Gaussian estimates for second order elliptic operators with boundary conditions


Document status and date:
Published: 01/01/1995

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
Gaussian estimates
for second order elliptic operators
with boundary conditions

by

W. Arendt and A.F.M. ter Elst
Gaussian estimates
for second order elliptic operators
with boundary conditions

W. Arendt and A.F.M. ter Elst

Laboratoire de Mathématiques
Université de Franche-Comté
F-25030 Besançon Cedex
France

June 1995

AMS Subject Classification: 35J25, 35J15, 47A60.

Home institution:

1. Department of Mathematics and Computing Science
   Eindhoven University of Technology
   P.O. Box 513
   5600 MB Eindhoven
   The Netherlands

Abstract

We prove Gaussian estimates for the kernel of the semigroup generated by a second order operator $A$ in divergence form with real, not necessarily symmetric, second order coefficients on an open subset $\Omega$ of $\mathbb{R}^d$ satisfying various boundary conditions. If the boundary $\partial \Omega$ of $\Omega$ is a null set, then $A + \omega I$ has a bounded $H_\infty$ functional calculus and has bounded imaginary powers if $\omega$ is large enough.
1 Introduction

A large literature has recently arisen on Gaussian estimates for kernels of semigroups generated by elliptic operators, including several books, see Davies [Dav89], Robinson [Rob91] and Varopoulos–Saloff-Coste–Coulhon [VSC92]. The starting point was a paper of Aronson [Aro67] for real non-symmetric elliptic operators on $\mathbb{R}^d$ with measurable coefficients, which used Moser’s parabolic Harnack inequality [Mos64]. New impetus to the subject came from Davies [Dav87], who introduced a perturbation method together with logarithmic Sobolev inequalities to deduce Gaussian upper bounds with optimal constants for symmetric pure second order operators with $L_\infty$ coefficients and Dirichlet boundary conditions or, if the region has the extension property, with Neumann boundary condition. (See for a coherent description [Dav89].) A new type of proof for Gaussian bounds for real symmetric pure second order operators with measurable coefficients has been introduced by Fabes and Stroock [FS86] using a Nash inequality. This inequality, together with a parametrix argument, has subsequently been used to derive Gaussian bounds for $m$-th order strongly elliptic or subelliptic operators on Lie groups of which the $m$-th order coefficients are $m$ times differentiable and the lower order coefficients merely measurable. (See [ER93].)

In this paper, we consider second order elliptic operators of the form

$$\mathcal{A}u = -\sum_{i,j=1}^{d} D_j a_{ij} D_i u + \sum_{i=1}^{d} b_i D_i u - \sum_{i=1}^{d} D_i (c_i u) + c_0 u$$

with real, not necessarily symmetric coefficients $a_{ij} \in L_\infty(\Omega)$ satisfying a uniform ellipticity condition, and lower order coefficients $b_i, c_i \in W^{1,\infty}(\Omega)$ and $c_0 \in L_\infty(\Omega)$ real or complex. We study realizations $\mathcal{A}$ of $\mathcal{A}$ in $L_2(\Omega)$ obtained by quadratic form methods. They correspond to various boundary conditions, for example, Dirichlet, Neumann, mixed, or Robin boundary conditions. Our main results show that, in each of these cases, $\mathcal{A}$ generates a semigroup $S = (e^{-t\mathcal{A}})_{t>0}$ given by a kernel $(K_t)_{t>0}$ which satisfies a Gaussian estimate

$$|K_t(x;y)| \leq ct^{-d/2} e^{-b|x-y|^2 t^{-1}} e^{ut} (x,y) \text{ a.e.}$$

for all $t > 0$. We establish this by two different methods.

The first method (Section 3) works for Dirichlet boundary conditions and once differentiable second order coefficients. The proof is very short and elementary and relies on the Beurling–Deny criterion for forms in a non-symmetric version recently given by Ouhabaz [Ouh92a], [Ouh92b]. Besides its simplicity, one advantage of the method is that complex lower order coefficients are allowed. This approach is, however, restricted to Dirichlet boundary conditions.

The second method (Section 4) is based on an iteration process of Fabes–Stroock [FS86], which is also used in Robinson [Rob91] for second order real symmetric operators on Lie groups with constant coefficients. The advantage of this more elaborate method is that we no longer need to assume the once differentiability of the second order coefficients. Moreover, it works for all boundary conditions considered here. On the other hand the lower order coefficients have to be real.

Gaussian estimates have various interesting consequences. In Section 5 we show that for each of the considered boundary conditions one obtains a holomorphic semigroup on all the $L_p$-spaces with $1 \leq p \leq \infty$ with the same sector as in $L_2(\Omega)$. Moreover, using recent
results of Duong-Robinson [DR95] we show that, for all boundary conditions considered here, the operator $A + \omega I$ has a bounded $H_\infty(\Sigma(\nu))$ functional calculus on $L_p(\Omega)$ for each $p \in (1,\infty)$ and large $\omega$, where $\nu > 0$ is such that $\Sigma(\nu)$ contains the numerical range of the matrix $(a_{ij}(x))$ for a.e. $x \in \Omega$. In particular, the fractional powers $(A + \omega I)^{\alpha}$ are bounded, which is of interest in view of the regularity theorem of Dore-Venni [DV87] (see also [PS93], [Prü93]). In this context it is interesting to determine the range of $\omega$ for which this is true. It turns out that $\omega > \omega_0$ is allowed where $\omega_0$ is such that $\|S_z\|_{2 \to 2} \leq e^{\omega|z|}$ for all $z \in \mathbb{C}$ with $|\arg z| \leq \pi/2 - \nu$.

2 Preliminaries

In this section we fix some notations and give some basic results on semigroups and Sobolev spaces as they are needed throughout this paper.

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $1 \leq p_1 < p_2 \leq \infty$. A family of operators $T^{(p)} \in \mathcal{L}(L_p(\Omega))$, $p_1 \leq p \leq p_2$, is called consistent if

$$T^{(p)}\varphi = T^{(q)}\varphi$$

for all $p, q \in [p_1, p_2]$ and $\varphi \in L_p(\Omega) \cap L_q(\Omega)$. Similarly we refer to a consistent family of semigroups $(S_t^{(p)})_{t>0}$ on $L_p(\Omega)$, $p_1 \leq p \leq p_2$, if for every fixed $t > 0$ the family $S_t^{(p)}$, $p_1 \leq p \leq p_2$, is consistent. We shall briefly say that $S$ is consistent on $L_p$, $p_1 \leq p \leq p_2$ and drop the suffix $p$ in $S^{(p)}$.

Let $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$. Let $S$ be a $C_0$-semigroup on $L_2(\Omega)$. We say that $S$ interpolates on $L_p(\Omega)$, $p_1 \leq p \leq p_2$, if there exists a consistent family of semigroups $(S_t^{(p)})_{t>0}$ on $L_p(\Omega)$, $p_1 \leq p \leq p_2$, such that $S^{(p)}$ is strongly continuous if $p \in [p_1, p_2]$, $p \neq \infty$, and in the case $p_2 = \infty$, $S^{(\infty)}$ is weakly* continuous, and, moreover, $S_t = S_t^{(2)}$ for all $t > 0$. In that case, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S_t^{(p)}\|_{p \to p} \leq Me^{\omega t}$$

uniformly for all $p \in [p_1, p_2]$ and $t > 0$. In order to show that a given semigroup $S$ on $L_2$ interpolates, frequently the strong continuity in the endpoints $p_1$, $p_2$ is not a trivial problem. In the following lemma we give some sufficient conditions.

**Lemma 2.1** Let $S$ be a $C_0$-semigroup on $L_2(\Omega)$ satisfying $S_t(L_1 \cap L_2) \subset L_1$ for all $t > 0$ and

$$\|S_t\varphi\|_1 \leq M\|\varphi\|_1$$

(1)

uniformly for all $t \in (0,1]$ and all $\varphi \in L_1 \cap L_2$. (We use $\|\varphi\|_p$ to denote the norm of $\varphi$ in $L_p(\Omega)$.) Then $S$ interpolates on $L_p(\Omega)$, $1 \leq p \leq 2$, if one of the following conditions is satisfied.

**I.** $M = 1$.

**II.** $\Omega$ has finite measure.

**III.** $S_t \geq 0$ for all $t > 0$.

**IV.** There exists $\omega \in \mathbb{R}$ such that $\|S_t\varphi\|_1 \leq e^{\omega t}\|\varphi\|_1$ for all $\varphi \in L_1 \cap L_2$ and $t > 0$. 

2
There exist $c > 0$, open $\Omega' \subset \mathbb{R}^d$ with $\Omega \subset \Omega'$ and an interpolating semigroup $T$ on $L_p(\Omega')$, $1 \leq p \leq 2$, such that $|S_t \varphi| \leq c |t| \varphi$ for all $t \in (0, 1]$ and $\varphi \in L_1(\Omega) \cap L_2(\Omega)$.

**Proof.** It is clear that one obtains consistent semigroups $(S_t(t))_{t \geq 0}$ on $L_p$, $1 \leq p \leq 2$, and it follows from the interpolation inequality [Bre83] p. 57 that $S^{(p)}$ is strongly continuous for $p > 1$. The strong continuity of $S^{(1)}$ demands further arguments and is proved in Voigt [Voi92] (see also Davies [Dav89] pp. 22–23) if one of the first four above conditions is satisfied.

The sufficiency of Condition V can be proved as follows: Let $p \in [1, 2]$ and $\varphi \in L_p(\Omega) \cap L_2(\Omega)$. We identify a function on $\Omega$ with the function on $\Omega'$ by extending it by $0$ on $\Omega' \setminus \Omega$. Moreover, let $t_1, t_2, \ldots \in (0, 1]$ and suppose that $\lim t_n = 0$. Then $\lim S_{t_n} \varphi = \varphi$ in $L_2(\Omega)$, so there exists a subsequence such that $\lim_{k \to \infty} S_{t_{n_k}} \varphi = \varphi$ a.e.. Since $\lim_{k \to \infty} T_{t_{n_k}} |\varphi| = |\varphi|$ in $L_p(\Omega')$, there exist a subsequence (which we can assume to be the subsequence) and $\psi \in L_p(\Omega')$ such that $T_{t_{n_k}} |\varphi| \leq \psi$ a.e. for all $k \in \mathbb{N}$. Then $|S_{t_{n_k}} \varphi| \leq c T_{t_{n_k}} |\varphi| \leq c \psi$ a.e. for all $k \in \mathbb{N}$. Therefore, $\lim_{k \to \infty} S_{t_{n_k}} \varphi = \varphi$ in $L_p(\Omega)$ by an application of the Lebesgue dominated convergence theorem, and $S$ is continuous on $L_p(\Omega)$.

Similarly, if $S_t(L_2 \cap L_{\infty}) \subset L_{\infty}$ and

$$\|S_t \varphi\|_{\infty} \leq M \|\varphi\|_{\infty}$$

uniformly for all $t \in (0, 1]$ and $\varphi \in L_2 \cap L_{\infty}$, then the semigroup interpolates on $L_p$ if one of the Conditions I–V of Lemma 2.1 is satisfied (with $L_1$ replaced by $L_{\infty}$). Note that in that case $S^*$ satisfies (1) and one can define $S^{(\infty)}$ by $S_t^{(\infty)} = (S_t(t))^*$.

An operator $T$ on $L_p$ is called positive, notation $T \geq 0$, if $T \varphi \geq 0$ a.e. for all $\varphi \in L_p$ with $\varphi \geq 0$ a.e.. We call $T$ $L_{\infty}$-contractive if $\|T \varphi\|_{\infty} \leq \|\varphi\|_{\infty}$ for all $\varphi \in L_p \cap L_{\infty}$. Thus, if $S$ is a $C_0$-semigroup on $L_2(\Omega)$ and $S_t$ and $S_t^*$ are $L_{\infty}$-contractive for all $t > 0$, then $S$ interpolates on $L_p(\Omega)$, $1 \leq p \leq \infty$. Finally, a semigroup $S$ on $L_2$ is called quasi­contractive on $L_\infty$ if there exists an $\omega \in \mathbb{R}$ such that $\|S_t \varphi\|_{\infty} \leq e^{\omega t} \|\varphi\|_{\infty}$ for all $\varphi \in L_2 \cap L_{\infty}$ and $t > 0$.

Next we give some results on Sobolev spaces. As before, $\Omega$ denotes an open set in $\mathbb{R}^d$. For $p \in [1, \infty]$ let $W^{1,p}(\Omega) = \{u \in L_p(\Omega) : D_i u \in L_p(\Omega) \text{ for all } i \in \{1, \ldots, d\}\}$. Here $D_i u = \partial u / \partial x_i$ is the distributional derivative in $D'(\Omega)$. If $p = 2$, then the space $H^1(\Omega) = W^{1,2}(\Omega)$ is a Hilbert space for the norm

$$\|u\|^2_{H^1(\Omega)} = \sum_{i=1}^d \|D_i u\|^2_2 + \|u\|^2_2.$$ 

Here and in Section 4 we consider real spaces. In Sections 3 and 5 the spaces are complex and the notation and field will be clear from the context.

The following results follow from [GT83] p. 152.

**Lemma 2.2** Let $u \in H^1(\Omega)$. Then $u^+ = u \vee 0 \in H^1(\Omega)$ and $D_i u^+ = 1_{\{u > 0\}} D_i u$ a.e.

for all $i \in \{1, \ldots, d\}$. As a consequence, $u^- = (-u)^+ \in H^1(\Omega)$ and $|u| = u^+ + u^- \in H^1(\Omega)$ and

$$D_i |u| = (\text{sgn } u) D_i u \quad \text{a.e. },$$

(2)
where
\[(\text{sgn } u)(x) = \begin{cases} 
1 & \text{if } u(x) > 0 \\
0 & \text{if } u(x) = 0 \\
-1 & \text{if } u(x) < 0
\end{cases} \]

Moreover, one has
\[D_i u = 0 \text{ a.e. on the set } \{x : u(x) = 0\} \quad (3)\]
for all \(i \in \{1, \ldots, d\}\).

We note some further consequences. Set \(L_2(\Omega)_+ = \{u \in L_2(\Omega) : u \geq 0 \text{ a.e.}\}\) and \(H^1(\Omega)_+ = H^1(\Omega) \cap L_2(\Omega)_+\).

**Lemma 2.3**

I. If \(v \in H^1(\Omega)\), then the mappings \(u \mapsto u \land v\) and \(u \mapsto u \lor v\), and in particular \(u \mapsto u^+\), \(u \mapsto u^-\) and \(u \mapsto |u|\) from \(H^1(\Omega)\) into \(H^1(\Omega)\) are continuous.

II. If \(u \in H^1(\Omega)\), then \(\|u\|_{H^1(\Omega)} = \|u\|_{H^1(\Omega)}\).

III. If \(0 \leq u \in H^1(\Omega)\), then \(u \land 1 \in H^1(\Omega)\) and the mapping \(u \mapsto u \land 1\) is continuous on \(H^1(\Omega)_+\).

IV. If \(u \in H^1_0(\Omega)\), then \(u^+, u^-, |u|, |u| \land 1 \in H^1_0(\Omega)\).

**Proof.** Since \(u \lor v = u + (v - u)^+\) and \(u \land v = -((-u) \lor (-v))\), it suffices to show that \(u \mapsto u^+\) is continuous. Let \(u, u_1, u_2, \ldots \in H^1(\Omega)\) and suppose that \(\lim u_n = u\) in \(H^1(\Omega)\). It suffices to show that every subsequence of \((u^+_n)\) has a subsubsequence which converges to \(u^+\). Therefore, we can assume that \(\lim u_n = u\) a.e., \(\lim D_i u_n = D_i u\) a.e. and, moreover, \(|u_n| \leq f\) and \(|D_i u_n| \leq f\) for some \(f \in L_2(\Omega)\), uniformly for all \(n \in \mathbb{N}\) and \(i \in \{1, \ldots, d\}\). Then \(\lim D_i u_n^+ = \lim 1_{[u_n > 0]} D_i u_n = 1_{[u > 0]} D_i u = D_i u^+\) a.e. in virtue of (3). Now Statement I follows from the Lebesgue dominated convergence theorem.

Statement II follows from (2) and (3).

It follows from [GT83] p. 152 that \(u \land 1 = u + (1 - u)^+ \in H^1_{\text{loc}}(\Omega)\) and
\[D_i (u \land 1) = D_i (u + (1 - u)^+) = 1_{[u < 1]} D_i u \in L_2(\Omega) \ .\]
Therefore, \(u \land 1 \in H^1(\Omega)\) whenever \(0 \leq u \in H^1(\Omega)\). It follows from (2) that \(D_i u = 0\) a.e. on \([u = 1]\). So the proof of continuity is as in Statement I.

Next we prove Statement IV. Let \(u \in H^1_0(\Omega)\) and \(u_1, u_2, \ldots \in C_c^\infty(\Omega)\) be such that \(\lim u_n = u\) in \(H^1(\Omega)\). Let \(e_1, e_2, \ldots \in C_c^\infty(\mathbb{R}^d)\) be a regularizing sequence. Fix \(n \in \mathbb{N}\). Then \(e_n \ast u_n^+ \in C_c^\infty(\Omega)\) for \(m\) sufficiently large and \(\lim_m e_n \ast u_n^+ = u_n^+\) in \(H^1(\Omega)\). Hence \(u_n^+ \in H^1_0(\Omega)\) and \(u^+ = \lim_n u_n^+ \in H^1_0(\Omega)\). The proof for \(|u| \land 1\) is similar. □

**Remark 2.4**

I. The assertions of Lemma 2.3 remain valid if \(H^1(\Omega)\) is replaced by \(W^{1,p}(\Omega)\) with \(p \in [1, \infty]\).

II. It should be noted that \(H^1(\Omega)\) is not a Banach lattice. In fact, the intervals \([0, u] = \{v \in H^1(\Omega) : 0 \leq v \leq u\}\) are not norm bounded, in general.
III. If $H^1(\Omega)$ is the complex space, then one has
\[ D_i |u| = \text{Re}(\text{sgn} u \, D_i u) \]
for all $u \in H^1(\Omega)$ (cf. [Nag86] B-II, Lemma 2.4 and C-II.2 p. 251). In particular, one has
\[ \| |u| \|_{H^1(\Omega)} \leq \| u \|_{H^1(\Omega)} \quad (4) \]
In general, however, the inequality in (4) is strict. An example is $\Omega = (0, 1)$ and $u(x) = e^{ix}$. Then $\| |u| \|_{H^1(\Omega)} = 1$ but $\| u \|_{H^1(\Omega)} = \sqrt{2}$.

Next we introduce the following space:
\[ \overline{H}^1(\Omega) = \{ u|_\Omega : u \in H^1(\mathbb{R}^d) \}^{H^1(\Omega)}, \quad (5) \]
which will be useful in the context of Neumann boundary conditions. Note that $\overline{H}^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ which contains $H^1_0(\Omega)$. If $\Omega$ has the extension property (e.g. if the boundary of $\Omega$ is Lipschitz) then $H^1(\Omega) = \overline{H}^1(\Omega)$ but, in general, the two spaces are different. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$, it follows that
\[ \overline{H}^1(\Omega) = \{ u|_\Omega : u \in C_c^\infty(\mathbb{R}^d) \}^{H^1(\Omega)} \quad (6) \]

Lemma 2.5

I. If $u \in \overline{H}^1(\Omega)$, then $|u|, u^+, u^-, |u| \wedge 1 \in \overline{H}^1(\Omega)$.

II. If $u \in \overline{H}^1(\Omega)$ and $v \in W^{1,\infty}(\mathbb{R}^d)$, then $v|_\Omega \cdot u \in \overline{H}^1(\Omega)$

III. Let $u \in L_2(\Omega)$. Then $u \in \overline{H}^1(\Omega)$ if, and only if, there exist $v_1, \ldots, v_d \in L_2(\Omega)$ and $\varphi_1, \varphi_2, \ldots \in C_c^\infty(\mathbb{R}^d)$ such that $\lim \varphi_i|_\Omega = u$ in $L_2(\Omega)$ and $\lim D_i \varphi_i|_\Omega = v_i$ in $L_2(\Omega)$ for all $i \in \{1, \ldots, d\}$. In that case $D_i u = v_i$.

IV. If $u \in \overline{H}^1(\Omega) \cap L_\infty(\Omega)$ and $p \in \mathbb{N}$, then $u^p \in \overline{H}^1(\Omega)$ and $D_i (u^p) = pu^{p-1}D_i u$ for all $i \in \{1, \ldots, d\}$.

Proof. Let $u \in \overline{H}^1(\Omega)$. There exists a sequence $u_1, u_2, \ldots \in C_c^\infty(\mathbb{R}^d)$ such that $u_n|_\Omega \to u$ in $H^1(\Omega)$. Then $|u_n| \in H^1(\mathbb{R}^d)$ and $\lim |u_n|_\Omega = |u|$ in $H^1(\Omega)$ by Lemma 2.3. Therefore, $|u| \in \overline{H}^1(\Omega)$. Similarly one obtains that $u^+, u^-, |u| \wedge 1 \in \overline{H}^1(\Omega)$. This proves Statement I.

Next let $u \in \overline{H}^1(\Omega)$ and $v \in W^{1,\infty}(\mathbb{R}^d)$. Since $v \in L_\infty(\mathbb{R}^d)$ one has $\lim (vu_n)|_\Omega = v|_\Omega u$ in $L_2(\Omega)$ and since $D_i v \in L_\infty(\mathbb{R}^d)$ one similarly has $\lim D_i ((vu_n)|_\Omega) = \lim D_i v|_\Omega \cdot u_n|_\Omega + v|_\Omega D_i u_n|_\Omega = D_i v|_\Omega \cdot u + v|_\Omega D_i u = D_i (v|_\Omega \cdot u)$ in $L_2(\Omega)$. Because $(vu_n)|_\Omega \in H^1(\mathbb{R}^d)$ for all $n \in \mathbb{N}$ it follows that $v|_\Omega \cdot u \in \overline{H}^1(\Omega)$.

The proof of Statement III follows immediately from (6).

Finally, if $u \in \overline{H}^1(\Omega) \cap L_\infty(\Omega)$ and $p \in \mathbb{N}$. Set $c = \| u \|_\infty$. Let $u_1, u_2, \ldots \in C_c^\infty(\mathbb{R}^d)$ be such that $\lim u_n|_\Omega = u$ in $H^1(\Omega)$. Replacing $u_n$ by $(u_n \vee c) \wedge (-c) \ast e_n$, with $e_n \in C_c^\infty(\mathbb{R}^d)$ suitable, if necessary, we can assume that $\| u_n \|_\infty \leq c$. Taking subsequences, we can assume that $\lim u|_\Omega = u$ a.e., $\lim D_i u_n|_\Omega = D_i u$ a.e., $|u_n| \leq f$ a.e. and $|D_i u_n| \leq f$ a.e. for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$, for some $f \in L_2(\Omega)$. Then $u^p_n \in C_c^\infty(\mathbb{R}^d)$, $\lim u^p_n|_\Omega = u^p$ a.e. and $|u^p_n| \leq c^{p-1} f$ a.e. Therefore $\lim u^p_n|_\Omega = u^p$ in $L_2(\Omega)$ by the Lebesgue dominated
convergence theorem. Moreover, \( \lim \|D_i u_n^p\|_{\Omega} = \lim p u_n^{p-1}\|D_i u_n\|_{\Omega} = pu^{p-1}D_i u \) a.e. and \( |D_i u_n^p|_{\Omega}| \leq pe^{p-1}f \) a.e. for all \( n \in \mathbb{N} \) and \( i \in \{1, \ldots, d\} \). Hence \( \lim D_i u_n^p|_{\Omega} = pu^{p-1}D_i u \) in \( L_2(\Omega) \) by a second application of the dominated convergence theorem. Now it follows from Statement III that \( u^p \in \bar{H}^1(\Omega) \) and \( D_i(u^p) = pu^{p-1}D_i u \) for all \( i \in \{1, \ldots, d\} \).

The reason why \( \bar{H}^1(\Omega) \) is a suitable space for our purposes is that certain properties of \( H^1(\mathbb{R}^d) \) are inherited by \( \bar{H}^1(\Omega) \). We will use the following inequality of Nash.

**Lemma 2.6** There exists a \( c_N > 0 \) such that

\[
\|\varphi\|_{2}^{2+4/d} \leq c_N \|\varphi\|_{\bar{H}^1(\Omega)}^{2} \|\varphi\|_{1}^{4/d}
\]  

for all \( \varphi \in \bar{H}^1(\Omega) \cap L_1(\Omega) \).

**Proof.** There exists a constant \( c_N > 0 \) such that

\[
\|\varphi\|_{2}^{2+4/d} \leq c_N \|\varphi\|_{H^1(\mathbb{R}^d)}^{2} \|\varphi\|_{1}^{4/d}
\]

for all \( \varphi \in H^1(\mathbb{R}^d) \). (See [Rob91] p. 169 for a short proof.) In order to prove (7) we can assume that \( \varphi \in \bar{H}^1(\Omega) \) is positive. (Otherwise we replace \( \varphi \) by \( |\varphi| \) observing that \( \|\varphi\|_{H^1(\Omega)} \leq \|\varphi\|_{\bar{H}^1(\Omega)} \).) Let \( \varphi_1, \varphi_2, \ldots \in H^1(\mathbb{R}^d) \) be such that \( \lim \varphi_n|_{\Omega} = \varphi \) in \( H^1(\Omega) \) and a.e.. Replacing \( \varphi_n \) by \( \varphi_n^+ \), we can assume that \( \varphi_n \geq 0 \). Then \( \lim(\varphi_n \wedge \varphi) = \varphi \) in \( H^1(\Omega) \) by Lemma 2.5 and in \( L_1(\Omega) \) by the Lebesgue dominated convergence theorem. Now we obtain (7) for \( \varphi \) from (8) for \( \varphi_n \) and taking limits.

**Remark.** Note that the Nash inequality does not hold in \( H^1(\Omega) \) for general \( \Omega \).

We frequently use the following proposition on semigroups associated with continuous coercive forms.

**Proposition 2.7** Let \( V, \mathcal{H} \) be Hilbert spaces, \( V \) dense and continuously embedded in \( \mathcal{H} \) and \( a: V \times V \to \mathbb{C} \) a continuous sesquilinear form. Suppose the form \( a \) is coercive, i.e., there exist \( \omega \in \mathbb{R} \) and \( \mu > 0 \) such that

\[
\text{Re } a(u, u) + \omega \|u\|_{\mathcal{H}}^2 \geq \mu \|u\|_{V}^2
\]

for all \( u \in V \). Define the operator \( A \) associated with the form \( a \) by

\[
D(A) = \{u \in V : \exists v \in \mathcal{H} \forall \varphi \in V[a(u, \varphi) = (v, \varphi)]\}
\]

and \( Au = v \) for all \( u \in D(A) \) if \( a(u, \varphi) = (v, \varphi)_{\mathcal{H}} \) for all \( \varphi \in V \). Then \( A \) generates a holomorphic semigroup \( S = (e^{-tA})_{t>0} \) on \( \mathcal{H} \).


In the last part of this preliminary section, we put together some basic properties of traces. For that we assume that \( \Omega \) is a bounded open subset of \( \mathbb{R}^d \) with Lipschitz boundary \( \Gamma = \partial \Omega \). Note that this implies that \( \bar{H}^1(\Omega) = H^1(\Omega) \) and, even more, \( \Omega \) has the extension property, i.e. for all \( u \in H^1(\Omega) \) there exists a \( v \in H^1(\mathbb{R}^d) \) such that \( v|_{\Omega} = u \).
There exists a unique linear bounded operator \( B : H^1(\Omega) \to L_2(\Gamma) \) such that \( Bu = u|_\Gamma \) for all \( u \in H^1(\Omega) \cap C(\overline{\Omega}) \). Here \( \Gamma \) is considered as a measure space with the surface measure. The operator \( B \) is called the **trace operator** and \( Bu \) the **trace** of \( u \). (See Adams [Ada75] or Alt [Alt85] p. 168 for trace properties.) The operator \( B \) is a lattice homomorphism, i.e.,

\[
B(u \vee v) = (Bu) \vee (Bv) \quad B(u \wedge v) = (Bu) \wedge (Bv)
\]

and in particular

\[
B(u^+) = (Bu)^+ \quad B(u \wedge 1) = (Bu) \wedge 1
\]

for all \( u, v \in H^1(\Omega) \). In fact (9) and (10) are trivially valid for \( u|_\Omega \) with \( u \in C^\infty_c(\mathbb{R}^d) \). Since the lattice operations are continuous in \( H^1(\Omega) \) and \( L_2(\Gamma) \), the claim follows by taking limits. Note that \( H^1_0(\Omega) = \{ u \in H^1(\Omega) : Bu = 0 \} \).

### 3 Dirichlet boundary conditions

Given an elliptic operator arising from a form with Dirichlet boundary conditions, then we show in this section that the corresponding semigroup has a kernel which satisfies Gaussian bounds, provided the second order coefficients are once differentiable. Since we do not assume that the lower order coefficients are real, all spaces are complex in this section. The method we use here consists in proving uniform \( L_\infty \)-estimates for the semigroup perturbed by the Davies’ method. This is done via a criterion of quasi \( L_\infty \)-contractivity for non-symmetric forms due to Ouhabaz. Then the Gaussian estimates follow easily from the Nash inequality. The main theorem of this section is the following.

**Theorem 3.1** Let \( \Omega \subset \mathbb{R}^d \) open, let \( a_{ij} \in W^{1,\infty}(\Omega) \) be real functions for all \( i, j \in \{ 1, \ldots, d \} \) and let \( b_i, c_i \in W^{1,\infty}(\Omega) \) (complex) for all \( i \in \{ 1, \ldots, d \} \). Let \( c_0 \in L_\infty(\Omega) \). Consider the form \( a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C} \) defined by

\[
a(u, v) = \sum_{i,j=1}^d \int_\Omega a_{ij} D_i u \overline{D_j v} + \sum_{i=1}^d \int_\Omega b_i D_i u \overline{v} + \sum_{i=1}^d \int_\Omega c_i u \overline{D_i v} + \int_\Omega c_0 u \overline{v}.
\]

Suppose there exists a \( \mu > 0 \) such that

\[
\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \mu |\xi|^2
\]

for all \( \xi \in \mathbb{R}^d \), for a.e. \( x \in \Omega \). Let \( A \) be the operator associated with the continuous coercive form \( a \) and \( S = (e^{-tA})_{t>0} \) the semigroup generated by \( A \) (see Proposition 2.7). Then \( S \) interpolates on \( L_p \), \( 1 \leq p \leq \infty \) and there exists \( b, c > 0 \), \( \omega \in \mathbb{R} \) and \( K_t \in L_\infty(\Omega \times \Omega) \) such that

\[
|K_t(x;y)| \leq ct^{-d/2}e^{-b|x-y|^2t^{-1}}e^{\omega t}\quad (x,y)\text{-a.e.}
\]

and

\[
(S_t\varphi)(x) = \int_\Omega K_t(x;y)\varphi(y)dy \quad x\text{-a.e.}
\]

for all \( t > 0 \) and \( \varphi \in L_2(\Omega) \).
The proof relies on the Davies perturbation method to obtain Gaussian upper bounds. In order to be complete we describe briefly this method. For $K \in L_{\infty}(\Omega \times \Omega)$ define the integral operator $T_K \in \mathcal{L}(L_1(\Omega), L_\infty(\Omega))$ by

$$(T_K \varphi)(x) = \int_{\Omega} K(x; y) \varphi(y) \, dy.$$ 

(11)

Then it is well known that $K \mapsto T_K$ is an isometric isomorphism from $L_{\infty}(\Omega \times \Omega)$ onto $\mathcal{L}(L_1(\Omega), L_\infty(\Omega))$. (See, e.g. [ABu94] Theorem 1.3 for a short proof.) In particular, if $T \in \mathcal{L}(L_2(\Omega))$ is such that

$$\|T\|_{1 \to \infty} = \sup\{\|T\varphi\|_\infty : \varphi \in L_1 \cap L_2\} < \infty$$

then there exists a $K \in L_{\infty}(\Omega \times \Omega)$ such that (11) holds $x$-a.e. for all $\varphi \in L_1 \cap L_2$.

Next, let $W = \{\psi \in C^\infty_0(\mathbb{R}^d) : \psi \text{ is real and } \|D_i\psi\|_\infty \leq 1, \|D_iD_j\psi\|_\infty \leq 1 \text{ for all } i, j \in \{1, \ldots, d\}\}$.

Then clearly $d(x; y) = \sup\{\psi(x) - \psi(y) : \psi \in W\}$ defines a distance on $\mathbb{R}^d$. This distance is equivalent to the Euclidean metric.

**Lemma 3.2** There exists an $\alpha > 0$ such that

$$\alpha|x - y| \leq d(x; y) \leq \alpha^{-1}|x - y|$$

(12)

for all $x, y \in \mathbb{R}^d$.


Now let $S$ be a semigroup on $L_2(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^d$. For $\rho \in \mathbb{R}$ and $\psi \in W$ we define the perturbed semigroup $S^\rho$ on $L_2$ by $S^\rho_t = U_\rho S_t U_\rho^{-1}$, where $(U_\rho \varphi)(x) = e^{-\rho \varphi(x)} \varphi(x)$. Here we deliberately omit the dependence of $S^\rho$ and $U_\rho$ on $\varphi$ in our notation.

Gaussian upper estimates for the kernel of $S$ can be obtained from ultracontractivity of $S^\rho$, uniformly in $\rho$ and $\psi$. The following useful device is due to Davies [Dav89]. We include a proof for the convenience of the reader, since only variations of the criterion are explicitly given in the literature, cf. [Rob91] Chapter III p. 189 ff. and the proof of Proposition IV.2.2, or [Dav89] Section 3.2.

**Proposition 3.3** Let $S$ be a semigroup on $L_2(\Omega)$ and $c, \omega_1 \in \mathbb{R}$. Then the following are equivalent.

I. There exists a constant $\omega_2 > 0$ such that

$$\|S^\rho\|_{1 \to \infty} \leq c t^{-d/2} e^{\omega_1 t + \omega_2 \rho^2 t}$$

(13)

uniformly for all $\rho \in \mathbb{R}$, $t > 0$ and $\psi \in W$.

II. There exists a constant $b > 0$ such that the operators $S_t$ have a kernel $K_t \in L_{\infty}(\Omega \times \Omega)$ which verifies

$$|K_t(x; y)| \leq c t^{-d/2} e^{-b|x - y|^2 t^{-1}} e^{\omega_1 t} \quad (x, y)\text{-a.e.}$$

(14)

for all $t > 0$. 

8
Moreover, if one of the two conditions is valid then $S$ interpolates on $L_p(\Omega)$, $1 \leq p \leq \infty$ and there exists a constant $c_1 > 0$, depending only on the constants $b$ and $c$ in (14) such that $\|S_t\|_{p \to p} \leq c_1 e^{\omega_1 t}$ uniformly for all $t > 0$ and $p \in [1, \infty]$.

Proof. "I$\Rightarrow$II". Taking $\rho = 0$ we see that $S_t$ has a kernel $K_t \in L_\infty(\Omega \times \Omega)$. Then for each $\rho$ and $t$ the operator $S_t^\rho$ has a kernel $K_t^\rho$, given by

$$K_t^\rho(x; y) = e^{-\rho(\psi(x) - \psi(y))} K_t(x; y) \quad (x, y)\text{-a.e.}$$

Then (13) implies that for all $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in W$ one has

$$|K_t(x; y)| \leq c t^{-d/2} e^{\omega_1 t + \omega_2 \rho^2 t} e^{\rho(\psi(x) - \psi(y))} \quad (x, y)\text{-a.e.}$$

Replacing $\rho$ by $-\rho$ one deduces that

$$|K_t(x; y)| \leq c t^{-d/2} e^{\omega_1 t + \omega_2 \rho^2 t} e^{-\rho(\psi(x) - \psi(y))} \quad (x, y)\text{-a.e.}$$

Next, Lemma 3.4 below implies that

$$|K_t(x; y)| \leq c t^{-d/2} e^{\omega_1 t + \omega_2 \rho^2 t} e^{-\rho d(x; y)} \quad (x, y)\text{-a.e.}$$

for each $t > 0$ and $\rho \in \mathbb{R}$. For fixed $t > 0$ and $x, y \in \Omega$ the minimum over $\rho$ of the right hand side is attained in $\rho = (2\omega_1 t)^{-1} d(x; y)$. Thus, applying Lemma 3.4 again we obtain

$$|K_t(x; y)| \leq c t^{-d/2} e^{-(4\omega_1 t)^{-1} d(x; y)^2} e^{\omega_1 t} \quad (x, y)\text{-a.e.}$$

Now (14) follows from Lemma 3.2 with $b = (4\omega_2)^{-1} \alpha^2$.

"II$\Rightarrow$I". Let $\alpha$ be as in Lemma 3.2. Then

$$\|S_t^\rho\|_{2 \to \infty} = \sup_{\|\varphi\| \leq 1} \|S_t^\rho \varphi\|_{\infty} = \sup_{\|\varphi\| \leq 1} \operatorname{esssup}_{x \in \Omega} \left| \int_{\Omega} K_t^\rho(x; y) \varphi(y) dy \right|$$

$$= \operatorname{esssup}_{x \in \Omega} \operatorname{esssup}_{y \in \Omega} |K_t^\rho(x; y)| \leq \operatorname{esssup}_{x, y \in \Omega} |K_t^\rho(x; y)| e^{b|x-y|} \leq \sup_{x, y \in \Omega} c t^{-d/2} e^{b|x-y|^2 t^{-1} + \alpha^{-1} |x-y| e^{\omega_1 t}} \leq c t^{-d/2} e^{\omega_2 \rho^2 t} e^{\omega_1 t}$$

with $\omega_2 = (4\alpha^2 b)^{-1}$.

Finally, suppose II is valid. Let $T$ be the semigroup on $L_2(\mathbb{R}^d)$ generated by the operator $-\sum_{i=1}^d \partial^2 / \partial x_i^2$. Then $T$ interpolates on $L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ and $T$ has the Gaussian kernel $K^\Delta$. Then

$$|K_t(x; y)| \leq c(\pi b^{-1})^{d/2} e^{\omega_1 t} K^\Delta_{(4\alpha b)^{-1}}(x; y) \quad \text{a.e.} (x, y) \in \Omega \times \Omega$$

for all $t > 0$. Therefore, $|S_t \varphi| \leq c(\pi b^{-1})^{d/2} e^{\omega_1 t} T_{(4\alpha b)^{-1}}(x; y)$ for all $\varphi \in L_1(\Omega) \cap L_2(\Omega)$ and $t > 0$. So by Lemma 2.1.V it follows that $S$ interpolates on $L_p(\Omega)$, $1 \leq p \leq 2$. By duality, $S$ interpolates on $L_p(\Omega)$, $2 \leq p \leq \infty$. Moreover, $\|S_t\|_{p \to p} \leq c(\pi b^{-1})^{d/2} e^{\omega_1 t}$ for all $t > 0$ and $p \in [1, \infty]$.

In the previous proposition we needed the following result on infima, which can be stated in a more general context.
Lemma 3.4 Let \( Y \) be a \( \sigma \)-compact topological space and let \( F \subset C(Y) \) and assume that \( f_0(x) = \inf_{f \in F} f(x) \) for all \( x \in Y \). Then there exist \( f_1, f_2, \ldots \in F \) such that \( f_0(x) = \inf_{n \in \mathbb{N}} f_n(x) \) for all \( x \in Y \). In particular, if \((Y, \Sigma, \mu)\) is a measure space and \( h: Y \to \mathbb{R} \) is a measurable function such that \( h \leq f - \mu \)-a.e. for all \( f \in F \) then \( h \leq f_0 \mu \)-a.e..

**Proof.** First we can assume that \( Y \) is compact. Secondly, replacing \( F \) by \( F - f_0 \) we can (and do) assume that \( f_0 = 0 \). Let \( m \in \mathbb{N} \). For all \( x \in Y \) there exists an \( f_{x,m} \in F \) such that \( f_{x,m}(x) < m^{-1} \) and hence \( f_{x,m} < m^{-1} \) on an open neighbourhood \( U_{x,m} \) of \( x \). By compactness we find \( x_{m,1}, \ldots, x_{m,n_m} \in Y \) such that \( Y = \bigcup_{j=1}^{n_m} U_{x_{m,j},m} \). Then \( \inf_j f_{x_{m,j},m}(x) < m^{-1} \) for all \( x \in Y \). Now the set \( F_0 = \{x_{m,j} : m \in \mathbb{N}, j \in \{1, \ldots, n_m\} \} \) is countable and \( \inf_{f \in F_0} f(x) = 0 \) for all \( x \in Y \).

In view of Proposition 3.3, we have to show (13) in order to prove Theorem 3.1. This will be done in two steps. At first we show \( L_\infty \)-contractivity with help of the following criterion.

**Proposition 3.5** Denote by \( S = (e^{-tA})_{t \geq 0} \) the semigroup on \( L_2(\Omega) \) generated by the operator \( A \) of Theorem 3.1. Assume that

\[
\text{Re} \left( \sum_{i,j=1}^{d} a_{ij} D_i u D_j v + \sum_{i=1}^{d} (b_i - c_i) D_i u \overline{v} + (c_0 + \sum_{i=1}^{d} D_i c_i) |u|^2 \right) \geq 0 \quad \text{a.e.} \tag{15}
\]

for all \( u \in H_0^1(\Omega) \). Then \( S \) is \( L_\infty \)-contractive. In particular, \( S \) interpolates on \( L_2, 2 \leq p \leq \infty \).

**Proof.** Using integration by parts we obtain

\[
a(u,v) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} D_i u D_j v + \sum_{i=1}^{d} \int_{\Omega} (b_i - c_i) D_i u \overline{v} + \int_{\Omega} (c_0 + \sum_{i=1}^{d} D_i c_i) u \overline{v}
\]

for all \( u, v \in H_0^1(\Omega) \). Moreover, \((1 \wedge |u|) \text{sgn} u \in H_0^1(\Omega) \) for all \( u \in H_0^1(\Omega) \). Therefore, the \( L_\infty \)-contractivity follows from [Ouh92b] Theorem 4.2(3). The last statement follows from Lemma 2.1.1.

**Lemma 3.6** Let \( \psi \in W \) be fixed and \( \rho \in \mathbb{R} \). Denote by \( S = (e^{-tA})_{t \geq 0} \) the semigroup on \( L_2(\Omega) \) generated by the operator \( A \) of Theorem 3.1. Then the generator \( A^\rho \) of the perturbed semigroup \( S^\rho \) is associated with the form \( a^\rho \) on \( H_0^1(\Omega) \times H_0^1(\Omega) \) given by

\[
a^\rho(u,v) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} D_i u D_j v + \sum_{i=1}^{d} \int_{\Omega} b_i^\rho D_i u \overline{v} + \sum_{i=1}^{d} \int_{\Omega} c_i^\rho u D_i v + \int_{\Omega} c_0^\rho u \overline{v},
\]

where

\[
b_i^\rho = b_i - \rho \sum_{j=1}^{d} a_{ij} \psi_j, \\
c_i^\rho = c_i + \rho \sum_{k=1}^{d} a_{ik} \psi_k, \\
c_0^\rho = c_0 - \rho^2 \sum_{i,j=1}^{d} a_{ij} \psi_i \psi_j + \rho \sum_{i=1}^{d} b_i \psi_i - \rho \sum_{i=1}^{d} c_i \psi_i
\]

and \( \psi_i = D_i \psi \) for all \( i \in \{1, \ldots, d\} \).
Proof. Note that \( A^\rho = U_\rho AU_\rho^{-1} \). Furthermore, one has \( e^{\rho\psi}H^1_0(\Omega) = H^1_0(\Omega) \) and \( U_\rho D_i U_\rho^{-1} = D_i + \rho \psi_i \). Therefore,

\[
a(U_\rho^{-1} u, U_\rho v) = \sum_{i,j=1}^d \int_{\Omega} a_{ij}(D_i + \rho \psi_i)u(D_j - \rho \psi_j)v \\
+ \sum_{i=1}^d \int_{\Omega} b_i(D_i + \rho \psi_i)u \overline{v} + \sum_{i=1}^d \int_{\Omega} c_i u(D_i - \rho \psi_i)\overline{v} + \int_{\Omega} c_0 u \overline{v}
\]

for all \( u, v \in H^1_0(\Omega) \). This proves the lemma. \( \square \)

The second statement in the following lemma shows again the well known fact that the form \( a \) is coercive, which we have used already. For the sequel we need a uniform coercivity estimate for the form \( a^\rho \).

**Lemma 3.7** Denote by \( S = (e^{-tA})_{t>0} \) the semigroup on \( L^2(\Omega) \) generated by the operator \( A \) of Theorem 3.1.

I. There exists an \( \omega > 0 \) such that

\[
\|S_t^\rho \varphi\|_\infty \leq e^{\omega(1+\rho^2)t}\|\varphi\|_\infty
\]

uniformly for all \( \rho \in \mathbb{R}, \varphi \in W, t > 0 \) and \( \varphi \in L^2 \cap L^\infty \). The constant \( \omega \) depends only on \( \mu, \|a_{ij}\|_{W^{1,\infty}}, \|b_i\|_\infty, \|c_i\|_{W^{1,\infty}} \) and \( \|c_0\|_\infty \).

II. There exists an \( \omega > 0 \) such that

\[
\text{Re} a^\rho(u, u) + \omega(1 + \rho^2)\|u\|_2^2 \geq 2^{-1} \mu\|u\|_{H^1_0(\Omega)}^2
\]

uniformly for all \( \rho \in \mathbb{R}, \varphi \in W, t > 0 \) and \( u \in H^1_0(\Omega) \). The constant \( \omega \) depends only on \( \mu, \|a_{ij}\|_{\infty}, \|b_i\|_\infty, \|c_i\|_\infty \) and \( \|c_0\|_\infty \).

Proof. We show that there exists an \( \omega \in \mathbb{R} \) such that

\[
\text{Re} \left( \sum_{i,j=1}^d a_{ij} D_i u \overline{D_j u} + \sum_{i=1}^d (b_i - c_i^\rho) D_i u \overline{\varphi} + (c_i^\rho + \sum_{i=1}^d D_i c_i^\rho) |u|^2 + \omega(1 + \rho^2)|u|^2 \right) \\
\geq 2^{-1} \mu \sum_{i=1}^d |D_i u|^2 \quad \text{a.e.} \quad (16)
\]

for all \( u \in H^1_0(\Omega), \rho \in \mathbb{R} \) and \( \varphi \in W \). Here \( b_i^\rho, c_i^\rho \) and \( c_0^\rho \) are as in Lemma 3.6. Let

\[
M = 1 + \max\{\|a_{ij}\|_{W^{1,\infty}}, \|b_i\|_\infty, \|c_i\|_{W^{1,\infty}}, \|c_0\|_\infty\}.
\]

The first term in (16) can be estimated by

\[
\text{Re} \sum_{i,j=1}^d a_{ij} D_i u \overline{D_j u} \geq \mu \sum_{i=1}^d |D_i u|^2 \quad \text{a.e.},
\]

11
for all \( u \in H_0^1(\Omega) \). The second term can be majorated in the following manner,

\[
\left| \Re \sum_{i=1}^{d}(b_i^e - c_i^e)(D_i u)\bar{u} \right| \leq \sum_{i=1}^{d} |b_i - c_i| |D_i u| |u| + |\rho| \sum_{i=1}^{d} \sum_{k=1}^{d} |a_{ki} + a_{ik}| \psi_k(D_i u)\bar{u} \\
\leq 2M \sum_{i=1}^{d} |D_i u| |u| + 2dM|\rho| \sum_{i=1}^{d} |D_i u| |u| = 2M(d + |\rho|) \sum_{i=1}^{d} |D_i u| |u|
\]

\[
\leq 2M(d + |\rho|) \varepsilon \sum_{i=1}^{d} |D_i u|^2 + (2\varepsilon)^{-1}dM(d + |\rho|)|u|^2 \\
\leq 2^{-1}\mu \sum_{i=1}^{d} |D_i u|^2 + 4d^3 M^2 \mu^{-1}(1 + \rho^2)|u|^2 \quad \text{a.e.} ,
\]

where we have chosen \( \varepsilon = (4M(d + |\rho|))^{-1}\mu \) and used the inequality \( xy \leq \delta x^2 + (4\delta)^{-1}y^2 \).

Finally, we majorate the coefficient in the third term in the following manner,

\[
|\Re(c_0^e + \sum_{i=1}^{d} D_ic_i^e)|
\]

\[
\leq M + d^2 M \rho^2 + dM|\rho| + dM|\rho| + \left| \sum_{i=1}^{d} \left( D_ic_i + \rho \sum_{k=1}^{d} ((D_ia_{ki}) \psi_k + a_{ki} D_i \psi_k) \right) \right|
\]

\[
\leq 2d^2 M(1 + \rho^2) + M(d + 2d^2|\rho|) .
\]

Here we have used the differentiability of the second order coefficients. Note that in case \( \rho = 0 \) these terms vanish. Hence, for all \( \rho \in \mathbb{R} \)

\[
|\Re(c_0^e + \sum_{i=1}^{d} D_ic_i^e)| \leq 4d^3 M(1 + \rho^2) \quad \text{a.e.} ,
\]

for all \( \rho \in \mathbb{R} \). Therefore, (16) holds if \( \omega = 4d^3 M^2 \mu^{-1} + 4d^2 M \). Now Statement I follows from Proposition 3.5.

Similarly one can estimate

\[
\Re \left( \sum_{i,j=1}^{d} a_{ij} D_i u D_j \bar{u} + \sum_{i=1}^{d} b_i^e D_i D_i u \bar{u} + \sum_{i=1}^{d} c_i^e D_i D_i \bar{u} + \omega(1 + \rho^2)|u|^2 \right)
\]

\[
\geq 2^{-1}\mu \sum_{i=1}^{d} |D_i u|^2 \quad \text{a.e.}
\]

if \( \omega = 4d^3 M_0^2 \mu^{-1} + 2d^2 M_0 \) and

\[
M_0 = 1 + \max\{\|a_{ij}\|_\infty, \|b_i\|_\infty, \|c_i\|_\infty, \|c_0\|_\infty\} .
\]

Integrating this inequality one obtains

\[
\Re a^\rho(u, u) + \omega(1 + \rho^2)\|u\|^2_2 \geq 2^{-1}\mu \sum_{i=1}^{d} |D_i u|^2
\]

for all \( u \in H_0^1(\Omega) \). Hence

\[
\Re a^\rho(u, u) + (\omega + 2^{-1}\mu)(1 + \rho^2)\|u\|^2_2 \geq 2^{-1}\mu\|u\|^2_{H_0^1(\Omega)} .
\]

12
Replacing $\omega$ by $\omega + 2^{-1}\mu$ proves Statement II.

We now know that the perturbed semigroup is quasi-contractive on $L_\infty$ and hence by duality one has a bound on $\mathcal{L}(L_1)$. Next we convert the $L_2$-ellipticity estimate and the $\mathcal{L}(L_1)$-bound in a $\mathcal{L}(L_1,L_2)$-bound for $S$ (cf. [Rob91] Step 2 of the proof of Proposition III.4.2, or [Dav89], Theorem 2.4.6). For our purposes, it is important to obtain independent constants.

**Proposition 3.8** Let $a$ be a continuous form with domain $D(a) = V$, with $V$ a Hilbert space which is continuous embedded in $L_2(X)$, where $(X, \Sigma, m)$ is a $\sigma$-finite measure space. Assume there exists a constant $\mu > 0$ such that $\text{Re}\, a(\varphi, \varphi) \geq \mu \|\varphi\|_2^2$ for all $\varphi \in V$. Let $S$ be the semigroup on $L_2$ generated by the operator associated with the form $a$. Suppose that $S$ interpolates on $L_p$, $1 \leq p \leq 2$. Assume there exists a $c_1 > 0$ such that $\|S_t\|_{1\to2} \leq c_1$ for all $t > 0$. Further, let $c_n, n > 0$ and suppose that the Nash inequality

$$\|\varphi\|_2^{2+4/n} \leq c_N \|\varphi\|_1^2 \|\varphi\|_1^4/n$$

is valid for all $\varphi \in L_1 \cap V$. Then there exists a constant $c > 0$, depending continuously on $\mu, c_1, c_N$ and $n$ and which is otherwise independent of $a$, such that

$$\|S_t\|_{1\to2} \leq ct^{-n/4}$$

uniformly for all $t > 0$.

**Proof.** Let $\varphi \in L_1(\Omega) \cap L_2(\Omega)$. Then

$$\frac{d}{dt} \|S_t \varphi\|_2^2 = -2 \text{Re} a(S_t \varphi, S_t \varphi) \leq -2\mu \|S_t \varphi\|_2^2 \leq -\frac{2\mu \|S_t \varphi\|_2^{2+4/n}}{c_N \|\varphi\|_1^{4/n}}.$$ 

Therefore,

$$\frac{d}{dt} (\|S_t \varphi\|_2^{2-2/n}) \leq -\frac{2}{n} (\|S_t \varphi\|_2^{1-2/n}) \frac{d}{dt} \|S_t \varphi\|_2 \geq \frac{4\mu}{nc_N c_1^{4/n}} \|\varphi\|_1^{-4/n}$$

and by integration

$$\|S_t \varphi\|_2^{4/n} = (\|S_t \varphi\|_2^{2-2/n}) \geq \frac{4\mu}{nc_N c_1^{4/n}} \|\varphi\|_1^{-4/n}.$$ 

Now the theorem follows if one takes $c = (4\mu)^{-n/4}(nc_N)^{n/4}c_1$. 

We continue the proof of Theorem 3.1.

**Corollary 3.9** Denote by $S = (e^{-tA})_{t>0}$ the semigroup on $L_2(\Omega)$ generated by the operator $A$ of Theorem 3.1. Then there exist $c, \omega > 0$ such that

$$\|S_t \psi\|_{1\to\infty} \leq ct^{-d/2} e^{\omega(t+\rho^2)t}$$

uniformly for all $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in W$. 

13
Proof. Since the form-adjoint of \( a \) is of the same form as the form \( a \) it follows from Lemma 3.7 that there exist \( \mu, \omega > 0 \) such that \( \Re a^\omega(\varphi, \varphi) + \omega(1 + \rho^2)\|\varphi\|_2^2 \geq \mu\|\varphi\|_2^2 \), and \( \|S_t^\rho e^{-\omega(1+\rho^2)t}\|_{1 \rightarrow 1} \leq 1 \) uniformly for all \( \rho \in \mathbb{R} \), \( \psi \in W \) and \( t > 0 \). Here \( a^\omega \) is as in Lemma 3.6. Moreover, by the Nash inequality (Lemma 2.6) there exists a \( c_N > 0 \) such that
\[
\|\varphi\|_2^{2+4/d} \leq c_N\|\varphi\|_2^d \|\varphi\|_4^{4/d}
\]
for all \( \varphi \in L_1(\Omega) \cap H_0^1(\Omega) \). Then by Proposition 3.8 there exists a \( c > 0 \) such that \( \|S_t^\rho e^{-\omega(1+\rho^2)t}\|_{1 \rightarrow 2} \leq ct^{-d/4} \) uniformly for all \( \rho \in \mathbb{R} \), \( \psi \in W \) and \( t > 0 \). So
\[
\|S_t^\rho\|_{1 \rightarrow 2} \leq ct^{-d/4} e^{\omega(1+\rho^2)t} .
\]
But by duality it then follows that
\[
\|S_t^\rho\|_{2 \rightarrow \infty} \leq ct^{-d/4} e^{\omega(1+\rho^2)t} ,
\]
possibly by enlarging \( c \) and \( \omega \). Then
\[
\|S_t^\rho\|_{1 \rightarrow \infty} \leq \|S_t^\rho\|_{1 \rightarrow 2} \|S_t^\rho\|_{2 \rightarrow \infty} \leq 2^{d/2} e^{d^2/2} \|\varphi\|_2 \leq \omega(1+\rho^2)t \]
uniformly for all \( t > 0, \rho \in \mathbb{R} \) and \( \psi \in W \).

Now Theorem 3.1 has been proved completely by an application of Proposition 3.3.

Remark 3.10

I. A version of Theorem 3.1 with somewhat complementary assumptions has been obtained by [ER93] for \( \Omega = \mathbb{R}^d \): if \( a_{ij} \in W^{2,\infty}(\mathbb{R}^d) \) are complex coefficients and satisfy
\[
\Re \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \mu |\xi|^2
\]
for all \( \xi \in \mathbb{R}^d \), for a.e. \( x \in \mathbb{R}^d \), with \( \mu > 0 \), and \( b_i, c_i, c_0 \in L_\infty \) then the assertions in Theorem 3.1 are valid.

II. If the coefficients \( a_{ij} \) in Theorem 3.1 are real and symmetric and \( b_i = c_i = 0 \), then one can deduce Theorem 3.1 for \( \Omega \) from the corresponding theorem for \( \mathbb{R}^d \) since the semigroup on \( L_p(\Omega) \) is dominated by the corresponding semigroup on \( L_p(\mathbb{R}^d) \) (see [ABa93] Examples 4.9 and 5.6 and Theorem 6.2).

4 General boundary conditions

In this section we consider second order operators in divergence form with real, \( L_\infty \), non-symmetric second order coefficients. Moreover, we drop the assumption that the operator satisfies Dirichlet boundary conditions. Since here all coefficients are supposed to be real we will only work over the real field in this section. So all spaces are real spaces. In general there are no Gaussian bounds for an elliptic operator defined on an open subset \( \Omega \subset \mathbb{R}^d \) with Neumann boundary conditions, even if the operator has constant coefficients.
An example is the Laplacian $\Delta$ on $\Omega = \bigcup_{n=1}^{\infty} (2^{-(n+1)}, 2^{-n}) \subset [0,1] \subset \mathbb{R}$. Then $1_{(2^{n+1}, 2^{-n})}$ is an eigenvector of $\Delta$ with eigenvalue 0 for all $n \in \mathbb{N}$. Therefore, $S_t$ has an eigenvalue with infinite multiplicity and $S_t$ is not compact for any $t > 0$. But the existence of a kernel for $S_t$ with Gaussian bounds on the pre-compact set $\Omega \times \Omega \subset [0,1] \times [0,1]$ implies that $S_t$ is a Hilbert–Schmidt operator and therefore compact. There are also examples of bounded connected domains $\Omega$ where $S_t$ is not compact on $L^2(\Omega)$, see Hempel–Seco–Simon [HSS91] for a systematic study of spectral properties of these kind of operators. Thus, in order to establish Gaussian estimates for the kernel one needs some kind of regularity of $\Omega$ or of the domain on which the sectorial form is defined. When the form domain equals $H^1(\Omega)$ there is never a problem, but in case the form domain equals $H^1(\Omega)$ one frequently demands in the literature the condition that $\Omega$ has the extension property, i.e., for all $u \in H^1(\Omega)$ there exists a $v \in H^1(\mathbb{R}^d)$ such that $v|_{\Omega} = u$. For example, if the boundary of $\Omega$ is Lipschitz continuous then $\Omega$ has the extension property. We use another way to avoid these difficulties and consider in this section "good Neumann boundary conditions" by considering as form domain the closed subspace

$$\widetilde{H}^1(\Omega) = \{u|_{\Omega} : u \in H^1(\mathbb{R}^d)\}^{H^1(\Omega)}$$

of $H^1(\Omega)$ instead of $H^1(\Omega)$ (see Section 2).

Now let $A$ be the (formal) elliptic operator

$$Au = - \sum_{i,j=1}^{d} D_j a_{ij} D_i u + \sum_{i=1}^{d} b_i D_i u - \sum_{i=1}^{d} D_i (c_i u) + c_0 u$$

(18)

with real coefficients. For the coefficients we suppose that $a_{ij} \in L_\infty(\Omega)$ $(i,j \in \{1, \ldots , d\})$, $b_i, c_i \in W^{1,\infty}(\Omega)$ $(i \in \{1, \ldots , d\})$ and $c_0 \in L_\infty(\Omega)$ are real valued functions such that

$$\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq \mu |\xi|^2$$

(19)

for all $\xi \in \mathbb{R}^d$, for a.e. $x \in \Omega$, where $\mu > 0$ is a fixed constant. We emphasize that the coefficients $a_{ij}$ need not be symmetric. We consider realizations of $A$ in $L^2(\Omega)$ with various boundary conditions. They will be defined by a form domain $V$ satisfying the following hypotheses:

1. $V$ is a closed subspace of $\widetilde{H}^1(\Omega)$ ,
2. $H^1_0(\Omega) \subset V$ ,
3. $v \in V$ implies $|v|, |v| \wedge 1 \in V$ ,
4. $v \in V, u \in \widetilde{H}^1(\Omega), |u| \leq v$ implies $u \in V$ .

Assumption (23) means that $V$ is an ideal in $\widetilde{H}^1(\Omega)$. Furthermore, we assume that the first order coefficients satisfy

$$i \in \{1, \ldots , d\} \text{ and } v \in V \text{ implies } b_i v, c_i v \in H^1_0(\Omega)$$

(24)

15
Now we consider the form $a: V \times V \rightarrow \mathbb{R}$ given by

$$a(u, v) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} D_i u D_j v + \sum_{i=1}^{d} \int_{\Omega} b_i D_i u v + \sum_{i=1}^{d} \int_{\Omega} c_i u D_i v + \int_{\Omega} c_0 u v .$$

Then $a$ is clearly continuous and coercive, i.e., there exists an $\omega \in \mathbb{R}$ such that

$$a(u, u) + \omega \|u\|_2^2 \geq 2^{-1} \mu \|u\|_2^2$$

for all $u \in V$. Let $A$ be the operator on $L_2(\Omega)$ associated with the form $a$ on $V$. It follows from Proposition 2.7 that the complexification of the operator $A$ associated with the complexified form $a$ generates a holomorphic semigroup $S$ on $L_2(\Omega)$. Recall that we assume throughout this section that the spaces are real.

If $V = H_0^1(\Omega)$ we say that $A$ is the realization of $A$ in $L_2(\Omega)$ with Dirichlet boundary conditions. In that case (24) is satisfied whenever $b_i, c_i \in W^{1,\infty}(\Omega)$.

If $V = \overline{H}^1(\Omega)$ we say that $A$ is the realization of $A$ in $L_2(\Omega)$ with good Neumann boundary conditions. In that case (24) is satisfied whenever $b_i, c_i \in W^{1,\infty}(\Omega)$. If $\Omega$ is bounded, then $b_i, c_i \in H_0^1(\Omega)$ is a necessarily condition for (24), since $1 \in V$.

**Example 4.1** If $a_{ij} = \delta_{ij}$, $V = H^1(\Omega)$ with $\Omega$ regular, then one obtains the Neumann-Laplacian with Neumann boundary conditions (cf. Example 4.8).

**Example 4.2** In general the boundary conditions depend on the coefficients. As an example we consider a concrete non-symmetric case. Let $\Omega = \{ r e^{i\theta} : r \in (0, 1), \theta \in \mathbb{R} \}$ be the open disk in $\mathbb{R}^2$ and let $V = H^1(\Omega) = \overline{H}^1(\Omega)$. Consider the pure second order operator with constant coefficients $(a_{ij}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Then one can easily see by Green's formula that $Au = -\Delta u$ for all $u \in D(A)$, and for $u \in C^2(\mathbb{R}^2)$ one has

$$u \in D(A) \Leftrightarrow u_\tau = u_\phi \text{ on } \partial \Omega .$$

Similarly, if we choose $\Omega = (0, 1) \times (0, 1)$ and the same matrix for the coefficients then

$$u \in D(A) \Leftrightarrow \begin{cases} u_x = -u_y & \text{on } (0, 1) \times \{0\} \cup \{0\} \times (0, 1) \\ u_x = u_y & \text{on } \{0\} \times (0, 1) \cup \{0\} \times (0, 1) \end{cases}$$

for all $u \in C^2(\mathbb{R}^2)$.

**Example 4.3** We may also consider what we call pseudo Dirichlet boundary conditions by choosing

$$V = \overline{H}^1_0(\Omega) = \{ u|_\Omega : u \in H^1(\mathbb{R}^d), u = 0 \text{ a.e. on } \Omega^c \} .$$

One has always $H^1_0(\Omega) \subset \overline{H}^1_0(\Omega)$. The spaces are equal if $\Omega$ is of class $C^1$, but they are different in general (see [ABA92], [ABA93]). It is clear that $\overline{H}^1_0(\Omega)$ satisfies assumptions (20), (21) and (22). We show the ideal property (23). Let $u|_\Omega \in H^1_0(\Omega)$, where $u \in H^1(\mathbb{R}^d)$, $u = 0$ a.e. on $\Omega^c$. Let $v \in \overline{H}^1(\Omega)$, $|v| \leq u$. There exist $w_1, w_2, \ldots \in H^1(\mathbb{R}^d)$ such that
lim \omega_n|\Omega = v \text{ in } H^1(\Omega) \text{ and a.e. on } \Omega. \text{ Let } v_n = (u_n \wedge u) \vee (-u). \text{ Then } v_n \in H^1(\mathbb{R}^d), v_n = 0 \text{ a.e. on } \Omega^c \text{ and } \lim v_n = \tilde{v} \text{ in } L^2(\mathbb{R}^d), \text{ where }

\tilde{v} = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}

It follows from (3) that \(D_i v_n = 0\) a.e. on \(\Omega^c\). Therefore, \(\int_{\mathbb{R}^d} |D_i v_n - D_i v_m|^2 = \int_{\Omega} |D_i v_n - D_i v_m|^2\). Hence, \(v_1, v_2, \ldots\) is a Cauchy sequence in \(H^1(\mathbb{R}^d)\). This shows that \(\tilde{v} = \lim v_n \in H^1(\mathbb{R}^d)\).

Finally one may consider mixed boundary conditions in the following way.

**Example 4.4** Let \(\Gamma_1 \subset \partial \Omega\) be a closed set and

\[ V = \{ u|_{\Omega} : u \in C^\infty_c(\mathbb{R}^d \setminus \Gamma_1) \}^{H^1(\Omega)}. \]

Let \(\Gamma_2 \subset \partial \Omega\) be closed such that \(\Gamma_1 \cup \Gamma_2 = \partial \Omega\) and \(b_i, c_i \in \{ \varphi|_{\Omega} : \varphi \in C^\infty_c(\mathbb{R}^d \setminus \Gamma_2) \}^{W^{1,\infty}(\Omega)}\). Then (20) - (24) is satisfied.

**Proof.** The domain \(V\) clearly satisfies (20) and (21). Let \(u \in V\). Then there exist \(u_1, u_2, \ldots \in C^\infty_c(\mathbb{R}^d \setminus \Gamma_1)\) such that \(\lim u_n = u\) in \(H^1(\Omega)\). Then \(\lim u_n^+ = u^+\) in \(H^1(\Omega)\). Let \(e_1, e_2, \ldots \in C^\infty_c(\mathbb{R}^d)\) be a regularizing sequence. Fix \(n \in \mathbb{N}\). Then for sufficiently large \(m\) one has \(e_m * u_n^+ \in C^\infty(\mathbb{R}^d \setminus \Gamma_1)\) and \(\lim_m e_m * u_n^+ = u_n^+\) in \(H^1(\mathbb{R}^d)\). Therefore, \(u^+ \in V\). It follows that \(|u| = u^+ \vee u^- \in V\). Using the regularizing sequence again one proves in a similar way that \(u \wedge 1 \in V\) whenever \(0 \leq u \in V\). This proves condition (22).

Next we prove the ideal condition (23). Let \(v \in V\), \(u \in H^1(\Omega)\) and suppose that \(|u| \leq v\). There exist \(v_1, v_2, \ldots \in C^\infty_c(\mathbb{R}^d \setminus \Gamma_1)\), \(u_1, u_2, \ldots \in C^\infty_c(\mathbb{R}^d)\) and \(f \in L^2(\Omega)\) such that \(\lim v_n|_{\Omega} = v\) in \(H^1(\Omega)\), \(\lim u_n|_{\Omega} = u\) in \(H^1(\Omega)\), \(\lim v_n|_{\Omega} = v\) a.e., \(\lim D_i v_n|_{\Omega} = D_i v\) a.e., \(\lim D_i u_n|_{\Omega} = D_i u\) a.e., and \(|v_n| \leq f\) a.e., \(|D_i v_n| \leq f\) a.e., \(|u_n| \leq f\) a.e. and \(|D_i u_n| \leq f\) a.e. on \(\Omega\) for all \(n \in \mathbb{N}\) and \(i \in \{1, \ldots, d\}\). Then \(\lim u_n = u^+\) in \(H^1(\Omega)\) and \(\lim(u_n^+ \vee v_n)|_{\Omega} = u^+ \vee v = u^+\) in \(H^1(\Omega)\). For all \(n \in \mathbb{N}\) one has \(e_m * (u_n^+ \vee v_n) \in C^\infty(\mathbb{R}^d \setminus \Gamma_1)\) for large \(m\) and \(\lim_m e_m * (u_n^+ \vee v_n) = u_n^+ \vee v_n\) in \(H^1(\mathbb{R}^d)\). So \(u^+ \in V\). Similarly \(u^- \in V\) and therefore \(u = u^+ - u^- \in V\).

Finally, let \(b \in \{ \varphi|_{\Omega} : \varphi \in C^\infty_c(\mathbb{R}^d \setminus \Gamma_2) \}^{W^{1,\infty}(\Omega)}\) and \(u \in V\). We show that \(b u \in H^1_0(\Omega)\). There exists \(b_1, b_2, \ldots \in C^\infty_c(\mathbb{R}^d \setminus \Gamma_2)\) and \(u_1, u_2, \ldots \in C^\infty_c(\mathbb{R}^d \setminus \Gamma_1)\) such that \(\lim b_n|_{\Omega} = b\) in \(W^{1,\infty}(\Omega)\) and \(\lim u_n|_{\Omega} = u\) in \(H^1(\Omega)\). Then \((b_n u_n)|_{\Omega} \in C^\infty(\Omega)\) and \(\lim(b_n u_n)|_{\Omega} = b u\) in \(H^1(\Omega)\). This shows condition (24).

**Theorem 4.5** Let \(V\) satisfy (20) - (23). Let \(A\) be the operator associated with the form a given by (25) with domain \(V\) and real coefficients \(a_{ij} \in L_\infty(\Omega)\), \(b, c, c_0 \in L^1(\Omega)\) satisfying the ellipticity condition (19) and the condition (24). Then \(A\) generates a positive semigroup \((e^{-tA})_{t>0}\) which interpolates on \(L_p(\Omega)\), \(1 \leq p \leq \infty\), and which is given by a kernel \(K\) for which \(K_t \in L_\infty(\Omega \times \Omega)\) for all \(t > 0\) satisfying

\[
0 \leq K_t(x; y) \leq c t^{-d/2} e^{-b|x-y|^2 t^{-1}} e^{\omega t} (x, y)\text{-a.e.}
\]

for some constants \(b, c > 0\) and \(\omega \in \mathbb{R}\), uniformly for all \(t > 0\).
In the proof of Theorem 4.5 we will again use Davies' perturbation method and prove
ultracontractivity of $S_t^\rho$ uniformly for all real $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\|D_t \psi\|_1 \leq 1$. In case of
(good) Neumann boundary conditions the method of Section 3 is, however, not applicable
since $(S_t^\rho e^{-\omega t})_{t>0}$ is not $L_\infty$-contractive for any $\omega \in \mathbb{R}$, in general, even for the Laplacian
as the following example shows.

**Example 4.6** Let $\Omega = (0,1) \subset \mathbb{R}$, $V = H^1(\Omega) = H^1(\Omega)$ and $a(u,v) = \int_0^1 u'v'$. Let $\rho = 1$
and $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(x) = x$ for all $x \in [-1,2]$. Then $S_t^\rho$ is associated with the form

$$a^\rho(u,v) = \int_0^1 (u' + u)(v' - v).$$

Let $\omega \in \mathbb{R}$ and suppose that $\|S_t^\rho e^{-\omega t}\|_{\infty \to \infty} \leq 1$ for all $t > 0$. Then $e^{-\omega t}S_t^\rho 1 \leq 1$ for all
$t > 0$ in $L_\infty(\Omega)$. Denote by $A^\rho$ the generator of $S^\rho$. Since $1 \in D(A^\rho)$ it follows that

$$(A^\rho + \omega I)1 = \lim_{t \to 0} \frac{(I - e^{-\omega t}S_t^\rho)1}{t} \geq 0.$$ 

Hence by density of $D(A)_+$ in $H^1(\Omega)_+$ one deduces that

$$a^\rho(1,u) + \omega(1,u)_{L^2} \geq 0$$

for all $u \in H^1(\Omega)_+$. Next for $n \in \mathbb{N}$ set $u_n(x) = (1 - x)^n$. Then $u_n \in H^1(\Omega)_+$ and

$$0 \leq a^\rho(1,u_n) + \omega(1,u_n)_{L^2} = \int_0^1 (u_n' - u_n) + \omega \int_0^1 u_n$$

$$= u_n(1) - u_n(0) + (\omega - 1) \int_0^1 u_n = -1 + \frac{\omega - 1}{n + 1}.$$

This gives a contractions if one chooses $n$ sufficiently large.

This example has been considered before by Ouhabaz [Ouh92b] Remark 4.3(b) in a
different context. \[\square\]

The method of proving ultracontractivity we use in this section is based on the following
proposition (cf. [Rob91] Chapter IV pp. 262–264). Again, it is important for us to obtain
constants which do not depend explicitly on the coefficients of the operator.

**Proposition 4.7** Let $S$ be a real continuous semigroup on $L_2(X)$ whose complexification
is a holomorphic semigroup, where $(X, \Sigma, m)$ is a $\sigma$-finite measure space. Assume that $S$
is consistent on $L_p(X)$, $2 \leq p \leq \infty$. Let $c_1, \mu > 0$ and $V$ be a Hilbert space which is
continuously embedded in $L_2$. Suppose that $(S_t\varphi)^p \in V$, $t \mapsto \|S_t\varphi\|_p^p$ is differentiable and

$$\frac{d}{dt}\|S_t\varphi\|_p^p \leq -\mu\|(S_t\varphi)^p\|_V^2 + c_1 p^2 \|S_t\varphi\|_p^2$$

for all $t > 0$, all real $\varphi \in L_2 \cap L_\infty$ and $p \in 2\mathbb{N}$. Let $c_N, n > 0$ and suppose that the Nash
inequality

$$\|\varphi\|_2^{2+4/n} \leq c_N\|\varphi\|_V^2\|\varphi\|_1^{4/n}$$

is valid for all $\varphi \in L_1 \cap V$. Moreover, let $M \geq 1$ and $\omega \geq 0$ be such that

$$\|S_t\|_{2 \to 2} \leq M e^{\omega t}$$

18
for all \( t > 0 \). Then there exists a constant \( c_2 > 0 \), depending only on \( c_N \) and \( n \), such that
\[
\|S_t\|_{2 \to \infty} \leq c_2 M \mu^{-n/4} t^{-n/4} e^{\omega t} e^{c_1 t / 2}
\]
for all \( t > 0 \).

**Proof.** Let \( \varphi \in L_1 \cap V \). Set \( \varphi_t = S_t \varphi \) for all \( t > 0 \). If \( \varphi_{t_0} = 0 \) for some \( t_0 > 0 \), then \( \varphi_t = 0 \) for all \( t > t_0 \) and by holomorphy of \( S \) it follows that \( \varphi_t = 0 \) for all \( t > 0 \) and hence \( \varphi = 0 \). So we may assume that \( \varphi_t \neq 0 \) for all \( t > 0 \). Then it follows from the Nash inequality that
\[
\frac{d}{dt} \|\varphi_t\|_{2p}^2 \leq -\frac{\mu}{c_N} \frac{\|\varphi_t\|_{2p}^{4+n/2}}{\|\varphi_t\|_1^{4+n/2}} + c_1 p^2 \|\varphi_t\|_{2p}^2 \]
\[
= -\frac{\mu}{c_N} \frac{\|\varphi_t\|_{2p}^{4+n/2}}{\|\varphi_t\|_p^{4+n/2}} + c_1 p^2 \|\varphi_t\|_{2p}^2 .
\]
Therefore,
\[
\frac{d}{dt} \|\varphi_t\|_{2p} \leq -\frac{\mu}{2c_N p} \|\varphi_t\|_{2p} \|\varphi_t\|_{2p} \|\varphi_t\|_{-4p/n} + 2^{-1} c_1 p \|\varphi_t\|_{2p}
\]
and
\[
\frac{d}{dt} \left( \|\varphi_t\|_{2p} e^{2^{-1} c_1 p t} \right)^{-4p/n} \geq 2 \mu (c_N n)^{-1} \left( \|\varphi_t\|_{p} e^{2^{-1} c_1 p t} \right)^{-4p/n} .
\tag{26}
\]
Since \( \lim_{p \to \infty} p \left( 1 - (1 - p^{-2})^{p-1} \right) = 1 \) there exists a \( \sigma > 0 \) such that
\[
p \left( 1 - (1 - p^{-2})^{p-1} \right) \geq \sigma
\]
for all \( p \geq 2 \). Next define
\[
f_2(t) = M e^{\omega t} \|\varphi\|_2
\]
and by induction for all \( p \in \{2^r : r \in \mathbb{N}\} \) define
\[
f_{2p}(t) = \left( c_3 \mu \right)^{-n/(4p)} e^{2^{-1} c_1 t / p^n (2p)} f_p(t)
\]
where \( c_3 = 2 \sigma (c_N n)^{-1} \).

Note that \( f_p \) is an increasing function. We shall prove by induction that
\[
\|\varphi_t\|_p \leq t^{2^{-1} n (2^{-1} - p^{-1})} f_p(t)
\tag{27}
\]
for all \( p \in \{2^r : r \in \mathbb{N}\} \) and \( t > 0 \).

Clearly (27) is valid if \( p = 2 \). Let \( p \in \{2^r : r \in \mathbb{N}\} \) and suppose that (27) is valid for all \( t > 0 \). Then it follows by integration from (26) that
\[
\left( \|\varphi_t\|_{2p} e^{2^{-1} c_1 p t} \right)^{-4p/n}
\geq 2 \mu (c_N n)^{-1} \int_0^t (s^{2^{-1} n (2^{-1} - p^{-1})} f_p(s)) e^{2^{-1} c_1 p s} \right)^{-4p/n} ds
\geq 2 \mu (c_N n)^{-1} f_p(t)^{-4p/n} \int_0^t s^p e^{2 c_1 p^2 s/n} ds
\]

19
for all $t > 0$. Therefore,

$$
\|\varphi_t\|_{2p} \leq (c_3 \mu)^{-n/(4p)} e^{-2-1 c_1 p(1-p^{-2})t} p^{n/(2p)} t^{-2-1 n(2^{-1} - (2p)^{-1})} f_p(t)
$$

and

$$
\|\varphi_t\|_{2p} \leq (c_3 \mu)^{-n/(4p)} e^{2-1 c_1 t/p} p^{n/(2p)} t^{-2-1 n(2^{-1} - (2p)^{-1})} f_p(t) = t^{-2-1 n(2^{-1} - (2p)^{-1})} f_{2p}(t).
$$

It follows from the definition of $f_p$ that

$$
f_{2^r}(t) = M \left( \prod_{k=1}^{r-1} (c_3 \mu)^{-2-k-2n} e^{2-k-1 c_1 t 2^{-k-1} n k} \right) e^{\omega t} \|\varphi\|_2
$$

$$
\leq c_3^{-n/4} M \left( \prod_{k=1}^{\infty} 2^{2-k-1} n k \right) \mu^{-n/4} e^{2-1 c_1 t} e^{\omega t} \|\varphi\|_2
$$

for all $r \in \mathbb{N}$. Hence by (27),

$$
\|S_t \varphi\|_{2^r} \leq c_2 M \mu^{-n/4} t^{-n/4} e^{2-1 c_1 t} e^{\omega t} \|\varphi\|_2,
$$

where $c_2 = c_3^{-n/4} \prod_{k=1}^{\infty} 2^{2-k-1} n k$. Thus

$$
\|S_t \varphi\|_{\infty} \leq \limsup_{r \to \infty} \|S_t \varphi\|_{2^r} \leq c_2 M \mu^{-n/4} t^{-n/4} e^{2-1 c_1 t} e^{\omega t} \|\varphi\|_2
$$

and the proposition has been proved. \(\square\)

**Proof of Theorem 4.5** It follows from Lemma 3.7.11 and Proposition 2.7 that the complexification of the operator $A$ associated with the complexified form a generates a holomorphic semigroup $S = (e^{-tA})_{t \geq 0}$ on $L^2(\Omega)$. Note that the proof of Lemma 3.7.11 is valid for $a_{ij} \in L_\infty(\Omega)$ and $u \in H^1(\Omega)$. Recall that we assume throughout this section that the spaces are real.

First we show that $S$ is positive. Let $\varphi \in V$. Since $D_i \varphi^+ = 1_{[\varphi > 0]} D_i \varphi$ and $D_i \varphi^- = -1_{[\varphi < 0]} D_i \varphi$ one has $a(\varphi^+, \varphi^-) = 0$. It then follows from [Ouh92b] Theorem 2.4 (which is also valid in case of real spaces) that $S$ is positive.

Secondly we show that there exists a constant $\omega \in \mathbb{R}$ such that

$$
\|S_t \varphi\|_{\infty} \leq e^{\omega t} \|\varphi\|_{\infty}
$$

for all $\varphi \in L_2(\Omega) \cap L_\infty(\Omega)$ and $t > 0$. Since the proof is very similar to a proof in Section 3 we discuss the critical steps. We wish to apply the proof of Lemma 3.7.1 in case $\rho = 0$. In that case we do not need the differentiability of the second order coefficients. Secondly, we used intergration by parts in the proof of Proposition 3.5. But by assumption (24) one
has \((c_i u) \in H^1_0(\Omega)\) for all \(u \in V\) and \(i \in \{1, \ldots, d\}\). Hence \(\int c_i u D_i v = -\int D_i (c_i u) v = -\int (D_i c_i u) v - \int c_i (D_i u) v\) for all \(u, v \in V\). Thirdly, one needs to verify that Theorem 4.2(3) (or Theorem 2.7) in [Ouh92b] is also valid for real spaces and that \((1 \wedge |u|) \text{sgn } u \in V\) for all \(u \in V\). But \((1 \wedge |u|) \text{sgn } u = u - (u - 1)^+ + (u - 1)^+ \in V\) for all \(u \in V\). Therefore, the semigroup \(S\) is quasi-contractive on \(L_{\infty}\).

Thirdly, replacing \(A\) by \(A^\ast\), \(S\) by \(S^\ast\), \(a(u, v)\) by \(a^\ast(u, v) = \overline{a(v, u)}\) one obtains by duality the \(L_1\)-bound

\[
\|S_t \varphi\|_1 \leq e^{\omega t} \|\varphi\|_1 \tag{29}
\]

for some \(\omega > 0\), uniformly for all \(t > 0\) and \(\varphi \in L_1 \cap L_2\). It follows from (28), (29) and Lemma 2.1.1 that \(S\) interpolates on \(L_p(\Omega), 1 \leq p \leq \infty\).

Fourthly, let \(\psi \in W\) (see Section 3), \(\rho \in \mathbb{R}\) and define \(U_\rho \varphi = e^{-\rho \psi} \varphi\) as before. We show that \(U_\rho \varphi \in V\) for all \(\varphi \in V\) and \(\rho \in \mathbb{R}\). It follows from Lemma 2.5.11 that \(e^{-\rho \psi} \varphi \in \overline{H}^1(\Omega)\) because \(\varphi \in V \subset \overline{H}^1(\Omega)\). Since \(|e^{-\rho \psi} \varphi| \leq c|\varphi|\) it follows from the ideal assumption (23) that \(U_\rho \varphi = e^{-\rho \psi} \varphi \in V\). Now define the form \(a^\rho : V \times V \rightarrow \mathbb{R}\) by

\[
a^\rho(u, v) = \sum_{i,j=1}^d \int_{\Omega} a_{ij}(D_i + \rho \psi_i)u(D_j - \rho \psi_j)v
\]

\[
+ \sum_{i=1}^d \int_{\Omega} b_i(D_i + \rho \psi_i)u v + \sum_{i=1}^d \int_{\Omega} c_i u(D_i - \rho \psi_i)v + \int_{\Omega} c_0 u v
\]

and let \(A^\rho\) be the operator associated with the form \(a^\rho\). Then \(a^\rho(u, v) = a(U_\rho^{-1} u, U_\rho v)\) for all \(u, v \in V\), so \(A^\rho = U_\rho A U_\rho^{-1}\). Hence \(S_t^\rho = U_\rho S_t U_\rho^{-1}\) for all \(t > 0\), where \(S^\rho\) is the holomorphic semigroup generated by \(A^\rho\). It then follows as in the proof of Lemma 3.7.11 that there exists an \(\omega > 0\) such that \(a^\rho(\varphi, \varphi) + \omega(1 + \rho^2) \|\varphi\|_2^2 \geq 0\) for all \(\varphi \in V\). Note that the second order coefficients \(a_{ij}\) need not be differentiable in Lemma 3.7.11. Hence

\[
\|S^\rho_t\|_{2-2} \leq e^{\omega(1 + \rho^2)t}
\]

for all \(t > 0\).

Fifthly one has \((S_t^\rho \varphi)^p \in V\) whenever \(t > 0\), \(\varphi \in V \cap L_{\infty}(\Omega)\) and \(p \in 2\mathbb{N}\). In fact, let \(f = S_t^\rho \varphi\). Then \(f \in V \subset \overline{H}^1(\Omega)\) and therefore \(f \in \overline{H}^1(\Omega) \cap L_{\infty}(\Omega)\). By Lemma 2.5.11 we have \(f^p \in \overline{H}^1(\Omega)\). But \(|f^p| \leq \|f\|_{p^{-1}} |f| = c|f|\). Therefore, it follows again from the ideal assumption (23) that \(f^p \in V\).

Sixthly, let \(\varphi \in V \cap L_{\infty}\) and \(p \in 2\mathbb{N}\). We show that \(t \mapsto \|S_t^\rho \varphi\|_{2p}^2\) is differentiable on \((0, \infty)\) and that

\[
\frac{d}{dt} \|S_t^\rho \varphi\|_{2p}^2 = -2p(A^\rho \varphi_t, \varphi_t^{2p-1}) = -2p a^\rho(\varphi_t, \varphi_t^{2p-1}) \tag{30}
\]

where we set \(\varphi_t = S_t^\rho \varphi\). Note that \(\varphi_t' = \frac{d}{dt} \varphi_t = -A^\rho \varphi_t\) exists in \(L_2(\Omega)\) since \(S^\rho\) is holomorphic and that \(\varphi_t \in L_2 \cap L_{\infty}\). Let \(t > 0\). Then

\[
|h^{-1}(\|\varphi_{t+h}\|_{2p}^2 - \|\varphi_t\|_{2p}^2) - 2p \int \varphi_t^{2p-1} \varphi_t'|
\]

\[
= \int h^{-1}(\varphi_{t+h}^{2p} - \varphi_t^{2p}) - 2p \int \varphi_t^{2p-1} \varphi_t'
\]

\[
= \int h^{-1}(\varphi_{t+h} - \varphi_t)(\varphi_{t+h}^{2p-2} \varphi_t + \ldots + \varphi_{t+h} \varphi_t^{2p-2} + \varphi_t^{2p-1}) - 2p \int \varphi_t^{2p-1} \varphi_t'
\]

21
\[
= \left| \int (h^{-1}(\phi_{t+h} - \phi_t) - \phi_t') (\phi_{t+h}^{2p-1} + \phi_{t+h}^{2p-2} \phi_t + \ldots + \phi_{t+h}^2 \phi_t^{2p-2} + \phi_t^{2p-1}) \\
+ \int \phi_t' ((\phi_{t+h}^{2p-1} - \phi_t^{2p-1}) + (\phi_{t+h}^{2p-2} \phi_t - \phi_t^{2p-2}) + \ldots \\
+ (\phi_{t+h} \phi_t^{2p-2} - \phi_t^{2p-1}) + (\phi_t^{2p-1} - \phi_t^{2p-1})) \right|
\]
\[
\leq \|h^{-1}(\phi_{t+h} - \phi_t) - \phi_t'||_2 \|(\phi_{t+h}^{2p-1} + \phi_{t+h}^{2p-2} \phi_t + \ldots + \phi_{t+h}^2 \phi_t^{2p-2} + \phi_t^{2p-1})_2 \\
+ \int \phi_t'((\phi_{t+h} - \phi_t)g_{t,h}) \\
\leq c\|h^{-1}(\phi_{t+h} - \phi_t) - \phi_t'||_2 + \|g_{t,h}\|_\infty \|\phi_t'||_2 \|\phi_{t+h} - \phi_t||_2 ,
\]
which tends to 0 if \( h \) tends to 0. Here \( g_{t,h} \) is an element of \( L_\infty(\Omega) \) which is uniformly bounded for small \( h \) by the estimates (28). (Note that \( t, \rho \) and \( \psi \) are fixed.)

Seventhly, we show that there exists a constant \( c > 0 \) such that
\[
\frac{d}{dt}\|S_t^p \varphi\|_{2p} \leq -2^{-1} \mu \sum_{i=1}^d \|D_i \varphi_t\|_2^2 + c(1 + \rho^2)p^2 \|\varphi_t^p\|_2^2
\]
uniformly for all \( t > 0, \rho \in \mathbb{R}, \psi \in W, \varphi \in V \cap L_\infty(\Omega) \) and \( p \in 2\mathbb{N} \). By (30) we have
\[
\frac{d}{dt}\|S_t^p \varphi\|_{2p} = -2p \sum_{i,j=1}^d (a_{ij}(D_i + \rho \psi_i) \varphi_t, (D_j - \rho \psi_j) \varphi_t^{2p-1}) \\\n- 2p \sum_{i=1}^d (b_i(D_i + \rho \psi_i) \varphi_t, \varphi_t^{2p-1}) - 2p \sum_{i=1}^d (c_i \varphi_t, (D_i - \rho \psi_i) \varphi_t^{2p-1}) - 2p \int c_0 \varphi_t^p \\
= -2p \sum_{i,j=1}^d (a_{ij} D_i \varphi_t, D_j \varphi_t^{2p-1}) + \tau_2 + \tau_3 + \tau_4 ,
\]
where \( \tau_2 \) is the sum of terms of the form \( pp(k_i D_i \varphi_t, \varphi_t^{2p-1}) \), \( \tau_3 \) is the sum of terms of the form \( pp(k'_i \varphi_t, D_i \varphi_t^{2p-1}) \), and \( \tau_4 \) is a term of the form \( p((k_0 + k'_0 \rho + k''_0 \rho^2) \varphi_t, \varphi_t^{2p-1}) \), with \( k_0, k'_0, k''_0, k_i, k'_i \in L_\infty(\Omega) \) functions of which the \( L_\infty \)-norm is bounded uniformly in \( \psi \in W \), and is independent of \( \rho, p, \varphi \) and \( t \). We estimate the first term.
\[
-2p \sum_{i,j=1}^d (a_{ij} D_i \varphi_t, D_j \varphi_t^{2p-1}) = -2p(2p - 1) \sum_{i,j=1}^d (a_{ij} D_i \varphi_t, \varphi_t^{2p-2} D_j \varphi_t) \\\n= -2p(2p - 1) \sum_{i,j=1}^d (a_{ij} \varphi_t^{2p-1} D_i \varphi_t, \varphi_t^{2p-1} D_j \varphi_t) \\\n= -2p^{-1}(2p - 1) \sum_{i,j=1}^d (a_{ij} D_i \varphi_t^p, D_j \varphi_t^p) \\\n\leq -2p^{-1}(2p - 1) \mu \sum_{i=1}^d \|D_i \varphi_t^p\|_2^2 \\\n\leq -2 \mu \sum_{i=1}^d \|D_i \varphi_t^p\|_2^2 .
\]
The second term can be estimated by
\[
|\tau_2| = \left| pp \sum_{i=1}^d (k_i D_i \varphi_t, \varphi_t^{2p-1}) \right| = |\rho| \left| \sum_{i=1}^d (k_i D_i \varphi_t^p, \varphi_t^p) \right|
\[ \leq c_2 |\rho| \sum_{i=1}^{d} \| D_i \varphi_i^2 \|_2 \| \varphi_i^2 \|_2 \leq \varepsilon \sum_{i=1}^{d} \| D_i \varphi_i^2 \|_2^2 + (4\varepsilon)^{-1} c_3^2 d \rho^2 \| \varphi_i^2 \|_2^2 \]

for all \( \varepsilon > 0 \). The third term can be estimated by

\[
|\tau_3| = \left| \rho \sum_{i=1}^{d} (k_i \varphi_i, D_i \varphi_i^{2p-1}) \right| = \rho (2p - 1) \rho \sum_{i=1}^{d} (k_i \varphi_i, \varphi_i^{2p-2} D_i \varphi_i) \\
= \left| (2p - 1) \rho \sum_{i=1}^{d} (k_i \varphi_i^p, D_i \varphi_i^p) \right| \leq c_3 \rho |\rho| \sum_{i=1}^{d} \| D_i \varphi_i^p \|_2 \| \varphi_i^p \|_2 \\
\leq \varepsilon \sum_{i=1}^{d} \| D_i \varphi_i^p \|_2^2 + (4\varepsilon)^{-1} c_3^2 d \rho^2 \| \varphi_i^p \|_2^2 .
\]

The fourth term is trivial:

\[ |p((k_0 + \rho k_0^2 + \rho^2) \varphi_1, \varphi_1^{2p-1})| \leq c_4 p (1 + \rho^2) \| \varphi_1^p \|_2^2 . \]

The constants \( c_2, c_3 \) and \( c_4 \) are independent of \( \rho, p, \psi \in W, \varphi \) and \( t \). Choosing \( \varepsilon \) appropriate one obtains that

\[ \frac{d}{dt} \| S^2 \varphi \|_{2p} \leq -\mu \sum_{i=1}^{d} \| D_i \varphi_i^p \|_2^2 + c' p^2 (1 + \rho^2) \| \varphi_i^p \|_2^2 \leq -\mu \| \varphi_i^p \|_2^2 + (c' + \mu) p^2 (1 + \rho^2) \| \varphi_i^p \|_2^2 \]

for some constant \( c' > 0 \), independent of \( \rho, p, \psi \in W, \varphi \) and \( t \).

Recall that one has the estimate \( \| S^p \|_{2 \to 2} \leq \varepsilon^{\omega(1+\rho^2)t} \) for some \( \omega' > 0 \), uniformly for all \( t > 0, \rho \in \mathbb{R} \) and \( \psi \in W \). Now one can apply Proposition 4.7 and deduce that

\[ \| S^p \|_{2 \to \infty} \leq c t^{-d/4} e^{\omega'(1+\rho^2)t} e^{2\omega'(c' + \mu)(1+\rho^2)t} = c t^{-d/4} e^{\omega(1+\rho^2)t} \]  \hspace{1cm} (31)

for a constant \( c > 0 \), independent of \( \rho, \psi \) and \( t \) and \( \omega = \omega' + (c' + \mu)/2 \). Since the adjoint of \( S^p \) is of the same form we obtain by duality

\[ \| S^p \|_{1 \to 2} \leq c t^{-d/4} e^{\omega(1+\rho^2)t} \]

possibly by enlarging \( c \) and \( \omega \). Hence

\[ \| S^p \|_{1 \to \infty} \leq 2^{d/2} c t^{-d/2} e^{\omega(1+\rho^2)t} \]

for all \( t > 0 \) and \( \rho \in \mathbb{R} \). Now the theorem follows from Proposition 3.3. \( \square \)

**Remark.**

**I.** One would expect to obtain the results of Theorem 4.5 also for coefficients \( b_i, c_i \in L_\infty \). The main point in the above argument is to prove that \( S \) operates consistently on \( L_1 \) and \( L_\infty \). This could be proved if the \( D_i \) are small perturbations of \( A \). However, this is not true in general. In fact, even the domain of the Dirichlet Laplacian on \( L_p(\Omega) \) is not contained in \( W^{1,p}(\Omega) \) for \( p \) sufficiently large, if \( \Omega \) is not regular, in general (see [Gri85]). This also shows that in general there are no Gaussian type bounds for the derivatives of the kernel if the domain is not regular (even if the coefficients are constant).
II. In general the theorem is false if all coefficients are complex. A counter example on a subset of $\mathbb{R}^d$ has been presented by Maz'ya-Nazarov-Plamenevskii [MNP85] and on $\mathbb{R}^d$ by Auscher-Tchamitchian [AT94] in case $d \geq 5$. Semigroups generated by complex operators on $\mathbb{R}^1$ and $\mathbb{R}^2$ have Gaussian kernel bounds by Auscher-McIntosh-Tchamitchian [AMT94].

Finally we consider the realization of $A$ (see (18)) with Robin boundary conditions. For this we assume that $\Omega$ is a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial \Omega$ and we let $\beta \in L_\infty(\Gamma')$ be a positive function. We still assume the conditions (20) – (23) on the form domain $V$ and the condition (24) on the coefficients. By $a$ we continue to denote the form (25) defined on $V$. Let $b: V \times V \to \mathbb{R}$ be defined by

$$b(u, v) = \int_\Gamma \beta(x) (Bu)(x)(Bv)(x) d\gamma(x),$$

where $B: H^1(\Omega) \to L^2(\Gamma')$ denotes the trace operator (see Section 2). Then $b$ is a continuous bilinear form on $V$. Set

$$q = a + b.$$

Then $q$ is a continuous bilinear form on $V$ which is coercive. Let $A$ be the operator associated with the form $q$. We call $A$ the realization of $A$ with Robin boundary conditions. Note that Robin boundary conditions coincide with Dirichlet boundary conditions if $V = H^1_0(\Omega)$ and with (good) Neumann boundary conditions if $V = H^1(\Omega)$ and $\beta = 0$.

Example 4.8 Let $a_{ij} = \delta_{ij}, b_i = c_i = 0, c_0 = 0$ and $V = H^1(\Omega)$. Assume that $u \in D(A) \cap C^2(\Omega)$. Then

$$\frac{\partial u}{\partial n} = -\beta u \text{ on } \Gamma. \tag{32}$$

Conversely, if $u \in C^2(\overline{\Omega})$ is such that (32) holds then $u \in D(A)$. This follows by applying Green’s formula. We call $A$ the Laplacian with Robin boundary conditions.

Theorem 4.9 Let $A$ be the realization of $A$ with Robin boundary conditions. Then $A$ generates a semigroup $S = (e^{-tA})_{t \geq 0}$ on $L^2(\Omega)$ which interpolates on $L^p(\Omega), 1 \leq p \leq \infty$. The semigroup $S$ is positive and is given by a kernel $K$. Moreover, there exist $b, c > 0$ and $\omega \in \mathbb{R}$ such that

$$0 \leq K_t(x, y) \leq ct^{-d/2}e^{-\omega t}t^{-1/2}e^{\omega t} \quad (x, y) \text{-a.e.}$$

uniformly for all $t > 0$.

Proof. First we show that $S$ is positive. Let $u \in V$. By [Ouh92b] Theorem 2.4 we have to show that $q(u^+, u^-) \leq 0$. Since $a(u^+, u^-) = 0$ (see the proof of Theorem 4.5) and $Bu^+ = (Bu)^+$ and $Bu^- = (Bu)^-$ (by (10)), we have

$$b(u^+, u^-) = \int_\Gamma \beta(x) (Bu)^+(x)(Bu)^-(x) d\gamma(x) = 0.$$

Thus $q(u^+, u^-) \leq 0$.

Secondly, it follows from Proposition 2.7 that $A$ generates a semigroup on $L^2(\Omega)$.  

24
Thirdly, we show that $S$ interpolates on $L_p(\Omega)$, $1 \leq p \leq \infty$. By the properties (10) of the trace operator we have $B((|u| - 1)^{+} \text{sgn } u) = B(u - 1)^{+} - B(-u - 1)^{+} = (Bu - 1)^{+} - (-Bu - 1)^{+}$ for all $u \in H^1(\Omega)$. Therefore,

$$b(u, (|u| - 1)^{+} \text{sgn } u) = \int \beta(Bu)((Bu - 1)^{+} - (-Bu - 1)^{+}) \, d\gamma \geq 0 .$$

Now one argues as in the proof of Theorem 4.5 and deduces that $S$ generates a quasi contraction semigroup on $L_\infty$ and by duality it interpolates.

Finally, let $S_t^\rho = U_\rho S_t U_{\rho}^{-1}$ where $\rho \in \mathbb{R}$ and $\psi \in W$. Then the associated form is given by

$$q^\rho(u,v) = q(U_{\rho}^{-1}u, U_{\rho}v) = a^\rho(u,v) + b(u,v)$$

since $b(U_{\rho}^{-1}u, U_{\rho}v) = b(u,v)$. Then the proof of Theorem 4.5 carries over to the present case.

Remark.

I. An alternative proof of Theorem 4.9 using the results of Theorem 4.5 can be given by domination. Denote by $A(a)$ the operator associated by the form $a$ and $S^{(a)} = (e^{-tA(a)})_{t \geq 0}$ the semigroup generated by $A(a)$. Then $S$ and $S^{(a)}$ are positive semigroups and $q(u,v) \geq a(u,v)$ for all $u,v \in V_4$. So it follows from [Ouh93] Proposition 3.2 and Theorem 3.7 that $S$ is dominated by $S^{(a)}$, i.e., $|S_t\varphi| \leq S^{(a)}_t|\varphi|$ for all $\varphi \in L_2(\Omega)$. Then $K_t \leq K^{(a)}_t$ and Gaussian estimates follow.

II. Similarly, one could prove Theorem 4.5 first for good Neumann boundary conditions (i.e. $V = \tilde{H}^1(\Omega)$) and then deduce the Gaussian estimates for the general $V$ by domination. However, this requires $b_i, c_i$ to be elements of $H_0^1(\Omega)$ which is stronger than our assumption (24).

5 Applications

In this section we give two kinds of applications of the previous results. They concern the holomorphy of the semigroup in $L_p$ and the bounded $H_\infty$ functional calculus.

If $T$ is a holomorphic semigroup on $L_2(\Omega)$ which interpolates on $L_p(\Omega)$, $1 \leq p \leq \infty$, then it follows from Stein's interpolation theorem that $T$ is also holomorphic on $L_p$, $1 < p < \infty$, but it may not be holomorphic on $L_1$. For elliptic operators with boundary conditions holomorphy in $L_1$ has first been proved by Amann [Ama83] for regular bounded domains and later for Dirichlet boundary conditions and no regularity assumptions on the domain in [ABa93] and [ABa92]. More recently Ouhabaz ([Ouh92a] and [Ouh95]) used Gaussian estimates and a Phragmen–Lindelöf argument (cf. [Dav89], Theorem 3.4.8) to show holomorphy for symmetric operators (see also [Dav93] Lemma 2). Here we prove holomorphy on $L_p(\Omega)$, $1 \leq p \leq \infty$ on a sector where $\|S_2\|_{L_2} \leq e^{\omega|\Omega|}$ by a direct short proof avoiding the Phragmen–Lindelöf theorem (see Theorem 5.2). In order to obtain a possibly larger sector, however, we adapt the Phragmen–Lindelöf argument to the non-symmetric case (see Theorem 5.3).
Adopt the notation and assumptions of Theorems 3.1, 4.5 or 4.9. In case of Theorems 4.5 and 4.9 we complexify the form domain $V$ and the form $a$. Set

$$\theta_a = \pi/2 - \inf\{\theta > 0 : \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \in \Sigma(\theta) \text{ for all } \xi \in \mathbb{C}^d, \text{ for a.e. } x \in \Omega\} .$$

Note that $\theta_a = \pi/2$ if the $a_{ij}$ are symmetric, i.e., $a_{ij}(x) = a_{ji}(x)$ for a.e. $x \in \Omega$ and all $i,j \in \{1, \ldots, d\}$.

It is a standard exercise to show that the semigroup $S = (e^{-tA})_{t>0}$ generated by the operator $A$ associated with the form is a holomorphic semigroup on $L_2$, with a holomorphy sector which contains at least $\Sigma(\theta_a)$. In fact one has the following.

**Lemma 5.1** Adopt the notation and assumptions of Theorems 3.1, 4.5 or 4.9. Then for all $\rho \in \mathbb{R}$ the operator $A^\rho$ generates a holomorphic semigroup $S^\rho$ on $L_2(\Omega)$, holomorphic in the sector $\Sigma(\theta_a)$. Moreover, for all $\theta \in (0, \theta_a)$ there exists an $\omega \in \mathbb{R}$, depending only on $\theta, \mu, ||a_{ij}||_{\infty}, \|b_i\|_{\infty}, \|c_i\|_{\infty}$ and $||c_0||_{\infty}$, such that

$$||S^\rho_z||_{2\rightarrow 2} \leq e^{\omega(1+\rho^2)|z|}$$

for all $z \in \Sigma(\theta), \rho \in \mathbb{R}$ and $\psi \in W$.

**Proof.** Let $\theta \in (0, \theta_a)$. There exists $\nu > 0$ such that $\mathrm{Re}\sum_{i,j=1}^d e^{i\alpha}a_{ij}(x) \xi_i \xi_j \geq \nu|\xi|^2$ uniformly for all $\alpha \in [-\theta, \theta], \xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Then one can argue as in the proof of Lemma 3.7 and deduce that

$$\mathrm{Re}\left(\sum_{i,j=1}^d a_{ij} D_i u D_j \overline{u} + \sum_{i=1}^d b_i u D_i u \overline{u} + \sum_{i=1}^d c_i u D_i u + c_0|u|^2\right) + \omega(1+\rho^2)|u|^2 \geq 2^{-1}\nu \sum_{i=1}^d |D_i u|^2 \text{ a.e.}$$

uniformly for all $\alpha \in [-\theta, \theta], u \in V, \rho \in \mathbb{R}$ and $\psi \in W$ if one chooses $\omega = 4d^3 M_0^2 \nu^{-1} + 2d^2 M_0$ and where

$$M_0 = 1 + \max\{||a_{ij}||_{\infty}, \|b_i||_{\infty}, \|c_i\|_{\infty}, \|c_0||_{\infty}\} ,$$

as before. Again integrating this inequality gives

$$\mathrm{Re}(e^{i\alpha}a^\rho(u,u)) + \omega(1+\rho^2)||u||_2^2 \geq 2^{-1}\mu \sum_{i=1}^d ||D_i u||_2^2 .$$

Hence $S^\rho$ is holomorphic on $\Sigma(\theta)$ and

$$||S^\rho_z||_{2\rightarrow 2} \leq e^{\omega(1+\rho^2)|z|}$$

uniformly for all $z \in \Sigma(\theta), \rho \in \mathbb{R}$ and $\psi \in W$. 

We next show the remarkable fact that $S$ is even holomorphic on any $L_p, 1 \leq p \leq \infty$, with a holomorphy sector which contains at least $\Sigma(\theta_a)$. 

26
Remark. Here a holomorphic semigroup $S$ on $L_\infty$ of angle $\theta \in (0, \pi/2]$ is by definition a holomorphic mapping $S: \Sigma(\theta) \to \mathcal{L}(L_\infty)$ such that $S_{z+z'} = S_z S_{z'}$ for all $z, z' \in \Sigma(\theta)$ and
\[
\lim_{\varepsilon \to 0} \left. \left( S_z \varphi, \psi \right) \right|_{z \in \Sigma(\theta-\varepsilon)} = (\varphi, \psi)
\]
for all $\varphi \in L_\infty$, $\psi \in L_1$ and $\varepsilon \in (0, \theta)$.

**Theorem 5.2** Adopt the notation and assumptions of Theorems 3.1, 4.5 or 4.9. Then the semigroup $S$ generated by the operator $A$ is holomorphic on any $L_p$, $1 \leq p \leq \infty$, with a holomorphy sector which contains at least $\Sigma(\theta_a)$. Moreover, $S_z$ has a kernel $K_z \in L_\infty(\Omega \times \Omega)$ for all $z \in \Sigma(\theta_a)$ and for all $\theta \in (0, \theta_a)$ there exist $b, c > 0$ and $\omega > 0$ such that
\[
|K_z(x, y)| \leq c(\Re z)^{-d/2} e^{-b|x-y|^2-t^{-1}} e^{\omega t} \quad (x, y)\text{-a.e.}
\]
uniformly for all $z \in \Sigma(\theta)$.

**Proof.** Let $\theta \in (0, \theta_a)$. Choose $\theta_1 \in (\theta, \theta_a)$. There exists a $\delta > 0$ such that $\delta t + is \in \Sigma(\theta_1)$ for all $t + is \in \Sigma(\theta)$. By Lemma 5.1 there exists $\omega_1 > 0$ such that
\[
\|S_t^\rho\|_{L^2} \leq e^{\omega_1(1+\rho^2)t}
\]
uniformly for all $\rho \in R$, $\psi \in W$ and $z \in \Sigma(\theta_1)$. By (17), (31) and duality there exist $c, \omega_2 > 0$ such that
\[
\|S_t^\rho\|_1 \leq c t^{-d/4} e^{\omega_2(1+\rho^2)t}, \quad \|S_t^\rho\|_\infty \leq ct^{-d/4} e^{\omega_2(1+\rho^2)t}
\]
uniformly for all $\rho \in R$, $\psi \in W$ and $t > 0$. Now let $z = t + is \in \Sigma(\theta)$. Then
\[
\|S_t^\rho\|_1 \leq \|S_{(1-\delta)t/2}^\rho\|_{L^2} \|S_{\delta t + is}^\rho\|_{L^2} \|S_{(1-\delta)t/2}^\rho\|_{L^\infty} \leq \left( c((1-\delta)t/2)^{-d/4} e^{\omega_2(1+\rho^2)(1-\delta)t/2} \right)^2 e^{\omega_1(1+\rho^2)|\delta t + is|} \\
\leq c t^{-d/2} e^{\omega(1+\rho^2)|z|}
\]
for some $c', \omega' > 0$, independent of $z$ and uniformly for all $\rho \in R$ and $\psi \in W$. Now the complex Gaussian bounds follow as in Proposition 3.3.

Moreover, by Proposition 3.3 there also exists a $c_1 > 0$ such that $\|S_{t+is}^\rho\|_{p \to p} \leq c_1 e^{\omega't}$, uniformly for all $t > 0$ and $\alpha \in [-\theta, \theta]$. The holomorphy now follows from Kato [Kat84], Theorem IX.1.23.

The above short proof for the complex Gaussian bounds works well for elliptic differential operators. More generally, any holomorphic semigroup on $L_2(\Omega)$ with real time Gaussian bounds is holomorphic on $L_p(\Omega)$, $1 \leq p \leq \infty$, with the same sector as in $L_2$. This is proved in the next theorem. It was known before for symmetric semigroups (see Ouhabaz [Ouh92a] and [Ouh95]).

**Theorem 5.3** Let $S$ be a holomorphic semigroup on $L_2(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^d$. Suppose $S$ is holomorphic in the sector $\Sigma(\theta_0)$, where $\theta_0 \leq \pi/2$ and suppose that $S_t$ ($t > 0$) has a kernel $K_t$ which satisfies Gaussian bounds
\[
|K_t(x; y)| \leq ct^{-d/2} e^{-b|x-y|^2+t^{-1}} e^{\omega t} \quad (x, y)\text{-a.e.}
\]
for some $b,c > 0$ and $\omega \in \mathbb{R}$, uniformly for all $t > 0$. Then $S$ interpolates on $L_p$, $1 \leq p \leq \infty$ and $S$ is a holomorphic semigroup on $L_p$, $1 \leq p \leq \infty$, with holomorphy sector $\Sigma(\theta_0)$. Moreover, for all $z \in \Sigma(\theta_0)$ the operator $S_z$ has a kernel $K_z \in L_\infty(\Omega \times \Omega)$ and for all $\theta \in (0,\theta_0)$ there are $b,c > 0$ and $\omega \in \mathbb{R}$ such that

$$|K_z(x; y)| \leq c|z|^{-d/2}e^{-bl|x-y|^2|z|^{-1}}e^{\omega|z|} \quad (x, y)\text{-a.e.}$$

(33)

uniformly for all $z \in \Sigma(\theta)$.

**Proof.** It follows from Proposition 3.3 that the Gaussian bounds imply that $S$ interpolates on $L_p$, $1 \leq p \leq \infty$. Moreover, one has bounds $\|S_t\|_{1\rightarrow 2} \leq c_1 t^{-d/4}e^{\omega t}$ and $\|S_t\|_{2\rightarrow \infty} \leq c_2 t^{-d/4}e^{\omega t}$, together with the bounds $\|S_z\|_{2\rightarrow 2} \leq M_\theta e^{\omega |z|}$ for all $z \in \Sigma(\theta)$, if $\theta \in (0,\theta_0)$.

Then one deduces as in the proof of Theorem 5.2 that $\|S_z\|_{1\rightarrow \infty} \leq c_3 (\text{Re } z)^{-d/2}e^{\omega |z|}$ for all $z \in \Sigma(\theta)$. Next one derives from [ABu94] Theorem 3.1 that there exists a measurable function $K: \Sigma(\theta) \times \Omega \times \Omega \rightarrow \mathbb{C}$ such that $z \mapsto K(z, x, y)$ is analytic from $\Sigma(\theta) \rightarrow \mathbb{C}$ for all $(x, y) \in \Omega \times \Omega$ and $K_z$ is the kernel of $S_z$, where $K_z(z; y) = K(z, x, y)$. By replacing $S_z$ by $e^{-\omega z^2}S_z$ we may assume that $\omega_1, \omega_2 < 0$. Now one can argue as in Davies [Dav89] Theorem 3.4.8 to deduce that $K_z$ has the complex Gaussian bounds (33) by an application of the Phragmén–Lindelöf theorem. Finally it can be proved as in the proof of Theorem 5.2 that $S$ is a holomorphic semigroup on $L_p$, holomorphic on a sector which contains $\Sigma(\theta_0)$.

**Remark.** By a similar argument one proves that if $S$ is holomorphic on $L_p$ in a sector $\Sigma(\theta_p)$ then the semigroup on $L_2$ is holomorphic on a sector which contains $\Sigma(\theta_p)$. Therefore, the maximal holomorphy sector is independent of $p, 1 \leq p \leq \infty$.

Now consider again the semigroup $S$ generated by an elliptic operator under the assumptions of Theorems 3.1, 4.5 or 4.9. We have proved that $S$ is a holomorphic semigroup and has complex Gaussian kernel estimates

$$|K_z(x; y)| \leq c|z|^{-d/2}e^{-lb|x-y|^2|z|^{-1}}e^{\omega|z|} \quad (x, y)\text{-a.e.}$$

(34)

uniformly on each closed sector

$$\tilde{\Sigma}(\theta) = \{z \in \mathbb{C} : z \neq 0, \ |\arg z| \leq \theta\}$$

for all $\theta \in (0,\theta_0)$. If the bounds (34) are valid, then

$$\|S_z\|_{2\rightarrow 2} \leq M e^{\omega |z|}$$

(35)

where $M$ depends on $b$ and $c_1$ but with the same $\omega$ as in (34). For applications to $H_\infty$-functional calculus given below, it is important to have a good control over the $\omega$ in (34). In general, if (35) is valid for some $\omega$ then there are no kernel bounds (34) with the same $\omega$. An example is minus the Laplace operator $-\Delta$ on a bounded regular open set $\Omega$ with Neumann boundary conditions and $\theta = 0$. Then the constant function 1 is in the domain of $-\Delta$ and $-\Delta 1 = 0$. Therefore, $S_1 1 = 1$ on $L_2(\Omega)$. Gaussian kernel bounds with $\omega \leq 0$, however, imply that $\lim_{t\rightarrow \infty} S_t 1 = 0$, which is impossible.

We have shown in Lemma 5.1 that there are always bounds (35) with $M = 1$. We next establish that there are complex kernel bounds with a slightly larger $\omega$ than the $\omega$ in (35) in case $M = 1$. 28
Theorem 5.4. Adopt the notation and assumptions of Theorems 3.1, 4.5 or 4.9. Let $\theta \in [0, \theta_0)$ and let $\omega_0 \in \mathbb{R}$ be such that

$$\|S_z\|_{2 \to 2} \leq e^{\omega_0|z|}$$

for all $z \in \Sigma(\theta) = \{ z \in \mathbb{C} : z \neq 0, |\arg z| \leq \theta \}$. Then for all $\omega > \omega_0$ there exist $b, c > 0$ such that

$$|K_z(x; y)| \leq c |x|^{-d/2} e^{-b|x| |x|^{-1}} e^{\omega|z|} \quad (x, y) \text{-a.e.}$$

uniformly for all $z \in \Sigma(\theta)$.

Proof. We have to give a better estimate for Lemma 5.1. There exists $\nu > 0$ such that $\Re \sum_{i,j=1} e^{i \alpha a_{ij}(x)} \xi_i \xi_j \geq \nu |\xi|^2$ uniformly for all $\alpha \in [-\theta, \theta]$, $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. It follows from the Lumer–Phillips theorem that

$$\Re e^{i \alpha a}(\varphi, \varphi) + \omega_0(\varphi, \varphi) \geq 0$$

for all $\varphi \in V$. Let $\omega > \omega_0$ and $\delta \in (0, 1]$. Note that $\Re e^{i \alpha} \int_\beta(x)(B \varphi)(x)(\overline{B \varphi})(x) d\gamma(x) \geq 0$ in case of Robin boundary conditions, since $\beta \geq 0$. Then

$$\Re e^{i \alpha} a^\delta(\varphi, \varphi) + \omega(\varphi, \varphi) = (1 - \delta) \left( \Re e^{i \alpha} a(\varphi, \varphi) + \omega_0(\varphi, \varphi) \right) + \delta \Re e^{i \alpha} a(\varphi, \varphi)$$

$$+ \Re e^{i \alpha} b_\rho(\varphi, \varphi) + (\omega - (1 - \delta)\omega_0)||\varphi||^2$$

$$\geq \delta \Re e^{i \alpha} a(\varphi, \varphi) + \Re e^{i \alpha} b_\rho(\varphi, \varphi) + (\omega - (1 - \delta)\omega_0)||\varphi||^2,$$

where

$$b_\rho(\varphi, \varphi) = -\rho \sum_{i,j=1} e^{i \alpha a_{ij}} (D_i \varphi) \psi_j \overline{\varphi} + \rho \sum_{i,j=1} e^{i \alpha a_{ij}} \psi_i \varphi \overline{D_j \varphi} - \rho^2 \sum_{i,j=1} e^{i \alpha a_{ij}} \psi_i \varphi \psi_j \overline{\varphi}$$

$$+ \rho \sum_{i=1} \int e^{i \alpha} b_i \psi_i \varphi \overline{\varphi} - \rho \sum_{i=1} \int e^{i \alpha} c_i \varphi \psi_i \overline{\varphi}.$$

Now

$$\delta \Re e^{i \alpha} a(\varphi, \varphi) \geq \delta \nu \sum_{i=1} ||D_i \varphi||^2 - \delta \left| \sum_{i=1} b_i D_i \varphi \overline{\varphi} \right| - \delta \left| \sum_{i=1} c_i \varphi \overline{D_i \varphi} \right| - \delta \left| |c_0| \varphi \right|^2$$

$$\geq \delta \nu \sum_{i=1} ||D_i \varphi||^2 - 2 \delta \eta \sum_{i=1} ||D_i \varphi||^2 - \delta (2\eta)^{-1} d M_0^2 ||\varphi||^2 - \delta M_0 ||\varphi||^2$$

$$\geq 2^{-1} \delta \nu \sum_{i=1} ||D_i \varphi||^2 - \rho \delta \varphi \overline{\varphi}^2$$

for some $c > 0$, independent of $\delta$ and an appropriate choice of $\eta$. Here $M_0$ is as in the proof of Lemma 5.1. As in the proof of Lemma 3.7 one proves that there exists a $c' > 0$ such that

$$|b_\rho(\varphi, \varphi)| \leq \epsilon \sum_{i=1} ||D_i \varphi||^2 + c'((1 + \epsilon^{-1})|\rho|^2 + ||\rho||) ||\varphi||^2$$

$$\leq \epsilon \sum_{i=1} ||D_i \varphi||^2 + c'((1 + \epsilon^{-1})|\rho|^2 + \delta + (4\delta)^{-1} |\rho|^2) ||\varphi||^2$$

29
for all $\varepsilon > 0$. Combining these estimates one obtains

$$
\text{Re} e^{i\alpha_0}(\varphi, \varphi) + \omega(\varphi, \varphi) \geq \sum_{i=1}^{d} \| D_i \varphi \|^2 + (\omega - (1 - \delta)\omega_0 - c\delta - c'\delta)\|\varphi\|_{L^2}^2
-(c'(1 + \varepsilon^{-1}) + (4\delta^{-1}c')\rho^2\|\varphi\|_{L^2}^2.
$$

Since $\lim_{\varepsilon \to 0} \omega - (1 - \delta)\omega_0 - c\delta - c'\delta = \omega - \omega_0 > 0$ there exists $\delta > 0$ such that $\omega - (1 - \delta)\omega_0 - c\delta - c'\delta > 0$. Next take $\varepsilon = 2^{-1}\delta\nu$. Then

$$
\text{Re} e^{i\alpha_0}(\varphi, \varphi) + \omega(\varphi, \varphi) \geq -\omega_1\rho^2\|\varphi\|_{L^2}^2
$$

for some $\omega_1 > 0$, uniformly for all $\alpha \in [-\theta, \theta]$ and $\rho \in \mathbb{R}$. Therefore,

$$
\|S_{\varepsilon}^\rho\|_{L^2} \leq e^{\omega|z|}e^{\omega_1\rho^2|z|}
$$

uniformly for all $z \in \tilde{\Sigma}(\theta)$ and $\rho \in \mathbb{R}$.

By Theorem 5.2 there exist $b, c > 0$ and $\omega_2 \in \mathbb{R}$ such

$$
|K_\varepsilon(x; y)| \leq c|z|^{-d/2}e^{-b|x-y|^2|z|^{-1}}e^{\omega_2|z|} (x, y) \text{-a.e.}
$$

uniformly for all $z \in \tilde{\Sigma}(\theta)$. Let $\alpha > 0$ be as in Lemma 3.2. Then

$$
\|S_{\varepsilon}^\rho\|_{L^\infty}^2 = \sup_{\|\varphi\|_{L^2} \leq 1} \|S_{\varepsilon}^\rho\varphi\|_{L^2}^2 = \sup_{\|\varphi\|_{L^2} \leq 1} \left( \int_{\Omega} K_\varepsilon^\rho(x; y) \varphi(y) dy \right)^2
$$

$$
= \sup_{\|\varphi\|_{L^2} \leq 1} \left( \int_{\Omega} K_\varepsilon^\rho(x; y) \varphi(y) dy \right)^2
$$

$$
\leq \sup_{\|\varphi\|_{L^2} \leq 1} \left( \int_{\Omega} \left| K_\varepsilon^\rho(x; y) \right|^2 dy \right)
$$

$$
\leq \left( \int_{\mathbb{R}^d} \left( c|z|^{-d/2}e^{-b|x-y|^2|z|^{-1}}e^{\omega_2|z|} \right)^2 dy \right.
$$

$$
= \left( c'|z|^{-d/4}e^{\omega_3\rho^2|z|}e^{\omega_2|z|} \right)^2
$$

for some $c', \omega_3 > 0$, uniformly for all $z \in \tilde{\Sigma}(\theta)$ and $\rho \in \mathbb{R}$. So

$$
\|S_{\varepsilon}^\rho\|_{L^\infty} \leq c'|z|^{-d/4}e^{\omega_3\rho^2|z|}e^{\omega_2|z|}
$$

and by duality

$$
\|S_{\varepsilon}^\rho\|_{L^1} \leq c'|z|^{-d/4}e^{\omega_3\rho^2|z|}e^{\omega_2|z|}
$$

possibly by enlarging $c'$ and $\omega_2$ and $\omega_3$. Then for all $\varepsilon > 0$ one establishes

$$
\|S_{\varepsilon}^\rho\|_{L^1} \leq \|S_{\varepsilon}^\rho\|_{L^\infty} \leq \|S_{\varepsilon}^\rho\|_{L^1} \leq \|S_{\varepsilon}^\rho\|_{L^1} \leq \|S_{\varepsilon}^\rho\|_{L^1} \leq \|S_{\varepsilon}^\rho\|_{L^1}
$$

$$
= \left( c'(e|z|^{-d/4}e^{\omega_3\rho^2|z|}e^{\omega_2|z|}) \right)^2 e^{(1 - 2\varepsilon)|z|}e^{(1 - 2\varepsilon)\omega_2\rho^2|z|}
$$

$$
= (c')^2 e^{-d/2|z|}e^{(\omega + \varepsilon(2\omega_2 - 2\omega))|z|}e^{(2\omega_3 + (1 - 2\varepsilon)\omega_1)\rho^2|z|}
$$
uniformly for all $p \in \mathbb{R}$. Since $\omega > \omega_0$ and $\varepsilon > 0$ are arbitrary, the theorem follows by a minimalization over $p$ and $\psi \in W$ as in the proof of Proposition 3.3.

Next we show that the operator $A + \omega I$ has a bounded $H_\infty$-functional calculus in $L_p$, $1 \leq p \leq \infty$. Frequently it is easy to establish a bounded $H_\infty$-functional calculus in $L_2$; for example, $m$-accretivity is a sufficient condition. Recently, Duong and Robinson [DR95] proved the remarkable fact that this functional calculus can be carried over to $L_p$, $1 < p < \infty$, whenever a complex Gaussian estimate is valid. Their result can be applied directly to $\Omega = \mathbb{R}^d$. In the following theorem we show how to extend it to fairly general open subsets of $\mathbb{R}^d$ by a simple direct sum argument. Concerning the definition and basic facts on $H_\infty$-functional calculus we refer to [DR95] and the references given there.

**Theorem 5.5** Let $\Omega \subset \mathbb{R}^d$ be open such that $\partial \Omega$ is a null set. Let $S = (e^{-tA})_{t>0}$ be a holomorphic semigroup on $L_2(\Omega)$ with generator $A$. Suppose that $S$ is holomorphic in the sector $\Sigma(\theta)$, where $\theta \in (0, \pi/2)$. Assume that

(a) $A$ is accretive in $L_2(\Omega)$,
(b) $S_z$ is given by a kernel $K_z \in L_\infty(\Omega \times \Omega)$ satisfying

$$|K_z(x;y)| \leq c|z|^{-d/2}e^{-b|x-y|^2|z|^{-1}}$$

uniformly for all $z \in \Sigma(\theta)$ and some $b, c > 0$.

Then $S$ interpolates on $L_p(\Omega)$, $1 \leq p \leq \infty$ and $A$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus for all $\nu > \pi/2 - \theta$ in $L_p(\Omega)$ for all $p \in (1, \infty)$. Moreover, $f(A)$ is of weak type $(1,1)$ for each $f \in H_\infty(\Sigma(\nu))$. Here $A$ denotes the generator of $S$ in $L_p(\Omega)$.

**Remark.**

I. Condition (a) implies that

(a') $A$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_2(\Omega)$ for some $\nu > \pi/2 - \theta$.

Theorem 5.5 remains valid if one replaces (a) by the more general condition (a').

II. A special case of Theorem 5.5 had been obtained by Hieber [Hie94] who applied it to a purely second order symmetric elliptic operator on a bounded domain with Lipschitz boundary.

**Proof.** It follows from (36) and Theorem 5.3 that $S$ interpolates in $L_p(\Omega)$, $1 \leq p \leq \infty$ and that $S$ is holomorphic on the sector $\Sigma(\theta)$ on $L_p$. Moreover, $S$ is bounded on $\Sigma(\theta)$ in $\mathcal{L}(L_p)$ by Proposition 3.3. Now, if $\Omega = \mathbb{R}^d$, the assertion follows from [DR95] Theorem 3.1.

The general case can be reduced to the case where the domain is $\mathbb{R}^d$ in the following way. Let $\Omega_1 = \mathbb{R}^d \setminus \overline{\Omega}$ and let $A_1 = -\sum_{i=1}^d \partial^2/\partial x_i^2$ with Dirichlet boundary conditions on $L_2(\Omega_1)$. Since $\partial \Omega$ is a null set one has $L_2(\mathbb{R}^d) = L_2(\Omega) \oplus L_2(\Omega_1)$, where the decomposition is given by $f = f_1 + f_1\Omega$. Let $\tilde{A} = A \oplus A_1$. Then $\tilde{A}$ satisfies the hypotheses of the theorem on $L_2(\mathbb{R}^d)$ and consequently, $\tilde{A}$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\mathbb{R}^d)$ for $p \in (1, \infty)$ whenever $\nu > \pi/2 - \theta$. Then $A$ has the same property.

Similarly the $(1,1)$-estimate follows from [DR95] Theorem 3.1.
In virtue of Theorem 5.3 one obtains a bounded $H_\infty$-functional calculus for $A + \omega I$ for some $\omega$ if one has merely real time Gaussian bounds. More precisely, assume that $\partial \Omega$ is a null set and assume that the hypotheses of Theorem 5.3 are satisfied. Denote the generator of $S$ in $L_p(\Omega)$ by $A$. Then for all $\nu > \pi/2 - \theta$ there exists an $\omega \in \mathbb{R}$ such that the operator $A + \omega I$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\Omega)$, $1 < p < \infty$. Of course, if $A + \omega_0 I$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus then the same is true for $A + \omega I$ for all $\omega > \omega_0$. For the elliptic operators obtained here, Theorem 5.4 allows us to consider the result for small $\omega$.

Theorem 5.6 Adopt the notation and assumptions of Theorems 3.1, 4.5 or 4.9. Moreover, suppose that $\partial \Omega$ is a null set in $\mathbb{R}^d$. Let $\nu > \pi/2 - \theta$, $\nu < \pi/2$ and $\omega_0 \in \mathbb{R}$ be such that

$$\|S_z\|_{2 \to 2} \leq e^{\omega_0 |z|}$$

for all $z \in \tilde{\Sigma}(\pi/2 - \nu) = \{z \in \mathbb{C} : z \neq 0, \ |\arg z| \leq \pi/2 - \nu\}$. Then for all $\omega > \omega_0$ the operator $A + \omega I$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\Omega)$ for each $p \in (1, \infty)$. Moreover, $f(A + \omega I)$ is of weak type $(1, 1)$ for each $f \in H_\infty(\Sigma(\nu))$.

Proof. This is a direct consequence of Theorems 5.4 and 5.5.

Remark. We had to suppose the very weak condition on $\Omega$ that $\partial \Omega$ is a null set in $\mathbb{R}^d$ in order to apply the result of Duong and Robinson on $\mathbb{R}^d$. We do not know whether this condition can be omitted.

Corollary 5.7 Adopt the notation and assumptions of Theorems 3.1, 4.5 or 4.9. Moreover, suppose that $\partial \Omega$ is a null set in $\mathbb{R}^d$. Let $\nu > \pi/2 - \theta$, $\nu < \pi/2$ and $\omega_0 \in \mathbb{R}$ be such that

$$\|S_z\|_{2 \to 2} \leq e^{\omega_0 |z|}$$

for all $z \in \tilde{\Sigma}(\pi/2 - \nu) = \{z \in \mathbb{C} : z \neq 0, \ |\arg z| \leq \pi/2 - \nu\}$. Then for all $\omega > \omega_0$ the operator $A + \omega I$ has bounded imaginary powers and there exists $c > 0$ such that

$$\|(A + \omega I)^{is}\|_{p \to p} \leq ce^{u|s|}$$

uniformly for all $s \in \mathbb{R}$ and $p \in (1, \infty)$.

Proof. Apply Theorem 5.6 to the holomorphic function $z \mapsto z^{is}$.

Note that the value of $\nu$ in the previous theorem is less than $\pi/2$. This is important in order to apply the Dore–Venni theorem [DV87] and its extensions (see [Prü93], Theorem 8.4, p. 218).

Example 5.8 Suppose the operator $A$ is pure second order (not necessarily symmetric) with $L_\infty$ coefficients and Dirichlet boundary conditions. Moreover, suppose that $\Omega$ is contained in a strip

$$\{x \in \mathbb{R}^d : l < x \cdot \xi < r\}$$

for some $l < r$ and $\xi \in \mathbb{R}^d, \xi \neq 0$. Then for all $\theta \in (0, \theta_a)$ there exists $\mu' > 0$ such that

$$\Re e^{i\alpha} a(\phi, \phi) \geq \mu' \sum_{i=1}^d \|D_i \phi\|_2^2$$

32
for all $\alpha \in [-\theta, \theta]$ and $\varphi \in H^1_0(\Omega)$. Therefore, by the Poincaré inequality, one deduces that

$$\text{Re} \ e^{i\alpha} a(\varphi, \varphi) \geq 2(r - l)^{-2} \mu \|\varphi\|_2^2$$

(see [DL87a] p. 920). So

$$\|S_z\|_{2 \rightarrow 2} \leq e^{-(r-\theta)^{-2} \mu |z|}$$

for all $z \in \Delta(\theta)$. As a result one obtains from Theorems 5.4, 5.6 and Corollary 5.7 that for all $\theta \in (0, \theta_a)$ there exist $b, c > 0$ and a negative $\omega < 0$ such that

$$|K_z(x; y)| \leq c |z|^{-d/2} e^{-b|x-y|^2|z|^{-1} e^{\omega |z|}} \quad (x, y) \text{-a.e.}$$

uniformly for all $z \in \Delta(\theta)$ and, if in addition $\partial \Omega$ is a null set, then $A$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\Omega)$ for all $p \in (1, \infty)$ and $\nu \in (\pi/2 - \theta_a, \pi/2)$. In particular, there exists a $c > 0$, depending on $\nu$, such that $\|A^s\|_{p \rightarrow p} \leq ce^{|s|}$ for all $s \in \mathbb{R}$.

The next remark clarifies the nature of the angle $\theta_a$.

**Remark.** Assume that $b_i = c_i = c_0 = 0$ for all $i \in \{1, \ldots, d\}$. Let $A$ be any of the operators considered in Theorems 3.1, 4.5 or 4.9. Then

$$\|S_z\|_{2 \rightarrow 2} \leq 1 \quad \text{for all } z \in \Sigma(\theta_a)$$

by the proof of Lemma 5.1 for $\rho = 0$. If $\Omega$ is bounded, $V = \overline{H^1}(\Omega)$ and the coefficients $a_{ij}$ are constant, then $\Sigma(\theta_a)$ is the largest sector on which $S_z$ is a contraction. In fact, by the Lumer–Phillips theorem we have to show that $\theta_a$ is the smallest angle in $(0, \pi/2)$ such that the numerical range $\Theta(A)$ of $A$ is included in $\{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \pi/2 - \theta_a\} \cup \{0\}$. We will show the following identity

$$\Theta(A) = \overline{\Theta(B)} = \left\{ z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \pi/2 - \theta_a \right\} \cup \{0\}, \quad (37)$$

where $B = (a_{ij})$ and $\Theta(B)$ is the numerical range of the matrix $B$. Obviously the second equality is valid by definition of $\theta_a$, the convexity of the numerical range of $B$ and the fact that $B$ is a real matrix. Let $\lambda \in \Theta(B)$ and $r \geq 0$. Let $\xi \in \mathbb{C}^d$ be such that $|\xi| = 1$ and $\lambda = (B\xi, \xi)$. Let $u \in C^\infty_c(\mathbb{R}^d)$ and $\alpha \in (0, \infty)$ be such that $u(x) = e^{\alpha r \xi \cdot x + \ldots + r\xi \cdot x}$ for all $x \in \Omega$ and $\|u\|_2 = 1$. Then $u|\Omega \in \overline{H^1}(\Omega)$ and $D_i u = r \xi_i u$ on $\Omega$ for all $i \in \{1, \ldots, d\}$. Therefore,

$$a(u, u) = \int_\Omega (B \nabla u, \nabla u) = \int_\Omega r^2(B\xi, \xi) |u|^2 = \lambda r^2$$

and $\overline{\Theta(B)} \subset \Theta(A)$. Conversely, if $u \in \overline{H^1}(\Omega)$ with $\|u\|_2 = 1$ then

$$a(u, u) = \int_\Omega (B \nabla u, \nabla u) = \int_\Omega (Bv, v) |\nabla u|^2 \in \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \pi/2 - \theta_a \} \cup \{0\}$$

since $(Bv(x), v(x)) \in \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \pi/2 - \theta_a \} \cup \{0\}$ for a.e. $x \in \Omega$, where

$$v(x) = \begin{cases} \frac{|\nabla u(x)|}{|\nabla u(x)|} & \text{if } (\nabla u)(x) \neq 0, \\ 0 & \text{if } (\nabla u)(x) = 0. \end{cases}$$

Now (37) follows.

The equality (37) even implies that $S$ cannot be holomorphic and quasi-contractive on $L_2$ on a sector strictly larger than $\Sigma(\theta_a)$.

We conclude by a consequence concerning the spectrum of the different realizations of $A$ in $L_p(\Omega)$.
Theorem 5.9 (p-independence of the spectrum.) Adopt the notation and assumptions of Theorems 3.1, 4.5 or 4.9, so $A$ is the realization of the elliptic operator $\mathcal{A}$ in $L_p(\Omega)$ with boundary conditions. Then the component $\rho_\infty(A)$ of the resolvent set of $A$ which contains a left half-plane is independent of $p$, $1 \leq p \leq \infty$. Moreover, $(\lambda I + A)^{-1}$ is a kernel operator for all $\lambda \in \rho_\infty(A)$.

Proof. This follows immediately from [Are94] Theorem 4.2, the remark following Corollary 4.3 in [Are94] and the Gaussian estimates established here.

Acknowledgements

This paper has been written while the second named author was visiting the Department of Mathematics at the University of Besançon. He wishes to thank the first named author for his generous hospitality and the department for financial support.

References


