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Existence of average optimal strategies in
Markovian decision problems with strictly
unbounded costs

by

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1. INTRODUCTION

The existence of average optimal strategies in Markovian decision processes is investigated by Blackwell [1] (finite state space), Derman [2], Ross [4], Hordijk [3] (countable state space), Tijms [7], Wijngaard [8] (general state space). Conditions sufficient for the existence of average optimal strategies consist in general of some recurrence conditions and some continuity and compactness conditions. The easiest recurrence conditions to work with are of course the conditions which guarantee recurrency to one point. (Derman, Ross, Hordijk). One can also use recurrency to a finite subset of the state space. The simultaneous Doeblin condition, introduced by Hordijk [3], is a strong form of this recurrency. The conditions used in [8] include the finiteness of the expected time and costs until the first visit to some subset of the state space.

In a lot of problems with a countable state space the condition of recurrency to a finite subset are not very strong. In inventory problems for instance the costs are high if the inventory level is far from zero. That means that "good" strategies have to bring back the inventory level near to zero. The unboundedness of the costs makes that the recurrence conditions are satisfied for all "good" strategies. In this paper we try to replace the recurrence conditions by an unboundedness condition. To guarantee that the set of "good" strategies is non void we state the existence of at least one strategy for which the average costs exist and are bounded on the state space.

2. PRELIMINARIES

We consider a Markovian decision process on a countable state space V . Let Ω be the set of all stationary strategies. The process under strategy $\alpha \in \Omega$ is described by a Markov chain P_α . The one-period costs under α are given by the function $c_\alpha(\cdot)$ on V , $c_\alpha(u)$ are the costs starting in u . We assume that $c_\alpha(u) \geq 0$ for all $u \in V$, $\alpha \in \Omega$. The functions c_α are assumed to be strictly unbounded, that means that there is a positive function $h(\cdot)$ on V such that $h(\cdot)$ has a positive lower bound on V ,

$h(\cdot)$ is unbounded from above on each infinite subset of V ,

$$c_\alpha(u) \geq h(u) \text{ for all } u \in V, \alpha \in \Omega.$$

To guarantee that the problem is interesting we assume the existence of a strategy $\alpha^* \in \Omega$ such that the average costs under α^* , g_{α^*} , exist and are bounded on V , $g_{\alpha^*}(u) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell=0}^{n-1} (P_{\alpha^*}^\ell c_{\alpha^*})(u) \leq b < \infty$

Define $\Omega' := \{\alpha \in \Omega \mid \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell=0}^{n-1} (P_\alpha^\ell c_\alpha)(u) < b, u \in V\}$

The set Ω' contains all "good" strategies.

It is clear, by the assumptions on c_α , that all strategies in Ω' have to bring back the state of the system to a finite set. This is worked out in lemma 1. First we have to give some definitions and notations.

$$(P_{\alpha B} f)(u) := \sum_{v \in B} P_\alpha(u, v) f(v) \text{ for each } u \in V, B \subset V, \text{ nonnegative function } f \text{ on } V.$$

$$A := \{u \in V \mid h(u) \leq 2b\}, A' := V \setminus A$$

Q_α is the embedded (sub-)Markov process of P_α on A ,

$$Q_\alpha(u, v) = \sum_{n=0}^{\infty} (P_{\alpha A}^n, P_{\alpha A} 1_v)(u), \text{ (where } 1_v = 1_{\{v\}})$$

$$(T_\alpha f)(u) := \sum_{n=0}^{\infty} (P_{\alpha A}^n, f)(u) \text{ for each } u \in V, \text{ nonnegative function } f \text{ on } V.$$

Hence $(T_\alpha c_\alpha)(u)$ is equal to the expected costs until the first visit to A , starting in u .

Lemma 1. For each $\alpha \in \Omega'$ the sub-Markov process Q_α is a Markov process $((P_{\alpha A}^n, 1)(u) \rightarrow 0 \text{ for } n \rightarrow \infty)$. Let π_α be an invariant probability of Q_α for $\alpha \in \Omega'$, then $(T_\alpha 1)(u) \leq \frac{2}{\pi_\alpha(u)}$ and $(T_\alpha c_\alpha)(u) \leq \frac{2b}{\pi_\alpha(u)}$ for all $u \in A$.

For the proof of this lemma we refer to [9].

Now, let $\alpha \in \Omega'$ and $u \in A$ be such that $\pi_\alpha(u) > 0$ for some invariant probability π_α of Q_α . Let $Q_{\alpha u} := Q_{\alpha D}$ with $D := A \setminus \{u\}$. Then the sum $\sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell, T_\alpha c_\alpha)(v)$ is equal to the expected costs until the first visit to $\{u\}$, starting in v and the average costs under strategy α , starting in u , are equal to

$$g_\alpha(u) = \frac{\sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell, T_\alpha c_\alpha)(u)}{\sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell, T_\alpha 1)(u)}$$

Define for each $u \in A$ the subset Ω_u of Ω' by

$$\Omega_u := \{\alpha \in \Omega' \mid Q_\alpha \text{ has an invariant probability } \pi_\alpha \text{ with } \pi_\alpha(u) > 0\}$$

Define for each subset $C \subset A$ the set of strategies Ω_C by $\Omega_C := \bigcup_{u \in C} \Omega_u$

Let $A_r := \{u \in A \mid \Omega_u \neq \emptyset\}$, $g_u := \inf_{\alpha \in \Omega_u} g_\alpha(u)$ for $u \in A_r$ and $g := \min_{u \in A_r} g_u$

Let C be an arbitrary subset of A such that $g_u = g$ for all $u \in C$ and let $C' := V \setminus C$. Consider the set of strategies Ω_C .

Let for each $\alpha \in \Omega_C$ the union of the maximal invariant sets of Q_α which have a non void intersection with C be denoted by $E_{\alpha C}$. Define $\bar{E}_{\alpha C} := \{u \in V \mid Q_\alpha(u, E_{\alpha C}) = 1\}$. Notice that $\bar{E}_{\alpha C}$ is an invariant subset of P_α and that $E_{\alpha C} \subset \bar{E}_{\alpha C}$.

For all $u \in \bar{E}_{\alpha C}$ the sum $\sum_{\ell=0}^{\infty} Q_{\alpha C}^\ell, T_\alpha(c_\alpha - g)(u)$ exists. This sum is equal to $+\infty$ if it is possible to reach, from u , points v with $T_\alpha(c_\alpha - g)(v) = +\infty$ and the sum is finite if that is not possible.

Define the function $v_{\alpha C}$ on V by

$$v_{\alpha C}(u) = \sum_{\ell=0}^{\infty} Q_{\alpha C}^\ell, T_\alpha(c_\alpha - g)(u), u \in \bar{E}_{\alpha C} \text{ and } v_{\alpha C}(u) = +\infty, u \notin \bar{E}_{\alpha C}$$

Notice that for all $u \in V$

$$v_{\alpha C}(u) = T_\alpha(c_\alpha - g)(u) + (Q_{\alpha C}, v_{\alpha C})(u) = c_\alpha(u) - g + (P_{\alpha C}, v_{\alpha C})(u)$$

In the next lemma it is proved that for all $\alpha_1, \alpha_2 \in \Omega_C$ there is an $\alpha_0 \in \Omega_C$ such that $v_{\alpha_0 C}(u) \leq \min\{v_{\alpha_1 C}(u), v_{\alpha_2 C}(u)\}$, $u \in V$

Lemma 2. Let $C \subset A$ such that $g_u = g$ for all $u \in C$. Let for all $\alpha \in \Omega_C$ the sets $E_{\alpha C}$ and $\bar{E}_{\alpha C}$ and the functions $v_{\alpha C}$ be defined as above. The strategy α_0 is a combination of α_1 and α_2 , apply α_1 on the subset of $\bar{E}_{\alpha_1 C} \cap \bar{E}_{\alpha_2 C}$ where $v_{\alpha_1 C} < v_{\alpha_2 C}$ and apply α_2 on the other points of V .

Then $\alpha_0 \in \Omega_C$ and $v_{\alpha_0 C}(u) \leq \min\{v_{\alpha_1 C}(u), v_{\alpha_2 C}(u)\}$, $u \in V$.

Proof. Let F be the subset of V where α_1 is applied and $G := V \setminus F$. Let F^* and G^* be the intersections of F and G with $V \setminus C$. The sub-Markov process $P_{\alpha C}$, describes the state of the system under strategy α until the first visit to C . The embedded sub-Markov processes of $P_{\alpha_1 C}$ on G^* and of $P_{\alpha_2 C}$ on F^* are denoted by S_1 and S_2 . The process $R := S_1 S_2$ describes the state of the system under strategy α_0 on the moments that the set F^* is entered (until the first visit to C).

Define $W := \{u \in V \mid \min\{v_{\alpha_1 C}(u), v_{\alpha_2 C}(u)\} < \infty\}$

If $v_{\alpha_1 C}(u) < \infty$ the sum $\sum_{n=0}^{\infty} P_{\alpha_1 D}^n (c_{\alpha_1} - g)(u)$ converges absolute for each $D \supset C$

and if $v_{\alpha_2 C}(u) < \infty$ the sum $\sum_{n=0}^{\infty} P_{\alpha_2 E}^n (c_{\alpha_2} - g)(u)$ converges absolute for each $E \supset C$

For $n = 1, 2, 3, \dots$ the functions $v_{\alpha_0 n \alpha_1}$ on W are defined by

$$v_{\alpha_0 1 \alpha_1}(u) = \sum_{n=0}^{\infty} P_{\alpha_2 G^*}^n (c_{\alpha_2} - g)(u) + (S_2 v_{\alpha_1 C})(u), \quad u \in W \cap G^*$$

$$v_{\alpha_0 1 \alpha_1}(u) = \sum_{n=0}^{\infty} P_{\alpha_1 F^*}^n (c_{\alpha_1} - g)(u) + (S_1 v_{\alpha_2 C})(u), \quad u \in W \cap F^*$$

and

$$v_{\alpha_0 n \alpha_1}(u) = \sum_{n=0}^{\infty} P_{\alpha_2 G^*}^n (c_{\alpha_2} - g)(u) + (S_2 v_{\alpha_0 n-1 \alpha_1})(u), \quad u \in W \cap G^*$$

$$v_{\alpha_0 n \alpha_1}(u) = \sum_{n=0}^{\infty} P_{\alpha_1 F^*}^n (c_{\alpha_1} - g)(u) + (S_1 v_{\alpha_0 n-1 \alpha_1})(u), \quad u \in W \cap F^*$$

The value $v_{\alpha_0 n \alpha_1}(u)$ can be interpreted as the costs until absorption in C of a process, starting in u , with one period costs $c_{\alpha} - g$ where the strategy α_0 is applied until the n th entrance in the set F^* and from then on the strategy α_1 .

It is easy to verify that

$$(1) \quad \min\{v_{\alpha_1 C}(u), v_{\alpha_2 C}(u)\} - v_{\alpha_0 n\alpha_1}(u) \geq \sum_{\ell=1}^n R^\ell(v_{\alpha_2 C} - v_{\alpha_1 C})(u), \quad u \in W$$

which implies that

$$v_{\alpha_0 n\alpha_1}(u) \leq \min\{v_{\alpha_1 C}(u), v_{\alpha_2 C}(u)\} < \infty \text{ for } u \in W$$

Define $v(u) := \min\{v_{\alpha_1 C}(u), v_{\alpha_2 C}(u)\}$

Since $v_{\alpha_1 C}(u) = T_{\alpha_1}(c_{\alpha_1} - g)(u) + (Q_{\alpha_1 C}, v_{\alpha_1 C})(u)$

and $v_{\alpha_2 C}(u) = T_{\alpha_2}(c_{\alpha_2} - g)(u) + (Q_{\alpha_2 C}, v_{\alpha_2 C})(u)$

we know that $T_{\alpha_0}(c_{\alpha_0} - g)(u) + (Q_{\alpha_0 C}, v)(u) \leq v(u)$ (see Strauch [6])

Hence $T_{\alpha_0}(c_{\alpha_0} - g)(u) < \infty$ for $u \in W$ and therefore also $(T_{\alpha_0} c_{\alpha_0})(u) < \infty$ and

$$(T_{\alpha_0} 1)(u) < \infty$$

We will prove now that all ergodic sets of Q_{α_0} which can be reached with positive probability from some point in W have a non void intersection with C .

Assume the existence of an ergodic set E_0 of Q_{α_0} with $E_0 \cap C = \emptyset$ which can be reached with positive probability from some point $u \in W$.

Let E_0^* be the set of all points in V which can be reached with positive probability from some point in E_0 . Then $E_0^* \subset W$. The average costs under

α_0 , g_{α_0} , exist and are constant on E_0^* and there is a function v_{α_0} on E_0^* such that $v_{\alpha_0} = c_{\alpha_0} - g_{\alpha_0} + P_{\alpha_0} v_{\alpha_0}$ (see [10]).

Using the definition of $v_{\alpha_0 n\alpha_1}$ it is possible to derive that

$$v_{\alpha_0 n\alpha_1}(u) = v_{\alpha_0}(u) + R^n(v_{\alpha_1 C} - v_{\alpha_0})(u) + f_n(g_{\alpha_0} - g)(u), \quad u \in E_0^*$$

, where $f_n(g_{\alpha_0} - g) \rightarrow +\infty$ for $n \rightarrow \infty$ if $g_{\alpha_0} > g$

$$\begin{aligned}
 \text{We have } v_{\alpha_0}(u) &= T_{\alpha_0}(c_{\alpha_0} - g_{\alpha_0})(u) + Q_{\alpha_0} v_{\alpha_0} \leq T_{\alpha_0}(c_{\alpha_0} - g)(u) + \\
 &+ (Q_{\alpha_0 C, v})(u) + (Q_{\alpha_0} v_{\alpha_0})(u) - (Q_{\alpha_0 C, v})(u) \leq \\
 &\leq v(u) + (Q_{\alpha_0} v_{\alpha_0})(u) - (Q_{\alpha_0 C, v})(u) \leq \\
 &\leq v_{\alpha_1 C}(u) + (Q_{\alpha_0} v_{\alpha_0})(u) - (Q_{\alpha_0 C, v})(u)
 \end{aligned}$$

Hence $v_{\alpha_1 C}(u) - v_{\alpha_0}(u) \geq (Q_{\alpha_0 C, v})(u) - (Q_{\alpha_0} v_{\alpha_0})(u)$ which implies that $R^n(v_{\alpha_1 C} - v_{\alpha_0})(u)$ is bounded from below in n and therefore $g_{\alpha_0} = g$

From (1) it follows that $\sum_{n=1}^{\infty} R^n(v_{\alpha_2 C} - v_{\alpha_1 C})(u) < \infty$ for $u \in E_0^*$, that means that starting in E_0^* the state of the system stays at last in $F \cap E_0^*$ or in $G \cap E_0^*$, the process R is absorbing. This is only possible if $E_0^* \subset F$ or $E_0^* \subset G$. (see [10] for details). The strategy α_0 is therefore on E_0^* identical to the strategy α_1 or identical to the strategy α_2 . This implies that $E_0 \cap C$ can not be empty which gives a contradiction.

It is proved now that all ergodic sets of Q_{α_0} which can be reached with positive probability from some point in W have a non void intersection with C .

Hence $\alpha_0 \in \Omega_C$. Since $T_{\alpha_0}(c_{\alpha_0} - g)(u) < \infty$ for $u \in W$ the function $v_{\alpha_0 C}$ on W

defined by $v_{\alpha_0 C}(u) = \sum_{n=0}^{\infty} Q_{\alpha_0 C}^n T_{\alpha_0}(c_{\alpha_0} - g)(u)$ is finite on W .

Further $v_{\alpha_0 n \alpha_1}(u) \rightarrow v_{\alpha_0 C}(u)$ for $n \rightarrow \infty$ and by (1) $v_{\alpha_0 C}(u) \leq \min \{v_{\alpha_1 C}(u), v_{\alpha_2 C}(u)\}$ $\square$

3. AVERAGE OPTIMAL STRATEGIES

To prove the existence of average optimal strategies we need two extra conditions.

Condition I. There is a positive number B such that for all $u \in A_r$ the sum $\sum_{\ell=0}^{\infty} (Q_{\alpha u}^{\ell}, 1)(v) \leq B$ for all $\alpha \in \Omega_u$, $v \in \bar{E}_{\alpha u}$ (where $\bar{E}_{\alpha u}$ is defined as in lemma 2 with $C = \{u\}$)

Notice that this condition is satisfied if there is a $\delta > 0$ such that for all $\alpha \in \Omega$ and $u, v \in A$ the transition probability $Q_{\alpha}(u, v) = 0$ or $Q_{\alpha}(u, v) > \delta$.

This is true for instance in problems with an inventory structure (see [9]).

Let $\pi_{\alpha}(u)$ be the unique invariant probability of Q_{α} with $\pi_{\alpha}(u) > 0$.

Since $\pi_{\alpha}(u) = \frac{1}{\sum_{\ell=0}^{\infty} (Q_{\alpha u}^{\ell}, 1)(u)}$ the condition I implies that $\pi_{\alpha}(u) \geq \frac{1}{B}$

Condition II. Let $B \subset V$, $\alpha \in \Omega$ and Ω_B be the subset of strategies which are identical to α on B . For each extended real valued function v on A , bounded from below, there is an $\alpha_0 \in \Omega_B$ such that

$$T_{\alpha_0}(c_{\alpha_0} - g) + Q_{\alpha_0} v = \inf_{\alpha \in \Omega_B} \{T_{\alpha}(c_{\alpha} - g) + Q_{\alpha} v\}$$

This condition guarantees the existence of an optimal stationary strategy

for a total costs decision process which stops as soon as A is entered,

the one-period costs are $c_{\alpha} - g$, the costs of stopping are v . If $B = \emptyset$, then $\Omega_B = \Omega$.

See Schäl [5] for various sets of continuity and compactness conditions

which guarantee that the condition II is satisfied (since $c_{\alpha} - g > 0$ on

$V \setminus A$ it is an almost negative dynamic programming problem).

Now, let $A_g := \{u \in A_r \mid g_u = g\}$

In the next theorem it is shown that there exists a strategy which is average optimal on A_g .

Theorem 3. There is a strategy $\alpha_p \in \Omega'$ and a set $C_p \supset A_g$ such that

C_p is invariant under Q_{α_p} and $g_{\alpha_p}(u) = g$ for all $u \in C_p$.

Proof. First the existence of a subset C_1 of A and a strategy $\alpha_1 \in \Omega'$ will be shown, such that $g_{\alpha_1}(u) = g$ for $u \in C_1$ and C_1 is invariant under Q_{α_1} . Then it will be shown that in case $A_g \setminus C_1 \neq \emptyset$ the pair (C_1, α_1) can be extended to a pair (C_2, α_2) with $C_2 \supset C_1$, $C_2 \setminus C_1 \neq \emptyset$, C_2 is invariant under Q_{α_2} , $g_{\alpha_2}(u) = g$ for $u \in C_2$ and α_2 is identical to α_1 on $\bar{C}_1$ ($\bar{C}_1 := \{u \in V \mid Q_{\alpha_1}(u, C_1) = 1\}$).

This extension procedure can be continued until we have a pair (C_n, α_n) with $C_n \supset A_g$.

To prove the first part, let $u \in A_g$. Let for $\alpha \in \Omega_u$ the functions $v_{\alpha u}$ be defined as the functions $v_{\alpha C}$ in lemma 2 with $C = \{u\}$.

Define $\omega(v) := \inf_{\alpha \in \Omega_u} \{v_{\alpha u}(v)\}$. Since $T_{\alpha}(c_{\alpha} - g)(v) \geq -g$ for all $v \in V$

and $v_{\alpha u}(v) = +\infty$ or $v_{\alpha u}(v) = \sum_{k=0}^{\infty} Q_{\alpha u}^k T_{\alpha}(c_{\alpha} - g)(v)$, the condition I yields $\omega(v) \geq -g$ for all $v \in V$. By condition II there is a strategy α_1 such that

$$(1) \quad T_{\alpha_1}(c_{\alpha_1} - g)(v) + (Q_{\alpha_1 u}, \omega)(v) \leq T_{\alpha}(c_{\alpha} - g)(v) + (Q_{\alpha u}, \omega)(v)$$

for all $v \in V$, $\alpha \in \Omega$

Lemma 2 implies for each $\varepsilon > 0$ the existence of a strategy $\alpha_{\varepsilon} \in \Omega_u$ such that $v_{\alpha_{\varepsilon} u}(v) \leq \omega(v) + \varepsilon$ for all $v \in A$.

Substitution of α_{ε} in (1) gives

$$\begin{aligned} T_{\alpha_1}(c_{\alpha_1} - g)(v) + (Q_{\alpha_1 u}, \omega)(v) &\leq T_{\alpha_{\varepsilon}}(c_{\alpha_{\varepsilon}} - g)(v) + (Q_{\alpha_{\varepsilon} u}, \omega)(v) \leq \\ &\leq T_{\alpha_{\varepsilon}}(c_{\alpha_{\varepsilon}} - g)(v) + (Q_{\alpha_{\varepsilon} u}, v_{\alpha_{\varepsilon} u})(v) = v_{\alpha_{\varepsilon} u}(v) \leq \omega(v) + \varepsilon \end{aligned}$$

Hence

$$(2) \quad T_{\alpha_1}(c_{\alpha_1} - g)(v) + (Q_{\alpha_1 u}, \omega)(v) \leq \omega(v) \quad \text{for all } v \in A$$

Let C_1 be the subset of A with all points where $\omega(\cdot)$ is finite. The point u is of course an element of C_1 and (2) implies that C_1 is invariant under Q_{α_1} .

Let E be an ergodic set of Q_{α_1} in C_1 . We consider two cases, $u \in E$ or $u \notin E$. If $u \in E$, repeated use of the inequality (2) yields

$$\omega(u) \geq \sum_{\ell=0}^{n-1} Q_{\alpha_1}^\ell T_{\alpha_1}(c_{\alpha_1} - g)(u) + (Q_{\alpha_1}^n \omega)(u)$$

But $u \in E$ implies that $\alpha_1 \in \Omega_u$, $(Q_{\alpha_1}^n \omega)(u) \rightarrow 0$ for $n \rightarrow \infty$ and $\omega(u) = v_{\alpha_1 u}(u)$

$$\begin{aligned} \text{Hence } \sum_{\ell=0}^{\infty} (Q_{\alpha_1}^\ell T_{\alpha_1} c_{\alpha_1})(u) - g \sum_{\ell=0}^{\infty} (Q_{\alpha_1}^\ell T_{\alpha_1} 1)(u) &\leq \\ &\leq \sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell T_{\alpha} c_{\alpha})(u) - g \sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell T_{\alpha} 1)(u), \alpha \in \Omega_u. \end{aligned}$$

Substitution of

$$g_{\alpha}(u) = \frac{\sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell T_{\alpha} c_{\alpha})(u)}{\sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell T_{\alpha} 1)(u)}, \alpha \in \Omega_u$$

yields $(g_{\alpha_1}(u) - g) \sum_{\ell=0}^{\infty} (Q_{\alpha_1}^\ell T_{\alpha_1} 1)(u) \leq (g_{\alpha}(u) - g) \sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell T_{\alpha} 1)(u)$, $\alpha \in \Omega_u$

Since $(T_{\alpha} 1)(u) \leq \frac{2}{\pi_{\alpha}(u)}$ (lemma 1), $\pi_{\alpha}(u) \geq \frac{1}{B}$ and $\sum_{\ell=0}^{\infty} (Q_{\alpha u}^\ell T_{\alpha} 1)(u) \leq B$

(condition I) this implies $g_{\alpha_1}(u) = g$ and therefore $g_{\alpha_1}(v) = g$ for all $v \in E$.

If $u \notin E$ the process $Q_{\alpha_1 u}$ is identical to the process Q_{α_1} on E and repeated use of the inequality (2) gives, for $v \in E$.

$$(3) \quad \omega(v) \geq \sum_{\ell=0}^{n-1} Q_{\alpha_1}^\ell T_{\alpha_1}(c_{\alpha_1} - g)(v) + (Q_{\alpha_1}^n \omega)(v)$$

Let g_{α_1} be the average costs of α_1 on E . The existence of this average cost follows from the finiteness of $T_{\alpha_1} c_{\alpha_1}$ and $T_{\alpha_1} 1$ on E , which follow from the finiteness of ω on E .

Furthermore the expression $\sum_{\ell=0}^{n-1} Q_{\alpha_1}^\ell T_{\alpha_1}(c_{\alpha_1} - g_{\alpha_1})(v)$ is bounded in n .

Hence, by (3), $g \geq g_{\alpha_1}$ and therefore $g_{\alpha_1} = g$.

We proved that the average costs of α_1 are equal to g on all ergodic sets of Q_{α_1} in C_1 . The finiteness of ω on C_1 implies that the average costs of α_1 are equal to g on the whole set C_1 .

If $A_g \setminus C_1 \neq \emptyset$, choose $u \in A_g \setminus C_1$ and let $C := C_1 \cup \{u\}$.

Let C_1^* be union of C_1 and the set of all points in V which can be reached with positive probability from some point in C_1 and define Ω^* as the set of all stationary strategies which are identical to α_1 on C_1^* .

Let $\Omega_v^* := \Omega^* \cap \Omega_v$, $v \in A$ and $\Omega_C^* := \Omega^* \cap \Omega_C$. Since Ω^* dominates Ω in the average costs sense we know that $\inf_{\alpha \in \Omega_v^*} g_\alpha(v) = g_v = g$ for $v \in A_g$.

Define $\omega^*(v) = \inf_{\alpha \in \Omega_C^*} \{v_{\alpha C}(v)\}$

By condition II there is a strategy α_2 which minimizes the expression

$T_{\alpha}(c_{\alpha} - g)(v) + (Q_{\alpha C}, \omega^*)(v)$ over the set of strategies Ω^* .

As for α_1 and ω it is possible to show that

$$T_{\alpha_2}(c_{\alpha_2} - g)(v) + (Q_{\alpha_2 C}, \omega^*)(v) \leq \omega^*(v) \text{ for all } v \in A.$$

and also that $g_{\alpha_2}(u) = g$ for all $u \in C_2$, where C_2 is the set of all points in A where $\omega^*(.)$ is finite.

If $A_g \setminus C_2 \neq \emptyset$ we can continue in the same way. □

The following corollary is a direct consequence of theorem 3.

Corollary 4. If it is possible to reach the set A_g from each point in V with finite expected costs then there exists a strategy which is average optimal on the whole state space V .

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