The classification of all (145,7,72) binary linear codes

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The classification of all (145,7,72) binary linear codes

by

Tor Helleseth and Henk C.A. van Tilborg

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1. Introduction

We let an \((n,k,d)\) code denote a binary linear \(k\)-dimensional code of length \(n\) and minimum distance at least \(d\).

We let

\[ n(k,d) := \min\{n \mid \text{there exist an } (n,k,d) \text{code}\} \]

Important results about \(n(k,d)\) which we will use repeatedly are:

**Theorem 1.1.** (Griesmer [2]). Let \(\left\lceil x \right\rceil\) denote the smallest integer greater than or equal to \(x\). Then

\[ n(k,d) \geq g(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil . \]

**Theorem 1.2.** (Logacev [4]). If \(3 \leq d \leq 2^{k-2} - 2\), then

\[ n(k,d) \geq g(k,d) + 1 . \]

**Theorem 1.3.** (van Tilborg [6]). If \(2^{k-2} + 3 \leq d \leq 2^{k-2} + 2^{k-3} - 4\), then

\[ n(k,d) \geq g(k,d) + 1 . \]

**Theorem 1.4.** (Belov [1]). Let \(s = \left\lceil \frac{d}{2^{k-1}} \right\rceil\) and \(s \cdot 2^{k-1} - d = \sum_{i=1}^{p} 2^{u_i - 1}\), where \(k > u_1 > u_2 > \ldots > u_p > 0\).

If

\[ \min(p, s+1) \sum_{i=1}^{s+1} u_i \leq s \cdot k \]

or

\[ u_{i+1} = u_i - 1, \text{ for } s \leq i \leq p - 1 \text{ and } u_p \in \{1, 2\} , \]

then

\[ n(k,d) = g(k,d) . \]

Belov gave constructions of codes with \(n(k,d) = g(k,d)\) for all parameters satisfying the conditions of Theorem 1.4. An open problem is to decide whether these sufficient conditions for \(n(k,d) = g(k,d)\) are necessary.
We will show the existence of new codes meeting the Griesmer bound without satisfying Belov's conditions. These codes are (145,7,72) codes. We will give a thorough analysis of codes with these parameters. We have several motivations for this. First of all the methods in this paper can in principle be applied to classify any \((n,k,d)\) code with \(n(k,d) = g(k,d)\), and thus we can either prove that such a code does not exist or (as in our case) find new codes meeting the Griesmer bound. Further, in a following paper [3] we will construct infinite sequences of \((n,k,d)\) codes based on our \((145,7,72)\) codes. To prove uniqueness results for these sequences of \((n,k,d)\) codes, as well as to prove nonexistence results for other \((n,k,d)\) codes with \(s > 1\) meeting the Griesmer bound, we will need a classification of all \((145,7,72)\) codes.
2. Preliminaries

Let $G$ denote a generator matrix of a binary $(n,k,d)$ code $C$.

Definition 2.1. The residual code of $C$ with respect to $c \in C$ is the code $C^0$ generated by the restriction of $G$ to the columns where $c$ has a zero.

We let $w(c)$ denote the Hamming weight of $c$. We denote $C^0$ by $\text{Res}(C,c)$ or $\text{Res}(C,w)$ if only the weight $w$ of $c$ is important.

If $c^0 \in C^0$ we let $c \in C$ denote a word whose restriction to $C^0$ is $c^0$. Also the restriction of $c \in C$ to $C^0$ is denoted by $c^0$.

We denote $\text{Res}(\text{Res}(C;c_1,c_2))$ by $\text{Res}(C;c_1,c_2)$ or $\text{Res}(C;w,w_0)$ if only the weights of $c_1 \in C$ and $c_2^0 \in C^0$ are important, and $w(c_1) = w$, $w(c_2^0) = w_0$, etc.

Let $[x]$ denote the greatest integer smaller than or equal to $x$. From Definition 2.1 we get:

Lemma 2.2. We have that $\text{Res}(C;w)$ is a $(n-w,k-1,d-[\frac{w}{2}])$ code.

Lemma 2.3. Let $c_1 \in C$ have weight $w$. Let $c_2^0 \in C^0 = \text{Res}(C;c_1)$ have weight $w_0$. Then \{w(c_2),w(c_1+c_2)\} = \{w_0 + [\frac{w}{2}], w_0 + [\frac{w}{2}] + i\}$ for some non-negative integer $i$.

Lemma 2.4. (van Tilborg [6]). Suppose that $C$ meets the Griesmer bound.

(i) There exists a generator matrix for $C$ such that all row vectors have weight $d$.

(ii) If $s = \left\lceil \frac{d}{2^{k-1}} \right\rceil$, the no column vector occurs more than $s$ times in $G$.

Lemma 2.5. (van Tilborg [6]). The following $(n,k,d)$ codes are unique:

(i) $C^{(i)}_{k;u} = (2^{k-2u},k,2^{k-1-2u-1})$, $1 \leq u \leq k-1$. The weight distribution is $A_0 = 1,A_2 = 2^{k-2u},A_2^1 = 2^{k-u-1}$

(ii) $C^{(ii)}_{k;k-2,2} = (2^{k-2k-2},k,2^{k-1-2k-3-2})$, $k \geq 5$. The weight distribution is $A_0 = 1,A_2 = 2^{k-1-2k-3-2} = 3(2^{k-2}),A_2^1 = 2^{k-2-1},A_2^2 = 3$.

(iii) $C^{(iii)}_k = (2^{k-1}+k,k,2^{k-2}+2)$, $k \geq 3$, $k \neq 5$.

We will also need the MacWilliams identities [5, p. 127].
Lemma 2.6. Let $A_w, B_w$ denote the number of codewords of weight $w$ in a binary linear code $C$ and in its dual code respectively. Then

$$B_w = \frac{1}{|C|} \sum_{i=0}^{n} A_i K_w(i), \quad 0 \leq w \leq n$$

where

$$K_w(i) = \sum_{\ell=0}^{w} (-1)^{\ell} \binom{n-i}{\ell} \binom{w-\ell}{\ell}, \quad 0 \leq i, w \leq n$$

In particular,

$$K_0(i) = 1,$$
$$K_1(i) = n - 2i,$$
$$K_2(i) = \binom{n}{2} - 2ni + 2i^2.$$
3. Weight distribution of (145,7,72) codes

Let $C$ denote a (145,7,72) code. We let $G$ be its generator matrix. The number of codewords of weight $w$ are $A_w$.

Our approach can briefly be sketched as follows. We first show that the only possible occurring nonzero weights of $C$ are 72, 80, or 88. Next we show that $C$ has only 5 possible weight distributions. Finally we prove that all (145,7,72) codes with a given weight distribution are isomorphic.

Lemma 3.1. If $A_w \neq 0$, then $w \in \{0, 72, 80, 82, 84, 86, 88, 90, 92, 96, 98, 100, 112, 114\}$.

Proof. From Lemma 2.4(i) we get $A_w = 0$ for $w$ odd. By Lemma 2.4(ii) we conclude $A_w = 0$ for $w > 128$ since $s = 2$ for $C$.

Suppose $C$ contains a codeword of weight $w \leq 128$ and $w$ even. The residual code $C^0$ of $C$ w.r.t. a word of weight $w$ has parameters given by Lemma 2.2, and these contradicts Theorem 1.1-1.3, except for the values of $w$ given in Lemma 3.1. The parameters of the $(n^0, k^0, d^0)$ code $C^0$ are given in Table 1.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$(n^0, k^0, d^0)$</th>
<th>contradicts</th>
<th>$w$</th>
<th>$(n^0, k^0, d^0)$</th>
<th>contradicts</th>
<th>$w$</th>
<th>$(n^0, k^0, d^0)$</th>
<th>contradicts</th>
</tr>
</thead>
<tbody>
<tr>
<td>72</td>
<td>73,6,36</td>
<td></td>
<td>96</td>
<td>49,6,24</td>
<td></td>
<td>120</td>
<td>25,6,12</td>
<td>T.1.2.</td>
</tr>
<tr>
<td>74</td>
<td>71,6,35</td>
<td>T.1.1.</td>
<td>98</td>
<td>47,6,23</td>
<td></td>
<td>122</td>
<td>23,6,11</td>
<td>T.1.1.</td>
</tr>
<tr>
<td>76</td>
<td>69,6,34</td>
<td>T.1.1.</td>
<td>100</td>
<td>45,6,22</td>
<td></td>
<td>124</td>
<td>21,6,10</td>
<td>T.1.1.</td>
</tr>
<tr>
<td>78</td>
<td>67,6,33</td>
<td>T.1.1.</td>
<td>102</td>
<td>43,6,21</td>
<td>T.1.1.</td>
<td>126</td>
<td>19,6,9</td>
<td>T.1.1.</td>
</tr>
<tr>
<td>80</td>
<td>65,6,32</td>
<td></td>
<td>104</td>
<td>41,6,20</td>
<td>T.1.3.</td>
<td>128</td>
<td>17,6,8</td>
<td>T.1.2.</td>
</tr>
<tr>
<td>82</td>
<td>63,6,31</td>
<td></td>
<td>106</td>
<td>39,6,19</td>
<td>T.1.1.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>84</td>
<td>61,6,30</td>
<td></td>
<td>108</td>
<td>37,6,18</td>
<td>T.1.1.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>86</td>
<td>59,6,29</td>
<td></td>
<td>110</td>
<td>35,6,17</td>
<td>T.1.1.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>88</td>
<td>57,6,28</td>
<td></td>
<td>112</td>
<td>33,6,16</td>
<td>T.1.1.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>55,6,27</td>
<td></td>
<td>114</td>
<td>31,6,15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>92</td>
<td>53,6,26</td>
<td></td>
<td>116</td>
<td>29,6,14</td>
<td>T.1.2.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>94</td>
<td>51,6,25</td>
<td>T.1.1.</td>
<td>118</td>
<td>27,6,13</td>
<td>T.1.1.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lemma 3.2. If $A_w \neq 0$, then $w \in \{0, 72, 80, 82, 84, 88, 92, 96, 112\}$.

Proof. In view of Lemma 3.1 we must prove that $A_{86} = A_{90} = A_{98} = A_{100} = A_{114} = 0$. To illustrate the method we prove that $A_{86} = 0$. Suppose $c \in C$ has weight $86$. Then $C^0 = \text{Res}(C; c_i)$ is a $(59, 6, 29)$ code. The extended code of $C^0$ is the unique $C_{6; 2}^{(i)}$ code of Lemma 2.5. Since $C_{6; 2}^{(i)}$ has 48 codewords of weight 30 it follows that $C^0$ has a codeword $c_0^0$ of weight 30. Using Lemma 2.3. with $w = 86$ and $w_0 = 30$ we get

$$\{w(c_2), w(c_1 + c_2)\} = \{73 - i, 73 + i\}$$

for some integer $i \geq 0$. This contradicts that $A_w = 0$ for $72 < w < 80$.

With the following choices of $w$ and $w_0$ the other cases can be proved in a similar way.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$C^0 = \text{Res}(C; c)$</th>
<th>$C^0$ (extended)</th>
<th>$w_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>$(55, 6, 27)$</td>
<td>$C_{6; 3}^{(i)}$</td>
<td>28</td>
</tr>
<tr>
<td>98</td>
<td>$(47, 7, 23)$</td>
<td>$C_{6; 4}^{(i)}$</td>
<td>24</td>
</tr>
<tr>
<td>100</td>
<td>$(45, 6, 22)$</td>
<td>$C_{6; 4, 2}^{(i)}$</td>
<td>-</td>
</tr>
<tr>
<td>114</td>
<td>$(31, 6, 15)$</td>
<td>$C_{6; 5}^{(i)}$</td>
<td>16</td>
</tr>
</tbody>
</table>

Lemma 3.3. If $A_w \neq 0$, then $w \in \{0, 72, 80, 88, 96, 112\}$.

Proof. According to Lemma 3.2 we have to prove that $A_{82} = A_{84} = A_{92} = 0$. We illustrate the method by proving that $A_{84} = 0$. Suppose $c_1 \in C$ has weight 84. Then $C^0 = \text{Res}(C; c_1)$ is a $(61, 6, 30)$ code. Since a $(61, 6, 31)$ code would contradict the Griesmer bound, $C^0$ has a codeword $c_2^0$ of weight 30. Then $C^{00} = \text{Res}(C^{00}; c_2)$ is a $(31, 5, 15)$ code. We claim that $C^{00}$ has a codeword of weight 16. If not, $C^{00}$ must have a word $c_0^{00}$ of weight 15 since a $(31, 5, 17)$ code contradicts the Griesmer bound. Then $C^{000} = \text{Res}(C^{00}; 15)$ is a $(16, 4, 8)$ code which has a codeword of weight 8 since a $(16, 4, 9)$ code would again contradict the Griesmer bound. Let $v^{00} \in C^{00}$ have weight 8 when restricted to $C^{00}$. Then $\{w(c^{00}), w(c^{00} + v^{00})\} = \{15, 16\}$ by Lemma 2.3.
By symmetry we may assume $w(c_0^0) = 16$. Then again either $c_0^0$ or $c_0^0 + c_2^0$ has weight 31 or 32, and we assume w.l.o.g. that it is $c_0^0$. Then $c$ or $c + c_1$ has weight $w$ such that $72 < w < 80$, a contradiction.

The proof that $A_{82} = A_{92} = 0$ is similar.

**Lemma 3.4.** If $A_w \neq 0$, then $w \in \{0, 72, 80, 88, 96\}$.

**Proof.** We have to prove that $A_{112} = 0$. Suppose $c \in C$ has weight 112. Then $c_0^0 = \text{Res}(C; c)$ is a $(33, 6, 16)$ code. Let $A_w^0$ denote the number of codewords of weight $w$ in $C_0$. Since a $(33-w, 5, 16-\left\lfloor \frac{w}{2} \right\rfloor)$ code will contradict at least one of the Theorems 1.1-1.3 for all $w$ such that $19 < w < 31$ we get that $A_w^0 = 0$ for $19 < w < 31$. Using Lemma 2.3 with $w = 112$ and $w_0 = 17$ or 18 we get $A_{17}^0 = A_{18}^0 = 0$. Since a $(33, 6, 17)$ code does not exist (contradicts the Griesmer bound), $A_{16}^0 > 0$. Therefore $A_{33}^0 = 0$, otherwise the complement of a word of weight 16 would have weight 17 which is excluded. From Lemma 2.6 we get, since $A_{32}^0 < 1$ that

$$64B_1^0 = 33A_{16}^0 + A_{16}^0 - 31A_{32}^0 = 64$$

leading to $B_1^0 = 1$. This means that one column of the generator matrix of $C_0$ is the zero vector, and therefore this is also the case for $G$. This contradicts that $C$ meets the Griesmer bound, and we conclude that $A_{112} = 0$.

**Lemma 3.5.** We have the following equations:

$$
\begin{bmatrix}
A_{72} & A_{80} & A_{88} & A_{96} \\
1 & 0 & 0 & 1 & 4(B_2 + 195) \\
0 & 1 & 0 & -3 & 42 - B_2 \\
0 & 0 & 1 & 3 & 4(B_2 - 25)
\end{bmatrix}
$$

where $B_2$ is the number of pairs of repeated columns of $G$.

**Proof.** This follows from Lemma 2.6. Note that $B_1 = 0$ since $C$ meets the Griesmer bound. Further $B_2$ is the number of codewords in the dual code of $C$ which is equal to the number of repeated columns of $G$ since no column is repeated more than twice according to Lemma 2.4 (ii).
Lemma 3.6. Let $A_w^0$ denote the number of codewords of weight $w$ in the $(49,6,24)$ code $Res(C; 96)$. If $A_w^0 \neq 0$, then $w \in \{0, 24, 28, 32\}$ and we have the following equations,

\[
\begin{array}{cccc}
A_{24}^0 & A_{28}^0 & A_{32}^0 & 49 \\
1 & 0 & -1 & 14 \\
0 & 1 & 2 & 14 \\
0 & 0 & 1 & B_{2}^0
\end{array}
\]

where $B_{2}^0$ is the number of repeated columns in a generator matrix for $Res(C; 96)$.

Proof. Using the same techniques as in Lemma 3.1-3.4 it is easily seen that the only possible weights are 0, 24, 28, and 32. The lemma follows then from Lemma 2.6.

Lemma 3.7. Let $A_w^0$ denote the number of codewords of weight $w$ in the $(57,6,28)$ code $Res(C; 88)$. If $A_w^0 \neq 0$, then $w \in \{0, 28, 32, 36, 40\}$ and we have the following equations,

(i) $A_{28}^0 A_{32}^0 A_{36}^0 A_{40}^0$

\[
\begin{array}{cccc}
1 & 0 & -1 & -2 & 48 \\
0 & 1 & 2 & 3 & 15 \\
0 & 0 & 1 & 3 & B_{2}^0
\end{array}
\]

where $B_{2}^0$ is the number of repeated columns in a generator matrix for $Res(C; 88)$.

(ii) $A_{36}^0 A_{40}^0 = 0$

Proof. (i) This goes along the same lines as the previous lemma.

(ii) Suppose $c^0 \in C^0 = Res(C; 88)$ has weight 36, and that $A_{40}^0 > 0$.

According to van Tilborg [6] there are two non-isomorphic $(21,5,10)$ codes with the weight distributions,
where $A_{w}^{00}$ denotes the number of codewords of weight $w$ in a $(21,5,10)$ code. Since $\text{Res}(C_{0};C_{0})$ is a $(21,5,10)$ code it has one of the two weight distributions in (3.1). We note that if $v_{0} \in C_{0}$ has weight 40 (resp. 36) then its restriction to $\text{Res}(C_{0};C_{0})$ has weight 16 (resp. at least 14), since otherwise $w(v_{0} + c_{0}) > 28$. Therefore $\text{Res}(C_{0};C_{0})$ must have $A_{16}^{00} = 1$ and $A_{14}^{00} = 0$, and therefore $A_{36}^{00} = A_{40}^{00} = 1$. From (i) we get $B_{2}^{0} = 7$, but since $C_{0}$ is a $(57,6,28)$ code with a codeword of weight 40 its generator matrix must have at least 8 repeated columns. This contradicts that $B_{2}^{0} = 7$ and (ii) is proved.

We next will prove that $A_{96} = 0$. This is the hardest case and cannot be done by the methods of Lemma 3.1-3.4 alone. We first introduce some notations. The inner product $(c_{1},c_{2})$ between $c_{1} = (a_{1},\ldots,a_{n})$ and $c_{2} = (b_{1},\ldots,b_{n})$ is defined as $(c_{1},c_{2}) := \sum_{i=1}^{n} a_{i}b_{i}$. We denote $c = (1\cdots1\ 0\cdots0)$ by $\overline{r}\overbrace{s}$.

**Lemma 3.8.** If $A_{w}^{0} \neq 0$, then $w \in \{0, 72, 80, 88\}$.

**Proof.** According to Lemma 3.4 we must show that $A_{96} = 0$. Suppose $c_{1} \in C$ has weight 96. We have $A_{96} = 1$ since if $v \in C$ is another codeword of weight 96, then $w(v + c_{1}) < 72$ since the restriction of $v$ to $C_{0} = \text{Res}(C_{0};c_{1})$ by Lemma 3.6 has weight at most 32.

From Lemma 3.5 we get $B_{2} = 31 + 2A_{88}$. Since $C$ has a codeword of weight 96 there are at least 32 pairs of repeated columns and therefore $A_{88} \geq 1$.

We next show that $A_{88} \geq 2$. Suppose $A_{88} = 1$ and let $c_{2} \in C$ have weight 88. Then $(c_{1},c_{2}) = 56$ since $w(c_{0}) \leq 32$ and $w(c_{1} + c_{2}) \geq 72$. Lemma 3.5 gives $A_{80} = 12$ and $B_{2} = 33$. Therefore we can choose $c_{3} \in C$ of weight 80, such that we have,
Since \( B_2 = 33 \) and \( A_{32}^0 \geq 1 \) we get from Lemma 3.6 that \( B_2^0 = A_{32}^0 = 1 \), and therefore \((c_1, c_3) = 52\). The code \( C^{00} = \text{Res}(C^0_1; c_2^0) \) has the possible weights \( 8, 10, 12 \), or \( 16 \), as can be proved using the methods of Lemma 3.1-3.4. Since \( \gamma_0 \) is the weight of a codeword in the \((17,5,8)\) code \( \text{Res}(C^0_1; 3) \) and \( \gamma_0 + \gamma_1 = 28 \) it follows from Lemma 3.6 that \( \gamma_0 = 10 \) or \( 12 \). If \( \gamma_0 = 10 \), then \( \gamma_1 = 18 \) leading to \( B_2^0 \geq 2 \), a contradiction.

Hence \( \gamma_0 = 12 \) and \( \gamma_1 = 16 \). The \((57,6,28)\) code \( \text{Res}(C; c_2) \) has a word of weight \( 40 \), namely the restriction of \( c_1 \). By Lemma 3.7 (i) we get \( \gamma_2 = 16, 20, 24 \), or \( 28 \). If \( \gamma_2 = 16 \) or \( 24 \), then the restriction of \( c_3 \) or \( c_1 + c_3 \) have weight \( 36 \), contradicting Lemma 3.7 (ii). Further \( \gamma_2 = 28 \) is impossible since this leads to \( B_2 > 33 \). Hence \( \gamma_2 = 20 \) and \( \gamma_3 = 32 \).

We choose \( c_4 \in C \) of weight \( 80 \) such that \( c_4 \not\equiv c_1, c_2, c_3 \). This is possible since \( A_{80} = 12 \). Therefore,

\[
\begin{array}{c|c|c}
  c_1 & 96 & 49 \\
  c_2 & 56 & 40 & 32 & 17 \\
  c_3 & 32 & 24 & 20 & 20 & 16 & 16 & 12 & 5 \\
  c_4 & \gamma_7 & \gamma_6 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \\
\end{array}
\]

From the above discussions we get \( \gamma_7 + \gamma_6 = 32 \) and therefore \( \gamma_7 = \gamma_6 = 16 \), or a column of \( G \) will be repeated more than twice, contradicting Lemma 2.4 (ii). Also \( \gamma_1 + \gamma_0 = 12 \) gives \( \gamma_1 = 8 \) and \( \gamma_0 = 4 \) since \( \gamma_0 \) is the weight of a codeword in the \((5,4,2)\) code \( \text{Res}(C^{00}_2; c_1) \). The \( \text{Res}(C; c_3 + c_2) \) is a \((73,6,36)\) code and the restriction of \( c_2 + c_3 + c_4 \) has weight \( 32 \), a contradiction. We conclude \( A_{88} = 1 \) is impossible.

We next show \( A_{88} \leq 2 \). Suppose \( c_1, c_2, c_3 \in C \) where \( w(c_1) = 96 \), \( w(c_2) = w(c_3) = 88 \). We have
Since $Y_1 + Y_0 = 32$ and $\text{Res}(C; c_1, c_2)$ has only weights 8, 10, 12, or 16, we get $Y_0 = 12$ or 16, otherwise $w(c_2^0 + c_3^0) < 24$. If $Y_0 = 12$, then $Y_3 = 20$ and $w(c_2^0 + c_3^0) = 24$ which means that $w(c_2 + c_3) = 72$ and therefore $Y_3 = 32$ and $Y_2 = 24$. Then the restrictions of $c_1$ and $c_3$ to $\text{Res}(C; c_2)$ have weights 40 and 36 respectively, contradicting Lemma 3.7 (ii). We conclude that $Y_0 = 16$. This also means that every $c$ of weight 88 would have weight 16 when restricted to $\text{Res}(C; c_1, c_2)$. Since this is a $(17, 5, 8)$ code this code can not contain more than one word of weight 16. We therefore conclude $A_{88} = 2$.

Finally we prove that $A_{88} = 2$ is impossible. Suppose $A_{88} = 2$ and let $c_4 \in C - \langle c_1, c_2, c_3 \rangle$, $w(c_4) = 80$ where $c_1, c_2, c_3$ are as in (3.2). We have by Lemma 3.5 that $B_2 = 35$ and $A_{80} = 10$, so $c_4$ exists. Therefore,

We have $(c_1, c_4) = 52$ and therefore $Y_1 + Y_0 = 10$ for $i = 1, 2, 3$, which means that $Y_0 = 1, Y_1 = Y_2 = Y_3 = 9$, otherwise we contradict the occurring weights of $c_0$ or $\text{Res}(C; c_0, 32)$. Further by Lemma 2.4 (ii) we have $Y_7 = 16$. Also $Y_4 = 8$ since otherwise $B_2 \geq 36$, which is impossible. Since $c_3 + c_4$ restricted to $\text{Res}(C_0; c_2)$ has weight 8, the restriction of $c_3 + c_4$ to $\text{Res}(C; c_2)$ has weight 28. Therefore $Y_5 = 12$ and $Y_6 = 16$. We now get a contradiction since $w(c_2 + c_4) = 68$. We therefore have proved that $A_{96} = 0$. \qed
Lemma 3.9. A (145,7,72) code C must have one of the following 5 weight distributions,

<p>| | | | | |</p>
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>115</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>II</td>
<td>1</td>
<td>113</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>112</td>
<td>13</td>
<td>2</td>
</tr>
<tr>
<td>IV</td>
<td>1</td>
<td>111</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>V</td>
<td>1</td>
<td>110</td>
<td>17</td>
<td>0</td>
</tr>
</tbody>
</table>

Proof. By Lemma 3.5 it is sufficient to prove that $A_{g8} \in \{0,1,2,3,5\}$. Suppose $A_{g8} \geq 2$ and let $c_1, c_2 \in C$ have weight 88. Then $(c_1, c_2) = 48$ or 52. We will exclude $(c_1, c_2) = 48$. If $(c_1, c_2) = 48$ then $B_2 \geq 32$ and therefore Lemma 3.5 gives $A_{g8} \geq 4$. We can therefore choose $c_3 \in C - \langle c_1, c_2 \rangle$, $w(c_3) = 88$ such that,

\[
\begin{align*}
\text{c}_1 & \quad 88 \quad 57 \\
\text{c}_2 & \quad 48 \quad 40 \quad 40 \quad 17 \\
\text{c}_3 & \quad Y_3 \quad Y_2 \quad Y_1 \quad Y_0.
\end{align*}
\]

By Lemma 3.7 (ii) we get $(c_1, c_3) = (c_2, c_3) = 48$. The (17,5,8) code $\text{Res}(C; c_1, c_2^0)$ has only 8,10,12, or 16 as possible weights and therefore we get the unique solution $Y_0 = 16$, $Y_1 = Y_2 = Y_3 = 24$. In $\langle c_1, c_2, c_3 \rangle$ there are 3 words of weight 88. The argument above proves that any further codeword of weight 88 has weight 16 when restricted to $\text{Res}(C; c_1, c_2^0)$. Since $\text{Res}(C; c_1, c_2^0)$ has at most one word of weight 16 this means that $A_{g8} \leq 3$, a contradiction. We conclude that any two codewords of weight 88 have innerproduct 52.

Suppose next that $A_{g8} \geq 3$, and let $c_1, c_2, c_3 \in C$ have weight 88. Then,
The (21,5,10) code $\text{Res}(C;88,36)$ has one of the two weight distributions in (3.1). Since $y_i + y_0 = 36$ for $i = 1, 2$ we get $y_0 = 14$ or 16, otherwise we contradict the minimum distance of $\text{Res}(C;88)$.

**Case 1.** ($y_0 = 16$).
Then we have the unique solution $y_1 = y_2 = 20$, $y_3 = 32$, and $<c_1, c_2, c_3>$ contains three words of weight 88. Further it is impossible to choose another word $c \in C - <c_1, c_2, c_3>$ of weight 88 since $c$ restricted to $\text{Res}(C; c_1, c_2)$ would have weight 14 or 16, contradicting (3.1). Hence we conclude that $A_{88} = 3$ in this case.

**Case 2.** ($y_0 = 14$).
Then we have the unique solution $y_1 = y_2 = 22$, $y_3 = 30$. Inspection of (3.4) reveals at least 32 pairs of repeated columns of $G$. Lemma 3.5 gives $A_{88} \geq 4$ and therefore there is a codeword $c_4 \in C - <c_1, c_2, c_3>$ of weight 88 since $<c_1, c_2, c_3>$ only contains 3 words of weight 88.

We have,

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
</tr>
</thead>
</table>
| 88   | 52   | 30   | $y_7$
| 57   | 36   | 22   | $y_6$
|      | 36   | 22   | $y_5$
|      |      | 14   | $y_4$
|      |      | 22   | $y_3$
|      |      | 14   | $y_2$
|      |      |      | $y_1$
|      |      |      | $y_0$

Here $y_0 = 3, 4, 7$ since $y_0$ is the weight of a codeword in $\text{Res}(C;88,36,14)$ which extended is a unique $(8,4,4)$ code by Lemma 2.5 (i). Since $y_1 + y_0 = 14$ we have $y_0 = 7$, otherwise the weight of $c_3 + c_4$ when restricted to $\text{Res}(C; c_1, c_2)$ gives a contradiction. Therefore we get $y_0 = y_1 = 7$. By symmetry (interchanging $c_2$ and $c_3$) we obtain $y_2 = 7$ and therefore $y_3 = 15$. Further
(interchanging $c_1$ and $c_2$) we get $\gamma_4 = 7$, $\gamma_5 = 15$. Finally since $(c_3,c_4) = 52$ we obtain $\gamma_7 = 15$ and therefore $\gamma_6 = 15$. We observe that $<c_1,c_2,c_3,c_4>$ contains exactly 5 codewords of weight 88, namely $c_1,c_2,c_3,c_4,$ and $c_1 + c_2 + c_3 + c_4$. Since $\text{Res}(C;e_1,e_2)$ by (3.1) only contains 3 words of weight 14 it follows that there are not more codewords of $C$ of weight 88 since a further codeword of weight 88 would have weight at least 14 when restricted to $\text{Res}(C;e_1,e_2)$. Hence we have $A_{88} = 5$ in this case.
4. The final classification

In this section we will prove that there are exactly 5 non-isomorphic $(145,7,72)$ codes. We will show that for each of the 5 listed weight distributions in Lemma 3.9, there is a unique code.

Let $(A|B)$ be a partitioning of a matrix $G$. Then we shall often denote $B$ by $G - A$. Further $F_{2}^{*}$ denotes the set of all $l$-dimensional binary column vectors and $F_{2}^{*} := F_{2}^{l} - \{0\}$.

Theorem 4.1. Let $C$ be a $(145,7,72)$ code with weight distribution,

$$
\begin{array}{cccc}
A_{0} & A_{72} & A_{80} & A_{88} \\
1 & 115 & 7 & 5 \\
\end{array}
$$

Then $C$ is isomorphic to the code generated by $G_{1}$, where

$$
G_{1} = \begin{pmatrix}
11\ldots1 & 11\ldots1 & 11\ldots1 & 11\ldots1 & 00\ldots0 \\
11\ldots1 & 11\ldots1 & 11\ldots1 & 00\ldots0 & 11\ldots1 \\
11\ldots1 & 11\ldots1 & 00\ldots0 & 11\ldots1 & 11\ldots1 \\
11\ldots1 & 00\ldots1 & 11\ldots1 & 11\ldots1 & 11\ldots1
\end{pmatrix}
$$

Proof. According to the proof of Lemma 3.9 (case 2) the first 4 rows of $G$ can be chosen as in (3.5) i.e.,

- $C_{1}$
  - $88$
- $C_{2}$
  - $52$
  - $36$
- $C_{3}$
  - $30$
  - $22$
  - $22$
  - $14$
- $C_{4}$
  - $15$
  - $15$
  - $7$
  - $15$
  - $7$
  - $15$
  - $7$
  - $15$
  - $7$
  - $7$
- $C_{5}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
  - $\gamma_{15}$
Since $A_{80} = 7$ and $< c_1, c_2, c_3, c_4 >$ contains no codewords of weight 80 we can choose $c_5 \in C - < c_1, c_2, c_3, c_4 >$ of weight 80. Inspection of the first 4 rows gives that $B_2 \geq 35$. By Lemma 3.5 we have $B_2 = 35$ and therefore $\gamma_i$ for $1 < i \leq 15$ must be chosen such that no more pairs of repeated columns are introduced, and also such that no columns occurs more than twice. This means $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{12} \leq 4$ and $\gamma_7, \gamma_11, \gamma_{13}, \gamma_{14}, \gamma_{15} \leq 8$. Since $w(c_5) = \sum_{i=1}^{15} \gamma_i = 80$, equality must hold in all cases. We next note that $< c_1, c_2, c_3, c_4, c_5 >$ contains only one word of weight 80, namely $c_5$. Therefore we can choose $c_6 \in C - < c_1, c_2, c_3, c_4, c_5 >$ of weight 80. Exactly as above it follows that $c_6$ is uniquely determined. Further $< c_1, c_2, c_3, c_4, c_5, c_6 >$ has 3 codewords of weight 80, namely $c_5, c_6$, and $c_5 + c_6$, and we can therefore choose $c_7 \in C - < c_1, c_2, c_3, c_4, c_5, c_6 >$ of weight 80. Again we are led to a unique choice of $c_7$ by the arguments above. Therefore a (145,7,72) code with $A_{88} = 5$ must be unique. Since the code generated by $G_1$ in Theorem 4.1 is easily seen to be a (145,7,72) code with $A_{88} = 5$, this must be the one.

Theorem 4.2. Let $C$ be a (145,7,72) code with weight distribution,

\[
\begin{array}{cccc}
A_0 & A_{72} & A_{80} & A_{88} \\
1 & 113 & 11 & 3 .
\end{array}
\]

Then $C$ is isomorphic to the code generated by $G_2$, where

\[
G_2 = \begin{array}{c}
F^* \\
2^7
\end{array}
\]

\[
\begin{array}{cccccccccccc}
11110 & 11110 & 11110 & 11110 & 11110 & 11110 & 11110 & 11110 & 11110 & 11110 & 11110 & 11110 \\
11101 & 11101 & 11101 & 11101 & 11101 & 11101 & 11101 & 11101 & 11101 & 11101 & 11101 & 11101 \\
11011 & 11011 & 11011 & 11011 & 11011 & 11011 & 11011 & 11011 & 11011 & 11011 & 11011 & 11011 \\
10111 & 10111 & 10111 & 10111 & 10111 & 10111 & 10111 & 10111 & 10111 & 10111 & 10111 & 10111 \\
110 & 101 & 011 & F^* \\
2^4 & 2^4
\end{array}
\]
Proof. First, note that the code generated by $G_2$ is a (145,7,72) code with the required weight distribution. It is therefore sufficient to prove that there is at most one such code. From the proof of Lemma 3.9 (case 1), the first 3 rows of $G$ can be chosen w.l.o.g. as follows,

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>88</th>
<th>57</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_2$</td>
<td>52</td>
<td>36</td>
</tr>
<tr>
<td>$c_3$</td>
<td>32</td>
<td>20</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$\gamma_7$</td>
<td>$\gamma_6$</td>
</tr>
</tbody>
</table>

Since $A_{80} = 11$ we can choose $c_4 \in C - \langle c_1, c_2, c_3 \rangle$ of weight 80. Let $C^0 = \text{Res}(C; c_1^0)$, and define the (21,5,10) code $C_0 = \text{Res}(C^0; c_2^0)$. We first observe that $w(c_4^0) \geq 32$ otherwise $w(c_1 + c_4) < 72$.

By Lemma 3.7 we get $w(c_4^0) = 32$ or 36. If $w(c_4^0) = 36$ then $w(c_1 + c_4) \geq 14$ contradicting (3.1). Therefore $w(c_4^0) = 32$ which gives $w(c_0^0) \geq 12$, otherwise $w(c_3 + c_4) < 28$. By (3.1) we get $w(c_4^0) = 12$. The code $C_0 = \text{Res}(C_0^0; c_4^0)$ is the unique (5,4,2) even weight code. Therefore we have $Y_0 = w(c_4^0) = 4$, since $w(c_4^0) = 2$ gives $w(c_3 + c_4^0) < 10$. Hence $\gamma_4 = 8$, and the symmetry in $c_1, c_2$, and $c_3$ leads to $\gamma_3 = \gamma_5 = 12$, $\gamma_1 = \gamma_2 = \gamma_4 = 8$. Therefore $\gamma_6 + \gamma_7 = 28$ and since no column of $G$ occurs more than twice according to Lemma 2.4 (ii) we get $\gamma_7 = 16$ and $\gamma_6 = 12$. Hence $c_4$ is uniquely determined.

We next note that $\langle c_1, c_2, c_3, c_4 \rangle$ contains less than 11 words of weight 80. Therefore we can choose $c_5 \in C - \langle c_1, c_2, c_3, c_4 \rangle$ of weight 80. Then,

<table>
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<tr>
<th>$c_1$</th>
<th>88</th>
<th>57</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_2$</td>
<td>52</td>
<td>36</td>
</tr>
<tr>
<td>$c_3$</td>
<td>32</td>
<td>20</td>
</tr>
<tr>
<td>$c_4$</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$\gamma_{15}$</td>
<td>$\gamma_{14}$</td>
</tr>
</tbody>
</table>
We will show that $c_5$ is uniquely determined. Since $w(c_4) = w(c_5) = 80$, and $c_4$ is uniquely determined we have $w(c_{00}) = 32$, $w(c_{000}) = 12$, and $w(c_{0000}) = 4$. Since $w(c_{000} + c_{000}) \geq 2$ we have $\gamma_6 = 1$ and $\gamma_1 = 3$. Further $\gamma_2 = \gamma_3 = 4$, otherwise $w(c_{00} + c_{00}) < 10$ or $w(c_{00} + c_{00} + c_{00}) < 10$. By symmetry we obtain $\gamma_4 = \gamma_5 = \gamma_8 = \gamma_9 = 4$. Next, $w(c_{00} + c_{00}) = 10$ leaves $w(c_{00} + c_{00}) = 28$, otherwise $w(c_{00} + c_{00} + c_{00}) < 28$. Therefore since $\gamma_6 + \gamma_7 = 12$ we have $\gamma_6 = 5$ and $\gamma_7 = 7$. Again by symmetry we get $\gamma_11 = \gamma_13 = 7$ and $\gamma_10 = \gamma_12 = 5$. Finally $\gamma_14 = \gamma_15 = 8$, otherwise a column of $G$ occurs more than twice contradicting Lemma 2.4 (ii).

With this unique choice of $c_5$, inspection of these 5 rows gives at least 31 pairs of repeated columns. Lemma 3.5 gives $B_2 = 31$. Since $< c_1, c_2, c_3, c_4, c_5 >$ contains less than 11 words of weight 80 we can choose $c_6 \in C - < c_1, c_2, c_3, c_4, c_5 >$ of weight 80. The same argument as we used to prove the uniqueness of $c_5, c_6$ and $c_7$ in Theorem 4.1 will now prove the existence of unique words $c_6$ and $c_7$ of weight 80. We therefore conclude that there is at most one $(145, 7, 72)$ code with the weight distribution given in Theorem 4.2. Since the code generated by $G_2$ is a $(145, 7, 72)$ code with this weight distribution, this must be the unique such code.

**Theorem 4.3.** Let $C$ be a $(145, 7, 72)$ code with weight distribution,

\[
\begin{array}{cccc}
A_0 & A_{72} & A_{80} & A_{88} \\
1 & 112 & 13 & 2 \\
\end{array}
\]

Then $C$ is isomorphic to the code generated by $G_3$, where

\[G_3 = F^{2^3}_{2^7} \quad F^{2^3}_{2^3} \quad F^{2^3}_{2^3} \quad F^{2^3}_{2^3}\]

\[
\begin{array}{cccc}
00...0 & 11111111 & 00001111 & 00000000 \\
11...1 & 11...1 & 11...1 & 11001100 \\
11...1 & 00...0 & 11...1 & 10101010 \\
\end{array}
\]
Proof. By the proof of Lemma 3.9 it follows from (3.4) that the first 3 rows of $G$ can be taken to be as follows,

\[
\begin{array}{c|c|c|c}
\mathbf{c}_1 & 88 & 57 \\
\mathbf{c}_2 & 52 & 36 & 36 & 21 \\
\mathbf{c}_3 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \\
\end{array}
\]

where $\mathbf{c}_3 \in C - \langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ is any codeword of weight 80. From Lemma 3.5 and Lemma 3.7 we get $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3) \in \{44, 48\}$. Let $C^0$ denote the $(57,6,28)$ code $C^0 = \text{Res}(C; \mathbf{c}_1^0)$ and define the $(21,5,10)$ code $C^{00} = \text{Res}(C; \mathbf{c}_1^0, \mathbf{c}_2^0)$.

We first prove that $(\mathbf{c}_2, \mathbf{c}_3) = (\mathbf{c}_2, \mathbf{c}_3) = 48$. By symmetry it is sufficient to show that $(\mathbf{c}_1, \mathbf{c}_3) = 48$. Suppose $(\mathbf{c}_1, \mathbf{c}_3) = 44$, i.e. $w(\mathbf{c}_3) = 36$. From (3.1) it follows that $\gamma_0 = w(\mathbf{c}_3) = 14$ or 16, otherwise $w(\mathbf{c}_2^0 + \mathbf{c}_3^0) < 28$. If $\gamma_0 = 14$, then $\gamma_1 = 22$ which leads to at least 30 pairs of repeated columns, contradicting Lemma 3.5 which says $B_2 = 29$. Therefore $\gamma_0 = 16$ and $\gamma_1 = 20$.

If $(\mathbf{c}_2, \mathbf{c}_3) = 44$ we get $\gamma_2 = 20$ and $\gamma_3 = 24$. If $(\mathbf{c}_2, \mathbf{c}_3) = 48$ we get $\gamma_2 = 16$ and $\gamma_3 = 28$. In both cases the two $(57,6,28)$ codes $\text{Res}(C; \mathbf{c}_1^0)$ and $\text{Res}(C; \mathbf{c}_2^0)$ contain at least two codewords of weight 36, as inspection of (4.1) reveals. From Lemma 3.7 this leads to at least 5 pairs of repeated columns in these two codes. Since $C^{00}$ is a $(21,5,10)$ code meeting the Griesmer bound, it follows from Lemma 2.4 (ii) that it has no pairs of repeated columns. Therefore this leads to at least 30 pairs of repeated columns of $G$, contradicting $B_2 = 29$. We therefore conclude that every codeword of weight 80 has innerproduct 48 with $\mathbf{c}_1$ and $\mathbf{c}_2$.

Suppose $\mathbf{c}^{00} \in C^{00}$ has weight 16. Then by Lemma 3.7 $w(\mathbf{c}^0 + \mathbf{c}_2^0) \geq 36$ or $w(\mathbf{c}^0) \geq 36$ so we can assume w.l.o.g. that $w(\mathbf{c}^0) \geq 36$. Then by Lemma 3.5 $w(\mathbf{c} + \mathbf{c}_1) \geq 80$ or $w(\mathbf{c}) \geq 80$ so we can assume w.l.o.g. that $w(\mathbf{c}) \geq 80$. Since $A_{88} = 2$ and $\mathbf{c} \neq \mathbf{c}_1$ and $\mathbf{c} \neq \mathbf{c}_2$ we get that $w(\mathbf{c}) = 80$. Further since $(\mathbf{c}, \mathbf{c}_1) = (\mathbf{c}, \mathbf{c}_2) = 48$ we get with $\mathbf{c} = \mathbf{c}_3$ that $\gamma_0 = \gamma_1 = \gamma_2 = 16$, $\gamma_3 = 32$. Again this leads to two codewords of weight 36 in the two $(57,6,28)$ codes $\text{Res}(C; \mathbf{c}_1^0)$ and $\text{Res}(C; \mathbf{c}_2^0)$ which as above contradicts $B_2 = 29$. Hence (3.1) shows that $C^{00}$ has 3 words of weight 14.

Let $\mathbf{c}^{00} \in C^{00}$ have weight 14. As above we show that there is a $\mathbf{c} \in C$ which restricted to $C^{00}$ is $\mathbf{c}^{00}$ and such that $w(\mathbf{c}) = 80$. We can there-
fore w.l.o.g. choose $c_3 = c$. Since $(c_1, c_3) = (c_2, c_3) = 48$ we have the unique solution $\gamma_1 = \gamma_2 = 18$, and $\gamma_3 = 30$. Since $c^{00}$ has 3 words of weight 14 we can similarly w.l.o.g. choose $c_4$ such that $c_4 \in C - \langle c_1, c_2, c_3 \rangle$, $w(c_4) = 80$, and $w(c_4^{00}) = 14$. Then we have,

\[
\begin{align*}
\gamma_1 &\quad 88  \\
\gamma_2 &\quad 52  \\
\gamma_3 &\quad 30  \\
\gamma_4 &\quad 18  \\
\gamma_5 &\quad 18  \\
\gamma_6 &\quad 18  \\
\gamma_7 &\quad 14  \\
\gamma_0 &\quad 7
\end{align*}
\]

Since $\gamma_0$ is the weight of a codeword in a $(7,4,3)$ code $\text{Res}(C^{00}; c_4^{00})$, we have $\gamma_0 = 3, 4, \text{ or } 7$. Since $\gamma_0 + \gamma_1 = 14$ we get $\gamma_0 = \gamma_1 = 7$, otherwise $w(c_0^{00} + c_4^{00}) < 10$. From Lemma 3.7 it is straightforward to show that a $(25,5,12)$ code $\text{Res}(C; 88, 32)$ has only even weights. Therefore since $\gamma_2 + \gamma_3 = \gamma_4 + \gamma_5 = 18$ we get $\gamma_2 = \gamma_4 = \gamma_3 = \gamma_5 = 9$, otherwise we contradict $B_2 = 29$. Since $w(c_3 + c_4) \in \{72, 80\}$ we get $\gamma_6 = \gamma_7 = 15$ if $w(c_3 + c_4) = 80$ and $\gamma_6 = 11$, $\gamma_7 = 19$ if $w(c_3 + c_4) = 72$. But $\gamma_7 = 19$ is impossible since this contradicts Lemma 2.4 (ii). Hence we have $\gamma_6 = \gamma_7 = 15$. Inspection of these 4 rows reveals at least 29 pairs of repeated columns. Since $B_2 = 29$ and $\langle c_1, c_2, c_3, c_4 \rangle$ contains less than 13 words of weight 80, the arguments used to prove the uniqueness of a $c_5, c_6,$ and $c_7$ in Theorem 4.1 can also be applied here. We conclude that all $(145,7,72)$ codes with $A_{88} = 2$ are isomorphic to the code generated by $G_3$, which generates such a code.

To treat the two remaining cases we need the following two lemmas, whose proofs follow from Lemma 2.6 and the techniques in Lemma 3.1-3.4 and will be omitted here.

**Lemma 4.4.** Let $\Lambda_w^0$ denote the number of codewords of weight $w$ in the $(65,6,32)$ code $\text{Res}(C; 80)$. If $\Lambda_w^0 \neq 0$, then $w \in \{0, 32, 36, 40, 44, 48\}$ and we have the following equations,
where $B_2^0$ is the number of repeated columns in a generator matrix for $\text{Res}(C; 72)$.

**Lemma 4.5.** Let $A_w^0$ denote the number of codewords of weight $w$ in the $(73,6,36)$ code $\text{Res}(C; 72)$. If $A_w^0 \neq 0$, then $w \in \{0, 36, 40, 44, 48, 52\}$ and we have the following equations,

\[
\begin{array}{c|c|c|c|c|c}
A_{36}^0 & A_{40}^0 & A_{44}^0 & A_{48}^0 & A_{52}^0 \\
1 & 0 & -1 & -2 & -3 & 46 \\
0 & 1 & 2 & 3 & 4 & 17 \\
0 & 0 & 1 & 3 & 6 & B_2^0 - 12
\end{array}
\]

where $B_2^0$ is the number of repeated columns in a generator matrix for $\text{Res}(C; 72)$.

**Theorem 4.6.** Let $C$ be a $(145,7,72)$ code with weight distribution,

\[
\begin{array}{c|c|c|c}
A_0^0 & A_7^0 & A_8^0 & A_8^8 \\
1 & 111 & 15 & 1 \\
\end{array}
\]

Then $C$ is isomorphic to the code generated by $G_4$, where

\[
G_4 = \begin{pmatrix}
0000 & 0000 & 0000 & 0000 & 11111111 & 0000 \\
1111 & 1111 & 1111 & 1111 & 00000000 & 0000 \\
1111 & 1111 & 0000 & 0000 & 00000000 & 1100 \\
1111 & 0000 & 1111 & 0000 & 00000000 & 1010 \\
0100 & 0100 & 0100 & 0100 & F_2^3 & 1111 \\
0010 & 0010 & 0010 & 0010 & F_2^3 & 1111 \\
0001 & 0001 & 0001 & 0001 & 1111 & 1111
\end{pmatrix}
\]
Lemma 3.5 which says $B_2$ of the weight distribution,

Proof. Let $c_1$ be the codeword of weight $88$. The $(57,6,28)$ code $C^0 = \text{Res}(C;c_1)$ has $B_2^0 \geq 3$ according to Lemma 3.7. Therefore $B_2^0 = 3$, otherwise we get at least 28 pairs of repeated columns of $G$, contradicting Lemma 3.5 which says $B_2 = 27$. From Lemma 3.7 it follows that $C^0$ has the weight distribution,

$$
\begin{array}{ccc}
A_0 & A_0 & A_0 \\
28 & 32 & \\
1 & 48 & 15.
\end{array}
$$

In particular if $c \in C$ and $w(c) = 80$, then $w(c^0) = 32$. Also since $A_80 = A_32 = 15$ we note that if $c^0 \in C^0$ and $w(c^0) = 32$, then $c^0$ is the restriction of some $c \in C$ of weight 80.

Let $c_2 \in C$, $w(c_2) = 80$ and define $C^{10}$ as the $(25,5,12)$ code $C^{10} = \text{Res}(C;c_2)$. If $c^{00} \in C^{10}$ and $w(c^{00}) > 16$, then $w(c^0 + c_2^0) > 32$ or $w(c^0) > 32$, which contradicts the weight distribution of $C^0$. Further if $w(c^{00}) = 13$ or 15 then again $w(c^0)$ or $w(c^0 + c_2^0)$ would contradict the weight distribution of $C^{10}$. Therefore the only possible occurring weights in $C^{10}$ are 12, 14, and 16. From Lemma 2.6 we obtain the equations,

$$
\begin{array}{ccc}
A_{12} & A_{14} & A_{16} \\
1 & 1 & 1 & 31 \\
0 & 1 & 2 & 14 \\
0 & 0 & 1 & 2B_2^{10} + 1
\end{array}
$$

where $B_2^{10}$ is the number of repeated columns in a generator matrix for $C^{10}$. From (4.2) we observe that $C^{10}$ has a codeword of weight 16. This must be the restriction of a word of weight 32 in $C^0$ and therefore there is a $c_3 \in C$ of weight 80 such that $w(c_3^{10}) = 16$. We can therefore assume w.l.o.g. that the first rows of $G$ can be chosen as in one of the two cases,

Case 1. $(w(c_1 + c_2 + c_3) = 80)$

$$
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
\hline
88 & | & & & & | & 57 & \\
48 & | & 40 & | & 32 & | & 25 & \\
28 & | & 20 & | & 20 & | & 16 & | & 16 & | & 9
\end{array}
$$
**Case 2.** \((w(c_2 + c_3) = 80)\)

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<th>40</th>
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<tbody>
<tr>
<td>(c_1)</td>
<td>88</td>
<td>57</td>
<td></td>
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</tr>
<tr>
<td>(c_2)</td>
<td></td>
<td>24</td>
<td>16</td>
<td>16</td>
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<tr>
<td>(c_3)</td>
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<td>24</td>
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We next show that we can transfer Case 1 to Case 2. Suppose we have the situation in Case 1. Since \(B_2 = 27\) the generator matrix of at least one of the \((65,6,32)\) codes \(\text{Res}(C; c_2), \text{Res}(C; c_3)\) or \(\text{Res}(C; c_1 + c_2 + c_3)\) has at least 9 pairs of repeated columns. By symmetry we can assume it is \(\text{Res}(C; c_2)\). This code has a word of weight 40, namely the restriction of \(c_1\). Since the existence of a word of weight 44 or 48 in \(\text{Res}(C; c_2)\) would lead to a codeword of weight at least 88, different from \(c_1\) it follows from Lemma 4.4 that \(\text{Res}(C; c_2)\) contains another word of weight 40. Let \(c \in C\) be the codeword with this restriction to \(\text{Res}(C; c_2)\), then \(w(c) = 80\) and,

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<tr>
<td>(c_1)</td>
<td>88</td>
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<tr>
<td>(c_2)</td>
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<td>16</td>
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<tr>
<td>(c)</td>
<td>(\gamma_3)</td>
<td>(\gamma_2)</td>
<td>(\gamma_1)</td>
<td>(\gamma_0)</td>
</tr>
</tbody>
</table>

where \(\gamma_0 + \gamma_2 = 40\). We have \(\gamma_2 \leq 24\), otherwise we contradict \(B_2 = 27\).

Further since \(\gamma_0 \leq 16\) by (4.2) we get \(\gamma_0 = \gamma_1 = 16, \gamma_2 = 24,\) and \(\gamma_3 = 24\).

We therefore only have to consider Case 2. Since \(\text{Res}(C; c_2)\) has two words of weight 40 we get from Lemma 4.4 that it has 9 pairs of repeated columns. Since \(B_2 = 3\) we have \(B_2^0 \geq 1\) otherwise we contradict \(B_2 = 27\).

Therefore \(A_{16}^0 \geq 3\) follows from (4.2) and we can choose \(c_4 \in C - \langle c_1, c_2, c_3 \rangle\) such that \(w(c_4) = 80\) and \(w(c_4^0) = 16\), i.e.
Define the $(9,4,4)$ code $C_{000} = \text{Res}(C^0;C^0_{-3})$. A codeword of weight 9 in $C_{000}$ would lead to a word of weight greater than 16 in $C^0$. Further $\text{Res}(C^0;7)$ would have parameters $(2,3,1)$ which is impossible. Hence $C_{000}$ has 4, 5, 6, and 8 as only possible weights. Suppose $C_{000}$ has no codeword of weight 8. Since $w(C^0_3) = w(C^0_4) = 16$ this means $y_0 = 6$ and $y_1 = 10$, otherwise $w(C^0_3 + C^0_4) < 12$. Further $\text{Res}(C;C_3)$ has also two codewords of weight 40. We can therefore repeat the above arguments with $C_2$ and $C_3$ interchanged and obtain the existence of a $C^*_4 \in C$ of weight 80 such that $y^*_2 = 10$. But then inspection of (4.3) leads to $B^0_2 \geq 4$, a contradiction. We therefore conclude that $C^0$ has a codeword of weight 8.

We can therefore assume w.l.o.g. that $y_0 = w(C^0_4) = 8$. Since no codeword of $C^0$ has weight greater than 32 we get $y_1 = y_2 = y_3 = 8$. Since $B_2 = 27$ we have $y_4 = 8$, and all $y_5, y_6, y_7 \leq 16$. From Lemma 4.4 applied to $\text{Res}(C;C_2)$, $\text{Res}(C;C_3)$, and $\text{Res}(C;C_2 + C_3)$ it follows $y_5, y_6, y_7 \in \{12,16\}$. Since $y_5 + y_6 + y_7 = 40$ we can assume w.l.o.g. that $y_5 = 16$ and $y_6 = y_7 = 12$. Inspection of (4.3) now reveals that $\text{Res}(C;C_2)$ has at least 4 words of weight 40, namely the restrictions of $C_1, C_3, C_4$, and $C_1 + C_3 + C_4$. From Lemma 4.4 we conclude that $\text{Res}(C;C_2)$ has at least 11 pairs of repeated columns in a generator matrix. Hence $C^0_{00}$ has $B^0_2 = 3$ which by (4.2) leads to $A^0_{16} = 7$. We can therefore choose $C_5 \in C - \langle C_1, C_2, C_3, C_4 \rangle$ such that $w(C^0_5) = 80$ and $w(C^0_5) = 16$, and we have,
We immediately get \( Y_4 = Y_5 = Y_6 = Y_7 = Y_8 = Y_9 = Y_{10} = 4 \), otherwise we contradict \( B_2 = 27 \). From Lemma 2.4 (ii) we obtain \( Y_{11} = 8 \). Since \( Y_0 + Y_1 + Y_2 + Y_3 = 16 \) we are led to \( Y_i + Y_0 = 6 \) for \( i = 1, 2, 3 \), so \( B_0^0 = 3 \) gives \( Y_0 = 1 \) and \( Y_1 = Y_2 = Y_3 = 5 \). Also \( w(c_4^0 + c_5^0) = 28 \) means that \( w(c_4 + c_5) = 72 \) and therefore \( Y_{12} + Y_{14} = Y_{13} + Y_{15} = 14 \). Similarly \( w(c_4^0 + c_5^0) = 28 \) gives \( Y_{14} + Y_{15} = Y_{12} + Y_{13} = 14 \). Finally since \( w(c_3^0 + c_4^0 + c_5^0) = 28 \) we get \( Y_{12} + Y_{15} = Y_{13} + Y_{14} = 14 \). Hence we have \( Y_{12} = Y_{13} = Y_{14} = Y_{15} = 7 \).

Now inspection of the first 5 rows above gives at least 27 pairs of repeated columns. Since \( < c_1, c_2, c_3, c_4, c_5 > \) contains less than 15 words of weight 80, there exist a \( c_6 \in C - < c_1, c_2, c_3, c_4, c_5 > \) of weight 80. To prove the uniqueness of \( c_6 \) we use the same method as for \( c_5 \) in the proof of Theorem 4.1. Similarly we also prove the uniqueness of \( c_7 \in C \) of weight 80 such that \( C = < c_1, c_2, c_3, c_4, c_5, c_6, c_7 > \).

It is straightforward to check that the code generated by \( G_4 \) is a \((145, 7, 72)\) code with \( A_{88} = 1 \), and therefore all other \((145, 7, 72)\) codes with the weight distribution in Theorem 4.6 are isomorphic to this code. 

**Theorem 4.7.** Let \( C \) be a \((145, 7, 72)\) code with weight distribution,

\[
\begin{array}{ccc}
A_0 & A_{72} & A_{80} \\
1 & 110 & 17 .
\end{array}
\]

Then \( C \) is isomorphic with the code generated by \( G_5 \), where

\[ G_5 = F^*_{27} \]

| 11111111 | 00000000 | \( \bigcirc \) |
| 00000000 | 11111111 | \( \bigcirc \) |
| 00001111 | \( \bigcirc \) | 00011111 |
| 00110011 | \( \bigcirc \) | 11100011 |
| \( \bigcirc \) | 00001111 | 01101101 |
| \( \bigcirc \) | 00110011 | 10110110 |
| 01010101 | 01010101 | 11111111 |

| \( F^*_{2^2} \) | \( \bigcirc \) | \( \bigcirc \) |
| \( \bigcirc \) | \( F^*_{2^2} \) | \( \bigcirc \) |
| \( \bigcirc \) | \( \bigcirc \) | \( \bigcirc \) |
| \( \bigcirc \) | \( \bigcirc \) | \( \bigcirc \) | 1 |

□
Proof. If \( u, v \in C, u \neq v, \) and \( w(u) = w(v) = 80, \) then \( (u, v) = 40 \) or \( 44 \) since \( w(u + v) \in \{72, 80\}. \) In the first part of the proof we will show the existence of two codewords of weight 80 with innerproduct 40.

Suppose the contrary, i.e. \( (u, v) = 44 \) for all \( u, v \in C, u \neq v, \) and \( w(u) = w(v) = 80. \) We will show this to be impossible. We let the first 3 rows of \( C \) be \( c_1, c_2, c_3 \) which we can choose to have weight 80. Then

\[
\begin{array}{c|c|c}
C_1 & 80 & 65 \\
\hline
C_2 & 44 & 36 & 29 \\
\hline
C_3 & y_3 & y_2 & y_1 & y_0 \\
\end{array}
\]

Let \( C^0 \) be the \((65, 6, 32)\) code \( C^0 = \text{Res}(C; c_1). \) From Lemma 4.4 we obtain

\[ B_2^0 = 7 \] since \( A_i^0 = 0 \) by the assumption above. Let \( C^{00} \) be the \((29, 5, 14)\) code \( C^{00} = \text{Res}(C^0; c_2^0). \) Then all \( C^{00} \subset C^{00} \) have even weights, otherwise \( w(C^0 + c_2^0) \) will contradict Lemma 4.4. Also \( A_i^{00} = 0 \) for \( i \geq 20 \) since \( B_2^0 = 7. \) From Lemma 2.6 we get the equations,

\[
\begin{array}{ccc}
A_{14}^{00} & A_{16}^{00} & A_{18}^{00} \\
1 & 1 & 1 & 31 \\
0 & 1 & 2 & 15 \\
0 & 0 & 1 & 2(B_2^{00} - 1) \\
\end{array}
\quad (4.4)
\]

where \( B_2^{00} \) is the number of pairs of repeated columns in a generator matrix for \( C^{002}. \) If \( A_{18}^{00} \geq 1 \) then we can assume w.l.o.g. that \( w(c_3^0) = 18, \) and since \( (c_1, c_3) = (c_2, c_3) = 44 \) we get the unique solution \( y_0 = y_1 = y_2 = 18 \) and \( y_3 = 26. \) Since \( B_2^0 = 7 \) one of the codes \( C^{00} = \text{Res}(C^0; c_2^0), \) \( \text{Res}(C^0; c_3^0), \) or \( \text{Res}(C^0; c_2^0 + c_3^0) \) has exactly 3 pairs of repeated columns in its generator matrix. By symmetry we can assume \( B_2^{00} = 3. \) Then \( (4.4) \) gives \( A_{18}^{00} = 4 \) and we can choose \( c_4 \in C - \langle c_1, c_2, c_3 \rangle \) such that \( w(c_4) = 80 \) and \( w(c_4^{00}) = 18. \)
We have,
\[
\begin{align*}
\text{C}_1 & : 80 \quad 65 \\
\text{C}_2 & : 44 \quad 36 \quad 36 \quad 29 \\
\text{C}_3 & : 26 \quad 18 \quad 18 \quad 18 \quad 18 \quad 18 \quad 18 \quad 11 \\
\text{C}_4 & : Y_7 \quad Y_6 \quad Y_5 \quad Y_4 \quad Y_3 \quad Y_2 \quad Y_1 \quad Y_0 \\
\end{align*}
\]

The \((11,4,5)\) code \(C^{000} = \text{Res}(C^{00}; C^{00})\) reach the Griesmer bound, and we get from Lemma 2.4 (ii) and Lemma 2.6 the equations,

\[
\begin{align*}
A_0^{000} & = 0 \\
A_1^{000} & = 1 \\
A_2^{000} & = 1 \\
A_3^{000} & = 1 \\
A_4^{000} & = 15 \\
0 & = 1 \\
1 & = 2 \\
2 & = 3 \\
3 & = 13 \\
0 & = 0 \\
0 & = 1 \\
1 & = 3 \\
2 & = 5.
\end{align*}
\]

Hence there is a \(c^{000} \in C^{000}\) of weight 7. The \((4,3,2)\) code \(\text{Res}(C^{000}; C^{0})\) is unique and it has a codeword of weight 4. One of the two codewords of \(C^{000}\) whose restriction to \(\text{Res}(C^{000}; C^{00})\) has weight 4 must have weight 8. We conclude that \(A_7^{000} = 2\) and \(A_8^{000} = 1\). Since \(A_{18}^{000} = 4\) it follows that every word of weight 7 or 8 in \(C^{000}\) is the restriction of some word of weight 18 in \(C^0\). We can therefore assume w.l.o.g. that \(\gamma_0 = 7\) and \(\gamma_1 = 11\). Since \(w(c_3^{000} + c_4^{000}) = 14\) we get \(w(c_3^0 + c_4^0) = 32\) and therefore \(\gamma_2 = \gamma_3 = 9\). Similarly we obtain \(\gamma_4 = \gamma_5 = 9\), \(\gamma_6 = 11\), and \(\gamma_7 = 15\). We further can choose \(c_5 \in C - \langle c_1, c_2, c_3, c_4 \rangle\) of weight 80, and w.l.o.g. we assume \(w(c_5^{000}) = 18\), and \(w(c_5^{000}) = 7\). Then we have,

\[
\begin{align*}
\text{C}_1 & : 80 \quad 65 \\
\text{C}_2 & : 44 \quad 36 \quad 36 \quad 29 \\
\text{C}_3 & : 26 \quad 18 \quad 18 \quad 18 \quad 18 \quad 18 \quad 18 \quad 11 \\
\text{C}_4 & : 15 \quad 11 \quad 11 \quad 7 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 11 \quad 7 \quad 1 \quad 4 \\
\text{C}_5 & : 15 \quad 14 \quad 13 \quad 12 \quad 11 \quad 10 \quad 9 \quad 8 \quad 7 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0.
\end{align*}
\]
Since \( \gamma_0 \) is even we have \( \gamma_0 = 4 \) and \( \gamma_1 = 3 \). Further since \( w(c_4^{000} + c_5^{000}) = 8 \) it follows from the remark above that \( w(c_4^{000} + c_5^{000}) = 18 \) or \( w(c_3^{000} + c_4^{000} + c_5^{000}) = 18 \). Since \( \gamma_2 + \gamma_3 = 11 \) we have \( \gamma_2 = 4 \) or \( 5 \). Here \( \gamma_2 = 5 \) is impossible because this contradicts \( B_2^{00} = 3 \). We conclude that \( \gamma_2 = 4 \) and \( \gamma_3 = 7 \). Similar arguments give \( \gamma_{12} = 4 \) and \( \gamma_{13} = 7 \). Since \( \gamma_{14} + \gamma_{15} = 15 \) we obtain from Lemma 2.4 (ii) that \( \gamma_{14}, \gamma_{15} \in \{7,8\} \).

If \( \gamma_{14} = 7 \) and \( \gamma_{15} = 8 \) then the restriction of \( c_3 + c_4 + c_5 \) to the \( (73,6,36) \) code \( \text{Res}(C;c_1 + c_2) \) has weight 38, contradicting Lemma 4.5.

If \( \gamma_{14} = 8 \) and \( \gamma_{15} = 7 \), then since \( \gamma_i \in \{4,5\} \) for \( 4 \leq i \leq 11 \) and \( (c_4,c_5) = 44 \) we get \( \gamma_4 = \gamma_6 = \gamma_8 = \gamma_{10} = 4 \) and \( \gamma_5 = \gamma_7 = \gamma_9 = \gamma_{11} = 5 \).

We now get a contradiction since \( w(c_1 + c_2 + c_4) = w(c_5) = 80 \) and \( (c_1 + c_2 + c_4,c_5) = 40 \).

We can therefore assume that the codewords in \( \text{Res}(C;80,36) \) have 14 and 16 as only possible non zero weights. Therefore we can choose \( c_1,c_2,c_3,c_4 \in C \) as follows,

\[
\begin{align*}
\text{c}_1 &| 80 & \text{c}_1 &| 65 \\
\text{c}_2 &| 44 & \text{c}_2 &| 36 & \text{c}_2 &| 29 \\
\text{c}_3 &| 24 & \text{c}_3 &| 20 & \text{c}_3 &| 16 \\
\text{c}_4 &| \gamma_7 & \text{c}_4 &| \gamma_6 & \text{c}_4 &| \gamma_5 & \text{c}_4 &| \gamma_4 & \text{c}_4 &| \gamma_3 & \text{c}_4 &| \gamma_2 & \text{c}_4 &| \gamma_1 & \text{c}_4 &| \gamma_0 \\
\end{align*}
\]

where \( w(c_i) = 80 \). Let \( C^0 = \text{Res}(C;c_1), C^{00} = \text{Res}(C^0;c_2), \) and \( C^{000} = \text{Res}(C^{00};c_3) \). By (4.4) we have \( A_i^{00} = 0 \) for \( i > 16 \), and therefore \( A_i^{00} = 0 \) for \( i > 8 \). Hence the \( (13,4,6) \) code \( C^{000} \) has only 6,7, and 8 as possible weights. From Lemma 2.6 we obtain in particular that \( A_8^{00} \geq 3 \). We can therefore w.l.o.g. assume \( w(c_4^{000}) = 8 \). Since \( (c_4,c_3) = 44 \) for \( 1 \leq i,j \leq 4 \) and \( \text{Res}(C;80,36) \) is a two-weight code we get the unique solution \( \gamma_1 = \gamma_2 = \gamma_4 = 8 \) and \( \gamma_3 = \gamma_5 = \gamma_6 = \gamma_7 = 12 \). We further can choose \( c_5 \in C - \langle c_1,c_2,c_3,c_4 \rangle, w(c_5) = 80, \) and \( w(c_5^{000}) = 8 \). Then,
The \((5,3,2)\) code defined as \(C^{0000} = \text{Res}(C^{000}; C_4^{000})\) has \(A_4^{000} \geq 1\) according to Lemma 2.6. Therefore we can assume \(w(C_5^{000}) = 4\), and using the uniqueness of \(C_4\) combined with the facts that \(\text{Res}(C; 80, 36)\) is a two-weight code and that \((c_i, c_j) = 44\) for \(1 \leq i,j \leq 4\), we get as above that \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_8 = \gamma_9 = \gamma_{10} = \gamma_{12} = \gamma_{15} = 4\) and \(\gamma_7 = \gamma_{11} = \gamma_{13} = \gamma_{14} = 8\). We now get a contradiction by observing that \(w(c_1 + c_2 + c_3 + c_4 + c_5) < 72\). Hence we have shown the existence of two codewords in \(C\) of weight 80 whose innerproduct is 40.

We can therefore choose \(c_1, c_2 \in C\) of weight 80 such that \((c_1, c_2) = 40\). Let \(C^0 = \text{Res}(C; c_1^0)\) and \(C^{00} = \text{Res}(C^0; c_2^0)\). Every \(c^{00} \in C^{00}\) must have even weight otherwise \(w(c^0 + c_2^0)\) or \(w(c^0)\) will contradict Lemma 4.4. Further \(A_i^{00} = 0\) for \(i \geq 18\) since \(B_2 = 25\) according to Lemma 3.5. Therefore Lemma 2.6 gives the same equations as (4.2). Since \(A_{16}^{00} \geq 1\) we can choose \(c_3 \in C - \langle c_1, c_2 \rangle\) such that \(w(c_3^{00}) = 16\) and \(w(c_3^{00}) = 80\). Then,

\[
\begin{align*}
C_1 & \quad 80 \quad 65 \\
C_2 & \quad 44 \quad 36 \quad 36 \quad 29 \\
C_3 & \quad 24 \quad 20 \quad 20 \quad 16 \quad 20 \quad 16 \quad 16 \quad 13 \\
C_4 & \quad 12 \quad 12 \quad 8 \quad 12 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 5 \\
C_5 & \quad 15 \quad 14 \quad 13 \quad 12 \quad 11 \quad 10 \quad 9 \quad 8 \quad 7 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0
\end{align*}
\]

We have \(B_2^0 = 8\) or 9 since \(B_2 = 25\). Lemma 4.4 gives \(A_4^{00} = 0\) so therefore \(\gamma_1, \gamma_2, \gamma_3 \in \{20, 24\}\) which combined with \(\gamma_1 + \gamma_2 + \gamma_3 = 64\) gives w.l.o.g. \(\gamma_1 = 24\) and \(\gamma_2 = \gamma_3 = 20\).

We next let \(c_4 \in C - \langle c_1, c_2, c_3 \rangle\). Then we have,
We define the $(9,4,4)$ code $C^{000} = \text{Res}(C^{000}_1; C^{000}_3)$. The codewords of $C^{000}$ have all even weights since if $w(C^{000}_4)$ is odd, then $\gamma_1$ and $\gamma_2$ are odd and this contradicts $B_2 = 25$. From Lemma 2.6 we get the equations:

\[
\begin{array}{cccc}
A^{000}_4 & A^{000}_6 & A^{000}_8 & A^{000}_8 \\
1 & 1 & 1 & 15 \\
0 & 1 & 2 & 6 \\
0 & 0 & 1 & B^{000}_2 \\
\end{array}
\] (4.5)

where $B^{000}_2$ is the number of repeated columns in a generator matrix for $C^{000}$. In particular $A^{000}_6 \geq 1$ since $B^{000}_2 \leq 1$. We can therefore assume w.l.o.g. that $w(C^{000}_4) = 80$ and $w(C^{000}_4) = 6$. From Lemma 4.4 we get that $B^{000}_2 = 9$. Therefore $\gamma_0 = 6$ leads to $\gamma_1 = \gamma_2 = 8$ and $\gamma_3 = 14$. We have $\gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 = 44$ but since $(C^{000}_1 + C^{000}_2, C^{000}_4) \in \{40,44\}$ we obtain $\gamma_4 + \gamma_5, \gamma_6 + \gamma_7 \in \{18,22\}$ and therefore $\gamma_4 + \gamma_5 = \gamma_6 + \gamma_7 = 22$. By Lemma 4.4 the restrictions of $C^{000}_3 + C^{000}_4$ to $\text{Res}(C; C^{000}_2)$ and $\text{Res}(C; C^{000}_1 + C^{000}_2)$ has weight $32$ or $36$ since $40$ is excluded because then the restriction of $C^{000}_1 + C^{000}_3 + C^{000}_4$ would have weight less than $32$. Therefore $\gamma_4, \gamma_5, \gamma_6, \gamma_7 \in \{10,12\}$. Because of the symmetry in $C^{000}_2$ and $C^{000}_1 + C^{000}_2$ we can assume $\gamma_4 = 10$. Then Lemma 4.4 applied to $\text{Res}(C; C^{000}_3)$ gives the solution $\gamma_5 = 12, \gamma_6 = 12,$ and $\gamma_7 = 10$. We therefore have,

\[
\begin{array}{cccc}
C^{000}_1 & 65 \\
C^{000}_2 & 25 \\
C^{000}_3 & 9 \\
C^{000}_4 & 3 \\
C^{000}_5 & \\
\end{array}
\]
where \( C_5 \subset C \subset \langle C_1, C_2, C_3, C_4 \rangle \). We next show that \( A^\infty_8 = 0 \). If \( A^\infty_8 \geq 1 \), then we can let \( w(C_{5}^0) = 8 \) and \( w(C_{5}) = 80 \). We get \( Y_0 = 3 \) and \( \gamma_1 = 5 \) and since \( B_2 = 25 \) it follows that \( Y_2 = Y_3 = Y_4 = Y_5 = Y_{10} = Y_{12} = 4 \). Since \( B^0_2 = 9 \) we have \( A_0 = 2 \) by Lemma 4.4 and therefore \( w(C_{5}^0) = 36 \). Further \( w(C_{4}^0 + C_{5}^0) = 4 \) gives \( w(C_{4}^0 + C_{5}) = 32 \) which leads to \( Y_6 = 5 \) and \( Y_7 = 7 \). The restriction of \( C_5 \) to \( \text{Res}(C_i; C_4) \) has weight at least 36, and since \( B_2 = 25 \) we get \( Y_8 = Y_{14} = 6 \). From the discussion of \( C_3 \) we have \( \gamma_9 + \gamma_{11} = 14 \) and \( \gamma_{13} + \gamma_{15} = 10 \) or vica versa. If \( \gamma_9 + \gamma_{11} = 14 \) then \( \gamma_9 = 6 \) and \( \gamma_{11} = 8 \), and by Lemma 4.4. applied to \( \text{Res}(C_i; C_3) \) we get \( \gamma_9 = \{4, 8\} \). The restriction of \( C_4 + C_5 \) to \( \text{Res}(C_i; C_3) \) has weight 30 or 34, a contradiction. The case \( \gamma_9 + \gamma_{11} = 10 \) gives a similar contradiction. We conclude \( A^\infty_8 = 0 \), and \( (4.5) \) gives \( A_6 = 6 \).

We define \( C^\infty_{0000} = \text{Res}(C_{0000}; C_{4}) \). Then \( C^\infty_{0000} \) is a \((3, 3, 1)\) code and we can assume w.l.o.g. that \( w(C_{5}^0) = 2 \) and \( w(C_{5}) = 80 \). From (4.2) and (4.5) we obtain \( Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = 4 \) because \( B^0_2 = 0 \). Since \( w(C_{4}^0 + C_{5}^0) = 4 \) one gets \( w(C_{4}^0 + C_{5}) = 32 \), leading to \( Y_6 = 6 \) and \( Y_7 = 8 \). Further \( B_2 = 25 \) gives \( Y_{10} = Y_{12} = 4 \) and also that \( \gamma_8 = \gamma_{14} = 6 \) because the restriction of \( C_5 \) to \( \text{Res}(C_i; C_4) \) has weight at least 36. From the discussion of \( C_4 \) we get \( \gamma_9 = 4 \) or 6. If \( \gamma_9 = 4 \) we get \( \gamma_{11} = \gamma_{13} = 8 \) and \( \gamma_{15} = 4 \) but this is impossible since \( w(C_1 + C_2 + C_3 + C_4 + C_5) < 72 \). Therefore \( \gamma_9 = 6 \) which gives \( \gamma_{11} = \gamma_{13} = \gamma_{15} = 6 \). We therefore have,
We can assume w.l.o.g. that $w(\mathbf{e}_6^{000}) = 2$ and $w(\mathbf{c}_6) = 80$. Since $B_2 = 25$, $B_2 = 9$, and $B_2^{00} = 0$ we immediately get from the discussion of $\mathbf{c}_5$ that $\gamma_0 = \gamma_1 = 1$, $\gamma_i = 2$ for $2 \leq i \leq 11$, $\gamma_{16} = \gamma_{18} = \gamma_{20} = \gamma_{21} = \gamma_{24} = \gamma_{25} = \gamma_{28} = \gamma_{30} = 2$, and $\gamma_{14} = \gamma_{15} = \gamma_{17} = \gamma_{19} = \gamma_{29} = \gamma_{31} = 4$. We have $w(\mathbf{e}_4^{000} + \mathbf{e}_5^{000} + \mathbf{e}_6^{000}) = 4$ and thus the restriction of $\mathbf{c}_4 + \mathbf{c}_5 + \mathbf{c}_6$ to $\text{Res}(\mathbf{c}; \mathbf{c}_1)$, $1 \leq i \leq 3$, has weight 32. Hence we get $\gamma_{12} = \gamma_{13} = \gamma_{22} = \gamma_{23} = \gamma_{26} = \gamma_{27} = 3$.

Finally we can use the same method as in Theorem 4.1 to prove the existence of a unique $\mathbf{e}_7 \in \mathbf{C} - \langle \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6 \rangle$ of weight 80. Since $\mathbf{c}_5$ can be shown to generate a $(145,7,72)$ code with $A_{88} = 0$, this concludes the classification.
REFERENCES


