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Frequency-Domain Analysis of Networked Control Systems Modeled by Markov Jump Linear Systems

Duarte J. Antunes and Haiming Qu

Abstract—This article addresses networked control systems that can be modeled by Markov jump linear systems, including linear control systems with correlated random delays and correlated random packet drops. The analysis focuses on the statistical moments (mean and variance) of the modeled system. It is first established that the map between the deterministic input of the system, and the statistical moments of the state and output can be described by a time-invariant system. This fact is then used to provide a frequency-domain analysis framework that allows for computing the mean and bounding the variance of the time response of the system to any deterministic input, based on a graphical method. For the special case of networked control systems with i.i.d. delays and drops, it is formally established that the variance bounds are tighter than the ones provided in the previous work. The results are applied to the remote control of a linear process. The delays are obtained experimentally in a setting where two remote processors communicate wirelessly using two XBee modules. Based on these experimental data, it is concluded that the delays associated with two consecutive transmissions are correlated. The provided tools are then used to analyze the input to output behavior of the system.

Index Terms—Computer networks, frequency response, linear feedback control systems, Markov processes, stochastic systems, ZigBee.

I. INTRODUCTION

NETWORKED control systems (NCSs) are control systems for which at least some sensors, actuators, and controllers are spatially distributed and exchange information via a communication network. Examples include cooperative robotics, Internet of Things, and vehicle platooning. It is expected that in the near future cloud-based computing and 5G networks will enable many more applications. However, controlling real-time systems over communication networks leads to time-varying effects in the loop, such as computational and communication delays and data drops. These undesired effects can disturb the control loop, even disrupt it. Taking into account these effects in the controller design requires a significantly more elaborate analysis than that for traditional sampled-data systems.

Much research has been devoted to NCSs over the last two decades. Here, only a few important papers are surveyed. In [1], the stability of NCSs is analyzed with respect to the sampling rate and network delays. In [2], a model predictive control approach is used to compensate for the forward delay (delay from the controller to actuator), combined with a model predictor to compensate for the backward delay (delay from sensor to controller). NCSs with delays can also be tackled through robust control methods capturing uncertainty [3], [4]. State estimation for plants with multiple sensors transmitting through independent channels is considered in [5], where conditions are given for the estimation error process to be almost surely stable. Time-varying Kalman filters (TVKF) for linear systems with packet drops have been studied in [6], where it was shown that a critical value for the arrival rate of the observations is required to guarantee a bounded state error covariance. In [7], it is suggested to use precomputed gain matrices for the TVKF to reduce the computational cost and the variance of the estimation error is shown to be comparable to that of the optimal TVKF, when the set of precomputed gain matrices is sufficiently large.

One of the most used modeling techniques is to capture the network communication effects, such as delays and packet drops, with a stochastic model, namely, a Markov chain. Typically, the resulting system, capturing the dynamic equations of plant and controller and the network communication effects, is a Markov jump linear system (MJLS) [8]. For instance, Lincoln and Cervin [9] propose a toolbox Jitterbug that allows to model real-time systems and NCSs as MJLSs. A packet-loss dependent Lyapunov function is introduced in [10] to establish stability conditions for NCSs with Markovian packet losses. Markovian models with an upper bound on the number of consecutive packet losses in NCSs are considered in [11], where sufficient conditions for stochastic stability are given. In [12], an $\mathcal{H}_\infty$ control problem is tackled for NCSs considering Markov packet losses and in [13], an $\mathcal{H}_\infty$ output feedback controller for NCSs with an upper bound on the number of consecutive packet losses is proposed. NCSs with random delays are also often modeled by Markov jump models [14]–[16]. In [17], a unified Markov

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chain model is used for a class of NCSs, which takes both random delays and data losses into consideration. More expressive models considering hidden modes [18] or even nonlinearities [19] have the potential to model more general classes of NCSs.

Despite the extensive work on NCSs, frequency-domain analysis tools, which are widely used in industry to analyze traditional linear time-invariant (LTI) control systems, are almost absent in the context of NCSs. This follows from the fact that delays and packet drops add time-varying features to the closed-loop model, whereas traditional frequency-domain analysis only applies to LTI systems. The motivation to consider frequency-domain analysis tools in the LTI setting is clear: i) a frequency response plot fully characterizes the input to output behavior of an LTI system, and can be used by a practitioner/engineer to infer the system’s response based on the frequency content of the input; ii) the open-loop frequency response can be used to infer closed-loop properties (e.g., stability) and allows a practitioner/engineer to design controllers that meet closed-loop specifications. Despite the limited research aiming at providing similar tools in the NCS setting, one can still find in the literature, two frequency-domain approaches that partially address i); see [20] for initial steps toward addressing ii). First, Lincoln and Cervin [9] propose to plot the spectral density of the output of an NCS when white noise is applied as input. A different approach is followed in [21], which proposes to compute the expected value and the variance of the output response of an NCS with stochastic packet losses when a deterministic signal is applied as input. A larger class of NCSs, addressing, for instance, delays, is considered in [22]. However, Antunes et al. [21] and [22] only model NCS captured by linear systems with independent and identically distributed (i.i.d.) parameters, such as packet loss processes following a Bernoulli distribution and i.i.d. delays.

The present article follows the approach in [21] and [22] but considers a larger class of systems. In fact, it considers NCSs with phenomena such as random delays and data losses modeled by finite-dimensional Markov chains, which can be captured by MJLs. This model can be applied to a broad range of NCSs and, in particular, take correlations between delays and correlations between delays in the network into account. As in [21] and [22], the approach in the present article mainly relies on the calculation of statistical moments (expected value and variance) of the state and the output of the original system. Here, it is shown that also for MJLs these can be computed by considering time-invariant systems, despite the fact that the original system is time-varying. As in [21] and [22], building upon this fact, the amplitudes of the mean and variance of the output response to sinusoidal input signals can be plotted as a function of the input frequency, similarly to the classical frequency response (Bode) plot. This plot allows for inferring the behavior of the output response, characterized now by its mean and variance, to an arbitrary deterministic input; the output mean can be exactly computed. While, as in [21] and [22], the output variance can be upper bounded by a graphical method, a different bound is provided here. This bound can be also used for the special case of NCSs with i.i.d. parameters considered in [21] and [22], since a Markov Chain can also model i.i.d. sequences of random variables. For such a special case, it is formally shown that the new bounds are tighter than the ones provided in [21] and [22].

The results are applied to the remote control of a linear process. The delays are obtained experimentally in a setting where two remote processors communicate wireless using two XBee modules. By analyzing the communication delays, it is concluded that the delays associated with two consecutive transmissions are correlated. Then, the provided tools are used to analyze the input to output behavior of the system taking into account the correlation in the delays.

The remainder of the article is organized as follows. Section II introduces the MJLS model and illustrates how NCSs with correlated packet drops and delays can be captured by this model. Section III provides the main analytical results, which extend the frequency-domain results of [21] and [22] to MJLs and shows that the provided variance bounds are tighter than the ones in [21] and [22]. Section IV provides two examples. The first considers correlated packet drops, following a Gilbert-Elliot model, and it is based on simulations. The second considers delays obtained experimentally. Finally, Section V provides the concluding remarks.

Notation: Given \( w \in \mathbb{C}, \mathbb{R}\{w\}, \mathbb{S}\{w\}, \) and \( \text{arg}(w), \) denote the real part, the imaginary part, and the argument, respectively. The z-transform of a signal \( r_t \in \mathbb{R}, t \in \mathbb{Z}, \) is denoted by \( \hat{r}(z) = \sum_{t=-\infty}^{\infty} r_t z^{-t}. \) A vector with \( m \) entries equal to one is denoted by \( 1_m \) and the \( n \times n \) identity matrix by \( I_n. \) Given matrices \( Z_1, \ldots, Z_m, \) the Kronecker product is denoted by \( Z_1 \otimes \cdots \otimes Z_m. \)

\[
\text{diag}[Z_i] := \begin{bmatrix} Z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Z_m \end{bmatrix}
\]

\( \sigma_t \in \{1,2,\ldots,m\}, t \in \mathbb{Z}, \) is a Markov chain, if, for \( i_t \in \mathcal{P}, j \in \mathcal{P}, \) and \( t \in \mathbb{Z} \)

\[
\text{Prob}[\sigma_{t+1} = j | \sigma_0 = i_0, \ldots, \sigma_t = i_t] = \text{Prob}[\sigma_{t+1} = j | \sigma_t = i_t].
\]

II. PROBLEM SETTING

Consider the following model:

\[
\xi_{t+1} = E_{\sigma_t} \xi_t + H_{\sigma_t} r_t
\]

\[
y_t = F_{\xi_t}
\]

where, for each discrete-time \( t \in \mathbb{Z}, \) \( \xi_t \in \mathbb{R}^n \) is a generalized state, \( r_t \in \mathbb{R} \) is an input of interest such as a reference or a disturbance, and \( y_t \in \mathbb{R} \) is an output of interest; \( \sigma_t \in \mathcal{P} := \{1,2,\ldots,m\}, t \in \mathbb{Z}, \) is a Markov chain with transition probabilities denoted by

\[
\text{Prob}[\sigma_{t+1} = j | \sigma_t = i] = p_{ij}
\]

where \( p_{ij} \in [0,1] \) for \( i,j \in \mathcal{P}. \) The input \( r_t, t \in \mathbb{Z}, \) is assumed to either belong to the class of signals with bounded energy, i.e., \( \sum_{t=-\infty}^{\infty} |r_t|^2 < \infty, \) or to the class of periodic bounded signals. Stability of (1) is taken as a starting point and the focus lies on analyzing the input to output behavior of a (mean square) stable system, as defined next.
Definition 1: The unforced system (1) (when \( r_t = 0 \) for every \( t \)) is said to be mean square stable (MSS) if for any initial condition \( \xi_0 \in \mathbb{R}^n \), \( \lim_{t \to \infty} \mathbb{E}[\xi_t^T \xi_t] = 0 \).

Necessary and sufficient conditions to test mean square stability for the unforced model (1) are given in [8]; see also (17) later. The assumption is stated next.

Assumption 1: The unforced system (1) (when \( r_t = 0 \) for every \( t \)) is MSS.

Besides mean square stability the finite-state Markov chain is required to be ergodic (see, e.g., [23, Ch. 2]).

Assumption 2: The Markov chain with transition probabilities (2) is ergodic.

This assumption implies that the Markov chain has a unique invariant (stationary) distribution \( \pi = [\pi_1 \ldots \pi_r]^T \), such that \( \pi^T = \pi^T P, 1^T \pi = 1, \) and \( \pi_i \in [0, 1] \) (see [23, Ch. 2]), where \( P \) is a matrix with entries \( p_{ij} \) in row \( i \) and column \( j \). In particular, if the Markov chain is initialized with \( \text{Prob}[\sigma_0 = i] = \pi_i \), for every \( i \in \mathcal{P} \), then, for every \( t \in \mathbb{N} \) and \( i \in \mathcal{P} \), \( \text{Prob}[\sigma_t = i] = \pi_i \). Since the focus lies on the forced response of the system for \( t \) ranging from \(-\infty \) to \( \infty \), the following assumption is directly taken as a starting point.

Assumption 3: Letting \( \pi = [\pi_1 \ldots \pi_n]^T \) be the unique vector such that \( \pi^T P = \pi^T \), the following holds: \( \text{Prob}[\sigma_t = i] = \pi_i \), for every \( t \in \mathbb{Z}, i \in \mathcal{P} \).

A broad range of real-time and NCSs can be captured by such a model, for example, NCSs with i.i.d. stochastic delays, packet drops, and other effects, [21], [22]. In fact, a sequence of i.i.d. random variables can be modeled by the sequence \( \{\sigma_t \mid t \in \mathbb{Z}\} \), provided that \( P = 1_{m \times m} \pi^T \), where \( \pi = [\pi_1 \ldots \pi_m]^T \) and \( \pi_i = \text{Prob}[\sigma_t = i] \) for every \( t \). However, the fact that a Markov chain model is considered, allows, for instance, to take into account correlations. Two simple examples of systems with correlated packed drops and systems with correlated delays are provided next, which will also be considered in Section IV. Both examples consider an emulation framework: the controllers are designed without taking into account the network effects and the system is analyzed taking into account these effects. More elaborate cases can be found in [9] and [24].

A. NCSs With Correlated Packet Losses

This example considers a linear plant and a linear controller communicating through lossy channels. The plant is described by

\[
x_{t+1} = Ax_t + Bu_t \\
y_t = x_t
\]

(3)

and the controller is described by

\[
x_{t+1} = Ax_t + B_1(r_t - \tilde{y}_t) \\
u_t = C_1x_t + D_1(r_t - \tilde{y}_t)
\]

(4)

where \( x_t \in \mathbb{R}^{n_x} \) and \( x_t' \in \mathbb{R}^{n_x} \) denote the state of the plant and of the controller at time \( t \in \mathbb{Z} \), respectively; \( u_t \in \mathbb{R} \) and \( y_t \in \mathbb{R} \) are the input and the output of the plant at time \( t \), respectively; \( e_t := r_t - \tilde{y}_t \in \mathbb{R} \) and \( u_t \in \mathbb{R} \) are the input and the output of the controller at time \( t \), respectively, where \( \tilde{y}_t \in \mathbb{R} \) is the latest received output of the plant and \( r_t \in \mathbb{R} \) is the reference signal at time \( t \). The lossy sensor-to-controller channel is modeled as a hold function of \( y_t \)

\[
\tilde{y}_t = (1 - \theta_t)\tilde{y}_{t-1} + \theta_t y_t
\]

where \( \theta_t \in \{0, 1\}, t \in \mathbb{Z} ; \theta_t = 1 \) indicates that the controller receives the output \( y_t \) of the plant at time \( t \) and \( \theta_t = 0 \) indicates that the data are lost. In the same way, the lossy controller-to-actuator channel is modeled as a hold function of \( u_t \)

\[
\tilde{u}_t = (1 - \rho_t)\tilde{u}_{t-1} + \rho_t u_t
\]

where \( \rho_t \in \{0, 1\}, t \in \mathbb{Z} ; \rho_t = 1 \) indicates that the actuator receives the control input \( u_t \) from the controller at time \( t \) and \( \rho_t = 0 \) indicates that the control input is lost.

Let \( \sigma_t \in \{1, 2, 3, 4\} \) indicate which of the following possible data loss cases occurred at time \( t \):

\[
\sigma_t = \begin{cases} 
1, & \text{if } (\theta_t, \rho_t) = (0, 0) \\
2, & \text{if } (\theta_t, \rho_t) = (1, 0) \\
3, & \text{if } (\theta_t, \rho_t) = (0, 1) \\
4, & \text{if } (\theta_t, \rho_t) = (1, 1).
\end{cases}
\]

Then, defining the state vector \( \xi_t := [x_t^T, x_t'^T, \tilde{u}_{t-1}^T, \tilde{y}_{t-1}^T]^T \in \mathbb{R}^n \), the system can be modeled as in (1)

\[
\xi_{t+1} = E_{\sigma_t} \xi_t + H_{\sigma_t} r_t
\]

(5)

for every \( t \in \mathbb{Z} \), where, for \( i \in \{1, 2, 3, 4\} \)

\[
E_i := \begin{pmatrix} 
A - \frac{\rho_i}{\bar{\rho}_i} B D C & \frac{\rho_i}{\bar{\rho}_i} B C_c & (1 - \rho_i)B & -\frac{\rho_i}{\bar{\rho}_i} (1 - \bar{\rho}_i) D C_c \\
\frac{\rho_i}{\bar{\rho}_i} D C & A_c & 0 & -(1 - \bar{\rho}_i) D_c \\
\frac{\rho_i}{\bar{\rho}_i} D C_c & 0 & 0 & (1 - \bar{\rho}_i) D_c \\
0 & 0 & 0 & 0 
\end{pmatrix}
\]

and

\[
\begin{align*}
(\bar{\rho}_1, \rho_1) := (0, 0), & \quad (\bar{\rho}_2, \rho_2) := (1, 0), \\
(\bar{\rho}_3, \rho_3) := (0, 1), & \quad (\bar{\rho}_4, \rho_4) := (1, 1).
\end{align*}
\]

Note that \( (\theta_t, \rho_t) = (\bar{\rho}_{\sigma_t}, \rho_{\sigma_t}) \). Moreover, we have

\[
y_t = F \xi_t, \quad F := \begin{pmatrix} C & 0 & 0 & 0 \end{pmatrix}.
\]

(6)

If \( \sigma_t, t \in \mathbb{Z} \), is described by a Markov chain, then (5) is an MJLS. Two special cases of interest are as follows.

1) Data losses in each channel follow independent and identical stochastic processes according to Gilbert–Elliot models. This models bursty data with long sequences of consecutive data losses and consecutive successful data receptions. In other words, if the current data are lost, then the next data are likely to also be lost. If the current data are received successfully, then the next data are likely to be received successfully. In this case, taking
into account that the drops in both sensor-to-controller and controller-to-actuator channels follow identical and independent stochastic processes, the probabilities in (2) can be summarized in a matrix \( P \) with component \( p_{ij} \) in row \( i \), column \( j \) given by

\[
P = \bar{P} \otimes \bar{P}, \quad \bar{P} = \begin{bmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{bmatrix}.
\]  

(7)

2) Data losses in each channel follow independent Bernoulli distributions, with successful transmission probability \( p \). Then, \( \sigma_t, \ t \in \mathbb{Z} \), is an i.i.d. sequence and

\[
P = \bar{P} \otimes \bar{P}, \quad \bar{P} = \begin{bmatrix} 1 - p & p \\ 1 - p & p \end{bmatrix}.
\]

B. NCSs With Correlated Network-Induced Delays

Consider the NCS depicted in Fig. 1. The continuous-time linear plant is modeled as

\[
\dot{x}(s) = Ax(s) + Bu(s)
\]

where \( x(s) \in \mathbb{R}^{n_x} \) is the state and \( u(s) \in \mathbb{R}^{n_u} \) is the control input at continuous time \( s \in \mathbb{R} \). The state is sampled at times \( \{th | t \in \mathbb{Z} \} \), where \( h \) is the sampling period, and transmitted through a network to a remote controller. The controller computes the control input \( u_t \) and sends it to the plant through the network. This control input is hold constant

\[
u_t = u_t, \ \text{for} \ s \in [th + \delta_t, (t + 1)h + \delta_{t+1})
\]

where \( \delta_t := \delta_{tp} + \delta_{tc} + \delta_{cp} \) denotes the total delay in the \( t \)th period, with \( \delta_{tp} \) the delay from the plant (sensor) to the controller, \( \delta_{tc} \) the delay from the controller to the plant (actuator), and \( \delta_{cp} \) the controller computational delay. The control input is modeled as

\[
u_t = -K x(th) + r_t
\]

where \(-K x(th)\) is a state feedback control law and \( r_t, t \in \mathbb{Z} \), is a disturbance signal. Suppose that \( \delta_t < h \), for every \( t \); the general case as well as the case where the controller is an output feedback controller is treated in [24]. Let \( x_t := x(th) \). Exact discretization of the dynamics leads to

\[
x_{t+1} = e^{Ah}x_t + \int_{h-\delta_t}^{h} e^{As}dB u_{t-1} + \int_{0}^{h-\delta_t} e^{As}dB u_t.
\]  

(8)

Let \( \xi_t = [x(th) \ u_{t-1}^T]^T \). Then, we have

\[
\xi_{t+1} = E(\delta_t)\xi_t + H(\delta_t)r_t
\]

(9)

where

\[
E(\delta_t) = \begin{bmatrix} e^{Ah} - M_{2,\delta_t}K & M_{1,\delta_t} \\ -K & 0 \end{bmatrix}, \quad H(\delta_t) = \begin{bmatrix} M_{2,\delta_t} \\ 1 \end{bmatrix}.
\]

Suppose that the delays satisfy the Markov property

\[
\text{Prob}[\delta_{t+1} = b|\delta_0 = a_0, \ldots, \delta_t = a_t] = \text{Prob}[\delta_{t+1} = b|\delta_t = a_t]
\]

for \( a_t, b \in (0, h) \). If the delay is quantized and assumed to only take values in a finite set \( \delta_t \in (0, h) \cap \{k \epsilon | k \in \mathbb{N} \} \), where \( \epsilon > 0 \) is the quantization step size, then (9) is a discrete MJLS. In such a case, it can be written in the form (5) by considering \( \delta_t = \sigma_t \epsilon \) and \( E_{\sigma_t} = E(\delta_t) \) and \( H_{\sigma_t} = H(\delta_t) \).

III. FREQUENCY-DOMAIN ANALYSIS

This section provides a frequency-domain analysis framework for NCSs modeled by MJLSs. Sections III-A and III-B discuss the computation of the first and second statistical moments of the output response, respectively. Section III-C proposes a frequency response plot and discusses how this plot allows for reasoning about the behavior of the output response to a deterministic input. In Section III-D, the bounds given for the covariance are specialized for and compared with the models considered in [21] and [22].

A. Expected Value

Although (5) is a time-varying system, under Assumptions 2 and 3, the expected values of the state and output can be computed by resorting to an LTI system as shown next. Let \( 1 \{ \sigma_t = i \} := 1, \sigma_t = i, \) and \( 1 \{ \sigma_t = i \} := 0 \) otherwise, \( \beta_{t,i} := E[\xi_{t+1}1\{\sigma_t = i\}] \) and note that

\[
E[\xi_t] = m \sum_{i=1}^{m} E[\xi_{t+1}1\{\sigma_t = i\}] = m \sum_{i=1}^{m} \beta_{t,i}.
\]

**Lemma 1:** Suppose that Assumptions 2 and 3 hold and let \( \beta_t := [\beta_{t,1} \ldots \beta_{t,m}]^T, t \in \mathbb{Z} \). Then, for every \( t \in \mathbb{Z} \), we have

\[
\beta_{t+1} = \bar{F} \beta_t + \bar{H} r_t
\]

(10)

where \( \bar{F} = [F \ F \ \cdots \ F] \)

\[
E = \begin{bmatrix} p_{11}E_1 \cdots p_{1m}E_m \\ \vdots \vdots \vdots \vdots \\ p_{m1}E_1 \cdots p_{mm}E_m \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} \sum_{i=1}^{m} p_{1i}H_i \pi_t \\ \vdots \vdots \vdots \vdots \\ \sum_{i=1}^{m} p_{mi}H_i \pi_t \end{bmatrix}.
\]
The result follows from the following steps, which provide a recursive equation for $\beta_{t,i,j}$, $j \in \mathcal{P}$:

$$
\begin{align*}
\beta_{t+1,j} &= E[I_{t+1} \{ \sigma_{t+1} = j \}]
= E[(E_{\sigma} \xi_t + H_{\sigma} r_t)I_{\{ \sigma_{t+1} = j \}}]
= \sum_{i=1}^{m} E[(E_i \xi_t + H_i r_t)I_{\{ \sigma_t = i \}}I_{\{ \sigma_{t+1} = j \}}]
= \sum_{i=1}^{m} p_{ij} E_i \beta_{t,i} + \sum_{i=1}^{m} p_{ij} H_i r_t E[I_{\{ \sigma_t = i \}}]
= \sum_{i=1}^{m} p_{ij} E_i \beta_{t,i} + \sum_{i=1}^{m} p_{ij} H_i r_t \pi_t
\end{align*}
$$

where the last equality follows from Assumption 3 (which in turn relies on Assumption 2). It is also clear that $E[y_t] = F E[\xi_t] = \sum_{i=1}^{m} F \beta_{t,i} = F \beta_t$. □

Note that

$$
\begin{align*}
\tilde{E} = (P^T \otimes I_n) \text{diag} \{E_i\}, \quad \tilde{H} = (P^T \otimes I_n) \text{diag} \{H_i\} \pi.
\end{align*}
$$

Moreover, based on (10), the z-transform of $E[y_t]$, $\hat{y}(z) := \sum_{t=-\infty}^{\infty} E[y_t] z^{-t}$ can be calculated by

$$
\hat{y}(z) = a(z) \hat{r}(z)
$$

where $a(z) := \tilde{F}(zI - \tilde{E})^{-1} \tilde{H}$, for $z$ in the intersection of the regions of convergence of $\hat{r}(z)$ and $a(z)$. In particular, if $r_t = 3 \{ \nu e^{j \omega t} \} = |\nu| \sin(\omega t + \phi)$, $t \in \mathbb{Z}$

is a sinusoid, where $\nu = |\nu| e^{j \phi} \in \mathbb{C}$ is the complex amplitude and $\omega \in [0, 2\pi)$. Then $E[y_t]$ is also a sinusoid

$$
E[y_t] = 3 \{ a(e^{j \omega t}) \nu e^{j \omega t} \}, \quad t \in \mathbb{Z}.
$$

**B. Variance**

The standard derivation of the output at time $t$ is given by

$$
\text{var}(y_t) = \sqrt{E[y_t]^2 - E[y_t]^2},
$$

where the variance of the output is

$$
\text{var}(y_t) := E[(y_t - E[y_t])^2] = E[y_t^2] - E[y_t]^2.
$$

The second term in this expression can be calculated by squaring the inverse z-transform of (11). Moreover, notice that

$$
E[y_t^2] = GE[\xi_t \otimes \xi_t]
$$

where $G := F \otimes F$ and the fact $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ for matrices with compatible dimensions was used.

A similar approach to compute $E[\xi_t \otimes \xi_t]$ is taken as the one used to compute $E[\xi_t]$. Let $\zeta_{t,i} := E[\xi_t \otimes \xi_t(\{ \sigma_t = i \})]$ and $\zeta_t = [\zeta_{t,1} \ zeta_{t,2} \ \ldots \ \zeta_{t,m}]^T$.

**Lemma 2:** Suppose that Assumptions 2 and 3 hold. Then, for every $t \in \mathbb{Z}$, we have

$$
\begin{align*}
\zeta_{t+1} &= \tilde{M} \zeta_t + \tilde{L} \beta_t r_t + \tilde{N} r_t^2
E[y_t^2] &= \tilde{G} \zeta_t
\end{align*}
$$

where, for $i \in \mathcal{P}$, we obtain

$$
\tilde{M} := (P^T \otimes I_n) \text{diag} \{M_i\}, \quad M_i = E_i \otimes E_i
\tilde{L} := (P^T \otimes I_n) \text{diag} \{L_i\}, \quad L_i = E_i \otimes H_i + H_i \otimes E_i
\tilde{N} := (P^T \otimes I_n) \text{diag} \{N_i\} \pi, \quad N_i = H_i \otimes H_i
\tilde{G} := [G_1 \ G_2 \ \ldots \ G_m], \quad G_i = F \otimes F.
$$

**Proof:** The proof follows by establishing the following recursive equation for $\zeta_{t,i}$:

$$
\begin{align*}
\zeta_{t+1,i} &= \sum_{i=1}^{m} p_{ij} (E_i \otimes E_i) \zeta_{t,i} + \sum_{i=1}^{m} p_{ij} (E_i \otimes H_i) \beta_{t,i} r_t
+ \sum_{i=1}^{m} p_{ij} (H_i \otimes E_i) \beta_{t,i} r_t + \sum_{i=1}^{m} p_{ij} (H_i \otimes H_i) r_t^2 \pi_{t,i}
\end{align*}
$$

and from similar arguments as in the proof of Lemma 1. As established in [8], Assumption 1 is equivalent to

$$
\text{r}(\tilde{M}) < 1 \quad (17)
$$

where $\text{r}(\tilde{M})$ denotes the spectral radius of $\tilde{M}$.

Contrary to the equation describing the first moment (10), system (16) depends nonlinearly on the input $r_t$, $t \in \mathbb{Z}$. However, the solution to (16) can be exactly characterized and, thus, (14) can be computed, when $r_t, t \in \mathbb{Z}$ is a sinusoid, described by (12), as summarized in the following result.

**Theorem 1:** Suppose that Assumptions 1, 2, and 3 hold and let $y$ be the output response (6) of (5) to the input (12). Then, we have

$$
\begin{align*}
\text{var}(y_t) &= b(e^{j \omega_c} \nu^2 - \Re\{c(e^{j \omega_c}) \nu^2 e^{j \omega_{c} t} \})
\end{align*}
$$

for every $t \in \mathbb{Z}$, and, for every $z \in \mathbb{C}$, we obtain

$$
\begin{align*}
b(z) := \frac{1}{2} \text{Re}\{\tilde{G}(I - \tilde{M})^{-1}(\tilde{N} + \tilde{L}(zI - \tilde{E})^{-1} \tilde{H})\} - \frac{|a(z)|^2}{2}
\quad \text{and}
\quad c(z) := \frac{1}{2} \tilde{G}(z^2 - \tilde{M})^{-1}(\tilde{N} + \tilde{L}(zI - \tilde{E})^{-1} \tilde{H}) - \frac{|a(z)|^2}{2}.
\end{align*}
$$

**Proof:** Under the stated assumptions, it follows from Lemma 1 and Assumption 1 that the expected value of the state response $\beta$ to (12) is a vector of sinusoids with frequency $\omega_c$ and complex amplitudes $w(e^{j \omega_c} \nu)$, $w(z) := (zI - E)^{-1} H$ [cf., (10)] and $E[y_t]$, $t \in \mathbb{Z}$, is a sinusoid with frequency $\omega_c$ and complex amplitude $a(e^{j \omega_c} \nu)$. Thus, for every $t \in \mathbb{Z}$, we have

$$
\begin{align*}
E[y_t^2] &= \frac{1}{2} \left( \{ a(e^{j \omega_c}) \nu^2 - \Re\{a(e^{j \omega_c}) \nu^2 e^{j \omega_{c} t} \} \right)
\beta_t r_t &= \frac{1}{2} \left( \{ w(e^{j \omega_c}) \nu^2 - \Re\{w(e^{j \omega_c}) \nu^2 e^{j \omega_{c} t} \} \right)
\end{align*}
$$

Since (16) is a linear system driven by the two inputs $\beta_t r_t$ and $r_t^2$, both with two pure sinusoidal components (with frequencies 0 and $2\omega_c$), the output (15) will also be a sum of two sinusoids. Computing the complex amplitudes of these sinusoidal components of $E[y_t^2]$ and replacing the resulting expression for $E[y_t^2]$ and (19) in (14), (18) is obtained. □
From Theorem 1, it follows that $b(1) = c(1)$, by simply replacing $z = 1$ in the expressions for $b$ and $c$. Moreover, since the variance is always positive and the variance of the output, for a given $\omega$, is a sum of a constant term $b(e^{j\omega})|v|^2$ and a sinusoid with amplitude $|c(e^{j\omega})||v|^2$ the following must hold $|c(e^{j\omega})| \leq b(e^{j\omega})$ for every $\omega$.

The previous result exploited the fact that, although the input to output map from reference to output is nonlinear, for a sinusoidal signal, the output is still a sum of sinusoids. For an arbitrary input signal, an explicit solution is not provided, but instead a bound for the standard deviation is given.

**Theorem 2:** Suppose that Assumptions 1, 2, and 3 hold and let $y$ be the output response (6) of (5) to a reference input $r_t$, $t \in \mathbb{Z}$, with Fourier transform $\hat{r}(e^{j\omega})$, $\omega \in [0, 2\pi]$. Then, for $r_t$, $t \in \mathbb{Z}$, with bounded energy, it holds that

$$\text{std}(y_t) \leq \frac{1}{T} \sqrt{\int_0^\pi d(e^{j\omega})|\hat{r}(e^{j\omega})|^2 d\omega}$$

(20)

for every $t \in \mathbb{Z}$, where

$$d(z) := \sqrt{|b(z)| + |c(z)|}.$$ 

Moreover, for $T$-periodic $r$ with Fourier transform $\hat{r}(e^{j\omega}) = \frac{2\pi}{T} \sum_{k=0}^{T-1} v_k \delta \left( \omega - \frac{2\pi k}{T} \right)$, $\omega \in [0, 2\pi)$

where $v_k := \sum_{t=0}^{T-1} r_t e^{-j\frac{2\pi k t}{T}}$ and $\delta$ is the Dirac function, it holds that

$$\text{std}(y_t) \leq \frac{2}{T} \sum_{k=0}^{T-1} d(e^{j\omega_k}) |v_k|$$

(22)

for every $t \in \mathbb{Z}$, where $\omega_k := \frac{2\pi k}{T}$, and $\left\lfloor \frac{T}{2} \right\rfloor = T$ if $T$ is even and $\left\lfloor \frac{T}{2} \right\rfloor = T-1$ if $T$ is odd.

**Proof:** Consider first a periodic $r_t$, $t \in \mathbb{Z}$, characterized by the Fourier transform (21). Suppose that the period $T$ is even (the proof for an odd $T$ follows similar arguments). Then, we have

$$r_t = \frac{1}{T} \sum_{k=0}^{T-1} v_k e^{j\omega_k t} = \frac{1}{T} \left( v_0 + 2 \sum_{k=1}^{T-1} \Re \{ v_k e^{j\omega_k t} \} + v_{T/2} (-1)^k \right)$$

(23)

for every $t \in \mathbb{Z}$, where $v_0 \in \mathbb{R}$, $v_{\frac{T}{2}} \in \mathbb{R}$, and for

$$k \in \mathcal{P} := \{ 1, 2, \ldots, \frac{T}{2} - 1 \}, v_k \in \mathbb{C}.$$ 

Let $y_{t_0 t}$, $y_{k,t}$, $k \in \mathcal{P}$, and $y_{\frac{T}{2}}$ denote the stochastic processes coinciding with the output responses of the closed-loop system to $v_0$, $\Re \{ v_k e^{j\omega_k t} \} = \Im \{ v_k e^{j\omega_k t} \}$ and $v_{\frac{T}{2}} (-1)^k$, respectively. Then, from Theorem 1, it follows that, for each $k \in \{ 0, \ldots, \frac{T}{2} \}$, $y_{k,t}$

$$\text{var}(y_{k,t}) = b(e^{j\omega_k}) |v_k|^2 + \Re \{ c(e^{j\omega_k}) e^{2j\omega_k t} \}$$

for every $t \in \mathbb{Z}$. Moreover, due to the linearity of the system, the response of the closed-loop system to (23) is a stochastic process given by

$$y_t = \frac{1}{T} \left( y_{0,t} + 2 \sum_{k=1}^{T-1} y_{k,t} + y_{\frac{T}{2},t} \right), \quad t \in \mathbb{Z}.$$ 

Then, letting $R_{t\kappa} := \mathbb{E}[y_{t,t} - \mathbb{E}[y_{t,t}]]y_{t,t} - \mathbb{E}[y_{t,t}]]$ (the dependence of $R_{t\kappa}$ and $\mu$ on $t$ is omitted)

$$\text{var}(y_t) = \mathbb{E} \left[ \left( \frac{1}{T} (y_{0,t} - \mathbb{E}[y_{0,t}]) + 2 \sum_{k=1}^{T-1} y_{k,t} - \mathbb{E}[y_{k,t}] + y_{\frac{T}{2},t} - \mathbb{E}[y_{\frac{T}{2},t}] \right)^2 \right]$$

$$\leq \frac{4}{T^2} \sum_{t=0}^{T-1} \sum_{\kappa=0}^{T-1} |R_{t\kappa}|$$

$$\leq \frac{4}{T^2} \sum_{t=0}^{T-1} \sum_{\kappa=0}^{T-1} \text{var}(y_{t,t}) \text{var}(y_{t,t})$$

$$= \left( \frac{2}{T} \sum_{t=0}^{T-1} \text{var}(y_{t,t}) \right)$$

$$\leq \left( \frac{2}{T} \sum_{t=0}^{T-1} \sqrt{\text{var}(y_{t,t})} \text{var}(y_{t,t}) \right)^2$$

$$\leq \left( \frac{2}{T} \sum_{k=0}^{T-1} \sqrt{b(e^{j\omega_k})} |v_k|^2 + \Re \{ c(e^{j\omega_k}) e^{2j\omega_k t} \} \right)^2$$

which is (22). To establish the second inequality, the Cauchy–Schwarz inequality was used

$$R_{t\kappa} \leq \sqrt{\text{var}(y_{t,t}) \text{var}(y_{t,t})}$$

(24)

for $\ell, \kappa \in \mathcal{P}$.

The result for systems with bounded energy follows by a limiting argument similar to the one provided in [21, proof of Th. 3] and is omitted for the sake of brevity.

**C. Reasoning in Terms of Frequency Response Plots**

This article proposes characterizing the map between the input and the mean and the variance of the output by: 1) a magnitude plot, which consists of the following two graphs $(\omega, |a(e^{j\omega})|)$, $(\omega, d(e^{j\omega}))$, $\omega \in [0, \pi]$ and 2) a phase plot, which consists of the following graph $(\omega, \arg(a(e^{j\omega}))$). Logarithmic scales may be used for convenience. Then, the following procedure allows to obtain insights on the mean and the variance of the output response to a reference $r$ with bounded energy.

1. Multiply $a(e^{j\omega})$ and $\hat{r}(e^{j\omega})$ and obtain the expected value of the output by inverting the Fourier transform $a(e^{j\omega}) \hat{r}(e^{j\omega})$, $\omega \in [0, 2\pi]$.

2. Multiply $|\hat{r}(e^{j\omega})|/|d(e^{j\omega})|$, for $\omega \in [0, \pi]$, and obtain a bound for the standard deviation at every time step $t \in \mathbb{Z}$ according to (20). Graphically, this amounts to
plotted $|d(e^{j\omega})||\hat{r}(j\omega)|$, for $\omega \in [0, \pi)$, and computing the average over frequency $\omega$. This provides a bound for the variance of the output for every time step.

Note that a related, but different, frequency response plot was proposed in [21], which contains the $b$ and $c$ functions. One of the advantages of such a plot is that the standard deviation of the output response to a sinusoidal input signal is fully characterized by $b$ and $c$. However, the proposed plot here is more suitable to compute the bounds (20), (22) and still provides the maximum amplitude of the standard deviation of the output response to a sinusoidal input signal.

From the expected value and the variance, one can infer the behavior of the sample paths of the output. In fact, from Chebychev’s inequality, it follows that, for $\alpha > 1$:  

$$\text{Prob}[|y_t - E[y_t]| \geq \alpha \text{ std}(y_t)] \leq \frac{1}{\alpha^2}, \quad t \in \mathbb{Z}. \quad (24)$$

One can then use the bound (20) to provide a guarantee on the probability that the output response is not far from its expected value.

### D. Comparison With Previous Bounds

Previous work in [21] and [22] considering the special case where $\{\sigma_t | t \in \mathbb{Z}\}$ in (1) is an i.i.d. sequence of random variables, provided the following bound for the variance:

$$\text{var}(y_t) \leq \bar{v}_{\text{old},p}, \quad \text{for every } t \in \mathbb{Z}$$

where $\bar{v}_{\text{old}} = \bar{v}_{\text{old},p}$

$$\bar{v}_{\text{old},p} := \frac{4}{T} \sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} \left( |b(e^{j\omega_k})| + |c(e^{j\omega_k})| \right) |v_k|^2$$

when the input is a periodic signal and $\bar{v}_{\text{old}} = \bar{v}_{\text{old},e}$

$$\bar{v}_{\text{old},e} := \frac{2}{\pi} \int_0^\pi \left( |b(e^{j\omega})| + |c(e^{j\omega})| \right) \hat{r}(e^{j\omega}) |d\omega|^2$$

for signals with finite energy. Note that Theorem 2 provides new bounds for this special case

$$\text{var}(y_t) \leq \bar{v}_{\text{new}}, \quad \text{for every } t \in \mathbb{Z}$$

where $\bar{v}_{\text{new}} = \bar{v}_{\text{new},p}$

$$\bar{v}_{\text{new},p} := \left( \frac{2}{\pi} \sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} d(e^{j\omega_k}) |v_k| \right)^2$$

when the input is a periodic signal and $\bar{v}_{\text{new}} = \bar{v}_{\text{new},e}$

$$\bar{v}_{\text{new},e} := \left( \frac{2}{\pi} \int_0^\pi d(e^{j\omega}) \hat{r}(e^{j\omega}) |d\omega|^2 \right)^2 \quad (25)$$

for signals with finite energy. The next results formally establish that the new bounds are tighter.

---

1In [21], it is incorrectly stated that $b(1) = 0$ and $c(1) = 0$. An example is given in Section IV-B with $b(1) = c(1) > 0$ (as in Fig. 8). This incorrectness leads to the omission of the first term ($\ell = 0$) in (22), which is corrected in the present expression.

### Proposition 3: The following holds:

$$\bar{v}_{\text{new},p} \leq \bar{v}_{\text{old},p}, \quad \bar{v}_{\text{new},e} \leq \bar{v}_{\text{old},e}.$$  

**Proof:** Only the result for periodic inputs is proved here (the result for inputs with bounded energy follows by a limiting argument similar to the one provided in [21, proof of Th. 3]). Suppose $T \geq 2$ and let $f_k := d(e^{j\omega_k})|v_k|$, $f = (f_0, f_1, \ldots, f_{\lfloor \frac{T}{2}\rfloor})$, and $\hat{f} = (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{\lfloor \frac{T}{2}\rfloor})$ be a reordering of vector $f$ such that $\hat{f}_i \geq \hat{f}_{i+1}$, $i \in \{0, 1, \ldots, \lfloor \frac{T}{2}\rfloor - 1\}$. Note that, if $i < j$, then $\hat{f}_i \geq \hat{f}_j$, which implies that

$$1^T \hat{f} \hat{f}^T 1 = \left( \sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} \hat{f}_k^2 + 2 \sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} \hat{f}_k \hat{f}_k \right) \leq T \sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} \hat{f}_k^2$$

where $\sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} \hat{f}_k := 0$. If $T = 1$, the input is constant, $f = \hat{f}_0$, and this inequality holds trivially. Then, we have

$$\bar{v}_{\text{new},p} = \frac{4}{T^2} (1^T f)^2 = \frac{4}{T^2} (1^T \hat{f})^2 = \frac{4}{T^2} 1^T \hat{f} \hat{f}^T 1$$

$$\leq \frac{4}{T^2} T \left( \sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} \hat{f}_k^2 \right) = \frac{4}{T} \left( \sum_{k=0}^{\lfloor \frac{T}{2}\rfloor} \hat{f}_k^2 \right) = \bar{v}_{\text{old},p}.$$  

Another interesting fact is that the new bound is exact when the input signal is a sinusoid (12) with frequency $w_c = \frac{2\pi}{T}$, for $\kappa, T \in \mathbb{N}, T > 2, \kappa < T/2$, that is, its Fourier transform is (21) with

$$v_k = \begin{cases} T \frac{\nu}{2j}, & \text{if } k = \kappa \\ -T \frac{\nu}{2j}, & \text{if } k = T - \kappa \\ 0, & \text{otherwise} \end{cases}$$

for $t \in \mathbb{Z}$. In fact, in such a case, we have

$$\bar{v}_{\text{new},p} = \frac{4}{T^2} d(e^{j\omega})^2 |v_c|^2 = d(e^{j\omega})^2$$

which is the maximum amplitude of (18). This fact holds independently of $T$. The old bound is strictly worse even for this simple case and the bound becomes less and less tight as $T$ increases.

### IV. Simulations and Network-in-the-Loop Experiments

#### A. Data Losses Captured by Gilbert–Elliot Model

In the setting of Section II-A, let the plant be a double integrator described by the transfer function $\frac{1}{s^2}$ and the controller be described by the transfer function $\frac{10(s+1)}{s^2+1}$. The plant and the controller are discretized with a sampling period $h = 0.05$, with the zero-order hold invariant method, leading to the following matrices in (3) and (4):

$$A = \begin{bmatrix} 1 & 0 \\ 0.05 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.05 \\ 0.0013 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
Fig. 2. Frequency response plot for the first example (taking into account drops). The complex function $a$ can be used to compute the mean of the output for any input signal and the nonnegative real function $d$ can be used to compute or bound the output variance according to the proposed approach in Section III-C.

Fig. 3. Plots of the absolute value of the complex functions $b(e^{j\omega})$, $c(e^{j\omega})$, for $\omega \in [0, \pi]$, which determine the output variance when the input is a sinusoid.

$$A_c = 0.7788, \quad B_c = 0.0442, \quad C_c = -40, \quad D_c = 10.$$ 

The transition matrix of the Gilbert model modeling data losses in the sensor-to-controller channel and, independently, in the controller-to-actuator channel is (7) with

$$\bar{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}.$$ 

Fig. 2 plots the frequency response pertaining to the mean (complex function $a$) and variance (non-negative real function $d$) following the proposed approach in Section III-C. Fig. 3 plots the magnitude of the complex functions $b(e^{j\omega})$ and $c(e^{j\omega})$, and $d(e^{j\omega})^2 = |b(e^{j\omega})| + |c(e^{j\omega})|$ to illustrate the properties of these functions, which are discussed after Theorem 1. Fig. 4 plots several output responses of the system to a reference signal, comprising an increasing ramp, a constant value, and a decreasing ramp. Also shown are the mean and $3\sigma_{\text{std}}$ bounds computed by the method indicated in Sections III-A and III-B, where $\sigma_{\text{std}}$ denotes the standard deviation, and by Monte Carlo (MC) simulations (bottom figure). (a) Expected values computed with the method indicated in Sections III-A and III-B. (b) Expected values computed with MC simulations.

Fig. 5. Output standard deviation computed according to the method indicated in Section III-B for the case of correlated packet drops (described by an MJLS) and assuming that the packet drops would be i.i.d.

$$\bar{P} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}$$

where $\pi = [\pi_1 \, \pi_2]^T = [0.3750 \, 0.5145]^T$ is such that $\pi^T \bar{P} = \pi^T$ (invariant distribution of the Markov chain). This is an i.i.d. model for the drops, not taking into account correlations where $\pi_1$ and $\pi_2$ correspond to the ratio of occurrences of drop and
no drops. Note how the plots differ significantly, illustrating the importance of taking into account the correlation between drops, which is only possible with the tools provided in the present article. Fig. 6 illustrates how to obtain the bound for the standard deviation provided by Theorem 2. Both $|\hat{r}(e^{j\omega})|$ and $|d(e^{j\omega})|$ are plotted, as well as their product and the mean value of the product, which provides such a bound. Following this method leads to a bound for the standard deviation of 0.1468, about three times larger than the maximum value of the standard deviation shown in Fig. 5. Therefore, for this example, the bound provided by the method given by Theorem 2 is reasonably close to the maximum value of the standard deviation.

### B. Round Trip Delay Modeled by Markov Chain

In the setting of Section II-B, consider an inverted pendulum model, with state $x_t = [\theta_t \ \dot{\theta}_t]^T$, where $\theta_t$ and $\dot{\theta}_t$ are the angle and angular velocity at time $t \in \mathbb{Z}$. The linearized system about its unstable equilibrium point is

$$\dot{x}(s) = \begin{bmatrix} 0 & 1 \\ -\frac{3(M+m)g}{4(M+m)} & 0 \end{bmatrix} x(s) + \begin{bmatrix} -\frac{3}{4(M+m)} \\ 1 \end{bmatrix} u(s), \ s \in \mathbb{R}. \tag{26}$$

Let $M = 8$ kg, $m = 2$ kg, $l = 0.5$ kg, and $g = 9.8$ m/s$^2$. Then, we have

$$A = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}. \tag{27}$$

The gains of the state-feedback controller are set to $K = [-110.2335 \ -26.5072]$. The output is set to $y = [1 \ 0]^T x$ as the angle of the pendulum.

The delays in a communication channel consisting of two XBee nodes connected to two raspberry pi computing modules are obtaining experimentally. The Baud rate is set to 57 600 and a unicast transmission between the two XBees based on the 16-b address is considered. A sample of 500 000 round trip delays is premeasured, using a sampling period of 0.2 s (the experiment took 100 000 s). The delays obtained over a period of 2100 s are shown in Fig. 7 and a closer look at the interval between 600 and 650 s is also shown. This pattern was observed throughout the entire sample of 500 000 delays: there are two modes: mode 1, where the delay is approximately constant and varies around 32 ms; and mode 2, where it takes values around three numbers: 16, 32, and 48 ms. The system dwells in the first mode for a random time around 500–700 s before transiting to the other mode. In mode 2, the system dwells also a random time around 100 s. In mode 2, the delays are correlated: for instance, when the system has a delay around 48 ms, it is more likely that the next delay is around 48 ms. This is clearly visible in the data in the interval 600 and 650 s.

The behavior of the system in the second mode is considered and the delays are divided in three classes: 1) small ($< 24$ ms), 2) medium ($\geq 24$ ms and $< 40$ ms), and 3) large ($\geq 40$ ms). Estimating the Markov chain probabilities leads to

$$P = \begin{bmatrix} 0.0486 & 0.7361 & 0.2153 \\ 0.0809 & 0.7721 & 0.1471 \\ 0.0284 & 0.4174 & 0.5542 \end{bmatrix},$$

where each entry $(i, j)$ of the transition matrix was estimated by

$$p_{ij} = \frac{\#\text{pairs}(i, j) \in \mathcal{I}}{\# i \in \mathcal{I}}$$

where $\mathcal{I}$ is the interval of data. It is also clear from this matrix that the delays are correlated.

Based on this model, the system can be written in the form described in Section II-B, and the frequency response pertaining to the mean (complex function $a$) and variance (positive function $d$) can be plotted following the approach proposed in Section III-C. The plot is shown in Fig. 8.

In Fig. 9(left), several periods of the output responses to a periodic reference

$$r_t = \frac{1}{10} \sin \left( \frac{2\pi}{15} t \right), \ t \in \mathbb{Z},$$

after the transitory has vanished are merged. Appealing to ergodicity, this is equivalent to plotting a fixed period of several independent experiments. The sample mean is also computed, which is compared with the theoretical mean in Fig. 9(center). The sample standard deviation is compared with the theoretical one in Fig. 9(right). Note that these plots are very similar, especially the one corresponding to the mean [see Fig. 9(center)], validating the results presented in the present article. The cause for the small discrepancy in the plot corresponding to the standard deviation [see Fig. 9(right)] is unknown but might be caused by the fact that the underlying stochastic process generating the delays is not exactly, but also approximately, captured by the proposed Markov model.

As a final note, if an i.i.d. model for the delays with the invariant distribution of the Markov chain had been used, a
standard deviation oscillating around 0.032 and 0.046 [contrast with Fig. 9(right)] would have been obtained, illustrating once again the importance of taking correlations into account.

V. CONCLUSION

This article tackled NCSs that can be captured by MJLs, including systems with random delays and packet drops modeled by a Markov chain. The proposed analytical framework relies on the statistical moments (mean and variance) of the state and output. Based on the fact that the map between the deterministic input, and the statistical moments of the state and output is time-invariant, a new state-space model is introduced to compute the statistical moments of the response to any deterministic input. A frequency plot is obtained to infer the expected time response considering continuous-time MJLs; and 2) understanding how the open-loop input to output map translates into properties of the closed loop.

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