \text{rank } E.
\] (5.2)
By using the equality (5.2), it can be easily checked that $Q = 0$ satisfies the rank conditions (b) and (d). The condition $\rho(PQ) < \gamma^2$ is trivially satisfied if $Q = 0$. We find that in this case condition (ii) of theorem 5.1 becomes:

There exists a positive semi definite solution $P$ of the quadratic matrix inequality $F_\gamma(P) \geq 0$ such that the following two rank conditions are satisfied:

\begin{align*}
(i) \quad \text{rank} \ F_\gamma(P) &= \text{rank}_{\mathcal{R}(s)} G_{ci}, \\
(ii) \quad \text{rank} \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} &= n + \text{rank}_{\mathcal{R}(s)} G_{ci}, \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+, 
\end{align*}

which is exactly the result obtained in the previous chapter (see theorem 4.1).

Regular case: $D_1$ surjective and $D_2$ injective

In this case it can be shown in the same way as in chapter 4 that $F_\gamma(P) \geq 0$ together with rank condition (a) is equivalent to the condition:

\begin{align*}
A^T P + PA + C_2^T C_2 + \gamma^{-2} P E E^T P \\
- (P B + C_2^T D_2)(D_2^T D_2)^{-1} (B^T P + D_2^T C_2) = 0.
\end{align*}

The dual version of this proof can be applied to the dual matrix inequality $G_\gamma(Q) \geq 0$ together with rank condition (b). These conditions turn out to be equivalent to the condition:

\begin{align*}
A Q + Q A^T + E E^T + \gamma^{-2} Q C_2^T C_2 Q \\
- (Q C_1^T + E D_1^T) (D_1^T D_1)^{-1} (C_1 Q + D_1 E^T) = 0.
\end{align*}

The two remaining rank conditions (c) and (d) turn out to be equivalent to the requirement that the following two matrices be asymptotically stable:

\begin{align*}
A + \gamma^{-2} E E^T P - B (D_2^T D_2)^{-1} (B^T P + D_2^T C_2), \\
A + \gamma^{-2} Q C_2^T C_2 - (Q C_1^T + E D_1^T) (D_1^T D_1)^{-1} C_1.
\end{align*}

Together with the remaining condition $\rho(PQ) < \gamma^2$, we thus obtain exactly the same conditions as [Do4, Gl3].
5.3 Reduction of the original problem to an almost disturbance decoupling problem

In this section the implication (i) $\Rightarrow$ (ii) in theorem 5.1 will be proven. Moreover, in the case that the condition (ii) of theorem 5.1 is satisfied, we shall show that the problem of finding a suitable compensator $\Sigma_F$ for the system (5.1) is equivalent to finding a suitable compensator $\Sigma_F$ for a new system which has some very nice structural properties. In the next section the $H_\infty$ problem for this new system will be tackled. We recall that in the remainder of this chapter we assume that $\gamma = 1$. Define $F(P)$, $G(Q)$, $L(P,s)$ and $M(Q,S)$ to be equal to $F_1(P)$, $G_1(Q)$, $L_1(P,s)$ and $M_1(Q,s)$, respectively.

Lemma 5.2: Assume that the systems $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ have no invariant zeros on the imaginary axis. If there exists a dynamic compensator $\Sigma_F$ such that the resulting closed loop system is internally stable and has $H_\infty$ norm less than 1, then the following two conditions are satisfied:

(i) There exists a symmetric solution $P \succeq 0$ of the quadratic matrix inequality $F(P) \succeq 0$ satisfying the following two rank conditions:

(a) $\text{rank } F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci}$,

(b) $\text{rank } \left( \begin{array}{c}
L(P,s) \\
F(P)
\end{array} \right) = n + \text{rank}_{\mathcal{R}(s)} G_{ci}$, \quad $\forall s \in C^0 \cup C^+$.

(ii) There exists a symmetric solution $Q \succeq 0$ of the dual quadratic matrix inequality $G(Q) \succeq 0$ satisfying the following two rank conditions:

(a) $\text{rank } G(Q) = \text{rank}_{\mathcal{R}(s)} G_{di}$,

(b) $\text{rank } \left( \begin{array}{cc}
M(Q,s) & G(Q) \\
G(Q) & M(Q,s)
\end{array} \right) = n + \text{rank}_{\mathcal{R}(s)} G_{di}$, \quad $\forall s \in C^0 \cup C^+$. \hfill $\square$

Proof: Since there exists an internally stabilizing feedback which makes the $H_\infty$ norm of the closed loop system less than 1 for the problem with measurement feedback, it is easily checked that condition (ii) of theorem 4.1 is satisfied. This implies, according to theorem 4.1, that there exists a matrix $P$ satisfying the conditions in part (i) of the above lemma. By dualization it
can be easily shown that there also exists a matrix $Q$ satisfying the conditions in part (ii) of the above lemma.

Assume that there exist $P$ and $Q$ satisfying conditions (i) and (ii) in lemma 5.2. We make the following factorization of $F(P)$:

$$F(P) = \begin{pmatrix} C_{1,p}^T \\ D_{1,p}^T \end{pmatrix} \begin{pmatrix} C_{2,p} & D_p \end{pmatrix}$$ (5.3)

where $C_{2,p}$ and $D_p$ are matrices of suitable dimensions. This can be done since $F(P) \geq 0$. We define the following system:

$$\Sigma_p : \begin{cases} \dot{x}_p = A_p x_p + B u_p + E w_p, \\
y_p = C_{1,p} x_p + D_1 w_p, \\
z_p = C_{2,p} x_p + D_p u_p, \end{cases}$$ (5.4)

where $A_p := (A + EE^T P)$ and $C_{1,p} := (C_1 + D_1 E^T P)$. We first derive the following lemma

**Lemma 5.3**: Assume that the systems $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ have no invariant zeros on the imaginary axis. In that case the systems $(A_p, B, C_{2,p}, D_p)$ and $(A_p, E, C_{1,p}, D_1)$ have no invariant zeros on the imaginary axis either.

**Proof**: Note that the system $(A_p, E, C_{1,p}, D_1)$ can be obtained from the system $(A, E, C_1, D_1)$ by applying the preliminary feedback $u = E^T P x + v$. Therefore, the invariant zeros of the two systems coincide. Hence the system $(A_p, E, C_{1,p}, D_1)$ has no invariant zeros on the imaginary axis.

The rank condition (b) of lemma 5.2 can be reformulated as

$$\text{rank} \begin{pmatrix} sI - A_p & -B \\ C_{2,p} & D_p \end{pmatrix} = n + \text{rank} \begin{pmatrix} C_{2,p} & D_p \end{pmatrix}, \ \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+.$$

This immediately yields that $(A_p, B, C_{2,p}, D_p)$ has no invariant zeros on the imaginary axis.

Next, we derive the following relation between the systems $\Sigma$ and $\Sigma_p$: 
Lemma 5.4: Let $P$ satisfy the conditions (i) of lemma 5.2. Moreover, let an arbitrary dynamic compensator $\Sigma_F$ be given, described by (2.4). Consider the following two systems, where the system on the left is the interconnection of (5.1) and (2.4) and the system on the right is the interconnection of (5.4) and (2.4):

\[
\begin{align*}
\Sigma & \quad \Sigma_F \\
\Sigma_P & \quad \Sigma_F
\end{align*}
\]

Then the following statements are equivalent:

(i) The system on the left is internally stable and its transfer matrix from $w$ to $z$ has $H_\infty$ norm less than 1.

(ii) The system on the right is internally stable and its transfer matrix from $w_p$ to $z_p$ has $H_\infty$ norm less than 1.

Proof: see appendix B for the proof.

We assumed that for the original system (5.1) there exists an internally stabilizing dynamic compensator such that the resulting closed loop matrix has $H_\infty$ norm less than 1. Hence, by applying lemma 5.4, we know that the same compensator is internally stabilizing for the new system (5.4) and yields a closed loop transfer matrix with $H_\infty$ norm less than 1. Moreover we know by lemma 5.3 that $\Sigma_F$ satisfies the assumptions on the invariant zeros needed to apply lemma 5.2. Therefore, if we consider for this new system the two quadratic matrix inequalities we know that there exist positive semi-definite solutions to these inequalities satisfying a number of rank conditions. We shall now formalize this in the following lemma.

For arbitrary $X$ and $Y$ in $\mathbb{R}^{n \times n}$ we define the following matrices:

\[
\bar{F}(X) := \begin{pmatrix}
A^*_p X + X A_p + C^*_p C_{2,p} + X E E^* X & X B + C^*_p D_p \\
B^T X + D^T_p C_{2,p} & D^T_p D_p
\end{pmatrix},
\]
\[ \bar{G}(Y) := \begin{pmatrix} A_P Y + Y A_P^T + E E^T + Y C_{2,p} C_{2,p} Y & Y C_{1,p}^T + E D_1^T \\ C_{1,p} Y + D_1 E^T & D_1 D_1^T \end{pmatrix}, \]

\[ \bar{L}(X, s) := \begin{pmatrix} s I - A_P - E E^T X & -B \end{pmatrix}, \]

\[ \bar{M}(Y, s) := \begin{pmatrix} s I - A_P - Y C_{2,p} C_{2,p} & -C_{1,p} \end{pmatrix}. \]

Moreover we define two new transfer matrices:

\[ \bar{G}_{ci}(s) := C_{2,p} (s I - A_P)^{-1} B + D_P, \]

\[ \bar{G}_{di}(s) := C_{1,p} (s I - A_P)^{-1} E + D_1. \]

**Lemma 5.5**: Let \( P \) and \( Q \) satisfy the conditions (i) and (ii) in lemma 5.2, respectively. Assume that both the system \((A, B, C_2, D_2)\) as well as the system \((A, E, C_1, D_1)\) have no invariant zeros on the imaginary axis. Then we have the following two results:

(i) \( X := 0 \) is a solution of the quadratic matrix inequality \( \bar{F}(X) \geq 0 \) and the following two rank conditions are satisfied:

(a) \( \text{rank } \bar{F}(X) = \text{rank}_{\mathcal{R}(s)} \bar{G}_{ci} \),

(b) \( \text{rank} \left( \begin{pmatrix} \bar{L}(X, s) & \bar{F}(X) \end{pmatrix} \right) = n + \text{rank}_{\mathcal{R}(s)} \bar{G}_{ci}, \quad \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+ \).

(ii) There exists a symmetric matrix \( Y \) which satisfies the quadratic matrix inequality \( \bar{G}(Y) \geq 0 \) together with the following two rank conditions:

(a) \( \text{rank } \bar{G}(Y) = \text{rank}_{\mathcal{R}(s)} \bar{G}_{di} \),

(b) \( \text{rank} \left( \begin{pmatrix} \bar{M}(Y, s) & \bar{G}(Y) \end{pmatrix} \right) = n + \text{rank}_{\mathcal{R}(s)} \bar{G}_{di}, \quad \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+ \),

if and only if \( I - Q P \) is invertible. Moreover in that case there is a unique solution \( Y := (I - Q P)^{-1} Q \). This matrix \( Y \) is positive semi-definite if and only if

\[ \rho(PQ) < 1. \]  

(5.6)
Proof: see appendix B for the proof.

Proof of (i) ⇒ (ii) in theorem 5.1: The existence of $P$ and $Q$ satisfying the quadratic matrix inequalities and the corresponding four rank conditions can be obtained directly from lemma 5.2. We know by lemma 5.3 that the two subsystems $(A_F, B, C_{2,F}, D_F)$ and $(A_F, E, C_{1,F}, D_1)$ have no invariant zeros on the imaginary axis. We also know by lemma 5.4 that for the transformed system $\Sigma_P$ there exists a dynamic compensator which internally stabilizes the system and makes the $H_\infty$ norm of the closed loop system less than 1. By applying lemma 5.2 to this new system we find that there exists a matrix $Y \geq 0$ satisfying part (ii) of lemma 5.5. Hence by lemma 5.5 we have (5.6) and therefore all the conditions in theorem 5.1, part (ii) are satisfied.

We are now going to prove the reverse implication (ii) ⇒ (i) in theorem 5.1. Therefore we no longer assume that there exists an internally stabilizing compensator which makes the $H_\infty$ norm less than 1. Instead we assume that there exist matrices $P$ and $Q$ satisfying part (ii) of theorem 5.1. In order to prove the implication (ii) ⇒ (i) in theorem 5.1 we transform the system (5.4) once again, this time however using the dualized version of the original transformation. By lemma 5.5 we know $Y = (I - QP)^{-1} Q \geq 0$ satisfies $\bar{G}(Y) \geq 0$. We factorize $\bar{G}(Y)$:

$$\bar{G}(Y) = \begin{pmatrix} E_{P,Q} & E_{P,Q}^T \end{pmatrix} \begin{pmatrix} D_{P,Q} & D_{P,Q}^T \end{pmatrix}.$$ (5.7)

where $E_{P,Q}$ and $D_{P,Q}$ are matrices of suitable dimensions. We define the following system:

$$\Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = A_{P,Q}x_{P,Q} + B_{P,Q}u_{P,Q} + E_{P,Q}w, \\ y_{P,Q} = C_{1,P}x_{P,Q} + D_{P,Q}w, \\ z_{P,Q} = C_{2,P}x_{P,Q} + D_{P}u_{P,Q}, \end{cases} \quad (5.8)$$

where

$$A_{P,Q} := A_P + YC_{2,P}^TC_{2,P}, \quad B_{P,Q} := B + YC_{2,P}D_P.$$ 

Note that the subsystems $(A_P, B, C_{2,P}, D_P)$ and $(A_P, E, C_{1,P}, D_1)$ of $\Sigma_P$ have no invariant zeros on the imaginary axis by lemma 5.3. By applying the dual version of lemma 5.5 (i.e. the combination of lemmas B.3 and B.4) to
the system $\Sigma_{F,Q}$ with the corresponding matrix inequalities we observe that $X_{P,Q} := 0$ and $Y_{P,Q} := 0$ satisfy the matrix inequalities and the corresponding rank conditions for this new system. It can be easily shown that this implies:

$$\text{rank} \begin{pmatrix} sI - A_{F,Q} & -B_{F,Q} \\ C_{2,P} & D_P \end{pmatrix} = n + \text{rank} \begin{pmatrix} C_{2,P} & D_P \end{pmatrix}, \forall s \in C^0 \cup C^+ \tag{5.9}$$

and

$$\text{rank} \begin{pmatrix} sI - A_{F,Q} & -E_{F,Q} \\ C_{1,P} & D_{P,Q} \end{pmatrix} = n + \text{rank} \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix}, \forall s \in C^0 \cup C^+. \tag{5.10}$$

By applying lemma 5.4 and its dualized version the following corollary can be derived:

**Corollary 5.6 :** Let an arbitrary compensator $\Sigma_F$ of the form (2.4) be given. The following two statements are equivalent:

(i) The compensator $\Sigma_F$ applied to the system $\Sigma$ described by (5.1), is internally stabilizing and the resulting closed loop transfer matrix has $H_\infty$ norm less than 1.

(ii) The compensator $\Sigma_F$ applied to the system $\Sigma_{P,Q}$ described by (5.8), is internally stabilizing and the resulting closed loop transfer matrix has $H_\infty$ norm less than 1. \qed

**Remark :** We note that even if for this new system we can make the $H_\infty$ norm arbitrarily small, for the original system we are only sure that the $H_\infty$ norm will be less than 1. It is possible that a compensator for the new system yields an $H_\infty$ norm of say 0.0001 while the same compensator makes the $H_\infty$ norm of the original plant only 0.9999.

In section 2.6 we have shown how to solve the $H_\infty$ problem for a system satisfying the extra conditions (5.9) and (5.10). It turned out that for such a system we can even make the $H_\infty$ norm arbitrarily small. We are now able to complete the proof of theorem 5.1:

**Proof of the implication (ii) $\Rightarrow$ (i) of theorem 5.1:** Since we can transform the original system into a system satisfying (5.9) and (5.10) we
know by lemma 2.20 that we can find an internally stabilizing dynamic compensator with McMillan degree $n - \text{rank } (C_1 \ D_1) + \text{rank } D_1$ for this new system such that the closed loop transfer matrix has $H_\infty$ norm less than 1. By applying corollary 5.6 we know that this compensator $\Sigma_F$ satisfies the requirements in theorem 5.1, part (i).

5.4 The design of a suitable compensator

In this section we shall give a method to calculate a dynamic compensator $\Sigma_F$ such that the closed loop system is internally stable and, moreover, the closed loop transfer matrix has $H_\infty$ norm less than 1. We shall derive this $\Sigma_F$ step by step, using the following conceptual algorithm.

(i) Calculate $P$ and $Q$ satisfying part (ii) of theorem 5.1. This can, for instance, be done using lemma A.6. If they do not exist or if $\rho(PQ) \geq 1$ then a feedback satisfying part (i) of theorem 5.1 does not exist and we stop.

(ii) Perform the factorizations (5.3) and (5.7). We can now construct the system $\Sigma_{P,Q}$ as given by (5.8).

We now start solving the almost disturbance decoupling problem for the system (5.8) we obtained in step (ii). We shall rename our variables and assume that we have a system in the form (5.1) which satisfies (2.41) and (2.42). We use the results as derived in section 2.6. We have to construct matrices $F$ and $G$ such that (2.43) and (2.44) are satisfied and moreover such that $A + BF$ and $A + GC_1$ are asymptotically stable. We shall only discuss the construction of $F$. The construction of $G$ can be obtained by duality.

(iii) Construct $V_g(\Sigma_{ci})$ by using lemma 2.5.

(iv) Construct an $\tilde{F}$ such that (2.31) and (2.32) are satisfied and, moreover, such that $A + B\tilde{F} | V_g(\Sigma_{ci})$ is asymptotically stable.

(v) Let $\Pi$ be the canonical projection $\mathbb{R}^n \to \mathbb{R}^n/V_g(\Sigma)$ and let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be such that (2.33)–(2.35) are satisfied. Construct the system $\Sigma_{f,s}$ as given by (2.36).

(vi) Construct $F_0$ and $M$ such that (2.37) and (2.38) are satisfied. Define the following matrices:
5.4 The design of a suitable compensator

(a) $\bar{A} := A + F_0 B$, 
(b) $\bar{B} := B M$, 
(c) $\bar{C} := C + D_2 F_0$.

and the system

$$\Sigma_{ht} : \begin{cases} \dot{x} = \bar{A} x + \bar{B} u \\ z = \bar{C} x \end{cases}$$

(5.11)

In this way we obtain a strongly controllable system (5.11), for which we have to find a static state feedback law $u = F x$ such that the closed loop system is internally stable and such that the closed loop impulse reponse has $L_1$ norm smaller than $\varepsilon/(3\|E\| + 1)$. We shall use a method for which this is an adapted version of the one given in [Tr].

(vii) We construct a new basis for the state space. We shall construct it by induction. Choose $0 \neq x_1 \in \ker \bar{C} \cap \im \bar{B}$ and, if such $x_1$ exists, choose $v_1$ such that $B v_1 = x_1$. If $x_1$ does not exist then set $i = 0$ and $S_i = \{0\}$ and go to item (viii).

Assume that $\{x_1, \ldots, x_i\}$ and $\{v_1, \ldots, v_i\}$ are given. Denote by $S_i$ the linear span of $\{x_1, \ldots, x_i\}$. If $(\bar{A} x_i + \im \bar{B}) \cap \ker \bar{C} \subset S_i$ and $\im \bar{B} \cap \ker \bar{C} \subset S_i$ then goto step (viii). Otherwise, if $(\bar{A} x_i + \im \bar{B}) \cap \ker \bar{C} \not\subset S_i$ then choose $v$ such that $\bar{A} x_i + \bar{B} v \in \ker \bar{C}$ and $\bar{A} x_i + \bar{B} v \not\in S_i$. Set $x_{i+1} = \bar{A} x_i + \bar{B} v$ and $v_{i+1} = v$. If $(\bar{A} x_i + \im \bar{B}) \cap \ker \bar{C} \subset S_i$, then choose $v$ such that $\bar{B} v \in \ker \bar{C}$ and $\bar{B} v \not\in S_i$. Set $x_{i+1} = \bar{B} v$ and $v_{i+1} = v$. Set $i := i + 1$ and repeat this paragraph again.

(viii) Define $\mathcal{R}_n^*(\ker \bar{C}) = S_i$. Define a linear mapping $F$ such that $F x_j = v_j$, $j = 1, \ldots, i$ and extend it arbitrarily to the whole state space. In [Tr] it has been shown that $\bar{A} \mathcal{R}_n^*(\ker \bar{C}) + \im \bar{B} = T(\Sigma_{ht}) = \mathcal{R}_n$. Therefore it is easily seen that we can extend the basis of $S_i$, $\{x_1, \ldots, x_i\}$ to a basis of $\mathcal{R}_n$ which can be written as

\[
\begin{align*}
\bar{B} v_1, A_F \bar{B} v_1, \ldots, A_F^{i-1} \bar{B} v_1, \\
\bar{B} v_2, A_F \bar{B} v_2, \ldots, A_F^{i-2} \bar{B} v_2, \\
\vdots & \quad \vdots \\
\bar{B} v_i, A_F \bar{B} v_i, \ldots, A_F^{i-i} \bar{B} v_i, \\
\bar{B} v_{i+1}, \ldots, \bar{B} v_k,
\end{align*}
\]
where \( A_F = \bar{A} + \bar{B}F \) and where we may have to extend our set \( \{v_1, \ldots, v_i\} \) with some vectors \( v_j, j = i + 1, \ldots, k \) in order to obtain a basis of \( \mathbb{R}^n \). Moreover, for \( j = 1, \ldots, i \) we should have

\[
\bar{B}v_j, A_F\bar{B}v_j, \ldots, A_F^{r_j-1}\bar{B}v_j \in \ker \bar{C}.
\]

We define \( r_j = 0 \) for \( j = i + 1, \ldots, k \).

(ix) We define the following sequence of vectors. For \( j = 1, \ldots, k \) we define:

\[
x_{j,1}(n) := \left(I + \frac{1}{n}A_F\right)^{-1}\bar{B}v_j
\]

\[
x_{j,2}(n) := \left(I + \frac{1}{n}A_F\right)^{-1}A_Fx_{j,1}(n)
\]

\[
\vdots
\]

\[
x_{j,r_j+1}(n) := \left(I + \frac{1}{n}A_F\right)^{-1}A_Fx_{j,r_j}(n)
\]

It is easily shown that \( x_{j,h}(n) \to A_F^{h-1}\bar{B}v_j \) as \( n \to \infty \) for \( j = 1, \ldots, k \) and \( h = 1, \ldots, r_j + 1 \). Therefore, for \( n \) sufficiently large, the vectors

\[
\{x_{j,h}(n), j = 1, \ldots, k; h = 1, \ldots, r_j + 1\}
\]

are linearly independent and hence form a basis of \( \mathbb{R}^n \) again. Let \( N \) be such that for all \( n > N \) these vectors indeed form a basis.

(x) For all \( n > N \) define a linear mapping \( \bar{F}_n \) by

\[
\bar{F}_nx_{j,1}(n) := -nv_j
\]

\[
\bar{F}_nx_{j,2}(n) := -n^2v_j
\]

\[
\vdots
\]

\[
\bar{F}_nx_{j,r_j+1}(n) := -n^{r_j+1}v_j
\]

for \( j = 1, \ldots, k \). This determines \( \bar{F}_n \) uniquely. Define \( F_n := F + \bar{F}_n \). It is shown in [Tr] that the spectrum of \( \bar{A} + \bar{B}F_n \) is the set \( \{-n\} \).

Moreover, we have

\[
\lim_{n \to \infty} \|\bar{C}e^{(A+ BF_n)t}\|_1 = 0
\]

Choose \( n \) such that the impulse response has \( L_1 \) norm smaller than \( \varepsilon/(3\|E\| + 1) \).
This $F_n$ is internally stabilizing and satisfies the $L_1$ bound. Now we can construct the $F$ we were looking for:

$F = F_n + (F_0 + MF_n)$. This $F$ is internally stabilizing and is such that (2.43) is satisfied.

We construct $G$ by dualizing the construction of $F$ and the required dynamic compensator is finally given by (2.45).

5.5 No assumptions on any direct feedthrough matrix

In this section we shall briefly discuss how we can extend our result in theorem 5.1 to the more general system (2.1). We set $\gamma = 1$ but the general result can be easily obtained by scaling. For this system we still have to assume that both the system $(A, B, C_2, D_{21})$ as well as the system $(A, E, C_1, D_{12})$ have no invariant zeros on the imaginary axis. We first tackle the extra feedthrough matrix $D_{22}$ and after that the extra feedthrough matrix $D_{11}$.

5.5.1 An extra direct feedthrough matrix from disturbance to output

The method we use for the case $D_{22} \neq 0$ is completely similar to the method presented in section 4.5 and stems from [Gl5]. If there exists an internally stabilizing feedback of the form (2.4) which makes the $H_\infty$ norm less than 1 for the system $\Sigma$ described by (2.1), then it is easily checked that there must exist a matrix $S$ such that $I - SD_{11}$ is invertible and

$$\|D_{22} + D_{21} (I - SD_{11})^{-1} SD_{12}\| < 1.$$  \hspace{1cm} (5.12)

We assume that an $S$ satisfying (5.12) exists. We define the following system:

$$\Sigma_S : \left\{ \begin{array}{l}
\dot{x}_s = A_s x_s + B_s u_s + E_s w_s,

y_s = C_{1,s} x_s + D_{11,s} u_s + D_{12,s} w_s,

z_s = C_{2,s} x_s + D_{21,s} u_s.
\end{array} \right.$$  \hspace{1cm} (5.13)

where

$$\tilde{D}_{22} := D_{22} + D_{21} (I - SD_{11})^{-1} SD_{12},$$

$$E_S := -(E + B (I - SD_{11})^{-1} SD_{12}) \left( I - \tilde{D}_{22}^T \tilde{D}_{22} \right)^{-1/2}.$$
\[ C_{2,s} := \left( I - \hat{D}_{22} \hat{D}_{22}^T \right)^{-1/2} \left( C_2 + D_{21} (I - S D_{11})^{-1} S C_1 \right), \]
\[ D_{12,s} := (I - D_{11} S)^{-1} D_{12} \left( I - \hat{D}_{22} \hat{D}_{22}^T \right)^{-1/2}, \]
\[ D_{21,s} := \left( I - \hat{D}_{22} \hat{D}_{22}^T \right)^{-1/2} D_{21} (I - S D_{11})^{-1}, \]
\[ A_s := \left( A + B (I - S D_{11})^{-1} S C_1 \right) - E_s \hat{D}_{22} C_{2,s}, \]
\[ B_s := B (I - S D_{11})^{-1} - E_s \hat{D}_{22} D_{21,s}, \]
\[ C_{1,s} := (I - D_{11} S)^{-1} C_1 - D_{12,s} \hat{D}_{22} C_{2,s}, \]
\[ D_{11,s} := (I - D_{11} S)^{-1} D_{11} - D_{12,s} \hat{D}_{22} D_{21,s}. \]

We can then derive the following lemma:

**Lemma 5.7**: Let \( \Sigma_F \) be a dynamic controller of the form (2.4) defining an operator \( G_F \) from \( y \) to \( u \). Consider the following two systems:

\[
\begin{array}{ccc}
& w & \\
\Sigma & \downarrow & \Sigma_F \\
y & \downarrow & y_s \\
& u & \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
& w_s & \\
\Sigma_S & \downarrow & \Sigma_F \\
y_s & \downarrow & u_s \\
\end{array}
\]

Here \( \Sigma_F \) is a compensator defined by the operator \( u = (G_F + S)y \). The system on the left is the interconnection of \( \Sigma \) described by (2.1) and \( \Sigma_F \). The system on the right is the interconnection of \( \Sigma_S \) described by (5.13) and the feedback compensator \( \Sigma_F \). The following two conditions are equivalent:

(i) The system on the left is well-posed, internally stable and the closed loop transfer matrix from \( w \) to \( z \) has \( H_\infty \) norm less than 1.

(ii) The system on the right is well-posed, internally stable and the closed loop transfer matrix from \( w \) to \( z \) has \( H_\infty \) norm less than 1.

\( \square \)

**Remark**: Note that we have to assume that the interconnection is well-posed. For the interconnection of \( \Sigma \) and \( \Sigma_F \) described by (2.1) and (2.4),
this requirement is equivalent to the condition that the matrix \( I - D_{11}N \) is invertible.

**Proof:** We define the following static system

\[
\Sigma_{\Theta} : \begin{pmatrix} z_{\Theta} \\ y_{\Theta} \end{pmatrix} = \begin{pmatrix} \tilde{D}_{22} \\ - (I - \tilde{D}_{22} \tilde{T}_{22})^{1/2} \tilde{T}_{22} \end{pmatrix} \begin{pmatrix} w_{\Theta} \\ u_{\Theta} \end{pmatrix}.
\]

Let \( \bar{\Sigma} \) be the system we obtain by applying to \( \Sigma \) the preliminary static output feedback \( u = Sy + v \), where \( S \) is such that (5.12) is satisfied. Then we note that in the picture

\[
\begin{array}{c}
\sum_{\Theta} \\
\sum_{S}
\end{array}
\]

the system on the left and the interconnection on the right have identical realizations. Moreover, note that the interconnection on the right is always well-posed.

The rest of the proof is completely similar to the proof of lemma 4.5.

In the same way as lemma 4.6 we are able to derive the following lemma:

**Lemma 5.8:** Let \( S \) be such that (5.12) is satisfied. Assume that \( \Sigma \) as described by (2.1) is such that the systems \((A, B, C_2, D_{21})\) and \((A, E, C_1, D_{12})\) have no invariant zeros on the imaginary axis. Then the new system \( \bar{\Sigma}_{S} \) as described by (5.13) has the property that the systems \((A_{S}, B_{S}, C_{2,S}, D_{21,S})\) and \((A_{S}, E_{S}, C_{1,S}, D_{12,S})\) have no invariant zeros on the imaginary axis.

Thus we are able to obtain the following theorem
Theorem 5.9: Consider the system $\Sigma$ described by (2.1). Assume that the systems $(A, B, C_2, D_{21})$ and $(A, E, C_1, D_{12})$ have no invariant zeros on the imaginary axis. Then the following statements are equivalent:

(i) There exists a compensator $\hat{\Sigma}_F$ of the form (2.4) such that the closed loop system $\Sigma \times \hat{\Sigma}_F$ is well-posed, internally stable and has $H_\infty$ norm less than 1.

(ii) There exists an $S$ such that (5.12) is satisfied and if for this $S$ we define the system $\Sigma_S$ by (5.13), then for $\Sigma_S$ there exists a controller $\Sigma_F$ of the form (2.4) such that the interconnection $\Sigma_S \times \Sigma_F$ is well-posed, internally stable and has $H_\infty$ norm less than 1. Moreover the subsystems $(A_S, B_S, C_{2,S}, D_{21,S})$ and $(A_S, E_S, C_{1,S}, D_{12,S})$ have no invariant zeros on the imaginary axis.

Remarks:

(i) Note that if we find a controller $\Sigma_F$ with associated input-output operator $\hat{G}_F$ satisfying part (ii), then we know that a controller with input-output operator $G_F + S$ satisfies part (i).

(ii) Remember that the existence of suitable controllers for the system $\Sigma_S$ is independent of our specific choice for the matrix $S$ satisfying (5.12).

(iii) In this way we have reduced the problem of finding necessary and sufficient conditions under which there exists an internally stabilizing controller which makes the $H_\infty$ norm of the closed loop system less than 1 to the same problem for the system $\Sigma_S$. However for the latter system $D_{22} = 0$. In the next subsection we show how we can reduce it to the same problem for a system with both $D_{11} = 0$ and $D_{22} = 0$. We can show that after these two steps for the system obtained in this way the two subsystems do not have invariant zeros on the imaginary axis if and only if the two subsystems for the original system (2.1) do not have invariant zeros on the imaginary axis. Hence we may apply theorem 5.1 to this new system.
5.5 General direct feedthrough matrices

5.5.2 An extra direct feedthrough matrix from control to measurement

In this section we investigate a system $\Sigma$, described by (2.1) with $D_{22} = 0$. It has been shown in the previous subsection how we can reduce the problem of finding internally stabilizing controllers which make the closed loop $H_\infty$ norm less than 1 to the same problem for a system with $D_{22} = 0$. In this subsection we shall show how we can reduce it even further to the case that both $D_{11} = 0$ as well as $D_{22} = 0$.

We define the following system:

$$
\tilde{\Sigma} : \left\{ \begin{array}{l}
\dot{x} = Ax + Bu + Ew, \\
y = C_1 x + D_{12} w, \\
z = C_2 x + D_{21} u.
\end{array} \right. \tag{5.14}
$$

Assume that for the system $\Sigma$ we have a controller $\Sigma_F$ of the form (2.4) such that the interconnection $\Sigma \times \Sigma_F$ is well-posed, i.e. $I - D_{11} N$ is invertible. Moreover, assume that this controller is internally stabilizing and makes the $H_\infty$ norm less than 1. We define the following controller

$$
\tilde{\Sigma}_F : \left\{ \begin{array}{l}
\dot{p} = \tilde{K} p + \tilde{L} y, \\
y = \tilde{M} p + \tilde{N} y, \\
\end{array} \right. \tag{5.15}
$$

where

$$
\tilde{K} := K + L (I - D_{11} N)^{-1} D_{11} M, \\
\tilde{L} := L (I - D_{11} N)^{-1}, \\
\tilde{M} := (I - N D_{11})^{-1} M, \\
\tilde{N} := N (I - D_{11} N)^{-1}.
$$

Then it is easily shown that $\tilde{\Sigma}_F$ is an internally stabilizing feedback for the system $\tilde{\Sigma}$. Moreover, the interconnection $\tilde{\Sigma} \times \tilde{\Sigma}_F$ and the interconnection $\Sigma \times \Sigma_F$ have identical realizations and hence the interconnection $\tilde{\Sigma} \times \tilde{\Sigma}_F$ has $H_\infty$ norm less than 1.

On the other hand, assume that we have a controller $\tilde{\Sigma}_F$ of the form (5.15) for $\tilde{\Sigma}$ which is internally stabilizing and makes the $H_\infty$ norm strictly less than 1. Note that this interconnection is always well-posed. We know that for all $\varepsilon > 0$ there exists a matrix $Z$ with $\|Z\| < \varepsilon$ such that

$$I + (\tilde{N} + Z) D_{11}$$
is invertible. Denote the controller we obtain by replacing $\tilde{N}$ by $\tilde{N} + Z$ by $\hat{\Sigma}_F(Z)$. It is easily seen that for $\varepsilon$ sufficiently small (and hence $Z$ sufficiently small) the closed loop system after applying the compensator $\hat{\Sigma}_F(Z)$ to $\hat{\Sigma}$ is still internally stable and still has $H_\infty$ norm less than 1. Assume that $Z$ is chosen like this then the controller $\hat{\Sigma}_F$ of the form (2.4) and defined by

$$K := \tilde{K} - \tilde{L} \left( I + D_{11} \left( \tilde{N} + Z \right) \right) D_{11} \tilde{M},$$

$$L := \tilde{L} \left( I + D_{11} \left( \tilde{N} + Z \right) \right)^{-1},$$

$$M := \left( I + \left( \tilde{N} + Z \right) D_{11} \right)^{-1} \tilde{M},$$

$$N := \left( I + \left( \tilde{N} + Z \right) D_{11} \right)^{-1} \left( \tilde{N} + Z \right),$$

is internally stabilizing for the system $\Sigma$ and the closed loop system $\Sigma \times \Sigma_F$ has $H_\infty$ norm less than 1.

In the above we derived the following theorem:

\textbf{Theorem 5.10:} Let the system $\Sigma$ described by (2.1) with $D_{22} = 0$ be given. Then the following statements are equivalent:

(i) There exists a compensator $\Sigma_F$ of the form (2.4) such that the closed loop system $\Sigma \times \Sigma_F$ is well-posed, internally stable and has $H_\infty$ norm less than 1.

(ii) For the system (5.14) there exists a controller of the form (5.15) which is internally stabilizing and makes the $H_\infty$ norm less than 1.

\textbf{Remark:} Thus we see that we can reduce the problem of finding a controller which is internally stabilizing and which makes the $H_\infty$ norm less than 1 (a suitable controller) for the system $\Sigma$ described by (2.1) with $D_{22} = 0$ to the problem of finding a suitable controller for the system we obtain by setting $D_{11} = 0$ in (2.1). The system we thus obtain then naturally has $D_{11} = 0$ as well as $D_{22} = 0$. In the previous subsection we showed how we could reduce the problem of finding a suitable controller for the general system (2.1) to the problem of finding a suitable controller for a system of the same form (2.1) but with $D_{22} = 0$. In both steps the property that the two subsystems have no invariant zeros on the imaginary axis is preserved. Hence, if for some general system $\Sigma$ of the form (2.1) we need to find a
suitable controller and if the subsystems \((A, B, C_2, D_{21})\) and \((A, E, C_1, D_{12})\) have no invariant zeros on the imaginary axis, then we can reduce this problem to the problem of finding a suitable controller for a system on which we may apply the main result of this chapter: theorem 5.1.

5.6 Conclusion

In this chapter we have given a complete treatment of the \(H_\infty\) problem with measurement feedback without restrictions on the direct feedthrough matrices. However, it remains an open problem how we can treat invariant zeros on the imaginary axis. This problem is studied in [SC4]. Other open problems are the determination of the minimally required dynamic order of the controller and the characterization of the behaviour of the feedbacks and closed loop system if we tighten the bound \(\gamma\). The latter problem has already been investigated. It is possible that the infimum over all stabilizing controllers of the closed loop \(H_\infty\) norm is never attained, attained by a non-proper controller or attained by a proper controller (see [Fr2]). Using the ideas of this chapter it might be possible to characterize whether we can attain the infimum by a proper controller.

Finally, it would be interesting to characterize all solutions. In our opinion it is not possible in general to obtain a characterization similar to the one obtained in [Do4]. This is due to the fact that the so-called central controller might be non-proper.

In our opinion this chapter again gives extra support to our claim that the approach to solve the \(H_\infty\) problem in the time-domain is a much more intuitive and appealing approach than the other methods used in recent papers. Only when using frequency related weighting are we losing the intuition of the meaning of the filters. Hence when discussing choices of the filters one might be better of in the frequency domain. But for intuition when actually solving the \(H_\infty\) control problem one is really better of in the time domain.
Chapter 6

The singular zero-sum differential game with stability

6.1 Introduction

In this chapter we shall consider the zero-sum linear quadratic finite-dimensional differential game. This is an area of research which was rather popular during the seventies (see e.g. [Ban, Ma, BO, Sw]).

In the last few years, the solution of the regular $H_\infty$ control problem (see chapter 3 and [Do4, Kh3, Pe]) turned out to contain the same kind of algebraic Riccati equation as the one appearing in the solution of the zero-sum differential game (see [Ban, Ma, Sw]). This Riccati equation has the special property that the quadratic term is, in general, indefinite, in contrast to, for instance, the equation appearing in linear quadratic optimal control theory (see [Wi4]), where the quadratic term in the Riccati equation is always definite.

Since in $H_\infty$ control theory the solution of the algebraic Riccati equation has no meaning in itself it is interesting to give a more intuitive characterization such as a Nash equilibrium in the theory of differential games. Recently, a number of papers appeared which studied a zero-sum differential game with the goal of obtaining such a characterization. (see [Pe2, Pe3, We])

In chapter 4 it has been shown that if the direct feedthrough matrix from the control input to the output is not injective then, instead of an algebraic Riccati equation, we get a quadratic matrix inequality. A similar phenomenon also occurs in linear quadratic optimal control theory, although in that case we get a linear matrix inequality (see [Wi4]).

This chapter is concerned with the zero-sum differential game in the case
that the direct feedthrough matrix is not injective. It will be shown that, as expected, we also get a quadratic matrix inequality. Moreover, by using results from $H_\infty$ control theory, we are able to derive necessary conditions for the existence of an equilibrium, which, to our knowledge, has not been done in previous papers. We shall study the differential game with certain stability requirements since it turns out to give results which indeed center around the same solution of the quadratic matrix inequality as the one which appears in $H_\infty$ control. If we assume detectability, then the problems with and without stability turn out to be equivalent.

We give this treatment of the differential game in this thesis although the formal proofs of the results on $H_\infty$ control theory do not depend on results derived in this chapter. The reason for nevertheless including this chapter lies in the fact that the intuition, which yielded the results on $H_\infty$ control, mostly stems from this chapter. The results of this chapter already appeared in [St3].

The outline of this chapter is as follows: in section 6.2 we formulate the problem and give our main results. In section 6.3 we prove the existence of an equilibrium under certain sufficient conditions. After that, in section 6.4 we derive necessary conditions for the existence of equilibria. The proofs in these two sections are very much concerned with the specific choice of bases and other results of appendix A. In section 6.5 we show that if the direct feedthrough matrix from control input to output is injective, then the necessary conditions of section 6.4 are also sufficient. We conclude in section 6.6 with some remarks.

### 6.2 Problem formulation and main results

We shall consider the zero-sum, infinite horizon, linear quadratic differential game with cost criterion

$$\mathcal{J}(u,w) = \int_0^\infty x^T(t)z(t) - w^T(t)w(t)dt,$$

(6.1)

and dynamics given by the following linear and finite-dimensional system

\[ \Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, & x(0) = \xi, \\ z = Cx + Du. \end{cases} \]

(6.2)

Here, for all $t$ we have $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^l$ and $z(t) \in \mathbb{R}^p$. $A, B, C, D$ and $E$ are matrices of appropriate dimensions. Note that contrary
6.2 Problem formulation and main results

to the other chapters we shall in general consider systems with initial state unequal to zero. We assume that \((A, B)\) is stabilizable. We define the following class of functions:

\[
U^k_{f_b} = \left\{ v : \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}^k \mid \forall x \in L^2_2 \quad v(x(.),.) \in L^k_2 \right\}.
\]

Note that we can consider \(L^k_2\) as a subset of \(U^k_{f_b}\) by identifying which each function \(v \in L^k_2\) a function \(\tilde{v} \in U^k_{f_b}\) as follows:

\[
\tilde{v}(x,t) := v(t) \quad \forall x \in \mathcal{R}^n, t \in \mathcal{R}^+.
\]

We call \((\tilde{u}, \tilde{w}) \in U^m_{f_b} \times U^l_{f_b}\) an admissible pair for initial value \(\xi\) if by setting \(u(.) = \tilde{u}(x(.),.)\) and \(w(.) = \tilde{w}(x(.),.)\) the differential equation (6.2) has a unique solution \(x_{u,w,\xi} \) on \([0, \infty)\) which satisfies \(x_{u,w,\xi} \in L^2_2\). Hence by the definition of \(U^k_{f_b}\) we have \(v \in L^2_2\) and \(w \in L^k_2\). This implies that the resulting output trajectory satisfies \(x_{u,w,\xi} \in L^2_2\) and hence \(J(u, w)\) as defined in (6.1) is well-defined. We shall only consider inputs of this form. Since \((A, B)\) is stabilizable the class of admissible pairs is non-empty for every initial state \(\xi\).

We also define the following class of functions:

\[
U^m_{f_t}(\xi) := \left\{ u \in U^m_{f_b} \mid \forall w \in L^2_2 : (u, w) \text{ is an admissible pair for initial value } \xi \right\}.
\]

Note that our assumption that \((A, B)\) is stabilizable also yields the stronger result that \(U^m_{f_t}(\xi)\) is non-empty for every initial condition \(\xi\). We shall call \(u\) a minimizing player and his goal is to minimize the cost criterion \(J(u, w)\). In the same way we shall call \(w\) a maximizing player who would like to maximize the cost criterion \(J(u, w)\).

**Definition 6.1**: The system (6.2) with criterion function (6.1) is said to have an equilibrium if for every initial value \(\xi\) there exist \(u_0 \in U^m_{f_t}(\xi)\) and \(w_0 \in U^l_{f_b}\) such that \((u_0, w_0)\) is an admissible pair and moreover,

\[
J(u_0, w) \leq J(u_0, w_0) \leq J(u, w_0),
\]

for all \(u \in U^m_{f_t}(\xi)\) and \(w \in U^l_{f_b}\) such that \((u, w_0)\) and \((u_0, w)\) are admissible pairs.
Note that in the theory of differential games, this is often called a Nash equilibrium. The existence of an equilibrium is, in general, too strong a condition. We shall define a weaker version.

**Definition 6.2 :** The system (6.2) with cost criterion (6.1) is said to have an almost equilibrium if there exists a function \( \mathcal{J}^* : \mathbb{R}^n \rightarrow \mathbb{R} \) such that for all \( \varepsilon > 0 \) and for every state \( \xi \in \mathbb{R}^n \) there exist \( u_0 \in \mathcal{U}_{f}^m(\xi) \) and \( w_0 \in \mathcal{U}_{b}^f(\xi) \) such that \((u_0, w_0)\) is admissible and moreover,

\[
\mathcal{J}(u_0, w) \leq \mathcal{J}^*(\xi) + \varepsilon, \\
\mathcal{J}(u, w_0) \geq \mathcal{J}^*(\xi) - \varepsilon,
\]

(6.3) for all \( u \in \mathcal{U}_{f}^m(\xi) \) and \( w \in \mathcal{U}_{b}^f(\xi) \) such that \((u, w_0)\) and \((u_0, w)\) are admissible pairs. \(\Box\)

**Remarks:**

(i) Note that an equilibrium certainly defines an almost equilibrium since we in that case we can find \( u_0 \) and \( w_0 \) such that \( \mathcal{J}^*(\xi) = \mathcal{J}(u_0, w_0) \) satisfies (6.3) for \( \varepsilon = 0 \).

(ii) Note that if either \( u_0 \) or \( w_0 \) is fixed, then choosing the other input in \( \mathcal{U}_{f}^m(\xi) \) such that we have an admissible pair, results in well-defined functions in \( L_2 \) for the state, the minimizing player and the maximizing player. Hence in definition 6.2 and also in definition 6.1 we can, without loss of generality, assume that \( u \) and \( w \) are in \( L_2 \) instead of \( \mathcal{U}_{f}^m(\xi) \). In that case there are no restrictions any more on \( w \) because \((u_0, w)\) is admissible for all \( w \in L_2 \) since \( u_0 \in \mathcal{U}_{f}^m(\xi) \). This condition is rather unusual in zero-sum linear quadratic differential games. Intuitively it means that we hand over the responsibility of the condition \( x \in L_2^n \) to the minimizing player. Without that assumption it can happen that there exist \( u_0 \) and \( w_0 \) such that

\[
\mathcal{J}(u_0, w) \leq M_1 \\
\mathcal{J}(u, w_0) \geq M_2
\]

for all \( u \) and \( w \) such that \((u_0, w)\) and \((u, w_0)\) are admissible pairs and \( M_2 > M_1 \). Clearly \((u_0, w_0)\) is not admissible but neither \( u_0 \) nor \( w_0 \) will change since that will be contrary to their objective of minimizing
respectively maximizing the cost-criterion. To prevent such a deadlock we hand over the responsibility for \( x \in \mathcal{L}^2 \) to one of the players. An example of this phenomenon is given by the system

\[
\begin{align*}
\dot{x} &= x + u + w, \\
z &= 10^3 u.
\end{align*}
\]

We shall derive conditions for the existence of an almost equilibrium. Since we do not assume that the \( D \) matrix is injective it is not surprising that, as in the singular LQ problem, we find a matrix inequality instead of a Riccati equation. We repeat a number of definitions already used in chapter 4:

\[
F(P) := \begin{pmatrix}
ATP + PA + C^T C + PEE^TP & PB + C^TD \\
B^TP + D^TC & D^TD
\end{pmatrix}.
\]

We call a symmetric matrix \( P \in \mathcal{R}^{n \times n} \) a solution of the quadratic matrix inequality if \( F(P) \geq 0 \). Furthermore, we define

\[
L(P, s) := (sI - A - EET P - B),
\]

and finally we define \( G_{ci}(s) := C(sI - A)^{-1}B + D \). We shall now present the main results of this chapter:

**Theorem 6.3**: Consider the system (6.2) with cost criterion (6.1) and assume that \((A, B)\) is stabilizable. There exists an almost equilibrium if the following condition is satisfied:

There exists a positive semi-definite solution \( P \) of \( F(P) \geq 0 \) such that

(i) \( \text{rank } F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci} \),

(ii) \( \text{rank } \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci}, \quad \forall s \in C^+ \cup C^0. \)

Moreover \( J^*(\xi) = \xi^TP\xi \) defines an almost equilibrium and for each bounded set of initial values and for each \( \varepsilon > 0 \) we can find static state feedbacks \( F_u, F_w \) such that \( u_0 = F_u x \) and \( w_0 = F_w x \) satisfy (6.3) for all initial values in that set.
Remark : We shall show that \( P \) also satisfies the following equality:

\[
\xi^T P \xi = \inf_{u \in \mathcal{U}_1^\infty(t)} \sup_{w \in \mathcal{U}_2^1} J(u, w).
\]

About the necessity of the above conditions we have the following result:

**Theorem 6.4** : Consider the system (6.2) with cost criterion (6.1). Assume that \((A, B)\) is stabilizable and assume that \((A, B, C, D)\) has no invariant zeros on the imaginary axis. If there exists an almost equilibrium, then the following condition is satisfied:

There exists a positive semi-definite solution \( P \) of \( F(P) \geq 0 \) such that

(i) \( \text{rank } F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci} \),

(ii) \( \text{rank} \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci} \), \( \forall s \in \mathcal{C}^+ \).

Moreover, in the case that \( D \) is injective, the above condition is also sufficient for the existence of an almost equilibrium.

Remark : Although in the case that \( D \) is injective we can prove the existence of an almost equilibrium under the assumptions of theorem 6.4, we have not been able to find static state feedback laws for \( u_0 \) and \( w_0 \) which we were able to find under the assumptions of theorem 6.3.

### 6.3 Existence of almost equilibria

In this section we assume that a \( P \) satisfying the conditions of theorem (6.3) exists. We shall show that there exists an *almost equilibrium*. The proof will be strongly related to the proof given in section 4.4. We first apply the preliminary feedback as defined in appendix A, i.e. \( u = F_0 x + v \) where \( F_0 \) is defined by (A.1). We obtain the system

\[
\hat{\Sigma} : \begin{cases} 
\dot{x} = (A + BF_0)x + Bv + Ew, \\
\dot{z} = (C + DF_0)x + Dw.
\end{cases}
\]

For this system we define the cost criterion:
6.3 Existence of almost equilibria

Clearly this system has an almost equilibrium if and only if the original system $\Sigma$ has an almost equilibrium. Moreover, for initial condition $\xi$, $(u, w)$ is an admissible pair if and only if $(v, w) = (u - F_0 x, w)$ is an admissible pair. Finally, if $v = u - F_0 x$, then we have $\tilde{J}(v, w) = \tilde{J}(u, w)$. Therefore in the remainder of this section we may investigate the system $\tilde{\Sigma}$, instead of $\Sigma$, with cost-criterion (6.4).

We shall use the following lemma which will give theorem 6.3 as an almost direct result.

Lemma 6.5: Let $P$ be given such that $F(P) \geq 0$. Choose the bases of appendix A so that $P$ has the form (A.15). Then for all admissible pairs $(v, w)$ we have:

$$
\tilde{J}(v, w) = \|z\|^2 - \|w\|^2 = \xi^T P \xi + \int_0^\infty x_1(\tau)^T R(P_1) x_1(\tau) d\tau + \|C_{23} q_3\|^2 + \|\tilde{D} v_1\|^2 - \|w_1\|^2.
$$

(6.5)

where

$$
q_3 := x_3 + (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{23}) x_1,
$$

(6.6)

$$
\tilde{v}_1 := v_1 + (\tilde{D}^T \tilde{D})^{-1} B_{11}^T P_1 x_1,
$$

(6.7)

$$
w_1 := w - E^T P x.
$$

(6.8)

Moreover the dynamics in these new coordinates are given by

$$
\dot{x}_1 = \tilde{A}_{11} x_1 + A_{13} q_3 + B_{11} \tilde{v}_1 + E_1 w_1,
$$

(6.9)

$$
\begin{pmatrix}
\dot{x}_2 \\
\dot{q}_3
\end{pmatrix} =
\begin{pmatrix}
A_{22} & A_{23} \\
A_{32} & \tilde{A}_{33}
\end{pmatrix}
\begin{pmatrix}
x_2 \\
q_3
\end{pmatrix} +
\begin{pmatrix}
B_{22} \\
B_{32}
\end{pmatrix} v_2 +
\begin{pmatrix}
\tilde{A}_{21} \\
\tilde{A}_{31}
\end{pmatrix} x_1 +
\begin{pmatrix}
B_{21} \\
B_{31}
\end{pmatrix} \tilde{v}_1 +
\begin{pmatrix}
E_2 \\
\tilde{E}_3
\end{pmatrix} w_1
$$

(6.10)

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{z}_2
\end{pmatrix} =
\begin{pmatrix}
\tilde{C}_1 \\
\tilde{C}_3
\end{pmatrix} z_1 +
\begin{pmatrix}
0 \\
C_{23}
\end{pmatrix} q_3
$$

(6.11)
Here we used the definitions given just after (4.13) in chapter 4. Moreover we used the following three definitions

\[ \tilde{A}_{11} := \tilde{A}_{11} + E_1 E_1^T P_1, \]
\[ \tilde{A}_{21} := \tilde{A}_{21} + E_2 E_1^T P_1, \]
\[ \tilde{A}_{31} := \tilde{A}_{31} + E_3 E_1^T P_1. \]

\[ \square \]

Remark: Note that the system dynamics given here are almost the same as the system dynamics given in (4.11)–(4.13). Only in chapter 4 we set \( \tilde{v}_1 = 0 \) and we did not make the transformation from \( w \) to \( w_1 \).

Proof: By using the system equations (6.2) we find

\[ \frac{d}{dt} \left[ x^T(t) P x(t) - \xi^T P \xi + \int_0^t z(\tau)^T z(\tau) - w(\tau)^T w(\tau) \, d\tau \right] \]

\[ = \begin{pmatrix} x \\ v \\ w \\ w_1 \end{pmatrix}^T \begin{pmatrix} A_{F_0}^T P + PA_{F_0} + C_{F_0}^T C_{F_0} & PB & PE \\ B^T P & D^T D & 0 \\ E^T P & 0 & -I \end{pmatrix} \begin{pmatrix} x \\ v \\ w_1 \end{pmatrix} \]

\[ = \begin{pmatrix} x \\ v \\ w_1 \end{pmatrix}^T \begin{pmatrix} A_{F_0}^T P + PA_{F_0} + C_{F_0}^T C_{F_0} + PEE^T P & PB & 0 \\ B^T P & D^T D & 0 \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} x \\ v \\ w_1 \end{pmatrix}. \]

where we have

\[ A_{F_0} := A + BF_0, \]
\[ C_{F_0} := C + DF_0, \]

and \( w_1 \) as given by (6.8) and \( F_0 \) as defined by (A.1). We can now use the decomposition as defined in (A.2) and we find that (6.13) is equal to:

\[ \begin{pmatrix} x_1 \\ x_3 \\ v_1 \\ w_1 \end{pmatrix}^T \begin{pmatrix} P_1 A_{11} + A_{11}^T P_1 + C_{21}^T C_{21} + P_1 E_1 E_1^T P_1 & P_1 A_{13} + C_{21}^T C_{23} & P_1 B_{11} & 0 \\ A_{F_0}^T P_1 + C_{F_0}^T C_{21} & C_{F_0}^T C_{23} & 0 & 0 \\ B_{11} P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ v_1 \\ w_1 \end{pmatrix}.
When we finally use the definitions (6.6) and (6.7) and integrate the equation (6.12) from 0 to \( \infty \) we find the equation (6.5). Here we used that

\[
\lim_{t \to \infty} x^T(t) Px(t) = 0,
\]

since the pair \((v, w)\) is admissible and hence \(x \in L_2^0\) and \(\dot{x} \in L_2^0\). Moreover \((v, w)\) admissible implies that the integral in (6.5) is well-defined.

Assume that we have a matrix \(P\) such that \(F(P) \geq 0\) and such that rank \(F(P) = \operatorname{rank} \mathcal{R}(\mathcal{S}) G_c\). Then in the bases of appendix A the matrix \(P\) can be written in the form (A.15) with \(R(P) = 0\). The minimizing player can control \(v_1\) and \(v_2\). As we did in chapter 4 we assume for the moment that the minimizing player can control \(v_1\) and \(v_3\) and hence also \(\delta_1\) and \(q_3\). The maximizing player can control \(w_1\). From the previous lemma it is then intuitively clear that the minimizing player should set \(\delta_1 = 0\) and \(q_3 = 0\) and the maximizing player should set \(w_1 = 0\) which would yield an equilibrium.

Two things remain to be done. First we have to work out how the minimizing player can control \(q_3\). This will be done using the following lemma. It will turn out that the minimizing player can control \(q_3\) arbitrarily well but not exactly, which is the reason why we only find an almost equilibrium instead of an equilibrium. Finally, it has to be shown that the above choices guarantee that the corresponding \(u\) and \(w\) are in the desired feedback classes. This will be done after the following lemma. For this we have to use the second rank condition that \(P\) satisfies.

We shall use lemma 6.5 together with lemma 2.18, which turns out to be extremely useful for singular differential games, to prove theorem 6.3:

**Proof of theorem 6.3**: Let the matrix \(P\) satisfying the conditions of theorem 6.3. Moreover, let \(\varepsilon > 0\) be given. First note that when we choose \(w_1 = 0\), i.e. \(w_0 = E^T P x\) then by lemma (6.5) we have:

\[
\hat{J}(v, w_0) \geq \xi^T P \xi \geq \xi^T P \xi - \varepsilon,
\]

for all \(v\) such that \((v, w_0)\) is an admissible pair. This is the second inequality in (6.3).

In order to prove the other inequality in (6.3) we have to do some preparatory work. We start by choosing \(\delta_1 = 0\), i.e.

\[
v_1 = -\left(\dot{D}^T \dot{D}\right)^{-1} B_{11}^T P_1 x_1.
\]
Since $P$ satisfies the conditions of theorem 6.3 we know by corollary A.8 that the matrix $A_{11}$ is asymptotically stable. Assume that we have an initial value in some bounded set $V$. Then the mapping from $q_3$ and $w_1$ to $x_1$ is bounded i.e. there are $M_1, M_2, M$ such that for all $q_3$ and $w_1$ in $L_2$ and $\xi \in V$ we have

$$\|x_1\|_2^2 \leq M_1\|q_3\|_2^2 + M_2\|w_1\|_2^2 + M.$$  \hfill (6.14)

Consider the system given by the differential equation (6.10) with input $v_2$, state $x_2, q_3$ and output $q_3$. (Note that $v_1 = 0$.) It was shown in the proof of theorem 4.3 that this system is strongly controllable. We assumed that $\xi \in V$ for some bounded set $V$. Therefore by lemma 2.18 we know we can find a feedback

$$v_2 = F_1 \begin{pmatrix} x_2 \\ q_3 \end{pmatrix},$$

such that by applying that feedback in (6.10) (with once again $v_1 = 0$) we have

$$\|q_3\|_2^2 \leq \|C_{23}\|^{-1} \min \left\{ \varepsilon (\varepsilon M_1 + M + 1)^{-1}, (M_1 + M_2 + 1)^{-1} \right\}$$

$$\times \left( \|x_1\|_2^2 + \|w_1\|_2^2 + 1 \right),$$

for all $\xi \in V$ and for all $w_1 \in L_2$ and $x_1 \in L_2$. Combining this with (6.14) we find

$$\|C_{23q_3}\|_2^2 \leq \|w_1\|_2^2 + \varepsilon.$$

By choosing

$$v_2 = F_1 \begin{pmatrix} x_2 \\ q_3 \end{pmatrix}, \quad v_0 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

we therefore find

$$\hat{J}(v_0, w) \leq \xi^T P \xi + \varepsilon.$$

This gives the first inequality in (6.3). By noting that for each bounded set of initial values the $u_0$ and $w_0$ are given by a static state feedback the proof is completed.
Corollary 6.6: Let $P \in \mathbb{R}^{n \times n}$ satisfy the conditions of theorem 6.3. We have the following equality,

$$\xi^T P \xi = \inf_{u \in \mathcal{U}^n_t(\varepsilon)} \sup_{w \in \mathcal{L}^n_2} J(u, w).$$

(6.15)

Proof: For any $\varepsilon > 0$ we can choose $u = u_0$ as defined in the proof of theorem 6.3. We have $\delta_1 = 0$ and $\|C_{23}q_3\| \leq \|w_1\| + \varepsilon$ and using the equality in (6.5) we find

$$\inf_{u \in \mathcal{U}^n_t(\varepsilon)} \sup_{w \in \mathcal{L}^n_2} J(u, w) \leq \xi^T P \xi + \varepsilon.$$ 

Since this is true for all $\varepsilon$ we find an inequality in (6.15). By choosing $w_1 = 0$ for arbitrary $u$ we find the opposite inequality and hence equality. \qed

6.4 Necessary conditions for the existence of almost equilibria

In this section we shall derive necessary conditions for the existence of an almost equilibrium. This will be done by showing that the existence of an almost equilibrium implies that there exists a feedback which makes the $H_\infty$ norm of the closed loop system less than or equal to 1. Our main tool will be theorem 4.1 which gives necessary and sufficient conditions under which we can make the $H_\infty$ norm of the closed loop system strictly less than $\gamma$. However, we have to do some work to find necessary conditions for "less than or equal to $\gamma$".

We define the following class of input functions:

$$\mathcal{U}(\Sigma, w, \xi) = \{ u \in L^m_2 | \text{By applying } u, w \text{ in } \Sigma \text{ we have } x_{u, w, \xi} \in L^m_2 \}.$$ 

We shall use the following lemmas.

Lemma 6.7: Assume that for the system (6.2) with cost-criterion (6.1) there exists an almost equilibrium. Moreover, let the initial condition $\xi = 0$. Under the above assumptions we have:
The singular differential game

\[ \inf_{u \in U(\Sigma, w, 0)} \mathcal{J}(u, w) \leq 0, \quad \forall w \in \mathcal{L}_2. \quad (6.16) \]

**Proof**: We know that for any \( \varepsilon > 0 \) there exists \( u_0 \in U_{fb} \) such that (6.3) is satisfied. Choose an arbitrary \( w \in \mathcal{L}_2 \). We have

\[ \inf_{u \in U(\Sigma, w, 0)} \mathcal{J}(u, w) \leq \mathcal{J}(u_0, w) \leq \mathcal{J}^*(0) + \varepsilon. \]

This implies that for arbitrary \( \lambda > 0 \) we have

\[ \inf_{u \in U(\Sigma, \lambda w, 0)} \mathcal{J}(\lambda u, \lambda w) = \inf_{u \in U(\Sigma, \lambda w, 0)} \mathcal{J}(u, \lambda w) \leq \mathcal{J}(u_0, \lambda w) \leq \mathcal{J}^*(0) + \varepsilon. \]

But since \( \xi = 0 \) we have \( \mathcal{J}(\lambda u, \lambda w) = \lambda^2 \mathcal{J}(u, w) \). Hence

\[ \lambda^2 \inf_{u \in U(\Sigma, w, 0)} \mathcal{J}(u, w) \leq \mathcal{J}^*(0) + \varepsilon. \]

Since this is true for all \( \lambda > 0 \) we find (6.16). \( \blacksquare \)

The following lemma is a straightforward consequence of the definition of an almost equilibrium and the definition of the feedback classes.

**Lemma 6.8**: Assume that \( \mathcal{J}^* \) defines an almost equilibrium for the system (6.2) with cost criterion (6.1). For all \( \xi \in \mathbb{R}^n \) we have:

\[ \inf_{u \in U_{fb}^*(\xi)} \sup_{w \in \mathcal{L}_2} \mathcal{J}(u, w) \leq \mathcal{J}^*(\xi). \quad (6.17) \]

**Proof of the necessity part of theorem 6.4**: We define:

\[ \mathcal{J}_\gamma(u, w) := \int_0^\infty z^\top(t)z(t) - \gamma^2 w^\top(t)w(t) \, dt = \|z\|^2 - \gamma^2 \|w\|^2. \]

It is easily seen that \( \mathcal{J}_\gamma(u, w) = \mathcal{J}(u, w) \) for \( \gamma = 1 \). Let \( \gamma > 1 \) be given. We have:
6.5 The regular differential game

\[ \inf_{u \in U(\Sigma,w,0)} J_\gamma(u,w) = \inf_{u \in U(\Sigma,w,0)} J(u,w) - (\gamma^2 - 1)\|w\|_2^2, \]

and hence by lemma 6.7 we have

\[ \inf_{u \in U(\Sigma,w,0)} \|z_{u,w,0}\|_2^2 - \gamma^2 \|w\|_2^2 \leq - (\gamma^2 - 1)\|w\|_2^2, \] (6.18)

for all \( w \in L_1^2 \). Therefore, by applying theorem 4.1 to the system \( \Sigma \), we find that there exists a positive semi-definite matrix \( P_\gamma \) such that \( F_\gamma(P_\gamma) \geq 0 \) where \( F_\gamma \) is defined by (4.2) (with \( D_1 \) replaced by \( D \)) and such that \( P_\gamma \) satisfies the following two rank conditions,

\[
\text{rank} \quad F_\gamma(P_\gamma) = \text{rank}_{R(s)} G_{ci},
\]

\[
\text{rank} \quad \left( \begin{array}{c}
L_\gamma(P_\gamma, s) \\
F_\gamma(P_\gamma)
\end{array} \right) = n + \text{rank}_{R(s)} G_{ci}, \quad \forall s \in C^+ \cup C^0,
\]

where \( L_\gamma \) is defined by (4.3). Since by theorem 6.3 \( P_\gamma \) is an almost equilibrium of a differential game with cost criterion \( J_\gamma \) which satisfies corollary 6.6 it is easily seen that if \( \gamma \downarrow 1 \) then \( P_\gamma \) increases i.e. \( P_{\gamma_1} - P_{\gamma_2} \geq 0 \) if \( 1 < \gamma_1 \leq \gamma_2 \). On the other hand by lemma 6.8 we have:

\[
\xi^T P_\gamma \xi = \inf_{u \in U_{\eta_1}(\xi)} \sup_{w \in L_2^1} J_\gamma(u,w)
\leq \inf_{u \in U_{\eta_1}(\xi)} \sup_{w \in L_2^1} J_1(u,w)
\leq J^*(\xi).
\]

Hence \( \lim_{\gamma \downarrow 1} P_\gamma = P \) exists. Since \( \text{rank} F_\gamma(X) \geq \text{rank}_{R(s)} G_{ci} \) for all symmetric matrices \( X \) (see the proof of theorem A.6), by a continuity argument it can be shown that our limit \( P \) satisfies the rank condition (i) of theorem 6.4. In lemma A.6, part (iii) it has been shown that the rank condition (ii) of theorem 6.3 implies that a certain matrix is asymptotically stable. Therefore, again by a continuity argument, we know that in the limit this matrix has all its eigenvalues in the closed right half plane. This is equivalent with the rank condition (ii) of theorem 6.4 by lemma A.6, part (iii).

6.5 The regular differential game

We shall now show the last part of theorem 6.4. The problem we have with the necessary conditions is that if we apply the \( u_0 \) and \( w_0 \) as we derived them
in section 6.3, then \((u_0, w_0)\) is no longer admissible. The trick we use is that we approximate \(w_0\) by a function \(\tilde{w}_0\) with compact support. Then \((u_0, \tilde{w}_0)\) is admissible and after some estimations it can be shown that we indeed have an almost equilibrium. This has been recapitulated in the following theorem,

**Theorem 6.9**: Assume that we have the system (6.2) with a cost criterion given by (6.1). Furthermore assume that \(D\) is injective and that there are no invariant zeros on the imaginary axis for the system \((A, B, C, D)\). If there exists a matrix \(P\) such that \(F(P) \geq 0\) and the rank conditions (i) and (ii) in theorem 6.4 are satisfied, then there exists an almost equilibrium. □

Remark: This is an extension of the results in [Ma]. However there is an essential difference because we require stability. In [Ma] one of the assumptions \(((C, A)\) detectable) is such that the problems with and without stability are equivalent. The problem in this chapter is that the set of admissible inputs is no longer a simple product space.

The proof will make use of two lemmas. The following lemma has been proven in chapter 4 after the statement of theorem 4.1.

**Lemma 6.10**: Assume that \(D\) is injective. Suppose a symmetric matrix \(P\) is given. Then the following two conditions are equivalent,

(i) \(F(P) \geq 0\) and \(\text{rank} F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci}\),

(ii) \(R(P) := PA + A^TP + PEETP + C^TC - (PB + C^TD)(D^TD)^{-1}(B^TP + D^TC) = 0\).

Moreover if \(P\) satisfies (i) (or equivalently (ii)), then the following two conditions are equivalent for all \(s \in \mathcal{C}\):

(iii) \(\text{rank} \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci}\),

(iv) the matrix \(A + EE^TP - B(D^TD)^{-1}(B^TP + D^TC)\) has no eigenvalue in \(s\). □
Note that in the case that $D$ is injective we have $\text{rank}_{R(z)} G_{ci} = \text{rank} D$. At this point we shall recall the following, known, result for the LQ-problem with stability (see [Wi4]).

**Lemma 6.11**: Consider the system (6.2) with cost-criterion (6.1). Let $w \equiv 0$. Assume that $(A, B)$ is stabilizable, that $(A, B, C, D)$ has no invariant zeros in $C^0$ and that $D$ injective. Then we have the following

$$\inf_{u \in U(\Sigma, 0, x)} J(u, 0) = \xi^T L \xi.$$  

Here $L \geq 0$ is the (unique) solution of the algebraic Riccati equation

$$A^T L + L A + C^T C - (P B + C^T D)(D^T D)^{-1} (B^T P + D^T C) = 0,$$

with the property that the matrix $A - B(D^T D)^{-1} (B^T P + D^T C)$ is asymptotically stable. $\Box$

**Proof of theorem 6.9**: We know that we have a solution of $R(P) = 0$ such that the matrix $A + EE^T P - B(D^T D)^{-1}(B^T P + D^T C)$ has all its eigenvalues in the closed left half plane. $P$ is related to the same linear quadratic control problem with stability as $L$ only the $C$ matrix should this time be replaced by $\tilde{C}$ satisfying $\tilde{C}^T \tilde{C} = C^T C + P E E^T P$. This immediately implies that $P \geq L$.

Next, consider the following Riccati differential equation (RDE),

$$\dot{K} + KA + A^T K + C^T C = (KB + C^T D)(D^T D)^{-1}(B^T K + D^T C)$$

$$-K E E^T K, \quad K(0) = L.$$  

Let $T > 0$ be such that the solution of the RDE exists on $[0, T]$. We know such a $T$ exists. For the system (6.2) we shall consider the finite horizon differential game with endpoint-penalty. The cost-criterion is given by

$$J_T(u, w) = \int_0^T z^T(t) z(t) - w^T(t) w(t) dt + x^T(T) L x(T).$$
It is well known (see [Ma]) that the optimal strategies for \( w \) and \( u \) are given by

\[
\begin{align*}
    w_0(t) &= -(D^T D)^{-1} (B^T K(T-t) + C^T D)x(t), \\
    u_0(t) &= E^T K(T-t)x(t),
\end{align*}
\]

and the corresponding equilibrium is \( J_T^u(\xi) = \xi^T K(T) \xi \). It can be seen (using the interpretation of \( L \) as the cost defined in lemma 6.11) that this problem is equivalent with the original problem with cost-criterion (6.1) when we add the additional constraint \( \forall t > T, \ w(t) = 0 \). This constraint is weakened for increasing \( T \) and hence it is clear that for increasing \( T \) the cost will increase since \( w \) is a maximizing player. That is, for \( T \geq t_1 \geq t_2 \) we have \( K(t_1) \geq K(t_2) \). Moreover, let \( S(t) \) be the solution of the RDE with endpoint \( S(T) = P \). Then \( S \) defines an equilibrium of a differential game over the same finite horizon but with endpoint-penalty \( x(T)^T P x(T) \). Since \( P \) is a stationary solution of the RDE we know \( S(t) = P \) for all \( t \in [0,T] \). Because the endpoint-penalties \( P \) and \( L \) of these two differential games satisfy \( P \geq L \) we know that \( P = S(t) \geq K(t) \ \forall t < T \). Since \( K(.) \) is an increasing solution of the RDE which is bounded from above we know that \( K(.) \) exists for all \( t \) and converges to a matrix \( K_\infty \) which is a stationary solution of the RDE, i.e. \( R(K_\infty) = 0 \). Note that \( K_\infty \geq L \geq 0 \).

Next we claim that the matrix \( A_1 := A - B (D^T D)^{-1} (B^T K_\infty + D^T C) \) is asymptotically stable. To show this we rewrite the ARE in the following form.

\[
K_\infty A_1 + A_1^T K_\infty + K_\infty E E^T K_\infty + C^T C + (K_\infty B + C^T D) (D^T D)^{-1} (B^T K_\infty + D^T C) = 0.
\]

By applying an eigenvector \( x \) corresponding to an unstable eigenvalue \( \lambda \) to both sides of this equation we find \( \text{Re} \lambda \neq 0 \), \( x^T K_\infty x = 0 \), \( E^T K_\infty x = 0 \), \( Cx = 0 \) and \( (B^T K_\infty + D^T C)x = 0 \). For \( \text{Re} \lambda > 0 \) we find that \( Ax = \lambda x \) and \( K_\infty x = 0 \). Since \( K_\infty \geq L \geq 0 \) this implies \( Lx = 0 \). This again implies that \( \lambda \) is also an unstable eigenvalue of \( A - B (D^T D)^{-1} (B^T L + D^T C) \). However since by lemma 6.11 this matrix is stable we have a contradiction.

If \( \text{Re} \lambda = 0 \), then we have:

\[
Ax = \lambda x, \quad Cx = 0. \tag{6.19}
\]

Because \( D \) is injective it can be shown that every unobservable eigenvalue of \( A \) is an invariant zero of \( (A, B, C, D) \). Hence (6.19) contradicts the fact
that we have no invariant zeros on the imaginary axis. Therefore we know that \( A_1 \) is stable.

We are now in the position to show that \( J^*(\xi) = \xi^T K_{\infty} \xi \) is an almost equilibrium of the system (6.2) with cost-criterion (6.1). Let \( \varepsilon > 0 \) be given. Choose \( T > 0 \) such that \( \xi^T K_{\infty} \xi - \xi^T K(T) \xi < \varepsilon \). The following \( u_0, w_0 \) turn out to satisfy (6.3):

\[
\begin{align*}
    u_0(t) &:= -(D^T D)^{-1} (B^T K_{\infty} + D^T C)x(t), \\
    w_0(t) &:= \begin{cases} 
        E^T K(T - t)x(t) & \text{for } t < T, \\
        0 & \text{otherwise}.
    \end{cases}
\end{align*}
\]

Indeed for admissible pairs \((u, w)\) we can now rewrite the cost-criterion in the following way

\[
J(u, w) = \xi^T K_{\infty} \xi - \int_0^\infty \| (w(t) - E^T K_{\infty} x(t)) \|^2 dt
\]

\[
+ \int_0^\infty \| D \left( u(t) + (D^T D)^{-1} (B^T K_{\infty} + D^T C)x(t) \right) \|^2 dt.
\]

Since \( u_0 \) is a stabilizing feedback it is easily seen from this equation that \( u_0 \) satisfies its requirements. Another way of rewriting the cost-criterion when \( w(t) = 0 \) \( \forall t > T \) is given by

\[
J(u, w) = \xi^T K(T) \xi - x^T(T) L x(T) + \int_T^\infty z^T(t) z(t) dt
\]

\[
+ \int_0^T \| D \left( u(t) + (D^T D)^{-1} (B^T K(T - t) + D^T C)x(t) \right) \|^2 dt
\]

\[- \int_0^T \| (w(t) - E^T K(T - t) x(t)) \|^2 dt.
\]

Since, by lemma 6.11, the sum of the second and third term is non-negative and the first term differs from \( J^*(\xi) \) less than \( \varepsilon \), it is easily seen that \( w_0 \) satisfies the second equation in (6.3). This proves that indeed an almost equilibrium exists. \[\blacksquare\]
6.6 Conclusions

In this chapter the linear quadratic differential game was solved. We could derive necessary conditions as well as sufficient conditions for the existence of equilibria. For the derivation of the necessary conditions we however made the extra assumption that there are no invariant zeros on the imaginary axis. Under this assumption we obtained necessary and sufficient conditions for the existence of equilibria for the case that $D$ is injective (but not for the case that $D$ is not injective).

Interesting points for future research would be to find necessary and sufficient conditions in the case that either $D$ is not injective or if there are invariant zeros on the imaginary axis. Another point is the uniqueness of equilibria. In our opinion the almost equilibrium is unique but we have not been able to prove this claim. The almost equilibrium we find in theorem 6.3 can be shown to be the smallest possible.

An interesting feature is that under the assumptions of theorem 6.3 the existence of an almost equilibrium guarantees the existence of an internally stabilizing controller which makes the $H_\infty$ norm of the closed loop system less than or equal to 1. Conversely, if there exists an internally stabilizing controller which makes the $H_\infty$ norm strictly less than 1, then there exists an almost equilibrium. This shows the strong relationship between $H_\infty$ control and differential games.
Chapter 7

The discrete time $H_\infty$ control problem: the full-information case

7.1 Introduction

As already mentioned in the previous chapters, in recent years a considerable number of papers have appeared on the $H_\infty$ optimal control problem. However most of these papers discuss the continuous time case. In this chapter we shall discuss the discrete time case.

In the papers on $H_\infty$ control with continuous time several methods were used to solve the $H_\infty$ control problem as discussed in section 5.1. Recently, a paper has appeared solving the discrete time $H_\infty$ control problem using frequency domain techniques (see [Gu]). Also the polynomial approach has been applied to discrete time systems (see [Gri]). In addition, a couple of papers have appeared using a time-domain approach (see [Bas, Li4, Ya]). Derivation of the results for the discrete time $H_\infty$ control problem could probably also be based on the work of [Wh].

With respect to the papers using a time-domain approach it should be noted that the references [Bas, Li4, Ya] do not contain a proof of the results obtained for the infinite horizon case and the papers [Bas, Ya] make a number of extra assumptions on the system under consideration. In [Bas, Li4, Ya] the authors first investigate the finite horizon problem and then derive a solution of the infinite horizon problem by considering it as a kind of limiting case as the endpoint tends to infinity.

In contrast, in the next two chapters we investigate the infinite horizon case using a direct approach. We shall use time-domain techniques which are reminiscent of those used in chapter 3 and the paper [Ta] which deal with
the continuous time case. The method used in the next two chapters was
derived independently of [Bas, Li4, Ya] and already appeared in [St5, St6].

In this chapter we assume that we deal with the special cases that either
both disturbance and state are available for feedback or only the state is
available for feedback. The more general case of measurement feedback is
discussed in the next chapter.

The assumptions we shall make are weaker than the assumptions made
in [Gu, Ya] and the same as the ones made in [Li4]. For the full-information
case we have to make two assumptions which are exactly the discrete time
analogues of the assumptions made in the regular continuous time $H_\infty$
control problem as discussed in chapter 3. Firstly, the subsystem from control
input to output should be left invertible and, secondly, this subsystem should
have no invariant zeros on the unit circle.

As in the regular continuous time $H_\infty$ problem the necessary and suffi­
cient conditions for the existence of an internally stabilizing controller such
that the closed loop system has $H_\infty$ norm less than 1 involve a positive semi­
definite stabilizing solution of a given algebraic Riccati equation. However,
other than for the continuous time case, $P$ has to satisfy an additional as­
sumption: a matrix depending on $P$ should be positive definite. (in the
continuous time case there also was a matrix which should be positive defi­
nite but in that case this matrix was independent of $P$).

Another difference with the continuous time case is that in the discrete
time case, even if $D_2 = 0$, we can not always achieve our goal with a static
state feedback. In general, we also need a static feedback depending on the
disturbance.

The outline of this chapter is as follows. In section 7.2 we shall formulate
the problem and give the main results. In section 7.3 we shall derive nec­
essary conditions under which there exists an internally stabilizing feedback
which makes the $H_\infty$ norm less than 1. In section 7.4 we shall show that
these conditions are also sufficient. After that, in section 7.5 we shall give but
not prove some results on the discrete algebraic Riccati equation. We shall
investigate properties like uniqueness of solutions and methods to calculate
the solutions. We shall end with some concluding remarks in section 7.6.

7.2 Problem formulation and main results

We consider the following system:
7.2 Problem formulation and main results

\[ \Sigma : \begin{cases} \sigma x = Ax + Bu + Ew, \\ z = Cx + D_1 u + D_2 w, \end{cases} \quad (7.1) \]

where for each \( k \), \( x(k) \in \mathbb{R}^n \) is the state, \( u(k) \in \mathbb{R}^m \) is the control input, \( w(k) \in \mathbb{R}^l \) is the unknown disturbance and \( z(k) \in \mathbb{R}^p \) is the output to be controlled. Moreover, \( A, B, E, C, D_1 \) and \( D_2 \) are matrices of appropriate dimensions. Our objective is to find a compensator \( \Sigma_F \) described by a static feedback law \( u(k) = F_1 x(k) + F_2 w(k) \) such that the closed loop system is internally stable and for the closed loop system the \( \ell_2 \)-induced operator norm from disturbance \( w \) to the output \( z \) is less than 1, i.e. the closed loop \( H_\infty \) norm is less than 1.

In this chapter we shall derive necessary and sufficient conditions for the existence of such a compensator \( \Sigma_F \). Moreover in the case that one exists we give an explicit formula for one static feedback law which yields a closed loop system which is internally stable and which has \( H_\infty \) norm less than 1.

We first give a definition:

Definition 7.1 : An function \( f : \ell_2 \rightarrow \ell_2, \ w \rightarrow f(w) \) is called causal if for any \( w_1, w_2 \in \ell_2 \) and \( k \in \mathbb{N} \):
\[
\| w_1 \|_{[0,k]} = \| w_2 \|_{[0,k]} \Rightarrow f(\| w_1 \|_{[0,k]}) = f(\| w_2 \|_{[0,k]}).
\]

Such a function \( f \) is called strictly causal if for any \( w_1, w_2 \in \ell_2 \) and \( k \in \mathbb{N} \) we have
\[
\| w_1 \|_{[0,k-1]} = \| w_2 \|_{[0,k-1]} \Rightarrow f(\| w_1 \|_{[0,k]}) = f(\| w_2 \|_{[0,k]}).
\]

A controller of the form (2.9) always defines a causal operator. In the case that \( N = 0 \) this operator is strictly causal.

We now formulate our main result:

Theorem 7.2 : Consider the system (7.1) and assume that the system \( (A, B, C, D_1) \) has no invariant zeros on the unit circle and is left invertible. The following three statements are equivalent:

(i) There exists a compensator \( \Sigma_F \) described by a static feedback law of the form \( u = F_1 x + F_2 w \) such that the closed loop system is internally stable and has \( H_\infty \) norm less than 1, i.e. the closed loop transfer matrix \( G_F \) satisfies \( \| G_F \|_\infty < 1 \).
(ii) \((A, B)\) stabilizable and for the system (7.1) there exists a causal operator \(f : \ell_2^1 \rightarrow \ell_2\) and \(\delta < 1\) such that for all \(w \in \ell_2^1\) with \(u = f(w)\) we have \(z_{u, w} \in \ell_2^1\) and \(\|z_{u, w}\|_2 \leq \delta\|w\|_2\).

(iii) There exists a symmetric matrix \(P \geq 0\) such that

(a) \(V > 0, \quad R > 0,\) \(7.2\)

where

\[
V := D_1^T D_1 + B^T P B, \\
R := I - D_2^T D_2 - E^T P E + (E^T P B + D_2^T D_1) V^{-1} (B^T P E + D_1^T D_2).
\]

This implies that the matrix \(G(P)\) is invertible, where:

\[
G(P) := \begin{pmatrix} D_1^T D_1 & D_1^T D_2 \\ D_2^T D_1 & D_2^T D_2 - I \end{pmatrix} + \begin{pmatrix} B^T \\ E^T \end{pmatrix} P \begin{pmatrix} B & E \end{pmatrix}. \quad 7.3
\]

(b) \(P\) satisfies the following discrete algebraic Riccati equation:

\[
P = A^T P A + C^T C - \begin{pmatrix} B^T P A + D_1^T C \\ E^T P A + D_2^T C \end{pmatrix}^T G(P)^{-1} \begin{pmatrix} B^T P A + D_1^T C \\ E^T P A + D_2^T C \end{pmatrix}. \quad 7.4
\]

(c) The matrix \(A_{cl}\) is asymptotically stable, where:

\[
A_{cl} := A - \begin{pmatrix} B & E \end{pmatrix} G(P)^{-1} \begin{pmatrix} B^T P A + D_1^T C \\ E^T P A + D_2^T C \end{pmatrix}. \quad 7.5
\]

Moreover, in the case that \(P\) satisfies part (iii), then the compensator \(\Sigma_F\) described by the static feedback law \(u(k) = F_1 x(k) + F_2 w(k)\) where

\[
F_1 := -(D_1^T D_1 + B^T P B)^{-1} (B^T P A + D_1^T C), \quad 7.6
\]

\[
F_2 := -(D_1^T D_1 + B^T P B)^{-1} (B^T P E + D_1^T D_2), \quad 7.7
\]

satisfies the requirements in part (i). \(\square\)
7.2 Problem formulation and main results

Remarks:

(i) Necessary and sufficient conditions under which we can find an internally stabilizing feedback which makes the $H_\infty$ norm of the closed loop system less than some, a priori given, upper bound $\gamma$ can be easily derived from theorem 7.2 by scaling.

(ii) Note that part (ii) of theorem 7.2 is equivalent to the requirement that there exists a causal operator $f$ such that the feedback law $u = f(x, w)$ satisfies part (ii). This follows from the fact that, after applying the feedback, there exists a causal operator $g$ mapping $w$ to $x$ and therefore we could have started with the causal operator $u = f(g(w), w)$ in the first place. Conversely if we have the feedback $u = f(w)$, then we define $f_1(x, w) := f(w)$ which then satisfies the requirements of the reformulated part (ii).

(iii) In the continuous time case (except for the singular case with $D_2 \neq 0$) we could not really do better when minimizing the $H_\infty$ norm of the closed loop system under the constraint of internal stability by allowing for non-causal feedbacks. This is not true in the discrete time case. Consider e.g. the following system:

\[
\begin{align*}
\sigma x &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 10 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w, \\
z &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.
\end{align*}
\]

The feedback $u = -\sigma w/10$ makes the $H_\infty$ norm of the closed loop system equal to 0.1. On the other hand, by a causal feedback we cannot make the $H_\infty$ norm of the closed loop system less than 1.

(iv) If we compare these conditions with the conditions for the continuous time case as given in chapter 3 we note that condition (7.2) (which is comparable to the condition (3.2) in theorem 3.1) now depends on $P$ and is still present if $D_2 = 0$. A simple example showing that this assumption is not superfluous is given by the system:

\[
\begin{align*}
\sigma x &= u + 2w, \\
z &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.
\end{align*}
\]
There exists no feedback $\Sigma F$ satisfying part (i) of theorem 7.2 but there does exist a positive semi-definite matrix $P$ satisfying (7.4) and such that $A_{cl} = 0$ and hence asymptotically stable, namely $P = 1$. However for this $P$ we have $R = -1$.

The general outline of the proof will be reminiscent of the proof given in [Ta] for the continuous time case. The extra condition (7.2), the invertibility of (7.3) and the requirement of left invertibility instead of assuming that $D_1$ is injective will give rise to a substantial increase in the amount of intricacies in the proof.

As in the continuous time case it is interesting to know whether we really need the disturbance feedback component or not. The following theorem gives necessary and sufficient conditions under which we can obtain internal stability and an $H_\infty$ norm of the closed loop system less than 1 by using a static state feedback.

**Theorem 7.3 :** Consider the system (7.1) and assume that $(A, B, C, D_1)$ has no invariant zeros on the unit circle and is left invertible. The following statements are equivalent:

(i) There exists a compensator $\Sigma F$ described by a static state feedback law such that the closed loop system is internally stable and has $H_\infty$ norm less than 1, i.e. the closed loop transfer matrix $G_F$ satisfies $\|G_F\|_\infty < 1$.

(ii) There exists a symmetric matrix $P \geq 0$ such that

(a) We have

\[
D_1^T D_1 + B^T P B > 0,
\]
\[
D_2^T D_2 + E^T P E < I.
\]  

This implies that the matrix $G(P)$ defined by (7.3) is invertible.

(b) $P$ satisfies the discrete algebraic Riccati equation (7.4).

(c) The matrix $A_{cl}$ defined by (7.5) is asymptotically stable.

Moreover, in the case that $P$ satisfies part (ii), then the compensator $\Sigma F$ described by the static feedback law $u(k) = Fx(k)$ where

\[
F := H^{-1} \left( B^T P A + D_1^T C + (B^T P E + D_2^T D_2) (I - D_1^T D_2 - E^T P E)^{-1} (E^T P A + D_2^T C) \right)
\]

satisfies the requirements in part (i) where

\[
138 \quad \text{The discrete time full-information case}
\]

\[
\text{There exists no feedback } \Sigma F \text{ satisfying part (i) of theorem 7.2 but there}
\]

\[
\text{does exist a positive semi-definite matrix } P \text{ satisfying (7.4) and such}
\]

\[
\text{that } A_{cl} = 0 \text{ and hence asymptotically stable, namely } P = 1. \text{ However}
\]

\[
\text{for this } P \text{ we have } R = -1.
\]

\[
\text{The general outline of the proof will be reminiscent of the proof given in [Ta]}
\]

\[
\text{for the continuous time case. The extra condition (7.2), the invertibility of}
\]

\[
\text{(7.3) and the requirement of left invertibility instead of assuming that } D_1 \text{ is}
\]

\[
\text{injective will give rise to a substantial increase in the amount of intricacies}
\]

\[
\text{in the proof.}
\]

\[
\text{As in the continuous time case it is interesting to know whether we really}
\]

\[
\text{need the disturbance feedback component or not. The following theorem}
\]

\[
\text{gives necessary and sufficient conditions under which we can obtain internal}
\]

\[
\text{stability and an } H_\infty \text{ norm of the closed loop system less than 1 by using a}
\]

\[
\text{static state feedback.}
\]

\[
\text{Theorem 7.3 : Consider the system (7.1) and assume that } (A, B, C, D_1)
\]

\[
\text{has no invariant zeros on the unit circle and is left invertible. The following}
\]

\[
\text{statements are equivalent:}
\]

(i) There exists a compensator $\Sigma F$ described by a static state feedback law

\[
\text{such that the closed loop system is internally stable and has } H_\infty \text{ norm}
\]

\[
\text{less than 1, i.e. the closed loop transfer matrix } G_F \text{ satisfies } \|G_F\|_\infty < 1.
\]

(ii) There exists a symmetric matrix $P \geq 0$ such that

(a) We have

\[
D_1^T D_1 + B^T P B > 0,
\]
\[
D_2^T D_2 + E^T P E < I.
\]  

This implies that the matrix $G(P)$ defined by (7.3) is invertible.

(b) $P$ satisfies the discrete algebraic Riccati equation (7.4).

(c) The matrix $A_{cl}$ defined by (7.5) is asymptotically stable.

Moreover, in the case that $P$ satisfies part (ii), then the compensator $\Sigma F$

\[
\text{described by the static feedback law } u(k) = Fx(k) \text{ where}
\]

\[
F := H^{-1} \left( B^T P A + D_1^T C + (B^T P E + D_2^T D_2) (I - D_1^T D_2 - E^T P E)^{-1} (E^T P A + D_2^T C) \right)
\]

satisfies the requirements in part (i) where

\[
138 \quad \text{The discrete time full-information case}
\]

\[
\text{There exists no feedback } \Sigma F \text{ satisfying part (i) of theorem 7.2 but there}
\]

\[
\text{does exist a positive semi-definite matrix } P \text{ satisfying (7.4) and such}
\]

\[
\text{that } A_{cl} = 0 \text{ and hence asymptotically stable, namely } P = 1. \text{ However}
\]

\[
\text{for this } P \text{ we have } R = -1.
\]

\[
\text{The general outline of the proof will be reminiscent of the proof given in [Ta]}
\]

\[
\text{for the continuous time case. The extra condition (7.2), the invertibility of}
\]

\[
\text{(7.3) and the requirement of left invertibility instead of assuming that } D_1 \text{ is}
\]

\[
\text{injective will give rise to a substantial increase in the amount of intricacies}
\]

\[
\text{in the proof.}
\]

\[
\text{As in the continuous time case it is interesting to know whether we really}
\]

\[
\text{need the disturbance feedback component or not. The following theorem}
\]

\[
\text{gives necessary and sufficient conditions under which we can obtain internal}
\]

\[
\text{stability and an } H_\infty \text{ norm of the closed loop system less than 1 by using a}
\]

\[
\text{static state feedback.}
\]

\[
\text{Theorem 7.3 : Consider the system (7.1) and assume that } (A, B, C, D_1)
\]

\[
\text{has no invariant zeros on the unit circle and is left invertible. The following}
\]

\[
\text{statements are equivalent:}
\]

(i) There exists a compensator $\Sigma F$ described by a static state feedback law

\[
\text{such that the closed loop system is internally stable and has } H_\infty \text{ norm}
\]

\[
\text{less than 1, i.e. the closed loop transfer matrix } G_F \text{ satisfies } \|G_F\|_\infty < 1.
\]

(ii) There exists a symmetric matrix $P \geq 0$ such that

(a) We have

\[
D_1^T D_1 + B^T P B > 0,
\]
\[
D_2^T D_2 + E^T P E < I.
\]  

This implies that the matrix $G(P)$ defined by (7.3) is invertible.

(b) $P$ satisfies the discrete algebraic Riccati equation (7.4).

(c) The matrix $A_{cl}$ defined by (7.5) is asymptotically stable.

Moreover, in the case that $P$ satisfies part (ii), then the compensator $\Sigma F$

\[
\text{described by the static feedback law } u(k) = Fx(k) \text{ where}
\]

\[
F := H^{-1} \left( B^T P A + D_1^T C + (B^T P E + D_2^T D_2) (I - D_1^T D_2 - E^T P E)^{-1} (E^T P A + D_2^T C) \right)
\]

satisfies the requirements in part (i) where

\[
138 \quad \text{The discrete time full-information case}
\]
7.3 A solution of the Riccati equation

\[ H := B^T P B + D_1^T D_2 + (B^T P E + D_2^T D_2) (I - D_2^T D_2 - E^T P E)^{-1} (E^T P B + D_2^T D_2). \]

Remark: Note that the only difference with the conditions of theorem 7.2 is that the condition \( R > 0 \) is replaced by the condition (7.8). Note that the latter condition indeed implies \( R > 0 \).

In the continuous time case we only needed the extra condition \( D_1^T D_2 < I \). This is not sufficient in discrete time as can be seen from the following system:

\[
\begin{align*}
\sigma x &= u + 1/2 w, \\
z &= 8x + u.
\end{align*}
\]

The matrix \( P = 84 \) satisfies the conditions in part (iii) of theorem 7.2 but condition (7.8) is not satisfied. By state feedback the \( H_\infty \) norm of the closed loop system can not be made less than 3 while maintaining internal stability. On the other hand with disturbance feedback we can make the \( H_\infty \) norm equal to 1/2.

7.3 Existence of a stabilizing solution of the Riccati equation

In this section we shall show that the existence of a causal operator \( f \) and \( \delta < 1 \) satisfying part (ii) of theorem 7.2 implies that there exist a positive semi-definite solution of the discrete algebraic Riccati equation (7.4) such that (7.5) is asymptotically stable and (7.2) is satisfied. We also prove the implication (i) \( \Rightarrow \) (ii) of theorem 7.3. The method used has a great similarity with the proof given in section 3.4. We shall assume that

\[ D_1^T[ C \ D_2 ] = 0, \quad (7.10) \]

for the time being and we shall derive the more general statement later. In order to prove the existence of the desired \( P \) we shall investigate the following sup-inf problem:

\[
C^*(\xi) := \sup_{w \in \ell_2^m} \inf_u \left\{ ||x_{u,w,\xi}||_2^2 - ||w||_2^2 : u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^2 \right\}, \]

(7.11)

for arbitrary initial state \( \xi \). It turns out that the conditions of part (ii) of theorem 7.2 imply that \( C^*(\xi) \) is finite for every \( \xi \). Moreover, it will be shown
The discrete time full-information case

that, as in section 3.4, there exists a $P \geq 0$ such that $C^*(\xi) = \xi^T P \xi$. At the end of this section we then prove that this $P$ exactly satisfies conditions (a)–(c) of theorem 7.2. We shall first minimize, for given $w \in \ell_2$ and $\xi \in \mathbb{R}^n$, the function $\|z_{u,w,\xi}\|_2^2 - \|w\|_2^2$ over all $u \in \ell_2$ for which $x_{u,w,\xi} \in \ell_2$. After that we shall maximize over $w \in \ell_2$.

As in chapter 3 our proof is based on Pontryagin's maximum principle. We shall use the ideas from [LM], together with our stability requirement $x_{u,w,\xi} \in \ell_2$ to adapt the proof to the infinite horizon discrete time case.

We start by constructing a solution of the adjoint Hamilton-Jacobi equation which is a natural starting point if one wants to use Pontryagin's maximum principle.

Let $L$ be such that $D_1^T D_1 + B^T L B$ is invertible and such that $L$ is the positive semi-definite solution of the following discrete algebraic Riccati equation:

$$ L = A^T L A + C^T C - A^T L B (D_1^T D_1 + B^T L B)^{-1} B^T L A, \quad (7.12) $$

for which

$$ A_L := A - B (D_1^T D_1 + B^T L B)^{-1} B^T L A, $$

is asymptotically stable. The existence of such $L$ is guaranteed under the assumption that $(A,B,C,D_1)$ has no invariant zeros on the unit circle and is left invertible and moreover $(A,B)$ is stabilizable (see [Si]). We define

$$ r(k) := - \sum_{i=k}^{\infty} [X_1 A^T]^{i-k} X_1 (L E w(i) + C^T D_2 w(i+1)), \quad (7.13) $$

where

$$ X_1 := I - L B (D_1^T D_1 + B^T L B)^{-1} B^T. $$

Note that $r$ is well-defined since the matrix $A_L = X_1^T A$ is asymptotically stable which implies that $X_1 A^T$ is asymptotically stable. Next we define the functions $y, \tilde{x}$ and $\eta$ by:

$$ y := M^{-1} B^T [A^T \sigma r - L E w - C^T D_2 \sigma w], \quad (7.14) $$

$$ \sigma \tilde{x} = A_L \tilde{x} + B y + E w, \quad \tilde{x}(0) = \xi, \quad (7.15) $$

$$ \eta := -X_1 L A \tilde{x} + r, \quad (7.16) $$

where $M := D_1^T D_1 + B^T L B$. Since $X_1 A^T$ is asymptotically stable, it can be checked straightforwardly that, given $\xi \in \mathbb{R}^n$ and $w \in \ell_2$, we have $r, \tilde{x}, \eta \in \ell_2$.

After some standard calculations, we find the following lemma:
Lemma 7.4: Let $\xi \in \mathbb{R}^n$ and $w \in \ell_2^1$ be given. The function $\eta \in \ell_2^1$ is a solution of the following backward difference equation:

$$\sigma^{-1} \eta = A^T \eta - C^T C \dot{x} - C^T D_2 w, \quad \lim_{k \to \infty} \eta(k) = 0. \tag{7.17}$$

Here $\eta$ is extended to a function from $\mathcal{N} \cup \{-1\}$ to $\mathbb{R}^n$ by choosing $\eta(-1)$ such that (7.17) is satisfied.

In the statement of Pontryagin’s maximum principle this equation is the so-called "adjoint Hamilton-Jacobi equation" and $\eta$ is called the "adjoint state variable". We have constructed a solution to this equation and we shall show that this $\eta$ indeed yields a minimizing $u$. Note the difference with the continuous time case where we could derive a differential equation forward in time, while in discrete time we can only derive a difference equation forward in time when $A$ is invertible. To prevent these kind of difficulties it is assumed in [Gu] that $A$ is invertible. The proof that $\eta$ yields a minimizing $u$ is adapted from the proof of lemma 3.4:

Lemma 7.5: Let the system (7.1) be given. Moreover let $w$ and $\xi$ be fixed. Then

$$\tilde{u} := -(D_1^T D_1 + B^T L B)^{-1} B^T L A \dot{x} + y$$

$$= \arg \inf_{u} \{ \| x_{u,w,\xi} \|_2 \mid u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^n \}. \tag*{□}$$

Proof: It can be easily checked that $\tilde{x} = x_{\tilde{u},w,\xi}$. Define

$$J_T(u) := \sum_{i=0}^{T} \| C x_{u,w,\xi}(i) + D_1 u(i) + D_2 w(i) \|_2^2.$$
Let $u \in \ell_2^n$ be an arbitrary control input such that $x_{u,w,\xi} \in \ell_2^n$. We find
\[
\mathcal{J}_T(u) - \mathcal{J}_{T-1}(u) - 2\eta^T(T)x(T + 1) + 2\eta^T(T - 1)x(T) =
\]
\[
||C x(T)||^2 + [D_1^T D_1 u(T) - 2B^T \eta(T)]^T u(T)
\]
\[
-2\eta^T(T)E w(T) - 2x^T(T)C^T C x(T).
\]
We also find
\[
\mathcal{J}_T(\bar{u}) - \mathcal{J}_{T-1}(\bar{u}) - 2\eta^T(T)\bar{x}(T + 1) + 2\eta^T(T - 1)\bar{x}(T) =
\]
\[
-||C \bar{x}(T)||^2 + [D_1^T D_1 \bar{u}(T) - 2B^T \eta(T)]^T \bar{u}(T) - 2\eta^T(T)E w(T).
\]
Hence if we sum the last two equations from zero to infinity and subtract from each other we find:
\[
||z_{u,w,\xi}||^2 - ||z_{u,w,\xi}||^2 = \sum_{i=0}^{\infty} -||C (x(i) - \bar{x}(i))||^2 +
\]
\[
+ \sum_{i=0}^{\infty} [D_1^T D_1 u(i) - 2B^T \eta(i)]^T u(i) - [D_1^T D_1 u(i) - 2B^T \eta(i)]^T u(i).
\]
It can easily be checked that $B^T \eta(i) = D_1^T D_1 \bar{u}(i)$ for all $i$. Therefore, for every $i$ we have
\[
[D_1^T D_1 u(i) - 2B^T \eta(i)]^T u(i) = \inf_u [D_1^T D_1 u - 2B^T \eta(i)]^T u.
\]
The last two equations together imply that:
\[
||z_{u,w,\xi}||^2 \leq ||z_{u,w,\xi}||^2,
\]
which is exactly what we had to prove. Since $(A, B, C, D_1)$ is left invertible it can easily be shown that the minimizing $u$ is unique.

We are now going to maximize over $w \in \ell_2^n$. This will then yield $C^*(\xi)$. Define $\mathcal{F}(\xi, w) := (\bar{x}, \bar{u}, \eta)$ and $\mathcal{G}(\xi, w) := z_{\bar{u},w,\xi} = C\bar{x} + D_1 \bar{u} + D_2 w$. It is clear from the previous lemma that $\mathcal{F}$ and $\mathcal{G}$ are bounded linear operators. Define
\[
C(\xi, w) := ||G(\xi, w)||^2 - ||w||^2,
\]
\[
||w||_C := (-C(0, w))^{1/2}.
\]
It can be easily shown that $||.||_C$ defines a norm on $\ell_2^n$. Using the conditions in part (ii) of theorem 7.2 it can be shown straightforwardly that
7.3 A solution of the Riccati equation

\[ \|w\|_2 \geq \|w\|_C \geq \rho \|w\|_2, \quad (7.18) \]

where \( \rho > 0 \) is such that \( \rho^2 = 1 - \delta^2 \) and \( \delta \) is such that the conditions of part (ii) of theorem 7.2 are satisfied. Hence \( \|\cdot\|_C \) and \( \|\cdot\|_2 \) are equivalent norms. We have

\[ C^*(\xi) = \sup_{\omega \in \ell^2_{\xi}} C(\xi, \omega). \]

We can derive the following properties of \( C^* \). The proof is identical to the proof of lemma 3.5 and is therefore omitted.

Lemma 7.6 :

(i) For all \( \xi \in \mathcal{R}^n \) we have

\[ 0 \leq \xi^T L \xi \leq C^*(\xi) \leq \frac{\xi^T L \xi}{1 - \rho^2}, \]

where \( \rho \) is such that (7.18) is satisfied.

(ii) For all \( \xi \in \mathcal{R}^n \) there exists an unique \( w_* \in \ell^1_2 \) such that \( C^*(\xi) = C(\xi, w_*) \).

Define \( \mathcal{H} : \mathcal{R}^n \rightarrow \ell^1_2, \xi \rightarrow w_* \). Unlike the explicit expression for \( \bar{u} \) we can only derive an implicit formula for \( w_* \). We can however show that \( w_* \) is the unique solution of a linear equation. Also this time we omit the proof. It is similar to the proof of lemma 3.7. The adaptations necessary are the same as the difference between the proofs of lemma 3.4 and lemma 7.5.

Lemma 7.7 : Let \( \xi \in \mathcal{R}^n \) be given. Then \( w_* = \mathcal{H}\xi \) is the unique \( \ell_2 \)-function \( w \) satisfying:

\[ (I - D_2^T D_2) w = -E^T \eta + D_2^T C x, \quad (7.19) \]

where \( (x, u, \eta) = \mathcal{F}(\xi, w) \).

Next, we shall show that \( C^*(\xi) = \xi^T P \xi \) for some matrix \( P \). In order to do that we first show that \( u_*, \eta_* \) and \( w_* \) are linear functions of \( x_* \):
Lemma 7.8: There exist constant matrices $K_1, K_2$ and $K_3$ such that

\[
\begin{align*}
\eta_* &= K_1 x_*, \\
\eta_* &= K_2 x_*, \\
\eta_* &= K_3 x_*.
\end{align*}
\] (7.20) (7.21) (7.22)

Proof: We shall first consider time 0. By lemma 7.7 it is easily seen that $\mathcal{H}: \xi \rightarrow \eta_*$ is linear. Hence also the mapping from $\xi$ to $\eta_*(0)$ is linear. This implies the existence of a matrix $K_3$ such that $\eta_*(0) = K_3 \xi$. From (7.17) and lemma 7.5 it is easily seen that $\eta_*$ and $\eta_*$ are linear functions of $\xi$ and $\eta_*$. This implies, since $\eta_*$ is a linear function of $\xi$, that $\eta_*(0)$ and $\eta_*(0)$ are linear functions of $\xi$ and hence there exist $K_1$ and $K_2$ such that $\eta_*(0) = K_1 \xi$ and $\eta_*(0) = K_2 \xi$. We shall now look at time $t$. The sup-inf problem starting at time $t$ with initial value $x(t)$ can now be solved. Due to the time invariance property we see that $\eta_*$ restricted to $[t, \infty)$ satisfies (7.19) and hence for this problem the optimal $x$ and $\eta$ are $x_*$ and $\eta_*$. But since $t$ is the initial time for this optimization problem, which is exactly equal to the original optimization problem, we find equations (7.20)-(7.22) at time $t$ with the same matrices $K_1, K_2$ and $K_3$ as at time 0. Since $t$ was arbitrary this completes the proof.

Lemma 7.9: There exists a matrix $P$ such that $\sigma^{-1} \eta_* = -P x_*$. Moreover for this $P$ we find

\[
C^*(\xi) = \xi \tau P \xi.
\] (7.23)

Proof: We have

\[
\sigma^{-1} \eta_* = [A^T \eta_* - C^T C x_* - C^T D_2 \eta_*] = (A^T K_2 - C^T C - C^T D_2 K_3) x_*
\]
We define \( P := -(A^T K_2 - C^T C - C^T D_2 K_3) \) using the matrices defined in lemma 7.8. We shall prove that this \( P \) satisfies (7.23). We can derive the following equation

\[
\|z_{\star, w_{\star}, \xi}(T)\|^2 - \|w_{\star}(T)\|^2 - 2\eta_{\star}^T(T)x_{\star}(T + 1) + 2\eta_{\star}(T - 1)^T x_{\star}(T) = \|w_{\star}(T)\|^2 - \|z_{\star, w_{\star}, \xi}(T)\|^2.
\]

We sum this equation from zero to infinity. Since \( \lim_{T \to \infty} \eta_{\star}(T) = 0 \) and \( \lim_{T \to \infty} x_{\star}(T) = 0 \) we find

\[
C(\xi, w_{\star}) + 2\eta_{\star}^T(-1)x_{\star}(0) = -C(\xi, w_{\star}).
\]

Since \( C(\xi, w_{\star}) = C^*(\xi) \) and \( \eta_{\star}(-1) = -P\xi \) we find (7.23). \( \blacksquare \)

We shall now show that this matrix \( P \) satisfies condition (a)–(c) of theorem 7.2. We first show part (a). Since we do not know yet if \( P \) is symmetric we have to be a little bit careful. This essential step in our derivation is new compared to the method used in chapter 3:

**Lemma 7.10**: Let \( P \) be given by lemma 7.9. The matrices \( V \) and \( R \) as defined in the part (iii) of theorem 7.2, condition (a) satisfy:

\[
V + V^T > 0, \quad R + R^T > 0.
\]

**Proof**: By lemma 7.6 and lemma 7.9, we know \((P + P^T)/2 \geq L\) and therefore we find \((V + V^T)/2 \geq D_1^TD_1 + B^TLB\). The latter matrix is positive definite and hence \((V + V^T)/2\) is positive definite, i.e. \( V + V^T > 0 \).

We shall now look at the following "sup-inf-sup-inf"-problem for initial condition 0:

\[
\mathcal{J}(0) := \sup_{w(0)} \inf_{u(0)} \sup_{w^+} \inf_{u^+} \|z_{u, w}\|^2 - \|w\|^2,
\]

(7.24)
The discrete time full-information case

where \( w^+ := w|_{[1, \infty)} \) and \( u^+ := u|_{[1, \infty)} \). We will always implicitly add the constraint that \( u^+ \) is such that the resulting state \( x \) is in \( L_2 \).

We know there exists a causal operator \( f \) satisfying part (ii) of theorem 7.2 and hence this function makes the \( L_2 \)-induced operator norm strictly less than 1 under the constraint \( x \in L_2^2 \). In (7.24) we set \( u = f(w) \). This is possible since by causality we know that \( u(0) \) only depends on \( w(0) \) and \( u^+ \) depends on the whole function \( w \). Thus we get:

\[
\mathcal{J}(\xi) = \sup_{w(0)} \inf_{u(0)} \sup_{w^+} \inf_{u^+} \| x_{u,w} \|_2^2 - \| w \|_2^2 \\
\leq \sup_{w} \| x_{f(w),w} \|_2^2 - \| w \|_2^2 \\
\leq 0. \tag{7.26}
\]

Since, by lemma 7.9, we have:

\[
\sup_{w^+} \inf_{u^+} \| x_{u^+,w^+,x(1)} \|_2^2 - \| w^+ \|_2^2 = x(1)^T P x(1), \tag{7.27}
\]

we can reduce (7.24) to the following "sup-inf" problem:

\[
\sup_{w(0)} \inf_{u(0)} \left( \begin{array}{c} u(0) \\ w(0) \end{array} \right)^T \left( \begin{array}{cc} V & B^T P E + D_1^T D_2 \\ E^T P B + D_2^T D_1 & E^T P E + D_2^T D_2 - I \end{array} \right) \left( \begin{array}{c} u(0) \\ w(0) \end{array} \right).
\]

When we define

\[
\tilde{u}(0) = u(0) - (E^T P B + D_2^T D_1) V^{-1} w(0),
\]

then we get

\[
\mathcal{J}(0) = \sup_{w(0)} \inf_{\tilde{u}(0)} \left( \begin{array}{c} \tilde{u}(0) \\ w(0) \end{array} \right)^T \left( \begin{array}{cc} V & 0 \\ 0 & -R \end{array} \right) \left( \begin{array}{c} \tilde{u}(0) \\ w(0) \end{array} \right). \tag{7.28}
\]

Since, by (7.26), \( \mathcal{J}(0) \) is finite we immediately find that a necessary condition is \( R + R^T \geq 0 \). We also note that \( \mathcal{J}(0) = 0 \).

Assume that \( R + R^T \) is not invertible. Then there exists a \( v \neq 0 \) such that \( v^T R v = 0 \). Let \( w^+(u(0)) \) be the \( L_2 \)-function which attains the optimum in the optimization (7.27) with initial state \( x(1) = B u(0) + E v \). We define the function \( w \) by

\[
[w(u(0))](t) := \begin{cases} v & \text{if } t = 0, \\ w^+(u(0))(t) & \text{otherwise}. \end{cases} \tag{7.29}
\]
7.3 A solution of the Riccati equation

Assume that \( \delta \) and \( f \) are such that part (ii) of theorem 7.2 is satisfied. Define \( u \) by

\[
    u = f[w(u(0))].
\]

(7.30)

Since the map from \( u \) to \( w \) defined by (7.29) is strictly causal and since \( f \) is causal, \( u \) is uniquely defined by (7.30). In order to prove this note that \( u(0) \) only depends on \( w(u(0))(0) = v \) and hence \( w^+ \) as a function of \( u(0) \) is uniquely defined which, in turn, yields \( u \). Denote \( u \) and \( w \) obtained in this way by \( u_1 \) and \( w_1 \). By (7.25), (7.28) and since \( Rv = 0 \) we find that, for this particular choice of \( w_1 \) and \( u_1 \), we have:

\[
    \|z_{u_1,w_1}\|_2^2 - \|w_1\|_2^2 \\
    \geq \inf_u \|z_{u,w(u(0))}\|_2^2 - \|w(u(0))\|_2^2 \\
    = \inf_{\tilde{u}(0)} \tilde{u}^T(0)V\tilde{u}(0) - v^T Rv \\
    = 0.
\]

(7.31)

On the other hand, using part (ii) of theorem 7.2 we find:

\[
    \|z_{u_1,w_1}\|_2^2 - \|w_1\|_2^2 < (\delta^2 - 1) \|w_1\|_2^2 < 0,
\]

since \( w_1(0) = v \neq 0 \). Therefore we have a contradiction and hence our assumption that \( R + R^T \) is not invertible was incorrect. Together with \( R + R^T \geq 0 \) this yields \( R + R^T > 0 \).

Lemma 7.11: Assume that \((A,B,C,D)\) has no invariant zeros on the unit circle and is left invertible. Moreover, assume that \( D_1^T[C \ D_2] = 0 \). If the statement in part (ii) of theorem 7.2 is satisfied, then there exists a symmetric matrix \( P \geq 0 \) satisfying (a)-(c) of part (iii) of theorem 7.2.

Proof: We define the matrices

\[
    M := D_1^T D_1 + B^T L B > 0, \\
    Z := I - D_2^T D_2 - E^T X_1 L E.
\]

We know that \(-(R + R^T)/2\) is the Schur complement of \((V + V^T)/2\) in \( G((P + P^T)/2) \). By lemma 7.10 we know that \( R + R^T > 0 \) and \( V + V^T > 0 \).
Therefore $G((P + P^T)/2)$ has $m$ eigenvalues on the positive real axis and $l$ eigenvalues on the negative real axis. We know $G((P + P^T)/2) - G(L) \geq 0$ since $(P + P^T)/2 \geq L$. An easy consequence of the theorem of Courant-Fischer (see [Bel]) then tells us that $G(L)$ has at least $l$ eigenvalues on the negative real axis. Since $-Z$ is the Schur complement of $M > 0$ in $G(L)$ this implies that $Z < 0$.

By lemma 7.9 we have $\eta_* = -\sigma Px_*$. By combining lemma 7.5 and lemma 7.7 and rewriting the equations we find that $u_*$ and $w_*$ satisfy the following equations:

\[
\begin{align*}
w_* &= Z^{-1} \left\{ E^T X_1 (P - L) \sigma x_* + (D_2^T C + E^T X_1 L A) x_* \right\}, \\
u_* &= M^{-1} B^T \left\{ (P - L) \sigma x_* + L A x_* + L E w_* \right\}.
\end{align*}
\]

Thus we get

\[
\begin{align*}
\left\{ I + \left[ BM^{-1} B^T - X_1^T E Z^{-1} E^T X_1 \right] (P - L) \right\} x_*(k + 1) = \\
X_1^T \left\{ A + E Z^{-1} E^T X_1 L A + E Z^{-1} D_2^T C \right\} x_*(k).
\end{align*}
\]

(7.32)

Since, by lemma 7.10, $R$ as defined in theorem 7.2 is invertible, it can be shown that the matrix on the left is invertible and hence (7.32) uniquely defines $x_*(k + 1)$ as a function of $x_*(k)$. It turns out that (7.32) can be rewritten in the form $\sigma x_* = A_{cl} x_*$ with $A_{cl}$ as defined by (7.5). Since $x_* \in \ell_2^\infty$ for every initial state $\xi$ we know that $A_{cl}$ is asymptotically stable. Next we show that $P$ satisfies the discrete algebraic Riccati equation (7.4). From the backwards difference equation in (7.17) combined with lemma 7.9 and the formula given above for $w_*$ we find:

\[
P = A^T P A_{cl} + C^T C + C^T D_2 Z^{-1} \left\{ E^T X_1 (P - L) A_{cl} + D_2^T C + E^T X_1 L A \right\}.
\]

By some extensive calculations this equation turns out to be equivalent to the discrete algebraic Riccati equation (7.4). Next we show that $P$ is symmetric. Note that both $P$ and $P^T$ satisfy the discrete algebraic Riccati equation. Using this we find that:

\[
(P - P^T) = A_{cl}^T (P - P^T) A_{cl}.
\]

Since $A_{cl}$ is asymptotically stable this implies that $P = P^T$. $P$ can be shown to be positive semi-definite by combining lemma 7.6 and (7.23). It remains to be shown that $P$ satisfies (7.2). Since $P$ is symmetric we know that $V$ and $R$ are symmetric. (7.2) is then an immediate consequence of lemma 7.10.
As with chapter 3 we can extend this result to systems which do not satisfy (7.10). A proof can be derived similarly to the proof of corollary 3.11.

**Corollary 7.12**: Assume that \((A, B, C, D_1)\) has no invariant zeros on the unit circle and is left invertible. If part (ii) of theorem 7.2 is satisfied, then there exists a symmetric matrix \(P \geq 0\) satisfying (a)-(c) of part (iii) of theorem 7.2.

Next, we show the implication (i) \(\Rightarrow\) (ii) in theorem 7.3.

**Proof of the implication (i) \(\Rightarrow\) (ii) in theorem 7.3**: First note that by corollary 7.12 we know there exists a matrix \(P\) satisfying (a)-(c) of part (iii) of theorem 7.2. It only remains to be shown that (7.8) is satisfied. We use the same kind of argument as used in lemma 7.10.

Note that a consequence of the fact that there exists an internally stabilizing feedback law \(u = Fx\) which makes the \(H_\infty\) norm less than 1 is that there exists a function \(f\) satisfying part (ii) of theorem 7.2 where \(f\) has the extra property that, for initial condition 0, the fact that \(x(t) = 0\) implies that \(u(t) = [f(w)](t) = 0\). One suitable choice for \(f\) is given by

\[
u(t) = [f(w)](t) := Fx(t) = \sum_{i=1}^{t} F(A + BF)^{t-i} Ew(i).
\]

Next, we investigate the criterion (7.24) once again with initial condition \(\xi = 0\) but this time we restrain \(u\) by requiring that \(u(0) = 0\). The inequalities (7.25) and (7.26) then still hold because of our extra requirement on \(f\). Using (7.27) we can then reduce our "sup-inf-sup-inf" problem to the following problem

\[
\mathcal{J}(0) = \sup_{w(0)} w^T(0) (E^T P E + D_2^T D_2 - I) w(0).
\]

Since we know by (7.26) that \(\mathcal{J}(0)\) is finite we find

\[
S := E^T P E + D_2^T D_2 - I \leq 0.
\]

Remains to be shown that this matrix is invertible. Assume not, then there exists a vector \(v \neq 0\) such that \(Sv = 0\). We define by \(w^+\) the function which attains the supremum in the optimization problem 7.27 with initial condition \(x(1) = Ev\). This function is well-defined by our previous results. Next, we define the function \(w_1\) by
The discrete time full-information case

\[ w_1(t) := \begin{cases} v & \text{if } t = 0, \\ w^+(t) & \text{otherwise.} \end{cases} \]  

(7.33)

Moreover, we define the function \( u_1 \) by \( u_1 := f(w_1) \). Because of our extra condition on \( f \) and since \( \xi = 0 \), we know that \( u_1(0) = 0 \). We know that \( \mathcal{J}(0) = 0 \) and hence by the inequality (7.25) we know that

\[ \inf_u \|z_{u_1,w_1}\|_2^2 - \|w\|_2^2 \geq 0. \]  

(7.34)

On the other hand, using part (ii) of theorem 7.2 we find:

\[ \|z_{u_1,w_1}\|_2^2 - \|w_1\|_2^2 < \left( \delta^2 - 1 \right) \|w_1\|_2^2. \]

Combined with (7.34) this implies that \( w_1 = 0 \). However \( w_1(0) = v \neq 0 \). This yields a contradiction and therefore \( S \) is invertible. •

7.4 Sufficient conditions for the existence of suboptimal controllers

In this section we shall show that if there exists a \( P \) satisfying the conditions of theorem 7.2, then the feedback as suggested by theorem 7.2 satisfies condition (i). In order to do this we first need a number of preliminary results.

We define the following system:

\[ \Sigma_U : \begin{cases} \sigma x_U = A_U x_U + B_U u_U + E_U w, \\ y_U = C_{1,u} x_U + D_{12,u} w, \\ z_U = C_{2,u} x_U + D_{21,u} u_U + D_{22,u} w, \end{cases} \]  

(7.35)

where

\[
\begin{align*}
A_U &:= A - B V^{-1} (B^T P A + D_1^T C), \\
B_U &:= B V^{-1/2}, \\
E_U &:= E - B V^{-1} (B^T P E + D_1^T D_2), \\
C_{1,u} &:= -R^{-1/2} (E^T P A + D_1^2 C - [E^T P B + D_1^2 D_1] V^{-1} [B^T P A + D_1^T C]), \\
C_{2,u} &:= C - D_1 V^{-1} (B^T P A + D_1^T C), \\
D_{12,u} &:= R^{1/2}, \\
D_{21,u} &:= D_1 V^{-1/2}, \\
D_{22,u} &:= D_2 - D_1 V^{-1} (B^T P E + D_1^T D_2),
\end{align*}
\]
and \( V \) and \( R \) as defined in part (iii) of theorem 7.2.

**Lemma 7.13**: The system \( \Sigma_V \) as defined by (7.35) is inner. Denote the transfer matrix of \( \Sigma_V \) by \( U \). We decompose \( U \):

\[
U \begin{pmatrix} w \\ u_v \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w \\ u_v \end{pmatrix} = \begin{pmatrix} z_v \\ y_v \end{pmatrix},
\]

compatible with the sizes of \( w, u_v, z_v \) and \( y_v \). Then \( U_{21} \) is invertible and its inverse is in \( H_\infty \). Moreover, \( U_{22} \) is strictly proper.

Proof: It can be easily checked that \( P \) as defined by conditions (a)-(c) of theorem 7.2 satisfies the conditions (i)-(iii) of lemma 2.13 for the system (7.35) with input \( (w, u_v) \) and output \( (z_v, y_v) \). Part (i) of lemma 2.13 turns out to be equivalent to the discrete algebraic Riccati equation (7.4). Parts (ii) and (iii) follow by simply writing out the equations in the original system parameters of system (7.1).

Next we note that \( P \geq 0 \) and

\[
P = A_v^T P A_u + \begin{pmatrix} C_{1,u}^T \\ C_{2,u}^T \end{pmatrix} \begin{pmatrix} C_{1,u} \\ C_{2,u} \end{pmatrix}. \quad (7.36)
\]

Since \( A_{cl} = A_u - E_u D_{12}^{-1} C_{1,u} \) and because we know that \( A_{cl} \) is asymptotically stable the pair \( (C_{1,u}, A_u) \) is detectable. Using standard Lyapunov theory it can then be shown that \( A_u \) is asymptotically stable.

To show that \( U_{21}^{-1} \) is an \( H_\infty \) function we write down a realization for \( U_{21}^{-1} \) and remember once again that \( A_{cl} \) is asymptotically stable. The proof is then trivial.

**Lemma 7.14**: Assume that there exists a \( P \) satisfying the conditions in part (iii) of theorem 7.2. In that case the compensator \( \Sigma_F \) described by the static feedback law \( u = F_1 x + F_2 w \) where \( F_1, F_2 \) are given by (7.6) and (7.7) satisfies condition (i) of theorem 7.2.
Proof: First note that $G_F$ as given by (2.7) for this particular $F$ is equal to $U_{11}$ and moreover $A + BF_1$ is equal to $A_F$. This implies that $\Sigma_F$ is internally stabilizing and $G_F$ as a submatrix of an inner matrix satisfies $\|G_F\| \leq 1$. Using the fact that $U_{21}$ is invertible in $H_\infty$, it can be shown that the inequality is strict.

Finally it can be quite easily seen that theorem 7.2 is simply a combination of corollary 7.12 and lemma 7.14. Therefore the main result has been proven.

Lemma 7.15: Assume that there exists a $P$ satisfying the conditions in part (ii) of theorem 7.3. In that case the compensator $\Sigma_F$ described by the static state feedback law $u = Fx$ where $F$ as given by (7.9) satisfies condition (i) of theorem 7.2.

Proof: We apply the feedback $u = \bar{F}x$ to the system $\Sigma_U$ where

$$\bar{F} := V^{-1/2} (B^TPE + D_1^T D_2) R^{-1/2}.$$ 

We have

$$\bar{F}^T \bar{F} = I - R^{-1/2} (I - D_2^T D_2 - E^T P E) R^{-1/2} < I.$$ 

By lemma 2.14 we then know that the resulting closed loop system is internally stable and has $H_\infty$ norm less than 1. Since this closed loop system is the same as the one we obtain by applying the feedback law $u = Fx$ to our original system, we conclude that $F$ indeed has the desired properties, i.e. $F$ satisfies part (i) of theorem 7.3.

The previous lemma yields the implication (ii) $\Rightarrow$ (i) in theorem 7.3 as a corollary. This also completes the proof of theorem 7.3.

7.5 Discrete algebraic Riccati equations

In this section we shall investigate uniqueness of stabilizing solutions of the discrete algebraic Riccati equation and methods to check whether such solutions exist and if they exist how we can find them. We shall not prove the results of this section because these results are not needed for our main results and to prevent a too extensive thesis.

We investigate the existence of a matrix $P$ such that
7.5 Riccati equations

- The matrix $G(P)$ defined by (7.3) is invertible.
- $P$ satisfies the discrete algebraic Riccati equation (7.4).
- The matrix $A_{cl}$ defined by (7.5) is asymptotically stable.

Next we list a number of properties for matrices satisfying these conditions.

**Lemma 7.16**: A matrix $P$ such that $G(P)$ is invertible, the discrete algebraic Riccati equation (7.4) is satisfied and such that $A_{cl}$ is asymptotically stable is uniquely defined by these three conditions.

The above result is more or less the discrete time analogon of corollary A.7 where we showed that a solution of the quadratic matrix inequality satisfying two corresponding rank conditions is unique. The continuous time result was proved by using the already known fact that in continuous time a stabilizing solution of an algebraic Riccati equation is unique. This is proven by reducing solvability to properties of modal subspaces of the associated Hamiltonian matrix. (a modal subspace of $H$ is simply the largest $H$-invariant subspace $S$ such that $\sigma(A|S)$ is contained in some prespecified area of the complex plane) In discrete time we have only been able to prove the following result:

**Lemma 7.17**: We assume that the matrices $S$ and $\tilde{A}$ are invertible where $S$ and $\tilde{A}$ are defined by:

$$
S := \begin{pmatrix} D_1^T D_1 & D_1^T D_2 \\ D_2^T D_1 & D_2^T D_2 - I \end{pmatrix}, \tag{7.37}
$$

$$
\tilde{A} := A - \begin{pmatrix} B & E \end{pmatrix} S^{-1} \begin{pmatrix} D_1^T \\ D_2^T \end{pmatrix} C. \tag{7.38}
$$

A matrix $P$ satisfies the above mentioned three conditions if and only if the subspace

$$
\mathcal{X} := \text{im} \begin{pmatrix} I \\ P \end{pmatrix}, \tag{7.39}
$$
is the modal subspace of $H$ associated to the open unit ball where $H$ is defined by

$$
H := \begin{pmatrix}
\tilde{A}^{-1} & \tilde{A}^{-1}XS^{-1}X^T \\
Q\tilde{A}^{-1} & \tilde{A}^T + Q\tilde{A}^{-1}XS^{-1}X^T
\end{pmatrix}.
$$

where

$$
X := \begin{pmatrix} B \\ E \end{pmatrix},
$$

$$
Q := C^T C - C^T XS^{-1}X^T C.
$$

The matrix $H$ is a symplectic matrix and therefore has the property that $\lambda$ is an eigenvalue of $H$ if and only if $\lambda^{-1}$ is an eigenvalue of $H$. □

Because of the symplectic property of $H$ we know that there exists at most one subspace which is $H$-invariant, for which the restriction of $H$ to this subspace is asymptotically stable and which is $n$-dimensional. We can then conclude that a result of the above lemma is that a solution $P$ of our three conditions is unique. But only under the assumption that the matrices $S$ and $\tilde{A}$ are invertible. For the proof of lemma 7.16 another kind of proof was needed.

In the case that the matrix $\tilde{A}$ is not necessarily invertible we can derive an extension of the previous lemma. We first need a definition:

**Definition 7.18**: A vector $v$ is called an eigenvector of the matrix pair $(H_1, H_2)$ with eigenvalue $\lambda$ if $v \neq 0$ and $H_1v = \lambda H_2v$. We call $v$ an eigenvector of the pair $(H_1, H_2)$ with eigenvalue $\infty$ if $v \neq 0$ and $H_2v = 0$.

A subspace $\mathcal{X}$ is called a modal subspace with respect to $D^-$ of the pair $(H_1, H_2)$ if $\mathcal{X}$ is the largest subspace $\mathcal{V}$ for which there exist a matrix $S$ and an asymptotically stable matrix $\tilde{A}$ such that $\mathcal{X} = \text{im } S$ and $H_1S = H_2S\tilde{A}$. □

Now we are able to give our extension of lemma 7.17:
Lemma 7.19: Assume that $S$, defined by (7.37), is invertible. A matrix $P$ satisfies the three above mentioned conditions if and only if the subspace defined by (7.39) is the modal subspace with respect to the open unit ball for the generalized symplectic pair $(H_1, H_2)$ where

$$
H_1 := \begin{pmatrix} \bar{A} & 0 \\ -Q & I \end{pmatrix},
$$

$$
H_2 := \begin{pmatrix} I & XS^{-1}X^T \\ 0 & \bar{A}^T \end{pmatrix},
$$

where $\bar{A}$ as defined by (7.38) and $X$ and $Q$ as defined in lemma 7.17. Moreover, $\lambda$ is an eigenvalue of $(H_1, H_2)$ if and only if $\lambda^{-1}$ is an eigenvalue of $(H_1, H_2)$. The same is true for $\lambda = 0$ and $\lambda = \infty$ respectively.

The last two lemmas give us the possibility of calculating the solution $P$ of our three conditions. However, for discrete time Riccati equations it is often more desirable to have recursive algorithms as e.g. is given in [Fa] for a different type of Riccati equation. Also an extension of our method to the case that the matrix $S$ is not invertible is useful. This is a subject of current research.

7.6 Concluding remarks

In this chapter the discrete time full information $H_\infty$ control problem has been investigated. As in the continuous time case the solvability is related to an algebraic Riccati equation. However, in contrast to the continuous time case, it turns out that, even in the case that $D_2 = 0$ the feedback law we find is in general not a state feedback but also requires a disturbance feedback part. Another interesting feature is the condition $R > 0$ where $R$ now depends on the solution of the algebraic Riccati equation. The assumptions made in this chapter are exactly the discrete time versions of the two main assumptions which are made in the chapter 3. Naturally, this chapter is a preliminary step towards the measurement feedback case which will be elaborated in the next chapter. More research needs to be done into the properties of this Riccati equation and the problems of finding solutions as discussed in the previous section should be solved.
The discrete time full-information case
Chapter 8

The discrete time $H_\infty$ control problem with measurement feedback

8.1 Introduction

The $H_\infty$ control problem with measurement feedback in continuous time has been thoroughly investigated, for example in chapter 5 and the references mentioned there. In this chapter we shall extend the result of the previous chapter to the discrete time $H_\infty$ control problem with measurement feedback.

One approach to this problem is to transform the system into a continuous time system, to derive controllers for the latter system and then transform back to discrete time. However, in our opinion, it is more natural to have the formulas directly available in terms of the original physical parameters, so their effect on the solution is transparent. This possibility might otherwise be blurred by the transformation to the continuous time. Also the numerical problems arising with this transformation and the fact that discrete time systems with a pole in 1 are transformed into non-proper systems are arguments in favour of a more direct approach.

The present chapter is reminiscent of chapter 5 which deals with the continuous time case. The results of this chapter already appeared in [St6].

We make the assumptions which yield an $H_\infty$ control problem which is exactly the exact discrete time analogue of the regular $H_\infty$ control problem with measurement feedback. The subsystem from the control input to the output should be left-invertible and should not have invariant zeros on the unit circle. Moreover, the subsystem from the disturbance to the measure-
The discrete time, measurement feedback case

The discrete time, measurement feedback case

... should be right-invertible and again should not have invariant zeros on the unit circle. Note that the left- and right-invertibility assumptions are automatically satisfied if the corresponding direct feedthrough matrices are injective and surjective, respectively.

As in the regular continuous time case, the necessary and sufficient conditions for the existence of suitable controllers involve positive semi-definite stabilizing solutions of two algebraic Riccati equations. As in the continuous time case the quadratic term in these algebraic Riccati equations is indefinite. However, other than in the continuous time case, the solutions of these equations have to satisfy another assumption: matrices depending on these solutions should be positive definite. Another complication is that this time the Riccati equations are coupled. The second Riccati equation is the discrete time analogue of the equation for $Y$ (see lemma 5.5). However, this time we did not succeed in expressing the existence and the solution in terms of the solutions of the full-information Riccati equation and its dual version. As in the regular $H_{\infty}$ control problem with measurement feedback (see remark (i) after theorem 5.1) we find an explicit expression for one internally stabilizing controller which makes the $H_{\infty}$ norm less than 1, if one exists.

The outline of this chapter is as follows. In section 8.2 we shall formulate the problem and give our main results. In section 8.3, we shall show the existence of stabilizing solutions of the two algebraic Riccati equations and complete the proof that our conditions are necessary. This is done by transforming the original system into a new system with the property that a controller "works" for the new system if and only if it "works" for the original system. In section 8.4 it is shown that our conditions are also sufficient. It turns out that the system transformation of section 8.3 repeated in a dual form exactly gives the desired results. We shall end with some concluding remarks in section 8.5.

8.2 Problem formulation and main results

We consider the following time-invariant system:

$$
\Sigma : \begin{cases}
\sigma x = Ax + Bu + Ew, \\
y = C_1x + D_{12}w, \\
z = C_2x + D_{21}u + D_{22}w,
\end{cases}
$$

(8.1)
where for each $k$ we have $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^l$ is the measurement, $w(k) \in \mathbb{R}^l$ the unknown disturbance and $z(k) \in \mathbb{R}^p$ the output to be controlled. $A, B, E, C_1, C_2, D_{12}, D_{21}$ and $D_{22}$ are matrices of appropriate dimension.

In this chapter we shall derive necessary and sufficient conditions for the existence of a dynamic compensator $\Sigma_F$ of the form (2.9) which is internally stabilizing and which is such that the closed loop transfer matrix $G_F$ satisfies $\|G_F\|_\infty < 1$. By scaling the plant we can thus, in principle, find the infimum of the closed loop $H_\infty$ norm over all stabilizing controllers. This will involve a search procedure. Furthermore, if a stabilizing $\Sigma_F$ exists which makes the $H_\infty$ norm less than 1, then we derive an explicit formula for one particular $\Sigma_F$ satisfying these requirements. Note that the loop-shifting arguments in section 5.5 allow us to extend our results to the more general system, defined by (2.8) (we could also assume for the moment that $D_{22} = 0$ and include it later via loop-shifting).

We can now formulate our main result:

**Theorem 8.1:** Consider the system (8.1). Assume that $(A, B, C_2, D_{21})$ has no invariant zeros on the unit circle and is left invertible. Moreover, assume that $(A, E, C_1, D_{12})$ has no invariant zeros on the unit circle and is right invertible. The following statements are equivalent:

(i) There exists a dynamic compensator $\Sigma_F$ of the form (2.9) such that the resulting closed loop transfer matrix $G_F$ satisfies $\|G_F\|_\infty < 1$ and the closed loop system is internally stable.

(ii) There exist symmetric matrices $P \succeq 0$ and $Y \succeq 0$ such that

(a) We have

$$V > 0, \quad R > 0,$$

where

$$V := B^T PB + D_{21}^T D_{21},$$

$$R := I - D_{22}^T D_{22} - (E^T PE)^{-1} (E^T PB + D_{22}^T D_{21}) V^{-1} (B^T PE + D_{21}^T D_{22}).$$

This implies that the matrix $G(P)$ is invertible where:
The discrete time, measurement feedback case

\[ G(P) := \begin{bmatrix} D_{21}^T D_{21} & D_{21}^T D_{22} \\ D_{22}^T D_{21} & D_{22}^T D_{22} - I \end{bmatrix} + \begin{bmatrix} B^T \\ E^T \end{bmatrix} P \begin{bmatrix} B & E \end{bmatrix}. \] (8.3)

(b) \( P \) satisfies the discrete algebraic Riccati equation:

\[ P = A^T P A + C_2^T C_2 - \left( \begin{bmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{bmatrix} \right)^T G(P)^{-1} \left( \begin{bmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{bmatrix} \right). \] (8.4)

(c) The matrix \( A_{cl,P} \) is asymptotically stable where:

\[ A_{cl,P} := A - \left( \begin{bmatrix} B & E \end{bmatrix} G(P)^{-1} \left( \begin{bmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{bmatrix} \right). \] (8.5)

Moreover if, given the matrix \( P \) satisfying (a)–(c), we define the following matrices:

\[ Z := E^T P A + D_{22}^T C_2 - \left[ E^T P B + D_{22}^T D_{21} \right] V^{-1} \left[ B^T P A + D_{21}^T C_2 \right], \]
\[ A_p := A + E R^{-1} V, \]
\[ E_p := E R^{-1/2}, \]
\[ C_{1,p} := C_1 + D_{12} R^{-1} V, \]
\[ C_{2,p} := V^{-1/2} \left[ B^T P A + D_{21}^T C_2 \right] + V^{-1/2} \left[ B^T P E + D_{21}^T D_{22} \right] R^{-1} V, \]
\[ D_{12,p} := D_{12} R^{-1/2}, \]
\[ D_{21,p} := V^{1/2}, \]
\[ D_{22,p} := V^{-1/2} \left( B^T P E + D_{21}^T D_{22} \right) R^{-1/2}, \]

then the matrix \( Y \) should satisfy:

(d) We have

\[ W > 0, \quad S > 0, \] (8.6)

where

\[ W := D_{12,p} D_{12,p}^T + C_{1,p} Y C_{1,p}^T, \]
\[ S := I - D_{22,p} D_{22,p}^T - C_{2,p} Y C_{2,p}^T \]
\[ + (C_{2,p} Y C_{1,p}^T + D_{22,p} D_{12,p}^T) W^{-1} (C_{1,p} Y C_{2,p}^T + D_{12,p} D_{22,p}^T). \]
This implies that the matrix \( H_P(Y) \) is invertible where:

\[
H(Y) := \begin{pmatrix}
D_{12, p} D^T_{12, p} & D_{12, p} D^T_{22, p} \\
D_{22, p} D^T_{12, p} & D_{22, p} D^T_{22, p} - I
\end{pmatrix} + \begin{pmatrix}
(C_{1, p}) Y (C^T_{1, p} & C^T_{2, p})
\end{pmatrix}.
\]

(8.7)

(e) \( Y \) satisfies the following discrete algebraic Riccati equation:

\[
Y = A_p Y A_p^T + E_p E_p^T - \begin{pmatrix}
(C_{1, p} Y A_p^T + D_{12, p} E_p^T) \\
(C_{2, p} Y A_p^T + D_{22, p} E_p^T)
\end{pmatrix}^T H(Y)^{-1} \begin{pmatrix}
(C_{1, p} Y A_p^T + D_{12, p} E_p^T) \\
(C_{2, p} Y A_p^T + D_{22, p} E_p^T)
\end{pmatrix}.
\]

(8.8)

(f) The matrix \( A_{d, p, y} \) is asymptotically stable where:

\[
A_{d, p, y} := A_p - \begin{pmatrix}
(C_{1, p} Y A_p^T + D_{12, p} E_p^T) \\
(C_{2, p} Y A_p^T + D_{22, p} E_p^T)
\end{pmatrix}^T H(Y)^{-1} \begin{pmatrix}
(C_{1, p}) \\
(C_{2, p})
\end{pmatrix}.
\]

(8.9)

In the case that there exist \( P \geq 0 \) and \( Y \geq 0 \) satisfying part (ii) then a controller of the form (2.9) satisfying the requirements in part (i) is given by:

\[
N := -D_{21, p}^{-1} (C_{2, p} Y C_{1, p}^T + D_{22, p} D_{12, p}^T) W^{-1},
M := -(D_{21, p}^{-1} C_{2, p} + N C_{1, p}),
L := BN + (A_p Y C_{1, p}^T + E_p D_{12, p}^T) W^{-1},
K := A_{d, p} - LC_{1, p}.
\]

Remark:

(i) Necessary and sufficient conditions for the existence of an internally stabilizing feedback compensator which makes the \( H_{\infty} \) norm less than some a priori given upper bound \( \gamma > 0 \) can be easily derived from theorem 8.1 by scaling.

(ii) If we compare these conditions with the conditions for the continuous time case (see [Do4, St2]), then we note that conditions (8.2) and (8.6) are this time depending on \( P \) and \( Y \). A simple example showing that
the assumption $G(P)$ invertible is not sufficient is given in the previous chapter. Note that if $R$ is not positive semi-definite, then matrices like $E_P$ are ill-defined and we can not even look for a matrix $Y$ satisfying (8.6)–(8.9).

The proofs in this chapter will be strongly reminiscent of the proofs given in chapter 5. They will depend on the results as derived in the previous chapter. The details of the proof are much easier since we do not need the bases from appendix A. This is a consequence of the fact that we work with the discrete time analogue of a regular problem.

8.3 A first system transformation

Using the results of the previous chapter we can derive the following result:

**Lemma 8.2 :** Assume that there exists a controller satisfying the conditions in part (i) of theorem 8.1. Moreover assume that $(A, B, C_2, D_{21})$ has no invariant zeros on the imaginary axis and is left invertible. Then there exists a positive semi-definite matrix $P$ satisfying conditions (a)–(c) of part (ii) of theorem 8.1.

**Proof :** If there exists a dynamic controller which is internally stabilizing and which makes the closed loop $H_\infty$ norm less than 1 for the problem with measurement feedback, then certainly condition (ii) of theorem 7.2 holds. This implies that also part (iii) of theorem 7.2 holds which exactly yields the desired result.

We assume throughout this section that a positive semi-definite matrix $P$ exists which satisfies the conditions (a)–(c) of part (ii) of theorem 8.1. By the above lemma we know that such a $P$ exists in the case that part (i) of theorem 8.1 is satisfied. But naturally also in the case that part (ii) of theorem 8.1 is satisfied this matrix $P$ with the desired properties exists.

In order to proceed with the proof of theorem 8.1 in this section we shall transform our original system (8.1) into a new system. The problem of finding an internally stabilizing feedback which makes the $H_\infty$ norm less than 1
for the original system is equivalent to the problem of finding an internally stabilizing feedback which makes the $H_\infty$ norm less than 1 for the new transformed system. However, this new system has some very desirable properties which make it much easier to work with. In particular, for this new system the disturbance decoupling problem with measurement feedback is solvable. We shall perform the transformation in two steps. First we shall perform a transformation related to the full-information $H_\infty$ problem and next a transformation related to the filtering problem. These two transformations are completely similar to the transformations used in chapter 5. Only, because our problem is regular, we do not need the technical hardware from appendices A and B.

We define the following system:

\[
\Sigma_P : \begin{cases}
\sigma x_P = A_P x_P + B u_P + E_P w_P, \\
y_P = C_{1,p} x_P + D_{12,p} w_P, \\
z_P = C_{2,p} x_P + D_{21,p} u_P + D_{22,p} w_P,
\end{cases}
\] (8.10)

where the matrices are as defined in the statement of theorem 8.1. Furthermore, we define the following system

\[
\Sigma_U : \begin{cases}
\sigma x_U = A_U x_U + B_U u_U + E_U w, \\
y_U = C_{1,u} x_U + D_{12,u} w, \\
z_U = C_{2,u} x_U + D_{21,u} u_U + D_{22,u} w,
\end{cases}
\] (8.11)

where

\[
A_U := A - B V^{-1} (B^T P A + D_{21}^T C_2), \\
B_U := B V^{-1/2}, \\
E_U := E - B V^{-1} (B^T P E + D_{21}^T D_{22}), \\
C_{1,u} := -R^{-1/2} Z, \\
C_{2,u} := C_2 - D_{21} V^{-1} (B^T P A + D_{21}^T C_2), \\
D_{12,u} := R^{1/2}, \\
D_{21,u} := D_{21} V^{-1/2}, \\
D_{22,u} := D_{22} - D_{21} V^{-1} (B^T P E + D_{21}^T D_{22}),
\]

and $V, R$ and $Z$ are as defined in theorem 8.1.

Note that this system $\Sigma_U$ is the same as the system $\Sigma_U$ as used in the previous chapter in section 7.4 where $C, D_1$ and $D_2$ are replaced by $C_2, D_{21}$ and $D_{22}$.
respectively. Hence \( \Sigma_U \) is inner and satisfies the other properties of lemma 7.13.

We shall now formulate our key lemma.

Lemma 8.3: Let \( P \) satisfy theorem 8.1 part (ii) (a)-(c). Moreover, let \( \Sigma_F \) be an arbitrary dynamic compensator in the form (2.9). Consider the following two systems, where the system on the left is the interconnection of (8.1) and (2.9) and the system on the right is the interconnection of (8.10) and (2.9):

\[
\begin{align*}
    & \begin{array}{c}
        z \\
        \Sigma \\
        \Sigma_F \\
        y \\
        u
    \end{array} \\
    & \begin{array}{c}
        w \\
        u
    \end{array} \\
\end{align*}
\]  

\[
\begin{align*}
    & \begin{array}{c}
        z_p \\
        \Sigma_P \\
        \Sigma_F \\
        y_p \\
        u_p
    \end{array} \\
    & \begin{array}{c}
        w_p
    \end{array}
\end{align*}
\]

Then the following statements are equivalent:

(i) The system on the left is internally stable and its transfer matrix from \( w \) to \( z \) has \( \| \Sigma \|_{\infty} \) norm less than 1.

(ii) The system on the right is internally stable and its transfer matrix from \( w_p \) to \( z_p \) has \( \| \Sigma \|_{\infty} \) norm less than 1.

Remark: Note the relation with lemma 5.4. Although the systems \( \Sigma_P \) and \( \Sigma_U \) in this chapter are discrete time systems and in chapter 5 we used continuous time systems the proof of lemma 5.4 can still be used to prove the discrete time version with some minor alterations. We shall add the proof for the sake of completeness.

Proof: We investigate the following systems:
8.3 A first system transformation

The system on the left is the same as the system on the left in (8.12) and the system on the right is described by the system (8.11) interconnected with the system on the right in (8.12). A realization for the system on the right is given by:

\[
\begin{bmatrix}

d(x_u - x_p) \\
x_u \\
p
\end{bmatrix} = \begin{bmatrix}
A_{cl,p} & 0 & 0 \\
* & A + BNC_1 & BM \\
* & LC_1 & K
\end{bmatrix} \begin{bmatrix}
xd \\
x_p \\
p
\end{bmatrix} + \begin{bmatrix}
0 \\
E + BND_{12} \\
LD_{12}
\end{bmatrix} w
\]

\[
z_u = (\star C_2 + D_{21} NC_1 D_{21} M) \begin{bmatrix}
x_u - x_p \\
x_p \\
p
\end{bmatrix} + (D_{22} + D_{21} N D_{12}) w
\]

where \( A_{cl,p} \) is defined by (8.5). The *'s denote matrices which are unimportant for this argument. The system on the right is internally stable if and only if the system described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (8.13), then we see immediately that, since \( A_{cl,p} \) is asymptotically stable, the system on the left is internally stable if and only if the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input \( w \), then we have \( z = z_u \) i.e. the input-output behaviour of both systems are equivalent. Hence the system on the left has \( H_\infty \) norm less than 1 if and only if the system on the right has \( H_\infty \) norm less than 1.

We know \( \Sigma_U \) is inner and satisfies the other properties necessary in order to apply lemma 2.14 to the system on the right in (8.13) and hence we find that the closed loop system is internally stable and has \( H_\infty \) norm less than 1 if and only if the dashed system is internally stable and has \( H_\infty \) norm less than 1.
Since the dashed system is exactly the system on the right in (8.12) and the system on the left in (8.13) is exactly equal to the system on the left in (8.12) we have completed the proof.

Using the previous lemma, we know that we only have to investigate the system $\Sigma_P$. This new system has some very nice properties which we shall exploit. First we shall look at the Riccati equation for the system $\Sigma_P$. It can be checked immediately that $X = 0$ satisfies conditions (a)-(c) of theorem 8.1 for the system $\Sigma_P$.

We now dualize $\Sigma_P$. We know that $(A, E, C_1, D_{12})$ is right-invertible and has no invariant zeros on the unit circle. It can be easily checked that this implies that $(A_p, E, C_{1,p}, D_{12})$ is right-invertible and has no invariant zeros on the unit circle. Hence for the dual of $\Sigma_p$ we know that $(A^\top_p, C^\top_{1,p}, E^\top, D^\top_{31})$ is left-invertible and has no invariant zeros on the unit circle. If there exists an internally stabilizing feedback for the system $\Sigma$ which makes the $H_\infty$ norm less than 1, then the same feedback is internally stabilizing and makes the $H_\infty$ norm less than 1 for the system $\Sigma_P$. If we dualize this feedback and apply it to the dual of $\Sigma_P$, then it is again internally stabilizing and again it makes the $H_\infty$ norm less than 1. We can now apply corollary 8.2 which exactly guarantees the existence of a matrix $Y$ satisfying conditions (d)-(f) of theorem 8.1. Thus we derived the following lemma which gives the necessity part of theorem 8.1:

**Lemma 8.4**: Let the system (8.1) be given with zero initial state. Assume that $(A, B, C_2, D_{21})$ has no invariant zeros on the unit circle and is left invertible. Moreover assume that $(A, E, C_1, D_{12})$ has no invariant zeros on the unit circle and is right-invertible. If part (i) of theorem 8.1 is satisfied, then there exist matrices $P$ and $Y$ satisfying (a)-(f) of part (ii) of theorem 8.1.

This completes the proof (i) $\Rightarrow$ (ii). In the next section we shall prove the reverse implication. Moreover in the case that the desired $\Sigma_P$ exists we shall derive an explicit formula for one choice for $F$ which satisfies all requirements.
8.4 The transformation into a disturbance decoupling problem with measurement feedback

In this section we shall assume that there exist matrices $P$ and $Y$ satisfying part (ii) of theorem 8.1 for the system (8.1). We shall transform our original system $\Sigma$ into another system $\Sigma_{P,Y}$. We shall show that a compensator is internally stabilizing and makes the $H_\infty$ norm less than 1 for the system $\Sigma$ if and only if the same compensator is internally stabilizing and makes the $H_\infty$ norm less than 1 for our transformed system $\Sigma_{P,Y}$. After that we shall show that $\Sigma_{P,Y}$ has a very special property: the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS) is solvable. We first define $\Sigma_{P,Y}$. First transform $\Sigma$ into $\Sigma_P$. Then we apply the dual transformation on $\Sigma_P$ to obtain $\Sigma_{P,Y}$:

$$\Sigma_{P,Y} : \begin{cases} 
\sigma_{P,Y} = A_{P,Y}x_{P,Y} + B_{P,Y}u_{P,Y} + E_{P,Y}w_{P,Y}, \\
y_{P,Y} = C_{1,P}x_{P,Y} + D_{12,P,Y}w_{P,Y}, \\
z_{P,Y} = C_{2,P,Y}x_{P,Y} + D_{21,P,Y}u_{P,Y} + D_{22,P,Y}w_{P,Y},
\end{cases} \quad (8.14)$$

where

$$\tilde{Z} := A_{P,Y}YC_{2,p}^T + E_{P,Y}D_{22,p}^T \hspace{1cm} - (A_{P,Y}YC_{1,p}^T + E_{P,Y}D_{12,p}^T) W^{-1} (C_{1,p}YC_{2,p}^T + D_{12,p}D_{22,p}^T),$$

$$A_{P,Y} := A_p + \tilde{Z}S^{-1}C_{2,p},$$

$$B_{P,Y} := B + \tilde{Z}S^{-1}D_{21,p},$$

$$E_{P,Y} := (A_{P,Y}YC_{1,p}^T + E_{P,Y}D_{12,p}^T) W^{-1/2} \hspace{1cm} + \tilde{Z}S^{-1} (C_{2,p}YC_{1,p}^T + D_{22,p}D_{12,p}^T) W^{-1/2},$$

$$C_{2,p,Y} := S^{-1/2}C_{2,p},$$

$$D_{12,p,Y} := W^{1/2},$$

$$D_{21,p,Y} := S^{-1/2}D_{21,p},$$

$$D_{22,p,Y} := S^{-1/2} (C_{2,p}YC_{1,p}^T + D_{22,p}D_{12,p}^T) W^{-1/2},$$

and $W$ and $S$ as defined in part (ii) of theorem 8.1. When we first apply lemma 8.3 on the transformation from $\Sigma$ to $\Sigma_P$ and then the dual version of lemma 8.3 on the transformation from $\Sigma_P$ to $\Sigma_{P,Y}$ we find:
Lemma 8.5: Let $P$ satisfy theorem 8.1 part (ii) (a)-(c). Moreover let an arbitrary dynamic compensator $\Sigma_F$ be given, described by (2.9). Consider the following two systems, where the system on the left is the interconnection of (8.1) and (2.9) and the system on the right is the interconnection of (8.14) and (2.9):

\[
\begin{array}{c}
\Sigma \\
\Sigma_F
\end{array}
\quad
\begin{array}{c}
\Sigma_{P,Y} \\
\Sigma_F
\end{array}
\]

Then the following statements are equivalent:

(i) The system on the left is internally stable and its transfer matrix from $w$ to $z$ has $H_\infty$ norm less than 1.

(ii) The system on the right is internally stable and its transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ has $H_\infty$ norm less than 1.

It remains to be shown that for $\Sigma_{P,Y}$ the disturbance decoupling problem with internal stability and measurement feedback is solvable:

Lemma 8.6: Let $\Sigma_F$ be given by:

\[
\Sigma_F : \begin{cases}
\sigma p = K_{P,Y} y_{P,Y} + L_{P,Y} y_{P,Y} \\
u_{P,Y} = M_{P,Y} y_{P,Y} + N_{P,Y} y_{P,Y}
\end{cases}
\]  

(8.15)

where

\[
N_{P,Y} := -D_{21,P,Y}^{-1} D_{22,P,Y} D_{12,P,Y}^{-1},
\]

\[
M_{P,Y} := -\left( D_{21,P,Y}^{-1} C_{2,P,Y} + N_{P,Y} C_{1,P} \right),
\]

\[
L_{P,Y} := B_{P,Y} N_{P,Y} + E_{P,Y} D_{12,P,Y}^{-1},
\]

\[
K_{P,Y} := A_{P,Y} + B_{P,Y} M_{P,Y} - E_{P,Y} D_{12,P,Y}^{-1} C_{1,P}.
\]

The interconnection of $\Sigma_F$ and $\Sigma_{P,Y}$ is internally stable and the closed loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero.
8.5 Conclusion

Proof: We can write out the formulas for a state space representation of the interconnection of $\Sigma_{P,Y}$ and $\Sigma_F$. We then apply the following basis transformation:

$$
\begin{pmatrix}
  x_{P,Y} - p \\
p
\end{pmatrix} =
\begin{pmatrix}
  I & -I \\
  0 & I
\end{pmatrix}
\begin{pmatrix}
x_{P,Y} \\
p
\end{pmatrix}.
$$

After this transformation one immediately sees that the closed loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero. Moreover the closed loop state matrix (2.6) after this transformation is given by:

$$
\begin{pmatrix}
  A_{cl,P,Y} & 0 \\
  L_{P,Y}C_{1,P} & A_{cl,P}
\end{pmatrix}.
$$

Since $A_{cl,P,Y}$ and $A_{cl,P}$ are asymptotically stable matrices, this implies that indeed $\Sigma_F$ is internally stabilizing.

This controller is the same as the controller described in the statement of theorem 8.1. We know $\Sigma_F$ is internally stabilizing and the resulting closed loop system has $H_\infty$ norm less than 1 for the system $\Sigma_{P,Y}$. Hence, by applying lemma 8.5, we find that $\Sigma_F$ satisfies part (i) of theorem 8.1. This completes the proof of (ii) ⇒ (i) of theorem 8.1. We have already shown the reverse implication and hence the proof of theorem 8.1 is completed.

8.5 Conclusions

In this chapter we have solved the discrete time $H_\infty$ problem with measurement feedback. It is shown that the techniques for the continuous time case can be applied to the discrete time case. Unfortunately the formulas are much more complex, but since we investigate the discrete time analogue of a regular system, it is possible to give an explicit formula for one controller satisfying all requirements. It would, however, be interesting to generalize this result and find a characterization of all controllers satisfying the requirements (this is given but not proven in [Li4]).

Another interesting problem is to derive recursive formulas for calculating the solutions to these algebraic Riccati equations; that is, to extend the results of section 7.5. It would also be interesting to find two dual Riccati equations and a coupling condition as in [Gl3]. Nevertheless the results presented in this chapter show that it is possible to solve discrete time $H_\infty$ problems.
directly, instead of transforming them to continuous time problems. The assumption of left-invertibility is not very restrictive. In case that the system is not left-invertible this implies that there are several inputs which have the same effect on the output and this non-uniqueness can be factored out (see for a continuous time treatment [SC]). The assumption of right invertibility can be removed by dualizing this reasoning. However at this moment it is unclear how to remove the assumptions concerning zeros on the unit-circle. Finally an interesting extension is the finite horizon discrete time case which is discussed in for example [Li4].
Chapter 9

Conclusion

In this thesis we have derived several results concerning the $H_\infty$ control problem. Naturally the research in this area is not completed by this thesis and several interesting open problems remain. In this chapter we first devote a section to a discussion of the main contributions of this thesis to the active research area of $H_\infty$ control. After that, in section 9.2, we apply the results of this thesis to a number of problems related to robustness. In section 9.3, we discuss several open problems which could be of interest for future research. Then, in the last section, we give our final concluding remarks on this thesis.

9.1 Summary of results obtained

The results derived in this thesis are related to three specific areas of research:

- Singular systems,
- Differential games,
- Discrete time systems.

We shall briefly discuss our contributions to these three subjects.

9.1.1 Singular systems

For systems which are not necessarily regular, we derived necessary and sufficient conditions for the existence of an internally stabilizing controller which makes the $H_\infty$ norm strictly less than some, a priori given, number $\gamma$. For the measurement-feedback case these conditions are in terms of solutions of
Conclusion

two quadratic matrix inequalities and four associated rank conditions. This can be shown to be equivalent to the existence of two stabilizing solutions of reduced order Riccati equations.

In our opinion, our proof yields a nice understanding of the structure of the $H_\infty$ control problem. However, our algorithm for finding a suitable controller, if one exists, is still not completely satisfactory.

At this point we shall outline the main steps of the proof of theorem 5.1, which is our main result for singular systems. This is done because the main steps give nice insight in the structure of the $H_\infty$ control problem as we see it.

We start with a system of the form (5.1). We first investigate how well we can regulate the system if we have all information of the state and the disturbance available. We can derive necessary and sufficient conditions under which we can find a controller which makes the $H_\infty$ norm less than some bound $\gamma$ (a so-called suitable controller), by investigating a sup-inf problem:

$$
\sup_w \inf_u \left\{ \|z_{u,w,\xi}\|_2^2 - \gamma^2\|w\|_2^2 \mid u \in \mathcal{L}_2^m, w \in \mathcal{L}_2^l \text{ such that } x_{u,w,\xi} \in \mathcal{L}_2^n \right\}
$$

(9.1)

for arbitrary initial state $x(0) = \xi$. We solved this problem using ideas supplied by Pontryagin's Maximum Principle. We did this explicitly for regular systems and we extended it to general systems by some decompositions of the state space, the input space and the output space (that this extension to the general case is related to the same sup-inf problem can be seen more easily by investigating the results of chapter 6). We find necessary and sufficient conditions under which a suitable controller exists: there should exist a positive semi-definite solution to a quadratic matrix inequality which satisfies two rank conditions. This can be shown to be equivalent to the existence of a stabilizing solution to an algebraic Riccati equation (a reduced order Riccati equation if the system is singular).

We now return to the general case where we might have only partial information on the state and the disturbance. If there exists a suitable controller with measurement feedback, then there certainly exists a suitable controller for the full-information feedback case. Hence, there exists a positive semi-definite solution to our quadratic matrix inequality and the corresponding rank conditions. Then we can transform our original system $\Sigma$ into a new system $\Sigma_p$. A controller is suitable for $\Sigma$ if and only if a controller is suitable for $\Sigma_p$ (this is true for controllers with measurement feedback and for
controllers with state feedback). However this new system $\Sigma_P$ has the interesting property that we can find state feedbacks which are internally stabilizing and which make the $H_\infty$ norm of the closed loop system arbitrarily small (ADDPS, as discussed in section 2.6, is solvable).

Remains the problem whether we can measure the state well enough in order to find a suitable controller with measurement feedback. It turns out that this problem is exactly dual to the problem of full-information $H_\infty$ control: minimizing the induced norm from $w$ to $x$ for some system $\Sigma_t$ by full-information feedback is essentially the same problem as building an observer for the state of $\Sigma_t'$ which minimizes the induced norm of the operator from $w$ to $x - \hat{x}$ where $\hat{x}$ denotes the estimated state. Hence, we immediately find necessary and sufficient conditions under which we can observe the state of $\Sigma_P$ well enough: there should exist a positive semi-definite solution to a second quadratic matrix inequality which has to satisfy two rank conditions (remember that we already had a solution to one quadratic matrix inequality before our transformation from $\Sigma$ to $\Sigma_P$). Thus we obtain a necessary condition for the existence of a suitable controller: we should have solutions of two quadratic matrix inequalities and four corresponding rank conditions. These conditions are also sufficient. To show this we apply a second transformation, dual to the first, from $\Sigma_P$ to $\Sigma_{P,Q}$. Again, a controller is suitable for $\Sigma_P$ if and only if this controller is suitable for $\Sigma_{P,Q}$. For $\Sigma_{P,Q}$ we can observe the state arbitrarily well. Surprisingly enough $\Sigma_{P,Q}$ still has the same nice property $\Sigma_P$ has: there exists an internally stabilizing state feedback which makes the $H_\infty$ norm of the closed loop system arbitrarily small (ADDPS is solvable). Together this implies that we can find a controller with measurement feedback which is internally stabilizing and which makes the $H_\infty$ norm of the closed loop system associated with $\Sigma_{P,Q}$ arbitrarily small (ADDPMS is solvable). We can make the $H_\infty$ norm of the closed loop system equal to 0 if the system $\Sigma$ is regular (DDPMS is solvable) but we cannot do this in general.

The conditions we thus obtained can be reformulated to yield the conditions of theorem 5.1. The problem is that our second quadratic matrix inequality as defined above is in terms of the system $\Sigma_P$. Rewriting this second matrix inequality with two corresponding rank conditions in terms of $\Sigma$ yields one quadratic matrix inequality, two rank conditions and a coupling condition.

### 9.1.2 Differential games
Differential games are not worked out in much detail in this thesis. However, some of the results obtained yield nice intuition.

Firstly we note that the quadratic form associated with the solution of our quadratic matrix inequality yields an *almost* Nash equilibrium while for the regular problem we obtain an (exact) Nash equilibrium.

Moreover, as we noted in the previous subsection, the results for the regular state feedback $H_\infty$ control problem were derived by investigating a sup-inf problem (9.1). The value of (9.1) is, as a function of $\xi$, equal to the quadratic form given by the solution of the algebraic Riccati equation. The general full-information $H_\infty$ control problem with state feedback was solved by reducing the general problem to the regular $H_\infty$ control problem. In chapter 6 while discussing differential games we find as a side result that the quadratic form associated with the solution of the quadratic matrix inequality again yields the value of (9.1) as a function of $\xi$. This is a nice result to complete the picture we have of the $H_\infty$ control problem.

Another interesting fact is that a necessary condition for the existence of an almost equilibrium is the possibility to make the $H_\infty$ norm less than or equal to 1 and a sufficient condition is the possibility to make the $H_\infty$ norm strictly less than 1. All of this holds under the constraint of internal stability and with state feedback. This provides a natural starting point for research for testing whether we can make the $H_\infty$ norm less than or equal to 1, as done in [Gl5] for the regular case.

### 9.1.3 Discrete time systems

For discrete time systems which are discrete time analogues of regular continuous time systems, we were able to derive nice conditions: there exists a suitable controller if and only if there exist positive semi-definite stabilizing solutions to two algebraic Riccati equations. This certainly shows that transformations to the continuous time case are not necessary and that $H_\infty$ control can be applied directly to discrete time systems. However, before a real application becomes possible we should spend some time on deriving numerically reliable methods to check whether these Riccati equations indeed have solutions or not and, if they exist, to calculate the solutions.

The techniques used to derive this result were completely similar to the techniques we used to derive the results for the regular continuous time $H_\infty$ control problem. Only for discrete time systems there are a lot of extra technicalities and the formulas are rather complex and cumbersome. On the other hand, there is one essential difference: for continuous time systems
we could do no better by allowing for non-causal feedbacks. This is not true for discrete time systems: it is possible to attain $H_\infty$ norms of the closed loop system via non-causal controllers which can not be attained nor approximated by causal controllers.

9.2 Application of our results to certain robustness problems

In section 1.3 we discussed stabilization of uncertain systems. In this section we are going to apply the results of this thesis to three different types of uncertainty:

- Additive perturbations,
- Multiplicative perturbations,
- Perturbations in the realization of a system.

Each time we find a problem which can be reduced to an $H_\infty$ control problem. The first two problems can be found e.g. in [MF, Vi]. The last problem is discussed in [Hi]. The second and third problem will in general yield singular $H_\infty$ control problems.

9.2.1 Additive perturbations

Assume that we have a continuous time system $\Sigma$ being an imperfect model of a certain plant. We assume that the error is additive, i.e. we assume that the plant can be exactly described by the following interconnection:

\begin{equation}
\begin{array}{c}
\Sigma \\
+ \\
\Sigma_{err} \\
\end{array}
\end{equation}

Here $\Sigma_{err}$ is some arbitrary system such that $\Sigma$ and $\Sigma + \Sigma_{err}$ have the same number of unstable poles. Thus we assume that the plant is described by the system $\Sigma$ interconnected as in diagram (9.2) with another system $\Sigma_{err}$. The
system $\Sigma_{err}$ represents the uncertainty and is hence, by definition, unknown. In this subsection we derive conditions under which there exists a controller $\Sigma_F$ of the form (2.4) from $y$ to $u$ such that the interconnection (9.2) is stabilized by this controller for all systems $\Sigma_{err}$ which do not change the number of unstable poles and which have $L_\infty$ norm less than or equal to some, a priori given, number $\gamma$. In [Vi] the following result is given:

Lemma 9.1: Let a controller $\Sigma_F$ of the form (2.4) be given. The following conditions are equivalent:

(i) If we apply the controller $\Sigma_F$ from $y$ to $u$ to the interconnection (9.2), then the closed loop system is well-posed and internally stable for every system $\Sigma_{err}$ such that

- $\Sigma_{err}$ has $L_\infty$ norm less than or equal to $\gamma$,
- $\Sigma$ and $\Sigma + \Sigma_{err}$ have the same number of unstable poles.

(ii) $\Sigma_F$ internally stabilizes $\Sigma$ and if $G_{ci}$ and $G_F$ denote the transfer matrices of $\Sigma$ and $\Sigma_F$, respectively, then $I - G_{ci}G_F$ is invertible as a proper rational matrix and $\|G_F(I - G_{ci}G_F)^{-1}\|_\infty < \gamma^{-1}$.

Assume that a minimal realization of $\Sigma$ is given. Hence $\Sigma$ is described by some quadruple $(A, B, C, D)$ with $(A, B)$ controllable and $(C, A)$ observable. We define a new system:

$$\Sigma_{na} : \begin{cases} \dot{x} = Ax + Bu, \\
y = Cx + Du + w, \\
z = u. \end{cases}$$ (9.3)

It is easily checked that a controller $\Sigma_F$ of the form (2.4) applied to $\Sigma$ yields a closed loop system that is well-posed and internally stable if and only if the same controller $\Sigma_F$ from $y$ to $u$ applied to $\Sigma_{na}$ yields a closed loop system which is well-posed and internally stable.

Moreover, assume that a controller $\Sigma_F$ from $y$ to $u$ of the form (2.4) is given which, when applied to $\Sigma_{na}$, yields a well-posed and internally stable closed loop system. Then the resulting closed loop transfer matrix is equal to $G_F(I - G_{ci}G_F)^{-1}$ where $G_{ci}$ and $G_F$ denote the transfer matrices of $\Sigma$ and $\Sigma_F$, respectively.
9.2 Robustness problems

Using the above reasoning we find that our original problem formulation is equivalent to the problem of finding an internally stabilizing controller for \( \Sigma_{na} \) which makes the \( H_\infty \) norm of the closed loop system less than \( \gamma^{-1} \). We first define \( P_a \) and \( Q_a \) as the unique positive semi-definite matrices satisfying the following two Riccati equations:

\[
P_a A + A^T P_a = P_a B B^T P_a, \]
\[
Q_a A^T + A Q_a = Q_a C^T C Q_a, \]

such that the matrices \( A - B B^T P_a \) and \( A - Q_a C^T C \) are asymptotically stable (existence is guaranteed by standard linear quadratic control, see [Wi4]).

After applying theorems 5.10, 5.1 and 9.1 we find the following theorem:

**Theorem 9.2**: Assume that a system \( \Sigma \) is given with minimal realization \( (A, B, C, D) \) such that \( A \) has no eigenvalues on the imaginary axis. We define the related system \( \Sigma_{na} \) by (9.3) and let \( \gamma > 0 \) be given. The following three conditions are equivalent:

(i) There exists a controller \( \Sigma_F \) from \( y \) to \( u \) of the form (2.4) which, when applied to the interconnection (9.2), yields a closed loop system that is well-posed and internally stable for all systems \( \Sigma_{err} \) such that:
  - \( \Sigma_{err} \) has \( L_\infty \) norm less than or equal to \( \gamma \),
  - \( \Sigma \) and \( \Sigma + \Sigma_{err} \) have the same number of unstable poles.

(ii) There exists a controller \( \Sigma_F \) from \( y \) to \( u \) of the form (2.4) which, when applied to the system \( \Sigma_{na} \), yields a closed loop system that is well-posed, internally stable and has \( H_\infty \) norm less than \( \gamma^{-1} \).

(iii) We have \( \rho(P_a Q_a) < \gamma^{-2} \).

Moreover, if \( P_a \) and \( Q_a \) satisfy part (iii), then a controller of the form (2.4) satisfying both part (i) as well as part (ii) is described by:

\[
N := 0, \\
M := -B^T P_a, \\
L := (I - \gamma^2 Q_a P_a)^{-1} Q_a C^T, \\
K := A - BB^T P_a - L(C - DB^T P_a). \]

\( \square \)
Remarks:

(i) Naturally the class of perturbations we have chosen is rather artificial. However, it can be easily shown that if we allow for perturbations which add an extra unstable pole, then there exist arbitrarily small perturbations which destabilize the closed loop system. On the other hand, our class of perturbations does include all stable systems \( \Sigma_{err} \) with \( H_\infty \) norm less than or equal to \( \gamma \).

(ii) We want to find a controller satisfying part (i) for a \( \gamma \) which is as large as possible. Note that part (iii) shows that for every \( \gamma \) smaller than the bound \( [\rho(P_aQ_a)]^{-1/2} \) we can find a suitable controller satisfying part (i).

(iii) It can be shown that the bound \( [\rho(P_aQ_a)]^{-1/2} \) only depends on the antistable part of \( \Sigma \). Hence we could assume a priori that \( A \) has only eigenvalues in the open right half complex plane. (we still have to exclude eigenvalues on the imaginary axis.) In that case it can be shown that \( P_a \) and \( Q_a \) are the inverses of \( X \) and \( Y \), respectively, where \( X \) and \( Y \) are the unique positive definite solutions of the following two Lyapunov equations:

\[
AX + XA^T = BB^T \\
A^TY + YA = C^TC
\]

Then it is easily derived that our bound is equal to the smallest Hankel singular value of \( \Sigma \). This result was already known (see e.g. [Gl2]).

9.2.2 Multiplicative perturbations

We assume that we have once again a continuous time system \( \Sigma \) being an imperfect model of a certain plant. This time however, we assume that the error is multiplicative, i.e. we assume that the plant is exactly described by the following interconnection:

\[
\begin{array}{c}
\Sigma_{err} \\
\downarrow u \\
\downarrow + \\
\Sigma \\
\uparrow y
\end{array}
\]
Here \( \Sigma_{err} \) is some arbitrary system such that the interconnection (9.4) has the same number of unstable poles as \( \Sigma \). Thus we assume that the plant is described by the system \( \Sigma \) interconnected as in diagram (9.4) with another system \( \Sigma_{err} \). The system \( \Sigma_{err} \) represents the uncertainty. As for additive perturbations, our goal is to find conditions under which there exists a controller \( \Sigma_F \) of the form (2.4) from \( y \) to \( u \) such that the interconnection (9.4) is stabilized by this controller for all systems \( \Sigma_{err} \) which do not change the number of unstable poles of the interconnection (9.4) and which have \( \mathcal{L}_\infty \) norm less than or equal to some, a priori given, number \( \gamma \). In [Vi] there is also a result for multiplicative perturbations:

**Lemma 9.3**: Let a controller \( \Sigma_F \) of the form (2.4) be given. The following conditions are equivalent:

(i) If we apply the controller \( \Sigma_F \) from \( y \) to \( u \) to the interconnection (9.4), then the closed loop system is well-posed and internally stable for every system \( \Sigma_{err} \) such that

- \( \Sigma_{err} \) has \( \mathcal{L}_\infty \) norm less than or equal to \( \gamma \),
- The interconnection (9.4) and \( \Sigma \) have the same number of unstable poles.

(ii) \( \Sigma_F \) internally stabilizes \( \Sigma \) and if \( G_{ci} \) and \( G_F \) denote the transfer matrices of \( \Sigma \) and \( \Sigma_F \), respectively, then \( I - G_{ci}G_F \) is invertible as a proper rational matrix and \( \|G_{ci}G_F(I - G_{ci}G_F)^{-1}\|_\infty < \gamma^{-1} \).

Assume that a minimal realization of \( \Sigma \) is given. Hence \( \Sigma \) is described by some quadruple \( (A, B, C, D) \) with \( (A, B) \) controllable and \( (C, A) \) observable. We define a new system:

\[
\Sigma_{nm}: \begin{cases} 
\dot{z} = Ax + Bu + Bw \\
y = Cx + Du + Dw \\
z = u.
\end{cases} \tag{9.5}
\]

As we did for additive perturbations we can once again rephrase our problem formulation in terms of our new system \( \Sigma_{nm} \). We first need some definitions:

\[
G(Q) := \begin{pmatrix} 
AQ + QA^T + BB^T & QC^T + BD^T \\
CQ + DB^T & DD^T
\end{pmatrix}
\]
\begin{align*}
M(Q, s) & := \begin{pmatrix} sI - A \\ -C \end{pmatrix} \\
G_{ci}(s) & := C(sI - A)^{-1}B + D
\end{align*}

Moreover, we define $P_m$ and $Q_m$ as the unique positive semidefinite matrices satisfying the following conditions:

(i) \hspace{1cm} \textit{A} \hspace{1cm} \begin{itemize}
\item $A^T P_m + P_m A = P_m B B^T P_m$,
\item $A - B B^T P_m$ is asymptotically stable.
\end{itemize}

(ii) \hspace{1cm} \begin{itemize}
\item $G(Q_m) \geq 0$,
\item $\text{rank } G(Q_m) = \text{rank}_{R(s)} G_{ci}$,
\item $\text{rank } \begin{pmatrix} M(Q_m, s) & G(Q_m) \end{pmatrix} = n + \text{rank}_{R(s)} G_{ci}$.
\end{itemize}

The existence of such a $P_m$ was already discussed in the previous subsection and follows from standard regular Linear Quadratic control. The first two conditions on $Q_m$ require that $Q_m$ is a rank-minimizing solution of a linear matrix inequality. The same linear matrix inequality is also appearing in the \textit{singular} filtering problem (see [SH4], this problem is dual to the singular linear quadratic control problem). Via the reduced order Riccati equation associated with this linear matrix inequality (see appendix A) it can be shown that the largest solution of the linear matrix inequality, whose existence is guaranteed since $(C,A)$ is detectable (dualize the results in [Gee, Wi4]), satisfies all the requirements on $Q_m$. This shows existence of $Q_m$. Uniqueness was already guaranteed by corollary A.7.

\textbf{Theorem 9.4} : Let a system $\Sigma$ be given with minimal realization $(A, B, C, D)$ and state space $\mathcal{R}^n$. Moreover, let $\gamma > 0$ be given. Assume that $A$ has no eigenvalues on the imaginary axis and assume that $(A, B, C, D)$ has no invariant zeros on the imaginary axis. We define the auxiliary system $\Sigma_{nm}$ by (9.9). Under the above assumptions the following three conditions are equivalent:

(i) There exists a controller $\Sigma_F$ from $y$ to $u$ of the form (2.4) which, when applied to the interconnection (9.4), yields a closed loop system that is well-posed and internally stable for all systems $\Sigma_{err}$ such that:

\begin{itemize}
\item $\Sigma_{err}$ has $L_\infty$ norm less than or equal to $\gamma$;
\end{itemize}
9.2 Robustness problems

- The interconnection (9.4) and $\Sigma$ have the same number of unstable poles.

(ii) There exists a controller $\Sigma_F$ from $y$ to $u$ of the form (2.4) which, when applied to the system $\Sigma_{nm}$, yields a closed loop system that is well-posed, internally stable and has $H_\infty$ norm less than $\gamma^{-1}$.

(iii) Either $A$ is stable or $1 + \rho(P_mQ_m) < \gamma^{-2}$.

Remarks:

(i) We already noted for additive perturbations that our class of perturbations is rather artificial. Our class of perturbations does include all stable systems $\Sigma_{err}$ with $H_\infty$ norm less than or equal to $\gamma$ if $\gamma < 1$. The restriction $\gamma < 1$ does not matter since we know that part (iii) from our theorem is only satisfied if $\gamma < 1$ (or $A$ is stable but in that case $\Sigma_{err}$ should be stable anyway).

(ii) As for additive perturbations we have an explicit bound for the allowable size of perturbations: part (iii) shows that for every $\gamma$ smaller than the bound $[1 + \rho(P_mQ)]^{-1/2}$ we can find a suitable controller satisfying part (i).

(iii) For additive perturbations it could be shown that the upper bound $[\rho(P_mQ_m)]^{-1/2}$ only depends on the antistable part of $\Sigma$. It should be noted that this is not true for the bound $[1 + \rho(P_mQ_m)]^{-1/2}$ which we obtained for multiplicative perturbations.

(iv) Note that because this is, in general, a singular problem we have not been able to find an explicit formula for a controller satisfying part (i). Note that we know that a controller satisfies part (i) if and only if this controller satisfies part (ii).

9.2.3 Perturbations in the realization of a system

The previous two classes of perturbations were directly concerned with the perturbations of input/output operators or, as an alternative formulation, with perturbations of transfer matrices. In contrast with the above, the theory of complex stability radii is concerned with perturbations of state space realizations (an interesting overview article is [Hi]).

Assume some autonomous system is given:
\[ \dot{x} = (A + D\Delta E)x. \]

\( A \in \mathcal{R}^{n \times n}, D \in \mathcal{R}^{n \times l} \) and \( E \in \mathcal{R}^{p \times n} \) are given matrices and \( \Delta \) expresses the uncertainty which is structured by the matrices \( D \) and \( E \). The complex stability radius of the triple \((A, D, E)\) is then defined as

\[ r_c(A, D, E) := \inf \left\{ \|\Delta\|_\infty \mid \Delta \in \mathcal{C}^{l \times p} \text{ such that } A + D\Delta E \text{ is not stable} \right\}. \]

Naturally by allowing for complex perturbations the class of perturbations is not very natural. But there are two good reasons for investigating this complex stability radius. First of all, we can derive very elegant results for the complex stability radius which we cannot obtain for the real stability radius (defined as the complex stability radius but with the restriction that \( \Delta \) should be a real matrix). Moreover, if we define \( \Sigma_\Delta \) as the feedback inter-connection of some stable system \( \Delta = (F, G, H, J) \) with \( \Sigma_{ci} = (A, D, E, 0) \), i.e.

\[ \Sigma_\Delta : \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A + DJE & DH \\ GE & F \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{9.6} \]

and the real dynamic stability radius \( r_{R,d} \) by

\[ r_{R,d}(A, D, E) := \inf \{ \|\Delta\|_\infty \mid \Delta = (F, G, H, J) \in S \text{ is such that } \Delta \times \Sigma_{ci} \text{ described by (9.6) is not stable} \}, \]

where \( S \) denotes the class of quadruples of real matrices which define an asymptotically stable system, then it is shown in [Hi] that the complex stability radius is equal to the real dynamic stability radius. This makes an investigation of the complex stability radius more important because we investigate the real dynamic stability radius at the same time.

In [Hi] the following relation between \( H_\infty \) control and the complex stability radius is given:

**Lemma 9.5**: We have

\[ r_c(A, D, E) = \|G\|_\infty^{-1}, \]

where \( G \) denotes the transfer matrix of \((A, D, E, 0)\). \( \Box \)
Next we investigate the problem of maximization of the complex stability radius. Let the system \( \Sigma_{ci} \) be given by:

\[
\Sigma_{ci} : \dot{x} = (A + D\Delta E)x + Bu.
\]

We search for a static state feedback \( u = Fx \) such that the closed loop stability radius \( r_c(A + BF, D, E) \) is larger than \( \gamma \). For any matrix \( P \in \mathbb{R}^{n \times n} \) we define:

\[
F_\gamma(P) := \begin{pmatrix}
PA + A^TP + E^TE + \gamma^2PD^TD^TP & PB \\
B^TP & 0
\end{pmatrix},
\]

\[
L_\gamma(P,s) := \begin{pmatrix}
sI - A - \gamma^2D^TD^TP & -B
\end{pmatrix},
\]

\[
G_{ci}(s) := E(sI - A)^{-1}B.
\]

Using lemma 9.5 and theorem 4.1 we find the following result:

**Theorem 9.6**: Assume that a system \( \Sigma_{ci} = (A, D, E, 0) \) is given which does not have invariant zeros on the imaginary axis. Let \( \gamma > 0 \). Then the following two conditions are equivalent:

(i) There exists a matrix \( F \in \mathbb{R}^{m \times n} \) such that

\[
rc(A + BF, D, E) > \gamma.
\]

(ii) There exists a positive semi-definite matrix \( P \) such that

(a) \( F_\gamma(P) \succeq 0 \),

(b) \( \text{rank } F_\gamma(P) = \text{rank} \mathcal{R}(s) G_{ci} \),

(c) \( \text{rank } \begin{pmatrix}
L_\gamma(P,s) \\
F_\gamma(P)
\end{pmatrix} = n + \text{rank} \mathcal{R}(s) G_{ci}, \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+ \). ☐

### 9.3 Open problems

#### 9.3.1 Invariant zeros

Throughout this thesis, we have excluded invariant zeros on the imaginary axis for continuous time systems. For discrete time systems, invariant zeros
on the unit disc have been excluded. In this subsection we want to discuss the
difficulty of these invariant zeros *intuitively*. The reader should not expect
formal proofs in this section.

A treatment of invariant zeros on the imaginary axis for continuous time
systems is given in several papers, [HSK, SC3, SC4]. For the regular state
feedback case (no problems at infinity) the conditions in [SC3] can, in prin­
ciple, be reformulated as: there exists a solution of a Riccati equation for
which the matrix $A_{cl}$ as given in theorem 3.1 has all eigenvalues in the open
left half plane or in points on the imaginary axis which are invariant zeros.
Besides that, separate extra conditions have to be satisfied for each invariant
zero on the imaginary axis.

However, it is worthwhile to look for alternative formulations of the results
in [HSK, SC3, SC4] in order to obtain a better insight of how the extra
conditions look like which have to be satisfied in the case that we have an
invariant zero on the imaginary axis.

Moreover, a numerical reliable way of finding a suitable controller, if one
exists, is needed. As in this thesis, in [SC3, SC4] a geometric approach is
chosen. This yields good understanding of the nature of the problems but
gives results which are in general numerically not very reliable.

Next, we shall try to show the difficulty of invariant zeros on the imaginary
axis for continuous time systems. Our approach is based on the paper [HSK],
which only treats the regular one block problem ($D_{12}$ and $D_{21}$ in (2.1) square
invertible matrices).

The $H_\infty$ control problem with measurement feedback can be reduced to the
so-called model-matching problem (see [Fr2]). This is the following problem:

$$\inf_{Q \in H_\infty} \|T_1 - T_2 QT_3\|_\infty,$$

(9.7)

where $T_1$, $T_2$ and $T_3$ are given matrices in $H_\infty$. For the sole reason of an
easy exposition of the problems we assume that $T_2$ and $T_3$ are, as rational
matrices, right- and left-invertible, respectively. In that case for $Q_1 :=
T_2^+ T_1 T_3^+$ ($+$ denotes a right cq. left inverse) we have

$$T_1 - T_2 Q_1 T_3 = 0.$$  

However, in general, $Q_1$ will not be in $H_\infty$. The invariant zeros of the
two subsystems $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ (using the definitions
of theorem 5.1) are the only points in the complex plane where $T_2$ and $T_3,$
respectively, might have a zero, i.e. lose rank. The right cq. left inverse will
then have a pole in these points. Hence if we have only invariant zeros in
the open left half plane, then $Q_1$ will be in $H_\infty$.
In invariant zeros in the open left half plane do not give rise to constraints on the attainable closed loop system $T_1 - T_2QT_3$. On the other hand, invariant zeros in the open right half plane do yield constraints on the attainable closed loop system. A zero on the imaginary axis can however be cancelled approximately by choosing $Q$ such that it has a pole in the left half plane arbitrarily close to the zero on the imaginary axis. It turns out that for these points the only constraint is a condition in terms of the transfer matrices evaluated in this point itself. If we have only invariant zeros on the imaginary axis we find the constraint that the infimum in (9.7) is less than 1 if and only if for all invariant zeros $s \in \mathbb{C}_0$ there exists a constant matrix $K_s$ such that

$$||T_1(s) - T_2(s)K_sT_3(s)|| < 1.$$ 

For invariant zeros in the open right half plane this condition is necessary but not sufficient for the infimum to be less than 1.

This shows that zeros on the imaginary axis and zeros in the right half plane need a different approach and that is exactly the difficulty we have in treating the most general case with both invariant zeros on the imaginary axis as well as zeros in the open right half plane.

We expect that for the general case without assumptions on the system and with continuous time, we can find the following necessary and sufficient conditions for the existence of a suitable controller (we shall use the notation as was already used in theorem 5.1): we think that the existence of matrices $P$ and $Q$ which satisfy the quadratic matrix inequalities $F_\gamma(P) \geq 0$ and $G_\gamma(Q) \geq 0$ and which satisfy the rank conditions (a) and (b) of theorem 5.1 are conditions which are always needed.

For any $s$ in the closed right half plane which is not an invariant zero of either $(A, B, C_2, D_2)$ or $(A, E, C_1, D_1)$, rank conditions (c) or (d) of theorem 5.1 should still be satisfied.

The condition for invariant zeros of $(A, B, C_2, D_2)$ or $(A, E, C_1, D_1)$ in the open right half plane is that rank conditions (c) and (d) of theorem 5.1 should still be satisfied. However, for invariant zeros of $(A, B, C_2, D_2)$ or $(A, E, C_1, D_1)$ on the imaginary axis the rank conditions (c) and (d) should be replaced by some weaker condition.

### 9.3.2 Practical applications

For practical applications of the theory as given in this thesis, the main problem is to choose weights. As already discussed in section 1.5, engineers
quite often think in terms of Bode plots. For single-input, single-output most of the performance criteria can be easily translated into frequency dependent upper bounds on the amplitude Bode plot of the closed loop system. Classical control engineers then try a PID controller and by choosing parameters in the P-, I- and D-parts of our controller they hope to achieve a closed loop system satisfying the upper bound. Because of their experience they often succeed. $H_\infty$ control gives the control engineer not only a larger class of controllers to choose from but can also guarantee whether a certain closed loop performance is attainable or not.

For multi-input, multi-output (MIMO) systems we lack a systematic approach to translate performance criteria into upper bounds for each frequency on the largest singular value of the transfer matrix evaluated in this specific frequency (the justification to investigate singular values for MIMO systems is e.g. given in [Do, Sa]).

Up to this moment we can, in general, only guarantee existence of controllers with McMillan degree equal to the sum of the McMillan degrees of the plant and all the weights. This implies that complex weights yield high-order controllers. However, in practice we would like to know whether there exists a controller of lower McMillan degree which still gives an acceptable closed loop performance. A main area of research therefore lies in reduced order $H_\infty$ controllers and in model reduction with an $H_\infty$ criterion (see [BH, HB, Mu2]).

Another problem is that certain practical constraints are not easily incorporated into the problem by a suitable choice of weights. This is because we have to add the assumption that our weights must be stable. That this extra assumption is needed is due to the fact that otherwise the system is not stabilizable or detectable. One approach for tackling this problem would be to work out the $H_\infty$ problem with only the requirement of input/output stability. This is a subject of current research.

The practical constraints we referred to in the previous paragraph are e.g. the following:

- **Steady state disturbance rejection**: the requirement that the closed loop transfer matrix is zero in zero.

- **High gain roll-off**: the requirement that the closed loop transfer matrix is strictly proper.

These kind of conditions are all based on the requirement that the closed loop transfer matrix satisfies certain constraints on the imaginary axis. Without
proof we shall give the following result. This result stems basically from [HSK] and [Vi, lemma 6.5.9].

Theorem 9.7: Let a system of the form (2.1) be given. Assume that there exists a compensator of the form (2.4) which is internally stabilizing and which is such that the closed loop system has $H_\infty$ norm less than one. Moreover, assume that for $s_i \in C^0$ with $i = 1, 2, \ldots, k$ there exists a static controller $K_i$ such that the resulting closed loop transfer matrix $G_i$ satisfies $G_i(s_i) = 0$. In this case there exists a time invariant compensator $\Sigma_F$ of the form (2.4) and with MacMillan degree less than $n + k$ such that the closed loop system is internally stable, the closed loop transfer matrix $G_{cl}$ has $H_\infty$ norm less than one and $G_{cl}(s_i) = 0$ for $i = 1, 2, \ldots, k$.

In the previous theorem $s_i = \infty$ is allowed but in this case we should replace $G_{cl}(s_i)$ by the corresponding direct feedthrough matrix of $G_{cl}$.

On the above result a lot of research can be based. Especially how to implement this result e.g. for the two special cases of steady state disturbance rejection and high gain roll-off.

9.3.3 Mixed $H_\infty/\ldots$ problems

As we already noted in section 1.1 robustness plays a key role in control theory. In this thesis we investigated $H_\infty$ control problems which play an essential role in robustness of internal stability as outlined in section 9.2. However, it often happens that we are not only interested in robustness of internal stability but in robustness analysis of other performance criteria as well.

Both mixed Linear Quadratic (LQ) and $H_\infty$ control problems as well as mixed Linear Quadratic Gaussian (LQG) and $H_\infty$ control problems have been studied in the literature (see e.g. [BH2, HB, HB2, RK]). In these cases one wants to minimize an LQ- and an LQG-criterion, respectively under the constraint that the closed loop $H_\infty$ norm be strictly less than 1. The constraint that the closed loop $H_\infty$ norm be bounded by 1 guarantees robustness of internal stability while at the same time our LQ- or LQG-criterion optimizes closed loop performance.

It should be noted that although the formal problem formulation of these two problems have completely different starting points, the solutions of these two problems are strongly correlated.
The general solution of the above mentioned problems is still an open problem. There are also other kinds of mixed problems which are of interest but which have not received much attention. For instance, in the above problems the sensitivity of the LQ- or LQG-criterion to system perturbations is not incorporated in the controller design. Only the sensitivity of internal stability to system perturbations is taken into account. The main problem of course remains the choice between robustness and performance. This also yields problems with respect to the required model accuracy and bounds on the McMillan degree of the controller.

9.4 Conclusion of this thesis

In this thesis we have generalized existing theorems on the standard $H_\infty$ control problem. Interesting open problems at this moment are more specific applications of $H_\infty$ control which are non-standard, e.g. the above mentioned mixed problems. The main open problem at this moment is related to the practical application of $H_\infty$ control. Although we feel that the theoretical $H_\infty$ control problem and its structure are well understood at the moment, the practical issue of the design of weights for multi-input, multi-output systems lacks a systematic approach. Besides that, the $H_\infty$ control problems with either invariant zeros on the imaginary axis (unit disc for discrete time systems) or with problems at infinity are now reasonably well understood but they need numerically more reliable algorithms to check existence of suitable controllers and to calculate them, if they exist.

We feel that this thesis and the large amount of papers mentioned in the references give a very thorough basic understanding of the $H_\infty$ control problem. Nevertheless the subject will remain an important research area in the near future.

This is due to the fact that we still need to make the step from basic understanding to design (how to apply all this nice theory in practice). Moreover, because of the extreme importance of robustness and its strong correlation with $H_\infty$, its relation with almost all aspects of system and control theory is interesting. We could think of identification with $H_\infty$ error bounds and other combinations which at the moment might be far fetched. However, control engineers working in practice will be the first to acknowledge that the main reason why they are sceptical about applying modern control schemes is their concern about robustness. They sometimes even claim that all modern schemes including controllers built with $H_\infty$ performance criteria
are not robust and only classical P(I)(D) controllers can be trusted to control their expensive machinery. The control engineer in practice should realize that, although we present this new tool of $H_{\infty}$ control in a very mathematical setting, in fact $H_{\infty}$ control is nothing else then applying the available theoretical machinery to the problem of obtaining the desired shape of the closed loop Bode plot.

It is our belief that in the future $H_{\infty}$ will lead to robust and reliable controllers but a lot of work has to be done in order to be ready to tackle real-life control problems (which will often be MIMO systems) in a structured and reliable way.
Appendix A

Preliminary basis transformations

A.1 A suitable choice of bases

In this appendix we show that by applying a suitable state feedback transformation \( u = F_0 x + v \) to the system \( \Sigma_{ci} = (A, B, C, D) \), it is transformed into a system \( \Sigma_{ci,F_0} := (A + BF_0, B, C + DF_0, D) \) with a very particular structure. We shall display this structure by writing down the matrices of the mappings \( A + BF_0, B, C + DF_0, D \) with respect to suitable bases in the input space \( \mathcal{R}^m \), the state space \( \mathcal{R}^n \) and the output space \( \mathcal{R}^p \). In this section we shall use the notation \( A_{F_0} := A + BF_0 \) and \( C_{F_0} := C + DF_0 \).

Our basic tool is the strongly controllable subspace. This subspace has been defined and some of its properties have been given in section 2.4. We now define the bases which will be used in the sequel.

First choose a basis of the input space \( \mathcal{R}^m \) as follows. Let \( u_1, u_2, \ldots, u_m \) be a basis such that \( u_1, u_2, \ldots, u_i \) is a basis of ker\( D \). (\( 0 \leq i \leq m \)). In other words, decompose \( \mathcal{R}^m = U_1 \oplus U_2 \) such that \( U_2 = \text{ker} \, D \) and \( U_1 \) arbitrary.

Next, choose a basis of the output space \( \mathcal{R}^p \) as follows. Let \( z_1, z_2, \ldots, z_p \) be an orthonormal basis such that \( z_1, \ldots, z_j \) is an orthonormal basis of \( \text{im} \, D \) and \( z_{j+1}, \ldots, z_p \) is an orthonormal basis of \( (\text{im} \, D)^\perp \). (\( 0 \leq j \leq p \)). In other words, write \( \mathcal{R}^p = Z_1 \oplus Z_2 \) with \( Z_1 = \text{im} \, D \) and \( Z_2 = (\text{im} \, D)^\perp \). Because this is an orthonormal basis this basis transformation does not change the norm \( \|z\| \).

With respect to these decompositions the mapping \( D \) has the form

\[
D = \begin{pmatrix}
\hat{D} & 0 \\
0 & 0
\end{pmatrix},
\]

with \( \hat{D} \) invertible. Moreover \( B \) and \( C \) can be partitioned as
Basis transformations

\[ B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix}. \]

It is easy to see that \( B_2 = B \ker D \) and \( \ker \hat{C}_2 = C^{-1} \im D \). Next define a linear mapping by

\[ F_0 := \begin{pmatrix} -\hat{D}^{-1} \hat{C}_1 \\ 0 \end{pmatrix}. \quad (A.1) \]

Then we have

\[ C + DF_0 = \begin{pmatrix} 0 \\ \hat{C}_2 \end{pmatrix}. \]

We now choose a basis of the state space \( \mathcal{R}^n \). Let \( x_1, x_2, \ldots, x_n \) be a basis such that \( x_{s+1}, \ldots, x_r \) is a basis of \( T(\Sigma_{ci}) \cap C^{-1} \im D \) and \( x_{s+1}, \ldots, x_n \) is a basis of \( T(\Sigma_{ci}) \). (0 \leq s \leq r \leq n) In other words, write \( \mathcal{R}^n = X_1 \oplus X_2 \oplus X_3 \) with \( X_2 = T(\Sigma_{ci}) \cap C^{-1} \im D, X_2 \oplus X_3 = T(\Sigma_{ci}) \) and \( X_1 \) arbitrary.

It turns out that with respect to the bases introduced above \( A_{F_0}, B \) and \( C_{F_0} \) have a particular form. This is a consequence of the following lemma:

**Lemma A.1**: Let \( F_0 \) be given by (A.1). Then we have

(i) \( (A + BF_0)(T(\Sigma_{ci}) \cap C^{-1} \im D) \subseteq T(\Sigma_{ci}) \),
(ii) \( \im B_2 \subseteq T(\Sigma_{ci}) \),
(iii) \( T(\Sigma_{ci}) \cap C^{-1} \im D \subseteq \ker \hat{C}_2 \).

**Proof**: (i) \( T(\Sigma_{ci}) \) is \( (C_{F_0}, A_{F_0}) \)-invariant by lemma 2.6. This implies that \( A_{F_0}(T(\Sigma_{ci}) \cap \ker C_{F_0}) \subseteq T(\Sigma_{ci}) \).

Since \( \ker C_{F_0} = \ker \hat{C}_2 = C^{-1} \im D \), the result follows.

(ii) Let \( T_i(\Sigma_{ci}) \) be the sequence defined by (2.15). Then we know that \( T_i(\Sigma_{ci}) = B \ker D = \im B_2 \). Since \( T_i(\Sigma_{ci}) \) is non-decreasing this proves our claim.

(iii) This follows immediately from the fact that \( C^{-1} \im D = \ker \hat{C}_2 \).

By applying this lemma we find that the matrices \( A + BF_0, B, C + DF_0 \) and \( D \) with respect to these bases have the following form.
A.1 A suitable choice of bases

\[
\begin{align*}
A + BF_0 &= \begin{pmatrix}
A_{11} & 0 & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix}, &
B &= \begin{pmatrix}
B_{11} & 0 \\
B_{21} & B_{22} \\
B_{31} & B_{32}
\end{pmatrix}, \\
C + DF_0 &= \begin{pmatrix}
0 & 0 & 0 \\
C_{21} & 0 & C_{23}
\end{pmatrix}, &
D &= \begin{pmatrix}
\hat{D} & 0 \\
0 & 0
\end{pmatrix}.
\end{align*}
\]

(A.2)

We apply the preliminary feedback \( u = F_0 x + v \) to the system \( \Sigma \), given by

\[
\Sigma : \begin{cases}
\dot{x} = Ax + Bu + Ew, \\
z = Cx + Du.
\end{cases}
\]

(A.3)

Denote the resulting system by \( \Sigma_{F_0} \). We decompose \( x,v \) and \( z \) corresponding to the bases in state, input and output space, i.e.

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.
\]

We also decompose \( E \) and \( \hat{C}_1 \) corresponding to the bases defined:

\[
\hat{C}_1 = \begin{pmatrix}
C_{11} & C_{12} & C_{13}
\end{pmatrix}, \quad E = \begin{pmatrix}
E_1 \\
E_2 \\
E_3
\end{pmatrix}.
\]

(A.4)

Hence we have

\[
C = \begin{pmatrix}
\hat{C}_1 \\ \hat{C}_2
\end{pmatrix} = \begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & 0 & C_{23}
\end{pmatrix}.
\]

In our new bases the system \( \Sigma_{F_0} \) then has the following form:

\[
\begin{align*}
\dot{x}_1 &= A_{11} x_1 + \begin{pmatrix} B_{11} & A_{13} \end{pmatrix} \begin{pmatrix} v_1 \\ x_3 \end{pmatrix} + E_1 w, \\
\dot{x}_2 &= \begin{pmatrix} A_{22} & A_{23} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_{22} & B_{23} \end{pmatrix} \begin{pmatrix} v_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_{21} & A_{21} \end{pmatrix} \begin{pmatrix} v_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ E_3 \end{pmatrix} w, \\
\dot{x}_3 &= \begin{pmatrix} 0 & 0 \end{pmatrix} x_1 + \begin{pmatrix} \hat{D} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} \hat{D} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ x_3 \end{pmatrix}.
\end{align*}
\]

(A.5) (A.6) (A.7)
As already suggested by the way in which we arranged these equations, the system \( \Sigma_{F_0} \) can be considered as the interconnection of two subsystems. This is depicted in the following diagram:

\[
\begin{array}{c}
v_1 \\
\Sigma \\
w \\
v_2 \\
\Sigma_0 \\
\end{array} \xrightarrow{\begin{array}{c} x_1 \\
x_3 \end{array}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

(A.8)

In the picture (A.8), \( \Sigma \) is the system given by the equations (A.5) and (A.7). It has input space \( U_1 \times X_3 \times \mathcal{R}^l \), state space \( X_1 \) and output space \( \mathcal{R}^p \). The system \( \Sigma_0 \) is given by equation (A.6). It has input space \( \mathcal{R}^m \times X_1 \times \mathcal{R}^l \), state space \( X_2 \oplus X_3 \) and output space \( X_3 \). The interconnection is made via \( x_1 \) and \( x_3 \) as in the diagram. Note that \( \Sigma \) and \( \Sigma_{F_0} \) have the same output equation. However, in \( \Sigma_{F_0} \) the variable \( x_3 \) is generated by \( \Sigma_0 \) while in \( \Sigma \) it is considered as an input and is free.

The systems \( \Sigma \) and \( \Sigma_0 \) turn out to have some nice structural properties:

Lemma A.2:

(i) \( C_{23} \) is injective,

(ii) The system,

\[
\Sigma_1 := \begin{bmatrix} (A_{22} A_{23}) & (B_{22}) & (0 \ I) & 0 \end{bmatrix}
\]

(A.9)

with input space \( U_2 \), state space \( X_2 \oplus X_3 \) \((= T(\Sigma_{ci})) \) and output space \( X_3 \) is strongly controllable.

Proof: (i) Let \((x_1^T, x_2^T, x_3^T)^T \) be the coordinate vector of a given \( x \in \mathcal{R}^n \). Assume that \( C_{23} x_3 = 0 \). Let \( \tilde{x} \in \mathcal{R}^n \) be the vector with coordinates \((0^T, 0^T, x_3^T)\). Then \( \tilde{x} \in X_3 \). In addition, \( \tilde{x} \in T(\Sigma_{ci}) \cap \ker C_2 = X_2 \). Thus \( \tilde{x} = 0 \) so \( x_3 = 0 \).
(ii) Let $T(\Sigma_1)$ be the strongly controllable subspace of the system $\Sigma_1$ given by (A.9). We shall prove $T(\Sigma_1) = \mathcal{X}_2 \oplus \mathcal{X}_3$. First note that there exists $G := \left( \begin{array}{cc} G_2^T & G_3^T \end{array} \right)^T$ such that

$$\left[ \begin{array}{cc} A_{22} & A_{23} \\ A_{32} & A_{33} \end{array} \right] + \left( \begin{array}{c} G_2 \\ G_3 \end{array} \right) \begin{pmatrix} 0 & I \end{pmatrix} \right] T(\Sigma_1) \subseteq T(\Sigma_1).$$

Also note that

$$\text{im} \left( \begin{array}{c} B_{22} \\ B_{32} \end{array} \right) \subseteq T(\Sigma_1).$$

Now assume that $T(\Sigma_1) \subset \mathcal{X}_2 \oplus \mathcal{X}_3$ with strict inclusion. Define $\mathcal{V} \subseteq \mathbb{R}^n$ by

$$\mathcal{V} := \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \mid \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in T(\Sigma_1) \right\}.$$

Clearly, $\mathcal{V} \subset T(\Sigma_{ci})$ with strict inclusion. We claim there exists a linear map $G_0 : \mathbb{R}^p \to \mathbb{R}^n$ such that

$$(A + G_0 C) \mathcal{V} \subseteq \mathcal{V},$$

and

$$\text{im} \left( B + G_0 D \right) \subseteq \mathcal{V}.$$

Indeed, let $C_{23}^+$ be any left inverse of $C_{23}$ and define

$$G_0 := \left( \begin{array}{ccc} B_{11} & -A_{13} \\ B_{21} & G_2 \\ B_{31} & G_3 \end{array} \right) \begin{pmatrix} -\hat{D}^{-1} & 0 \\ 0 & C_{23}^+ \end{pmatrix}.$$

It is then straightforward to check (A.10) and (A.11). This however contradicts the fact that $T(\Sigma_{ci})$ is the smallest subspace $\mathcal{V}$ for which (A.10) and (A.11) hold (see definition 2.3). We conclude that $\mathcal{X}_2 \oplus \mathcal{X}_3 = T(\Sigma_1)$. 

Our next result states that the zero structure of the system $\Sigma_{ci} = (A, B, C, D)$ is completely determined by the zero structure of $\Sigma_{ci}$ defined by

$$\Sigma_{ci} := \left[ A_{11}, \left( \begin{array}{cc} B_{11} & A_{13} \end{array} \right), \left( \begin{array}{c} 0 \\ C_{21} \end{array} \right) \right], \left( \begin{array}{cc} A_{22} & A_{23} \\ A_{32} & A_{33} \end{array} \right), \left( \begin{array}{cc} D & 0 \\ 0 & C_{23} \end{array} \right)].$$

(A.12)
Note that $\tilde{\Sigma}_{ci}$ is a subsystem of $\tilde{\Sigma}$ (where $\tilde{\Sigma}$ is described by (A.5) and (A.7)) in the same way as $\Sigma_{ci}$ is a subsystem of $\Sigma$ (where $\Sigma$ is described by (A.3)).

**Lemma A.3**: The non-trivial transmission polynomials of $\Sigma_{ci}$ and $\tilde{\Sigma}_{ci}$, respectively, coincide.

**Proof**: According to section 2.3 the transmission polynomials of $\Sigma_{ci} = (A, B, C, D)$ and $\Sigma_{ci,F_0} = (A_{F_0}, B, C_{F_0}, D)$ coincide. Thus, in order to prove the lemma it suffices to show that the system matrix $P_{ci,F_0}$ of $\Sigma_{ci,F_0}$ is unimodularly equivalent to a polynomial matrix of the form

$$
\begin{pmatrix}
\tilde{P}_{ci} & 0 & 0 \\
0 & I & 0
\end{pmatrix},
$$

where $\tilde{P}_{ci}$ is the system matrix of $\tilde{\Sigma}_{ci}$. Since $\Sigma_1$, as defined by (A.9), is strongly controllable and $(0 \ I)$ is surjective, by lemma 2.8 the Smith form of the system matrix $P_1$ of $\Sigma_1$ is equal to $(I \ 0)$. In addition we clearly have

$$
P_1 \sim \begin{pmatrix}
sI - A_{22} & 0 & -B_{22} \\
-A_{22} & 0 & -B_{32} \\
0 & I & 0
\end{pmatrix} \sim \begin{pmatrix}
sI - A_{22} & -B_{22} & 0 \\
-A_{22} & -B_{32} & 0 \\
0 & 0 & I
\end{pmatrix},
$$

so we conclude that

$$
\begin{pmatrix}
sI - A_{22} & -B_{22} \\
-A_{22} & -B_{32}
\end{pmatrix},
$$
is unimodularly equivalent to $(I \ 0)$. The proof is then completed by noting that
A.1  A suitable choice of bases

A consequence of the above lemma is that the invariant zeros of $\Sigma_{ci}$ and $\hat{\Sigma}_{ci}$, respectively, coincide.

Our next lemma states that the normal rank of the transfer matrix

$$G_{ci}(s) := C(sI - A)^{-1}B + D, \tag{A.13}$$

of the system $\Sigma_{ci}$ is equal to the number ($\text{rank } D + \text{dim } X_3$) or, equivalently,

Lemma A.4 : We have

$$\text{rank}_{R(s)} G_{ci} = \text{rank} \left( \begin{array}{cc} C_{23} & 0 \\ 0 & \hat{D} \end{array} \right).$$

Proof : Define $L(s) := sI - A$. Then we have

$$\text{rank}_{R(s)} \left( \begin{array}{cc} L & 0 \\ 0 & G_{ci} \end{array} \right) = n + \text{rank}_{R(s)} G_{ci}. \tag{A.14}$$

We also have

$$\left( \begin{array}{cc} L(s) & 0 \\ 0 & G_{ci}(s) \end{array} \right) \sim \left( \begin{array}{cc} sI - A_{F_0} & -B \\ C_{F_0} & D \end{array} \right)$$

$$= \left( \begin{array}{cccc} sI - A_{11} & 0 & -A_{13} & -B_{11} & 0 \\ -A_{21} & sI - A_{22} & -A_{23} & -B_{21} & -B_{22} \\ -A_{31} & -A_{32} & sI - A_{33} & -B_{31} & -B_{32} \\ 0 & 0 & 0 & \hat{D} & 0 \\ C_{21} & 0 & C_{23} & 0 & 0 \end{array} \right).$$
Since $C_{23}$ and $\hat{D}$ are injective we can make the (1,3), (1,4), (2,4) and (3,4) blocks zero by unimodular transformations. Furthermore we make a basis transformation on the output such that $C_{23}$ has the form $(I_r \ 0)^T$ where $r = \text{rank } C_{23}$. Thus, after suitable permutation of blocks, the normal rank of the latter matrix turns out to be equal to the normal rank of

$$\begin{pmatrix}
    sI - \tilde{A}_{11} & 0 & 0 & 0 & 0 \\
    -A_{21} & sI - A_{22} & -A_{23} & -B_{22} & 0 \\
    -A_{31} & -A_{32} & sI - A_{33} & -B_{32} & 0 \\
    C_{211} & 0 & I_r & 0 & 0 \\
    C_{212} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Here $\tilde{A}_{11}$ is a given matrix. Since, by lemma 4.2, the matrix in the center box has full row rank for all $s \in C$ and $\text{rank}_{R(s)}(sI - \tilde{A}_{11}) = \dim X_1$ we find

$$\text{rank}_{R(s)}\begin{pmatrix} L & 0 \\ 0 & G_{ci} \end{pmatrix} = n + \text{rank} \begin{pmatrix} C_{23} & 0 \\ 0 & \hat{D} \end{pmatrix}.$$  

Combining this with (A.14) gives the desired result.

To conclude we want to note that if $D$ is injective, then the subspace $U_2$ in the decomposition of $R^m$ vanishes. Consequently, the partitioning of $B$ reduces to a single block and the partitioning of $D$ reduces to $(\hat{D}^T \ 0)^T$ with $\hat{D}$ invertible. It is left as an exercise to the reader to show that $T(S_{ci}) = \{0\}$ if and only if $\ker D \subset \ker B$. Thus, if $D$ is injective, then also $T(S_{ci}) = \{0\}$. In that case the subspaces $X_2$ and $X_3$ appearing in the decomposition of the state space $R^n$ both vanish and the partitioning of $A_{F_0}$ reduces to a single block.

### A.2 The quadratic matrix inequality

We shall now derive some properties for matrices satisfying the *quadratic matrix inequality*, i.e. matrices $P$ such that

$$F(P) := \begin{pmatrix} A^TP + PA + PE^TP + C^TP + C^TD + PB \\ D^TC + B^TP \\ D^TD \end{pmatrix} \geq 0.$$  

We first derive a result which is one of the key lemmas of this thesis.
Lemma A.5 : Assume that a symmetric $P$ is a solution of $F(P) \geq 0$. We have $PT(\Sigma_{ci}) = 0$, i.e in our decomposition $P$ can be written as

$$
P = \begin{pmatrix}
P_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. \tag{A.15}
$$

Proof : Let $F_0$ be given by (A.1). Since $D^\tau C_{F_0} = 0$ (this can be checked easily) we may apply lemma 2.7. We define

$$
M(P) := \begin{pmatrix}
I & F_0^\tau \\
0 & I
\end{pmatrix} F(P) \begin{pmatrix}
I & 0 \\
F_0 & I
\end{pmatrix}. \tag{A.16}
$$

If $F(P) \geq 0$, then also

$$
M(P) = \begin{pmatrix}
PA_{F_0} + A_{F_0}^\tau P + PEET^\tau P + C_{F_0}^\tau C_{F_0} & PB \\
B^\tau P & D^\tau D
\end{pmatrix} \geq 0
$$

We claim $B \ker D \subseteq \ker P$. Let $u \in \mathcal{R}^m$ be such that $Du = 0$. Then we find

$$
\begin{pmatrix}
0^\tau & u^\tau
\end{pmatrix} M(P) \begin{pmatrix}
0 \\
u
\end{pmatrix} = 0
$$

and hence, since $M(P) \geq 0$, we find

$$
M(P) \begin{pmatrix}
0 \\
u
\end{pmatrix} = 0.
$$

This implies $PB = 0$. Next we have to show that $\ker P$ is $(C_{F_0}, A_{F_0})$ invariant. Assume that $x \in \ker P \cap \ker C_{F_0}$. Then

$$
x^\tau (PA_{F_0} + A_{F_0}^\tau P + PEET^\tau P + C_{F_0}^\tau C_{F_0}) x = 0.
$$

Hence, by applying $x$ to one side only, we find $PA_{F_0} = 0$ and therefore $A_{F_0} x \in \ker P$. Since $T(\Sigma_{ci})$ is the smallest space with these two properties we have $T(\Sigma_{ci}) \subseteq \ker P$. \hfill \blacksquare

Using the above we can derive the following result. Note that $G_{ci}$ is the transfer matrix defined by (A.13).
Theorem A.6: Let $P \in \mathbb{R}^{n \times n}$ be symmetric. The following two statements are equivalent:

(i) $P$ is a symmetric solution to the quadratic matrix inequality, i.e. we have $F(P) \geq 0$. Moreover, $\text{rank } F(P) = \text{rank}_{R(a)} G_{ci}$.

(ii) If $P$ is in the form (A.15) and if we define

$$R(P) := P_{1} A_{11} + A_{11}^{T} P_{1} + P_{1} \left( E_{1} E_{1}^{T} - B_{11} \left( \hat{D}^{T} \hat{D} \right)^{-1} B_{11}^{T} \right) P_{1} +$$

$$C_{21}^{T} C_{21} - (A_{13}^{T} P_{1} + C_{23}^{T} C_{21})^{T} (C_{23}^{T} C_{23})^{-1} (A_{13}^{T} P_{1} + C_{23}^{T} C_{21}),$$

(A.17)

then we have $R(P) = 0$.

Let $s_0 \in \mathbb{C}$ be given. If (i) holds (or equivalently (ii)), then the following statements are equivalent:

(iii) $P$ satisfies

$$\text{rank} \left( \begin{array}{c} L(P, s_0) \\ F(P) \end{array} \right) = n + \text{rank}_{R(a)} G_{ci}.$$  

(iv) The matrix $Z(P)$ defined by

$$Z(P) := A_{11} + E_{1} E_{1}^{T} P_{1} - B_{11} \left( \hat{D}^{T} \hat{D} \right)^{-1} B_{11}^{T} P_{1}$$

$$- A_{13} (C_{23}^{T} C_{23})^{-1} (A_{13}^{T} P_{1} + C_{23}^{T} C_{21}),$$

(A.18)

has no eigenvalue in $s_0$. \hfill \Box

Proof: By (A.16) we have $M(P) \geq 0$ if and only if $F(P) \geq 0$ and we also know that these matrices have the same rank. Assume that a symmetric $P$ satisfies $M(P) \geq 0$ and $\text{rank } M(P) = \text{rank}_{R(a)} G_{ci}$. By lemma A.5 we know that in our new basis we can write $P$ in the form (A.15). If we also use the decompositions (A.2) and (A.4) for the other matrices we find that $M(P)$ is equal to
A.2 The quadratic matrix inequality

\[
\begin{pmatrix}
P_{1}A_{11} + A_{11}^{T}P_{1} + C_{21}^{T}C_{21} + P_{1}E_{1}E_{1}^{T}P_{1} & 0 & P_{1}A_{13} + C_{21}^{T}C_{23} & P_{1}B_{11} & 0 \\
0 & 0 & 0 & 0 & 0 \\
A_{12}^{T}P_{1} + C_{22}^{T}C_{21} & 0 & C_{22}^{T}C_{23} & 0 & 0 \\
B_{11}^{T}P_{1} & 0 & 0 & D^{T}\hat{D} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

According to lemma A.4 the rank of this matrix equals the rank of the encircled matrix. Thus the Schur complement of this matrix must be equal to 0. Since this condition exactly yields the algebraic Riccati equation \( R(P_{1}) = 0 \) where \( R \) is defined by (A.17), we find that \( P_{1} \) is a solution of \( R(P_{1}) = 0 \). Conversely, if \( P_{1} \) is a solution of \( R(P_{1}) = 0 \), then the Schur complement of the encircled submatrix of the above matrix is 0. Therefore it satisfies the matrix inequality \( M(P) \geq 0 \) and the rank of the matrix is equal to the normal rank of \( G \). Hence \( P \) given by (A.15) satisfies the required properties. Now assume that (i) or (ii) holds. We will prove the equivalence of (iii) and (iv). Define \( Z(P_{1}) \) by (A.18). We will apply the following unimodular transformation to the matrix in (iii):

\[
\begin{pmatrix}
I & 0 & 0 \\
0 & I & F_{0}^{T} \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
L(P,s) \\
F(P)
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
F_{0}^{T} & I
\end{pmatrix}
\]

Using the decomposition (A.2) the latter matrix turns out to be equal to

\[
\begin{pmatrix}
sI - A_{11} - E_{1}E_{1}^{T}P_{1} & 0 & -A_{13} & -B_{11} & 0 \\
-A_{21} - E_{2}E_{2}^{T}P_{1} & sI - A_{22} & -A_{23} & -B_{21} & -B_{22} \\
-A_{31} - E_{3}E_{3}^{T}P_{1} & -A_{32} & sI - A_{33} & -B_{31} & -B_{32} \\
P_{1}A_{11} + A_{11}^{T}P_{1} + C_{21}^{T}C_{21} + P_{1}E_{1}E_{1}^{T}P_{1} & 0 & P_{1}A_{13} + C_{21}^{T}C_{23} & P_{1}B_{11} & 0 \\
0 & 0 & 0 & 0 & 0 \\
A_{12}^{T}P_{1} + C_{22}^{T}C_{21} & 0 & C_{22}^{T}C_{23} & 0 & 0 \\
B_{11}^{T}P_{1} & 0 & 0 & D^{T}\hat{D} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

By using Schur complements we can get the Riccati equation \( R(P_{1}) = 0 \) in the 4,1 position and the matrix \( Z(P_{1}) \) in the 1,1 position of the above matrix. Furthermore, since \( \hat{D}^{T}\hat{D} \) is invertible we can make the 2,4 and 3,4 blocks equal to zero by a unimodular transformation. Since \( P_{1} \) is a solution of \( R(P_{1}) = 0 \), the 4,1 block becomes 0. Thus we find that the above matrix is unimodularly equivalent to
where * denotes matrices which are unimportant for this argument.

Now, by lemma A.2 the encircled matrices together form the system matrix of a strongly controllable system. Hence this system matrix is unimodularly equivalent to a constant matrix \((I \ 0)\), where \(I\) denotes the identity matrix of appropriate size. Therefore we can make the 2,1 and 3,1 blocks zero by a unimodular transformation.

Thus after reordering we find,

\[
\begin{pmatrix}
  sI - Z(P_1) & 0 & 0 & 0 \\
  * & sI - A_{22} & -A_{23} & -B_{22} \\
  * & -A_{32} & sI - A_{33} & -B_{32} \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & C_{23}^T C_{23} & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix} \sim \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  sI - A_{22} & -A_{23} & -B_{22} & -B_{22} \\
  -A_{32} & sI - A_{33} & -B_{32} & -B_{32} \\
  0 & 0 & C_{23}^T C_{23} & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

It follows that the matrix on the left has rank \(n + \text{rank}_{\mathcal{R}(s)} G\) for some \(s_0 \in \mathcal{C}\) if and only if \(s_0\) is not an eigenvalue of \(Z(P_1)\). This proves that (iii) and (iv) are equivalent.

\[\text{Corollary A.7 : If there exists a matrix symmetric matrix } P \text{ such that } F(P) \geq 0 \text{ and moreover:}
\]

\[(i) \quad \text{rank } F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci},
\]

\[(ii) \quad \text{rank } \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci}, \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+,
\]
A.2 The quadratic matrix inequality

then this matrix is uniquely defined by the above inequality and the corresponding two rank conditions. \[ \square \]

Proof: By lemma A.6 a solution $P$ must be of the form (A.15) where $P_1$ is a solution of the algebraic Riccati equation $R(P_1) = 0$ such that $Z(P_1)$ is asymptotically stable. Denote the Hamiltonian matrix corresponding to this algebraic Riccati equation by $H$. Then we have:

$$H \begin{pmatrix} I \\ P_1 \end{pmatrix} = \begin{pmatrix} I \\ P_1 \end{pmatrix} Z(P_1).$$

Since a Hamiltonian matrix has the property that $\lambda$ is an eigenvalue if and only if $-\lambda$ is an eigenvalue of $H$ we know that an $n$ dimensional invariant subspace $\mathcal{W}$ of $H$ such that $H | \mathcal{W}$ is asymptotically stable, must be unique. This implies that $P_1$ is unique and hence also $P$ is unique. \[ \square \]

We also have the following corollary of theorem A.6. The result is strongly related to the result that $A_u$ is stable in lemma 3.13. Only we have a different and notationally more difficult Riccati equation.

Corollary A.8: If there exists a matrix $P \geq 0$ such that $F(P) \geq 0$ and moreover:

(i) $\text{rank } F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci},$

(ii) $\text{rank} \begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci}, \quad \forall s \in C^0 \cup C^+, $

then $P$ is in the form (A.15) and $P_1$ is such that the matrix

$$A_{11} - B_{11} \left( \hat{D}^T \hat{D} \right)^{-1} B_{11}^T P_1 - A_{13} (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{23}) \quad (A.19)$$

is asymptotically stable. \[ \square \]

Proof: By theorem A.6 we know $P_1$ is such that $R(P_1) = 0$ and $Z(P_1)$ is asymptotically stable where $R$ is defined by (A.17) and $Z$ is defined by (A.18). Denote the matrix in (A.19) by $\hat{A}$. It is easily checked that:
\[ P_1 \tilde{A} + \tilde{A}^T P_1 + P_1 N_1 P_1 + N_2 = 0, \]

where

\[
N_1 := E_1 E_1^T + B_{11} \left( \tilde{D}^T \tilde{D} \right)^{-1} B_{11}^T + A_{13} \left( C_{23} C_{23}^T \right)^{-1} A_{13}^T \geq 0,
\]

\[
N_2 := C_{21}^T \left( I - C_{23} \left( C_{23}^T C_{23} \right)^{-1} C_{23}^T \right) C_{21} \geq 0.
\]

Assume that \( \lambda \) is an eigenvalue of \( \tilde{A} \) with corresponding eigenvector \( x \neq 0 \). Then

\[
2 \text{Re} \lambda x^T P_1 x = -x^T (P_1 N_1 P_1 + N_2) x.
\]

Since \( P_1 \geq 0 \), \( N_1 \geq 0 \) and \( N_2 \geq 0 \) this implies that if \( \text{Re} \lambda \geq 0 \), then \( N_1 P_1 x = 0 \) which implies that \( E_1 E_1^T P_1 x = 0 \). Thus, if \( \text{Re} \lambda \geq 0 \) we find

\[
\lambda x = \tilde{A} x = (\tilde{A} + E_1 E_1^T P_1) x = Z(P_1) x.
\]

However, since \( Z(P_1) \) is asymptotically stable, this yields a contradiction. Thus we have established that \( \text{Re} \lambda < 0 \) which in turn yields that \( \tilde{A} \) is asymptotically stable.
Appendix B

Proofs concerning the system transformations

In this appendix we shall prove two key lemmas of chapter 5 for which the proofs are rather technical and are therefore deferred to this appendix. Throughout this appendix we shall assume that we have chosen the bases described in appendix A with $D$ replaced by $D_2$ and $C$ replaced by $C_2$. Thus we know the matrices have the special form as given in (A.2) and (A.4).

B.1 Proof of lemma 5.4

We first have to do some preparatory work.

Let the matrix $P$ satisfy the conditions of lemma 5.2 part (i). Hence we know that in our new bases $P$ has the form (A.15). It is easily shown that it is sufficient to prove lemma 5.4 for one specific choice of $C_2, P$ and $D_P$. We define the following matrices:

$$ C_{2,P} := \begin{pmatrix} \hat{D} \left( \hat{D}^T \hat{D} \right)^{-1} B_{11}^T P_1 + C_{11} & C_{12} & C_{13} \\ C_{23} \left( C_{23}^T C_{23} \right)^{-1} \left( A_{T3}^T P_1 + C_{23}^T C_{21} \right) & 0 & C_{23} \end{pmatrix}, \quad (B.1) $$

$$ D_P := \begin{pmatrix} \hat{D} & 0 \\ 0 & 0 \end{pmatrix} \quad (= D_2). \quad (B.2) $$

By writing down $F(P)$ in terms of the chosen bases and by using the fact that $P_1$ satisfies the algebraic Riccati equation $R(P_1) = 0$ where $R(P_1)$ is
defined by (A.17), it can be checked after some effort that these matrices indeed satisfy (5.3). We define the following matrices:

\[
\begin{align*}
\tilde{A} & := A_{11} - A_{13} (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{21}) \\
\tilde{B}_1 & := - (\tilde{D}^T)^{-1} B_{11}^T P_1, \\
\tilde{C}_1 & := C_{21} - C_{23} (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{21}), \\
\tilde{B}_{12} & := A_{13} (C_{23}^T C_{23})^{-1} C_{23}^T - P_1^T C_{21}^T (I - C_{23} (C_{23}^T C_{23})^{-1} C_{23}^T),
\end{align*}
\]

where \( \dagger \) denotes the Moore-Penrose inverse. We now define the following system:

\[
\begin{align*}
\dot{x}_u & = \tilde{A} x_u + (\tilde{B}_{11} \tilde{B}_{12}) u + E_1 w, \\
y_u & = -E_1^T P_1 x_u + w, \\
z_u & = (\tilde{C}_1 \tilde{C}_2) x_u + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} u.
\end{align*}
\]

We have the following properties of the system \( \Sigma_u \):

**Lemma B.1**: The system \( \Sigma_u \) is inner. Let \( U \) denote the transfer matrix of \( \Sigma_u \). If we decompose \( U \):

\[
U \begin{pmatrix} w_u \\ u_u \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w_u \\ u_u \end{pmatrix} = \begin{pmatrix} z_u \\ y_u \end{pmatrix},
\]

compatible with the sizes of \( w_u, u_u, y_u, \) and \( z_u \) then we have \( U_{21}^{-1} \in H_\infty \) and \( U_{22} \) is strictly proper.

**Proof**: By corollary A.8 we know that \( \tilde{A} \) is asymptotically stable and hence \( \Sigma_u \) is internally stable. Moreover by theorem A.6 the matrix \( Z(P_1) = \tilde{A} + E_1 E_1^T P_1 \) is asymptotically stable and therefore we have \( U_{21}^{-1} \in H_\infty \). The fact that \( U_{22} \) is strictly proper is trivial. It can be easily checked using lemma A.6 part (ii) that \( P_1 \) is the controllability gramian of \( \Sigma_u \). Moreover we have
B.1 Proof of lemma 5.4

\[
\begin{pmatrix}
0 & I & 0 \\
0 & 0 & I \\
I & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-E_1^T P_1 \\
\tilde{C}_1 \\
\tilde{C}_2
\end{pmatrix}
+ \begin{pmatrix}
\tilde{B}_1^T \\
\tilde{B}_2^T \\
E_1^T
\end{pmatrix} P_1 = 0.
\]

This can be checked by simply writing out and using the fact that

\[
\ker P_1 \subseteq \ker \left( I - C_{23} (C_{23}^T C_{23})^{-1} C_{23}^T \right) C_{21}.
\]

The result that \( \Sigma_U \) is inner then follows by applying lemma 2.10. ■

Proof of lemma 5.4 : We have our special choice of \( C_{2, P} \) and \( D_P \) given by (B.1) and (B.2). As already noticed taking this special choice for \( C_P \) and \( D_P \) is not essential. We shall first compare the following two systems:

The system on the left is the same as the system on the left in (5.5) and the system on the right is described by the system (B.3) interconnected with the system on the right in (5.5). We decompose the state of \( \Sigma \), \( x \) into \( x_1, x_2 \) and \( x_3 \) according to the choice of bases described in appendix A and decompose the state of \( \Sigma_P \) into \( x_{1, P}, x_{2, P}, x_{3, P} \) with respect to the same basis. (Note that \( \Sigma \) and \( \Sigma_P \) have the same state space \( \mathbb{R}^n \)). Writing out all the differential equations using the decompositions of the matrices given in (A.2),(A.4) and we find:

\[
\begin{pmatrix}
\dot{z}_U - \dot{x}_{1, P} \\
\dot{z}_P \\
\dot{\varphi}
\end{pmatrix}
= \begin{pmatrix}
\tilde{A} + E_1 E_1^T P_1 & 0 & 0 \\
* & A + BNC_1 & BM \\
* & LC_1 & K
\end{pmatrix}
\begin{pmatrix}
x_U - x_{1, P} \\
x_P \\
\varphi
\end{pmatrix}
+ \begin{pmatrix}
0 \\
E + BND_1 \\
LD_1
\end{pmatrix}
\begin{pmatrix}
w_U \\
0
\end{pmatrix}
\]

\[
z = \begin{pmatrix}
C_2 + D_2 NC_1 & D_2 M
\end{pmatrix}
\begin{pmatrix}
x_U - x_{1, P} \\
x_P \\
\varphi
\end{pmatrix}
+ D_2 ND_1 w_U
\]
The * denotes matrices which are unimportant for this argument. The system on the right is internally stable if and only if the system described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (B.4) we see immediately that, since \( \tilde{A} + E \Gamma \tilde{P} \) is asymptotically stable, the system on the left is internally stable if and only if the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input \( w \), then we have \( z = z_u \) i.e. the input-output behaviour of both systems are identical. Hence the system on the left has \( H_\infty \) norm less than 1 if and only if the system on the right has \( H_\infty \) norm less than 1.

By lemma B.1 we may apply lemma 2.12 to the system on the right in (B.4) and hence we find that the closed loop system is internally stable and has \( H_\infty \) norm less than 1 if and only if the dashed system is internally stable and has \( H_\infty \) norm less than 1.

Since the dashed system is exactly the system on the right in (5.5) and the system on the left in (B.4) is exactly equal to the system on the left in (5.5) we have completed the proof.

B.2 Proof of lemma 5.5

We are now going to prove lemma 5.5. In fact, we shall prove the dual version of this lemma since this is much more convenient for us. Otherwise we would have to introduce a decomposition dual to the one used in appendix A and based on the weakly unobservable subspace. We first factorize \( G(Q) \):

\[
G(Q) := \begin{pmatrix} E_Q^T & D_Q^T \\ E_Q & D_Q \end{pmatrix}.
\]

Define \( A_Q := A + QC_2^T C_2 \) and \( B_Q := B + QC_2^T D_2 \) and the system:

\[
\Sigma_Q := \begin{cases} 
\dot{x}_Q = A_Q x_Q + B_Q u_Q + E_Q w_Q, \\
y_Q = C_1 x_Q + D_Q w_Q, \\
z_Q = C_2 x_Q + D_2 u_Q.
\end{cases}
\]  

(B.5)

First of all, we know that \( \Sigma_F \) stabilizes \( \Sigma \) if and only if \( \Sigma_F^T \) stabilizes \( \Sigma^T \). Moreover, we know that \( \|G\|_\infty = \|G^T\|_\infty \). Hence we can derive the following dualized version of lemma 5.4 for this dual system \( \Sigma_Q \):
Lemma B.2 : Let $Q$ satisfy lemma 5.2 part (ii). Moreover let an arbitrary dynamic compensator $\Sigma_F$ be given, described by (2.4). Let the following two systems be given where the system on the left is the interconnection of (5.1) and (2.4) and the system on the right is the interconnection of (B.5) and (2.4).

![Diagram of systems](image)

Then the following statements are equivalent:

(i) The system on the left is internally stable and its transfer matrix has $H_\infty$ norm less than 1.

(ii) The system on the right is internally stable and its transfer matrix has $H_\infty$ norm less than 1.

We shall now investigate how the matrices appearing in the matrix inequality and the rank conditions look like for this new system $\Sigma_q$: 

\[
\tilde{F}(X) := \begin{pmatrix}
A^T_Q X + X A_Q + C^T_2 C_2 + X E_Q E^T_Q X & X B_Q + C^T_2 D_2 \\
B^T_Q X + D^T_2 C_2 & D^T_2 D_2
\end{pmatrix},
\]

\[
\tilde{G}(Y) := \begin{pmatrix}
A_Q Y + Y A^T_Q + E_Q E^T_Q + Y C^T_2 C_2 & Y C^T_1 + E_Q D^T_Q \\
C^T_1 Y + D_Q E^T_Q & D_Q D^T_Q
\end{pmatrix},
\]

\[
\tilde{L}(X, s) := \begin{pmatrix}
sI - A_Q - E_Q E^T_Q X & -B_Q
\end{pmatrix},
\]

\[
\tilde{M}(Y, s) := \begin{pmatrix}
sI - A_Q - Y C^T_2 C_2 & -C_1
\end{pmatrix}.
\]

Moreover, we define two new transfer matrices:

\[
\tilde{G}_{ci}(s) := C_2 (sI - A_Q)^{-1} B_Q + D_2,
\]

\[
\tilde{G}_{di}(s) := C_1 (sI - A_Q)^{-1} E_Q + D_Q.
\]
Proofs concerning the system transformations

Using these definitions we have the following result:

Lemma B.3: Let $Q$ satisfy lemma 5.2 part (ii). Then $Y = 0$ is the unique solution of the quadratic matrix inequality $\tilde{G}(Y) \succeq 0$ satisfying the following rank conditions

(i) $\text{rank } \tilde{G}(Y) = \text{rank}_{R(s)} \tilde{G}_{di},$

(ii) $\text{rank } \left( \tilde{M}(Y, s) \tilde{G}(Y) \right) = n + \text{rank}_{R(s)} \tilde{G}_{di}, \quad \forall s \in C^0 \cup C^+.$

Proof: It is trivial to check that $\tilde{G}(0) \succeq 0.$ Moreover, since $\tilde{G}(0) = G(Q)$ and $\tilde{M}(0, s) = M(Q, s)$ it remains to show that $\tilde{G}_{di}$ and $G_{di}$ have the same normal rank. We have

$$\text{rank}_{R(s)} \tilde{G}_{di} = \text{rank}_{R(s)} \begin{pmatrix} sI - A_Q & E_Q \\ -C_1 & D_Q \end{pmatrix} - n$$

$$= \text{rank}_{R(s)} \begin{pmatrix} sI - A_Q & E_Q E_T^Q & D_Q E_T^Q \\ -C_1 & D_Q E_T^Q & D_Q D_T^Q \end{pmatrix} - n$$

$$= \text{rank}_{R(s)} \begin{pmatrix} M(Q, s) & G(Q) \end{pmatrix} - n$$

$$= \text{rank}_{R(s)} G_{di}.$$

The matrix $Y$ is unique by the dualized version of corollary A.7. This is exactly what we had to prove. ■

Lemma B.4: There exists a solution $X$ of the matrix inequality $\tilde{F}(X) \succeq 0$ satisfying the following two rank conditions:

(i) $\text{rank } \tilde{F}(X) = \text{rank}_{R(s)} \tilde{G}_{ci},$

(ii) $\text{rank } \begin{pmatrix} \tilde{L}(X, s) \\ \tilde{F}(X) \end{pmatrix} = n + \text{rank}_{R(s)} \tilde{G}_{ci}, \quad \forall s \in C^0 \cup C^+,$
if and only if $I - PQ$ is invertible. Moreover in that case the solution is unique and is given by $X = (I - PQ)^{-1} P$. We have $X \succeq 0$ if and only if:

$$
\rho(PQ) < 1.
$$

\[ \square \]

**Proof:** We first make a transformation on $\tilde{F}(X)$:

$$
F_{tr}(X) := \begin{pmatrix}
I & (I + XQ) F_0^T \\
0 & I
\end{pmatrix} \tilde{F}(X) \begin{pmatrix}
I & 0 \\
F_0(I + QX) & I
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\tilde{A}^T X + X \tilde{A} + \tilde{C}_2^T \tilde{C}_2 + XM X & XB \\
B^T X & D_2^T D_2
\end{pmatrix}
$$

where,

$$
\tilde{A} := A + BF_0 + Q(C_2 + D_2 F_0)^T(C_2 + D_2 F_0),
$$

$$
\tilde{C}_2 := C_2 + D_2 F_0,
$$

$$
M := (A + BF_0)Q + Q(A^T + F_0^T B^T) + EE^T + Q\tilde{C}_2^T \tilde{C}_2 Q,
$$

and $F_0$ as defined in (A.1). We also transform the second matrix appearing in the rank conditions:

$$
W(X, s) := \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
-I & -QF_0^T \\
0 & I
\end{pmatrix} \begin{pmatrix}
\tilde{L}(X, s) \\
\tilde{F}(X)
\end{pmatrix} \begin{pmatrix}
I & 0 \\
F_0(I + QX) & I
\end{pmatrix}
$$

$$
= \begin{pmatrix}
sI - \tilde{A} - MX & -B \\
\tilde{A}^T X + X \tilde{A} + \tilde{C}_2^T \tilde{C}_2 + XM X & XB \\
B^T X & D_2^T D_2
\end{pmatrix}
$$

We have the following equality:

$$
\text{rank}_{\mathcal{R}(s)} \tilde{G}_{ci} = \text{rank}_{\mathcal{R}(s)} \begin{pmatrix}
sI - A_Q & -B_Q \\
C_2 & D_2
\end{pmatrix} - n
$$

$$
= \text{rank}_{\mathcal{R}(s)} \begin{pmatrix}
I & QC_2^T \\
0 & I
\end{pmatrix} \begin{pmatrix}
sI - A_Q & -B_Q \\
C_2 & D_2
\end{pmatrix} - n
$$

$$
= \text{rank}_{\mathcal{R}(s)} \begin{pmatrix}
sI - A & -B \\
C_2 & D_2
\end{pmatrix} - n = \text{rank}_{\mathcal{R}(s)} G_{ci}.
$$
Therefore the conditions that $X \geq 0$ has to satisfy can be reformulated as:

(i) $F_{tr}(X) \geq 0,$

(ii) $\text{rank } F_{tr}(X) = \text{rank}_{\mathcal{R}(s)} G_{ci},$

(iii) $\text{rank } W(X, s) = \text{rank}_{\mathcal{R}(s)} G_{ci} + n, \quad \forall s \in C^0 \cup C^+.$

Moreover, we note that $T(A, B, C_2, D_2) = T(\bar{A}, B, C_2, D_2).$ This can be shown by using that the new system is obtained by a state feedback and an output injection (note that $B = B + Q(C_2 + D_2 F_0)^T D_2$) and the well known fact that the strongly controllable subspace is invariant under feedback and output injection. This can be easily shown using the algorithm (2.15). We now choose the bases from appendix A where again $C$ is replaced by $C_2$ and $D$ is replaced by $D_2.$ By lemma A.5 we know that if $X$ exists, then it will have the form:

$$X = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (B.6)$$

for some positive semi-definite matrix $X_1.$ Note that there is small difference since $M$ is not necessarily positive semi-definite but it can be easily seen from the proof of lemma A.5 that this difference is not important. We use this decomposition for $X$ and the corresponding decompositions for $P$ and $Q:

$$P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}.$$ 

Together with the decompositions for the other matrices as given in (A.2) and (A.4) we can decompose $F_{tr}(X)$ correspondingly:

$$\begin{pmatrix} X_1 \bar{A}_{11} + \bar{A}_{11} X_1 + C_{T_1}^T C_{21} + X_1 M_{11} X_1 & 0 & X_1 \bar{A}_{13} + C_{T_1}^T C_{23} & X_1 B_{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
B.2 Proof of lemma 5.5

where

\[
\begin{align*}
\tilde{A}_{11} & := A_{11} + Q_{11}C^T_{21}C_{21} + Q_{13}C^T_{23}C_{21}, \\
\tilde{A}_{13} & := A_{13} + Q_{11}C^T_{21}C_{23} + Q_{13}C^T_{23}C_{23}, \\
M_{11} & := A_{11}Q_{11} + A_{13}Q^T_{13} + Q_{11}A^T_{11} + Q_{13}A^T_{13} + E^T_1E^T_1 \\
& + Q_{11}C^T_{21}(C_{21}Q_{11} + C_{23}Q_{13}) + Q_{13}C^T_{23}(C_{21}Q_{11} + C_{23}Q_{13}).
\end{align*}
\]

The rank condition: \(\text{rank } F_{r}(X) = \text{rank } R_{(s)}G_{ci}\) is according to lemma A.4 equivalent with the condition that the rank of the above matrix is equal to the rank of the submatrix:

\[
\begin{pmatrix}
C^T_{23}C_{23} & 0 \\
0 & D^T\hat{D}
\end{pmatrix}.
\]

Therefore the Schur complement with respect to this submatrix should be zero. This implies that if we define:

\[
\tilde{R}(X_1) := X_1\tilde{A}_{11} + \tilde{A}^T_{11}X_1 + C^T_{21}C_{21}
\]

\[
+X_1\left(M_{11} - B_{11}(D^T\hat{D})^{-1}B^T_{11}\right)X_1
\]

\[
-\left(X_1\tilde{A}_{13} + C^T_{21}C_{23}\right)(C^T_{23}C_{23})^{-1}\left(\tilde{A}^T_{13}X_1 + C^T_{23}C_{21}\right)
\]

then \(X_1\) should satisfy \(\tilde{R}(X_1) = 0\). Moreover, if we decompose \(W(X,s)\) correspondingly, then we can show by using elementary row and column operations that for any matrix \(X\) in the form (B.6), where \(X_1\) satisfies \(\tilde{R}(X_1) = 0\), and for all \(s \in \mathcal{C}\), the matrix \(W(X,s)\) has the same rank as the following matrix:

\[
\begin{pmatrix}
 sI - \tilde{Z}(X_1) & 0 & 0 & 0 & 0 \\
* & sI - A_{22} & -A_{23} & 0 & -B_{22} \\
* & -A_{32} & sI - A_{33} & 0 & -B_{32} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{B.7}
\]

where

\[
\tilde{Z}(X_1) := \tilde{A}_{11} + M_{11}X_1 - B_{11}(D^T\hat{D})^{-1}B^T_{11}X_1
\]

\[
-\tilde{A}_{13}(C^T_{23}C_{23})^{-1}\left(\tilde{A}^T_{13}X_1 + C^T_{23}C_{21}\right).
\]
The matrix:
\[
\begin{pmatrix}
    sI - A_{22} & -A_{23} & -B_{22} \\
    -A_{32} & sI - A_{33} & -B_{32} \\
    0 & I & 0
\end{pmatrix},
\]
has full row rank for all \( s \in \mathbb{C} \) by lemma A.2 part (ii) and lemma 2.8. Hence
the rank of the matrix (B.7) is \( n + \text{rank}_{\mathbb{C}^+ \cup \mathbb{C}^0} G_{ci} \) if and
only if the matrix \( \tilde{Z}(X_1) \) is asymptotically stable. Using this we can now
reformulate the conditions that \( X_1 \geq 0 \) has to satisfy:

(i) \( \tilde{R}(X_1) = 0 \),

(ii) \( \tilde{Z}(X_1) \) is asymptotically stable.

In other words \( X_1 \) should be the positive semi-definite stabilizing solution
of the algebraic Riccati equation \( \tilde{R}(X_1) = 0 \). Denote the Hamiltonian corresponding to this ARE by \( H_{\text{new}} \). We know that \( P_1 \) is the stabilizing solution
of the algebraic Riccati equation \( R(P_1) = 0 \) as given by (A.17). Denote the Hamiltonian corresponding to this algebraic Riccati equation by \( H_{\text{old}} \). Then
it can be checked that:
\[
H_{\text{old}} = \begin{pmatrix} I & Q_{11} \\ 0 & I \end{pmatrix} H_{\text{new}} \begin{pmatrix} I & -Q_{11} \\ 0 & I \end{pmatrix}.
\] (B.8)
Since \( P_1 \) is the stabilizing solution of the Riccati equation corresponding to the Hamiltonian \( H_{\text{old}} \) we know that the modal subspace of \( H_{\text{old}} \) corresponding to the open left half plane is given by:
\[
\mathcal{X}_g(H_{\text{old}}) = \text{Im} \begin{pmatrix} I \\ P_1 \end{pmatrix}.
\] (B.9)
Combining (B.8) and (B.9) we find:
\[
\mathcal{X}_g(H_{\text{new}}) = \text{Im} \begin{pmatrix} I & -Q_{11} \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ P_1 \end{pmatrix} = \text{Im} \begin{pmatrix} I - Q_{11}P_1 \\ P_1 \end{pmatrix}.
\]
Therefore we know that there exists a stabilizing solution to the algebraic Riccati equation \( \tilde{R}(X_1) = 0 \) if and only if \( I - Q_{11}P_1 \) is invertible and in
that case the solution is given by \( X_1 = P_1 (I - Q_{11}P_1)^{-1} \). This implies that \( X = P (I - QP)^{-1} = (I - PQ)^{-1} P \). The requirement \( X \geq 0 \) is satisfied if and only if \( \rho(PQ) < 1 \), which can be checked straightforwardly. This completes the proof.
Samenvatting

Het $H_\infty$ regelprobleem: een toestandsruimte aanpak.

In dit proefschrift wordt het standaard $H_\infty$ probleem behandeld, waarbij voor een zogenaamde toestandsaanpak werd gekozen. Bij het $H_\infty$ probleem probeert men het effect van verstoringen op de uitgang te minimaliseren door een geschikte terugkoppeling van de metingen op het systeem naar de regeling terwijl men bovendien eist dat gesloten-lus systeem intern stabiel is. Het gesloten-lus effect van verstoringen op de uitgang wordt tot uitdrukking gebracht in de $H_\infty$ norm. De $H_\infty$ norm is gelijk aan de $L_2$-geïnduceerde operator norm. Wij behandelen dit probleem zowel voor systemen in continue tijd als ook voor systemen in discrete tijd.

- **Systemen in continue tijd.** Voor systemen in continue tijd was er al een baanbrekend artikel [Do4] beschikbaar. Nodige en voldoende voorwaarden waaronder wij regelaars kunnen vinden die de $H_\infty$ norm kleiner dan een vooraf gegeven waarde $\gamma$ maken, worden gegeven in termen van twee algebraïsche Riccati vergelijkingen. In dit proefschrift wordt aangetoond hoe een van de twee essentiële aannames in dit artikel verwijderd kunnen worden. Mogelijkerwijze bestaan de algebraïsche Riccati vergelijkingen dan niet meer en vandaar dat onze condities in termen van twee kwadratische matrix ongelijkheden en vier rang condities worden geformuleerd.

- **Systemen in discrete tijd.** Voor systemen in discrete tijd was nog maar zeer weinig bekend. In dit proefschrift worden aannames gemaakt die precies de discrete tijd analogon zijn van de aannames in [Do4]. Nodige en voldoende condities waaronder er een terugkoppeling bestaat die de $H_\infty$ norm onder een vooraf gegeven grens brengt worden, net als in de continue tijd, gegeven in termen van twee algebraïsche Riccati vergelijkingen.
Naast deze hoofdproblemen wordt in dit proefschrift nog geprobeerd het belang aan te geven van de geschetste problemen. Daarnaast wordt geprobeerd om de lezer wat extra intuïtie te geven.
Bibliography


A.A. Stoorvogel, "The singular $H_{\infty}$ control problem with dynamic measurement feedback", To appear in *SIAM J. Contr. & Opt.*.


A.A. Stoorvogel, "The discrete time $H_{\infty}$ control problem with measurement feedback", To appear in *SIAM J. Contr. & Opt.*.

A.A. Stoorvogel, H.L. Trentelman, "The finite horizon singular time-varying $H_{\infty}$ control problem with dynamic measurement feedback", Submitted to *J. on Operator theory and Integral equations*.


### Index

<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sim)</td>
<td>20</td>
</tr>
<tr>
<td>(|M|)</td>
<td>29</td>
</tr>
<tr>
<td>(|M|_1)</td>
<td>39</td>
</tr>
<tr>
<td>(|M|_2)</td>
<td>29, 34</td>
</tr>
<tr>
<td>(|M|_\infty)</td>
<td>7, 28, 33</td>
</tr>
<tr>
<td>((A, B))-invariant</td>
<td>22</td>
</tr>
<tr>
<td>ADDPS</td>
<td>37</td>
</tr>
<tr>
<td>ADDPMS</td>
<td>37</td>
</tr>
<tr>
<td>(A_{F_0})</td>
<td>191</td>
</tr>
<tr>
<td>additive perturbations</td>
<td>175</td>
</tr>
<tr>
<td>admissible pair</td>
<td>117</td>
</tr>
<tr>
<td>asymptotically stable</td>
<td>17, 18, 19</td>
</tr>
<tr>
<td>backwards difference equation</td>
<td>19</td>
</tr>
<tr>
<td>(C)</td>
<td>15</td>
</tr>
<tr>
<td>(C^+ (C^0, C^-))</td>
<td>15</td>
</tr>
<tr>
<td>(C_0^\infty)</td>
<td>22</td>
</tr>
<tr>
<td>((C, A))-invariant</td>
<td>22</td>
</tr>
<tr>
<td>(C_{F_0})</td>
<td>191</td>
</tr>
<tr>
<td>causal</td>
<td>135</td>
</tr>
<tr>
<td>strictly</td>
<td>135</td>
</tr>
<tr>
<td>central controller</td>
<td>12</td>
</tr>
<tr>
<td>complex stability radius</td>
<td>182</td>
</tr>
<tr>
<td>conditioned invariant</td>
<td>22</td>
</tr>
<tr>
<td>control sensitivity function</td>
<td>9</td>
</tr>
<tr>
<td>controllability pencil</td>
<td>21</td>
</tr>
<tr>
<td>coprime factor perturbations</td>
<td>7</td>
</tr>
<tr>
<td>normalized</td>
<td>7</td>
</tr>
<tr>
<td>(D (\delta D, D^+))</td>
<td>15</td>
</tr>
<tr>
<td>DDPMS</td>
<td>37</td>
</tr>
<tr>
<td>DDPS</td>
<td>37</td>
</tr>
<tr>
<td>detectable strongly</td>
<td></td>
</tr>
<tr>
<td>controllable subspace</td>
<td>23</td>
</tr>
<tr>
<td>deterministic error</td>
<td>2</td>
</tr>
<tr>
<td>differential game</td>
<td>13</td>
</tr>
<tr>
<td>direct feedthrough matrix</td>
<td>18, 20</td>
</tr>
<tr>
<td>discrete time systems</td>
<td>14</td>
</tr>
<tr>
<td>distance of systems</td>
<td>6</td>
</tr>
<tr>
<td>disturbance decoupling</td>
<td>37, 167</td>
</tr>
<tr>
<td>almost</td>
<td>12, 37</td>
</tr>
<tr>
<td>equilibrium</td>
<td>18</td>
</tr>
<tr>
<td>almost</td>
<td>118</td>
</tr>
<tr>
<td>frequency spectrum</td>
<td>9</td>
</tr>
<tr>
<td>gap metric</td>
<td>7</td>
</tr>
<tr>
<td>gap topology</td>
<td>7</td>
</tr>
<tr>
<td>Gaussian process</td>
<td>2</td>
</tr>
<tr>
<td>graph topology</td>
<td>7</td>
</tr>
<tr>
<td>(H_1^1)</td>
<td>27, 33</td>
</tr>
<tr>
<td>(H_{\infty})</td>
<td>28, 33</td>
</tr>
<tr>
<td>(H_{\infty}) norm</td>
<td>3, 9</td>
</tr>
<tr>
<td>ill-posed</td>
<td>17</td>
</tr>
<tr>
<td>inner</td>
<td>29, 34</td>
</tr>
<tr>
<td>input matrix</td>
<td>18</td>
</tr>
<tr>
<td>input-output operator</td>
<td>17</td>
</tr>
<tr>
<td>input/output stability</td>
<td>186</td>
</tr>
<tr>
<td>internally stable</td>
<td>17, 18, 19</td>
</tr>
<tr>
<td>invariant factors</td>
<td>21</td>
</tr>
<tr>
<td>invariant zeros</td>
<td>11, 21</td>
</tr>
<tr>
<td>invertible</td>
<td></td>
</tr>
<tr>
<td>left, right</td>
<td>22</td>
</tr>
<tr>
<td>(\ell_2^\infty)</td>
<td>34</td>
</tr>
<tr>
<td>(\ell_2)-induced operator norm</td>
<td>34</td>
</tr>
</tbody>
</table>
Index

$\mathcal{L}_2^n$ ................................................. 28
$\mathcal{L}_2$-induced operator norm ................................ 29
$\mathcal{L}_\infty$ ................................................. 28
LQ .......................................................... 187
LQG design method ........................................... 2,187
Laplace transform ............................................. 9
linear matrix inequality .................................. 12
linear quadratic
  optimal control .......................................... 12
loop transfer recovery ................................... 13
Lyapunov equation ........................................... 178
$\mu$-synthesis ............................................. 6
matrix pencil ................................................. 20
maximizing player ........................................... 117
measurement error ........................................... 2
minimum-phase .............................................. 42
minimizing player .......................................... 117
mixed-sensitivity function ................................ 8
modal subspace ............................................ 24,153,154
model-matching problem ................................... 184
multiplicative perturbations ............................. 178
$\mathcal{N}$ ......................................................... 15
Nash equilibrium ........................................... 13,118
  almost ......................................................... 13
normal rank .................................................. 20,21
Nyquist contour ............................................. 9,30
observability gramian ...................................... 30
observability pencil ....................................... 21
one block $H_\infty$ problem ................................. 89,184
output matrix ................................................ 18
parameter uncertainty ..................................... 2,4
proper ......................................................... 20
  strictly ...................................................... 20
quadratic matrix inequality ............................ 12,69
$\mathcal{R}$ ......................................................... 15
$\mathcal{R}[s]$ ..................................................... 20
$\mathcal{R}(s)$ ..................................................... 20
$\rho(M)$ ......................................................... 94
random process ............................................. 2
rankx .......................................................... 20
real stability radius ....................................... 6,182
regular systems ............................................ 11
Riccati equation ........................................... 12
Riemann's mapping theorem ................................ 28
robustness ...................................................... 2
$\sigma$ ....................................................... 19
$\sigma(M)$ ..................................................... 29
$\sigma_1(M)$ ................................................... 28
sensitivity function ....................................... 9
singular systems .......................................... 11
Smith form ..................................................... 21
stability ......................................................... 2
stabilizable weakly
  unobservable subspace ................................... 24
state matrix .................................................. 18
strongly controllable ...................................... 23
strongly controllable subspace ................................ 23
strongly observable ....................................... 24
system matrix ................................................. 21
$\mathcal{T}(\Sigma_{ci})$ .......................................... 23
$\mathcal{T}_g(\Sigma_{ci})$ ....................................... 23
time-invariant ............................................. 16
tracking problem .......................................... 8
transmission polynomial ................................... 21
  non-trivial .................................................. 21
two block $H_\infty$ problem .................................. 89
$U_{f_b}^k$ .................................................. 117
$U_{g_t}^k(\xi)$ .................................................. 117
uncertain systems ........................................... 4
$V(\Sigma_{ci})$ ................................................. 24
$V_g(\Sigma_{ci})$ ................................................. 24
weakly unobservable subspace ................................ 24
weights ......................................................... 9
well posed ..................................................... 17
$z_{u,w}$, $z_{u,w}$ .............................................. 16
$z_{u,w,\xi}$, $z_{u,w,\xi}$ ........................................... 16
zeros ......................................................... 21
zero polynomial ............................................ 21
Curriculum Vitae

24 juni 1964 Geboren te Hengelo (O).
1976-1982 Atheneum B, Hertog Jan College, Valkenswaard;
Eindexamen juni 1982.
1982-1987 Rijksuniversiteit Leiden;
   1983 Propadeuse Natuurkunde,
   1983 Propadeuse Wiskunde,
   1987 Doctoraalexamen Wiskunde.
   Afstudeerrichting Functionaal Analyse, bijvak Informatica.
1986-1987 6 maanden stage in het kader van afstudeerwerk
   bij Philips’ Nat.Lab. te Eindhoven.
1987-1989 Postdoctorale opleiding in het kader van het
   Netwerk Systeem- en Regeltheorie.
1987-1990 Assistent in opleiding aan de faculteit Wiskunde en
   Informatica van de Technische Universiteit Eindhoven.
1991 Onderzoeker aan het Department of Electrical Engineering,
   University of Michigan, Ann Arbor, U.S.A. ondersteund
   door een beurs van de Nederlandse organisatie voor
   wetenschappelijk onderzoek, NWO.
PROPOSITIONS
accompanying the dissertation
The $H_\infty$ control problem: a state space approach
by
A.A. Stoorvogel

Eindhoven, The Netherlands, July 1990
1. A Szegő limit theorem for exponentially weighted Toeplitz matrices.
Let $Q$ be a real positive-definite operator from $\ell_2$ to $\ell_2$ represented by the following infinite-dimensional Toeplitz matrix:

$$
Q := \begin{pmatrix}
q_0 & q_1 & q_2 & \cdots \\
q_1 & q_0 & q_1 & \cdots \\
q_2 & q_1 & q_0 & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}
$$

We associate to this Toeplitz matrix the function $f \in L_2([0, 1])$:

$$
f(x) := q_0 + \sum_{k=1}^{\infty} q_k \cos(2\pi k x).
$$

Moreover, let $D(\beta)$ be the diagonal matrix defined by $D(\beta) := \text{diag}(1, \beta, \beta^2, \ldots)$. We define $V(\beta) := D(\beta)^{1/2} Q D(\beta)^{1/2}$ and let $\sigma_1(\beta) \geq \sigma_2(\beta) \geq \cdots$ be the eigenvalues of $V(\beta)$. Finally, let $g \in C([0, p])$, where $p$ denotes the spectral radius of $Q$, be such that $\lim_{\lambda \to 0} \frac{\sigma(\lambda)}{\lambda}$ exists. Then we have

$$
(1 - \beta) \sum_{k=1}^{\infty} g(\sigma_k(\beta)) \to \int_0^1 \int_{-1}^{1(0)} \frac{g(\sigma)}{\sigma} d\sigma d\theta
$$
as $\beta \uparrow 1$.

References


2. The finite-horizon $H_\infty$ control problem
Consider the following linear time-invariant system:

$$
\Sigma : \begin{cases}
\dot{z} = Ax + Bu + Ew, \\
y = C_1 z + D_1 w, \\
z = C_2 x + D_2 u,
\end{cases}
$$

(1)

For differentiable matrix functions $P, Q : [0, T] \to \mathcal{R}^{n \times n}$ we define

$$
F(P) := \begin{pmatrix}
P + A^T P + PA + C_2^T C_1 + PEE^T P & PB + C_2^T D_2 \\
B^T P + D_2^T C_2 & D_2^T D_2
\end{pmatrix},
$$

and
\[
G(Q) := \begin{pmatrix}
-\dot{Q} + AQ + QA^T + EE^T + QC_2^T C_2 Q & QC_1^T + ED_1^T \\
C_1 Q + D_1 E^T & D_1 D_1^T
\end{pmatrix}.
\]

We also define two transfer matrices:

\[
G(s) := C_2(sI - A)^{-1} B + D_2
\]
\[
H(s) := C_1(sI - A)^{-1} E + D_1
\]

With the above definitions the following statements are equivalent:

(i) There exists a linear time-varying finite-dimensional dynamic compensator \( \tilde{F} \) from \( y \) to \( u \) such that the closed loop system \( \Sigma \times \Sigma_F \) has \( \mathcal{L}_2[0, T] \)-induced operator norm strictly less than 1

(ii) There exist differentiable matrix functions \( P, Q : [0, T] \to \mathbb{R}^{n \times n} \) such that

\begin{enumerate}
\item \( F(P)(t) \geq 0 \) for all \( t \in [0, T] \) and \( P(T) = 0 \),
\item \( \text{rank} \ F(P)(t) = \text{rank}_{\mathbb{R}(t)} G_{cl} \) for all \( t \in [0, T] \),
\item \( G(Q)(t) \geq 0 \) for all \( t \in [0, T] \) and \( Q(0) = 0 \),
\item \( \text{rank} \ G(Q)(t) = \text{rank}_{\mathbb{R}(t)} G_{di} \) for all \( t \in [0, T] \),
\item \( I - Q(t)P(t) \) is invertible for all \( t \in [0, T] \).
\end{enumerate}

References


3. The minimum entropy \( H_\infty \) control problem

Assume that the system (1) of proposition 2 is given. We define the following entropy function:

\[
\mathcal{J}(G) := \lim_{s_0 \to \infty} -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln |\det (I - G^*(j\omega)G(j\omega))| \left( \frac{s_0^2}{s_0^2 + \omega^2} \right) d\omega
\]

for any strictly proper rational matrix \( G \in \mathcal{L}_\infty \) such that \( \|G\|_\infty < 1 \).

The minimum entropy \( H_\infty \) control problem is then defined as:

\[
\inf_{\mathcal{G}_{cl}} \mathcal{J}(G_{cl})
\]

over all controllers which yield a strictly proper, internally stable closed loop transfer matrix \( G_{cl} \) with \( H_\infty \) norm strictly less than 1.

If \( (A, B, C_2, D_2) \) and \( (A, E, C_1, D_1) \) have no invariant zeros on the imaginary axis and if there exists a controller which is such that the closed loop system is internally stable and has \( H_\infty \) norm strictly less than 1 then the above defined infimum is equal to
\[ \text{Trace } (E^tPE + (A^tP + PA + C_1^tC_2 + PEE^tP)(I - QP)^{-1}Q) \]

where \(P\) and \(Q\) are such that part (ii) of theorem 5.1 in [2] is satisfied. Moreover, if we define the system \(\Sigma_{p,q}\) by (5.8) in [2] then the infimum is attained if and only if the disturbance decoupling problem with internal stability (DDPS) is solvable for \(\Sigma_{p,q}\) and any controller solving DDPS is a controller minimizing the entropy function. If DDPS is not solvable then a sequence of controllers \(\Sigma_{f,n}\) is a minimizing sequence if and only if \(\Sigma_{p,q} \times \Sigma_{f,n}\) is internally stable for all \(n\) and the \(H_\infty\) norm of the associated transfer matrices goes to zero as \(n \to \infty\).

References


4. A mathematician who is receiving a Ph.D. degree in control theory should at least be able to communicate in this area with his colleagues in electrical engineering.

5. To enable long-term planning in a democracy we need people with power who do not have to run for re-election every so many years. This is however in contradiction with the essence of democracy.

6. The fact that the EEC is using the year 1992 as an easy round off for December 31, 1992, which is the official completion date of the internal market, is characteristic for the everlasting optimism and idealism of the EEC but shows a lack of realism.

7. A new oil crisis is the only method to obtain radical changes towards the use of more environment-friendly sources of energy. Therefore, it might well be a blessing instead of a crisis.

8. Social security is a very good thing, but too good it becomes a poison for the welfare state.

9. It is in the interest of every citizen of the European Community if Germany becomes big and strong, but only under the proviso that all borders within the EEC will disappear.

10. Neither the expression "all is well that ends well" nor the expression "there is an end to everything" is suitable for this dissertation.