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THE ALGEBRAIC RICCATI EQUATION
AND SINGULAR OPTIMAL CONTROL
(preliminary draft)

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THE ALGEBRAIC RICCATI EQUATION AND SINGULAR OPTIMAL CONTROL

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ABSTRACT

The paper links the class of non-negative definite linear-quadratic control problems to a subset of the set \( \Gamma \) of all real symmetric matrices \( K \) that satisfy the dissipation inequality \( F(K) \geq 0 \). This subset is formed by those \( K \in \Gamma \) for which the rank of the dissipation matrix \( F(K) \) attains the minimal rank of \( F(K) \) over \( \Gamma \). Since symmetric matrices that represent optimal costs for linear-quadratic problems necessarily turn out to be elements of this subset, it is of interest.

We present a straightforward characterization of the set of rank minimizing solutions of the dissipation inequality in terms of an Algebraic Riccati Equation and a linear condition. Moreover, we attach every positive semi-definite element of this set in a one-to-one way to a certain subspace of the factor space \( \mathbb{R}^n := \mathbb{R}^n/_{W} \), where \( W \) stands for the strongly reachable subspace. It is easily seen that \( W = 0 \) if and only if the input weighting matrix in the cost functional is positive definite. Assuming this to be the case, then the rank minimizing solutions of the dissipation inequality are solutions of the ordinary Algebraic Riccati Equation, and, moreover, the known results on bijective relations between positive semi-definite ARE solutions and certain subspaces of \( \mathbb{R}^n \) are recovered.

KEYWORDS: Linear-quadratic control problems, dissipation inequality, Algebraic Riccati Equation, strongly reachable subspace, induced map.
THE ALGEBRAIC RICCATI EQUATION AND SINGULAR OPTIMAL CONTROL

1. Introduction.

It is widely known that there exist strong ties between the Algebraic Riccati Equation (ARE) and infinite horizon linear–quadratic (LQ) optimal control (e.g. the optimal regulator problem). In 1971 ([1]), the real symmetric solutions $K$ of the ARE were seen to be certain solutions of the so-called dissipation inequality (DI) $F(K) \geq 0$, where $F(K)$ stands for the real and symmetric dissipation matrix ([2]). It was noted there, that for every solution $K^0$ of the ARE the rank of $F(K^0)$ is minimal in the sense that it equals $\rho := \min_{K \in \Gamma} \text{rank}(F(K))$, where $\Gamma := \{K = K' \mid F(K) \geq 0\}$.

For singular LQ problems (i.e. LQ problems where the input weighting matrix, appearing in the cost criterion, is not positive definite), the ARE is not defined. However, it was proven in 1983 ([2]) that also for the matrix $K^+$, representing the optimal cost for the zero end–point non–negative definite singular LQ problem, the rank of $F(K^+)$ is minimal. Hence the conjecture, made in [1], that optimal costs for LQ problems are rank minimizing solutions of the DI (and thus, in case of regular problems, solutions of the ARE), had been partially confirmed. Moreover, it was shown recently ([3]) that the matrix $K^-$, characterizing the optimal cost for the free end–point non–negative definite problem, is a rank minimizing solution of the DI. The subset of $\Gamma$, $\Gamma_{\text{min}} := \{K \in \Gamma \mid \text{rank}(F(K)) = \rho\}$ thus seems to be of interest.

Indeed, in [4] it is stated that, given a well–defined infinite horizon regular LQ problem with linear end–point constraints, then the optimal cost is determined by a real symmetric solution of the ARE. More generally, it can be shown ([5]) that for any of these problems (regular as well as singular) the optimal cost is characterized by a certain rank minimizing solution of the DI. Therefore it is justified to conclude that $\Gamma_{\text{min}}$ rather than $\Gamma$ is of importance when trying to solve LQ optimal control problems.

In this paper we will consider for the non–negative definite LQ problems the issue of computing all solutions of $\Gamma_{\text{min}}$ as well as a way of representing all $K \geq 0 \in \Gamma_{\text{min}}$ in terms of $K^-$ and $K^+$, the smallest and the largest positive semi–definite rank minimizing solutions of the DI.
We will show that $\Gamma_{\text{min}} = \{K = K' \mid \Phi(K) = 0, W \subseteq \ker(K)\}$, where $\Phi(K) = 0$ denotes a certain ARE and $W$ stands for the strongly reachable subspace, the dual of the weakly unobservable subspace (the space of initial states for which there exists an output nulling input), see e.g. [6] – [7]. In particular, if the input weighting matrix in the cost functional is regular, then $W = 0$ (and conversely) and $\Phi(K) = 0$ equals the ordinary ARE – and that is what we must expect.

A number of articles (e.g. [8] – [9]) have appeared on the representation of all positive semi-definite solutions of the ARE in terms of the smallest and the largest of these solutions. Although many of these papers take as a starting-point the Hamiltonian matrix, whereas others merely show the influence of the geometric approach initiated by Willems ([1], also [10]), the key observation is that there exists a one-to-one correspondence between these solutions and certain subspaces.

Here, we will link every positive semi-definite $K \in \Gamma_{\text{min}}$ in a bijective manner to a certain subspace of the factor space $\mathbb{R}^n := \mathbb{R}^n/W$. This makes sense, because ([3]) if $F(K) \geq 0$, then $W \subseteq \ker(K)$. In other words, two solutions of the DI can only differ "outside" $W$. Since it is found that in case of regularity ($W = 0$) the ordinary correspondence between solutions of the ARE and subspaces of $\mathbb{R}^n$ is recovered, we thus have generalized [8] – [9].

2. Outline of our results.

Consider the finite dimensional linear time-invariant system $\Sigma$
\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x_0, \quad (2.1a) \\
y &= Cx + Du,
\end{align*}
\]
and the quadratic cost criterion
\[
J(x_0, u) = \int_0^\infty y'y \, dt. \quad (2.2)
\]
The state, input, output variable are assumed to be $n$-, $m$-, $r$-dimensional, respectively, and $[B]$ $[C \, D]$ are left, right invertible. If $K$ is a real symmetric matrix of order $n$, then we say that $K$ satisfies the dissipation inequality if
\[
F(K) \geq 0 \text{ with } F(K) = \begin{bmatrix}
C'C + A'K + KA & KB + C'D \\
B'K + C'D & D'D
\end{bmatrix}, \quad (2.3)
\]
see [1], [2]. Let $T$ be any subspace and assume that $(A, B)$ is stabilizable. We
state the linear–quadratic control problem with stability modulo $T$ ($\text{(LQCP)}_T$): For all $x_0$, determine $J^T(x_0) := \inf\{J(x_0, u) | u \in L^m_{2, \text{loc}}(\mathbb{R}^+) \text{ and } (x/T)(\infty) = 0\}.$

Lemma 2.1 ([2]).

Let $\Gamma := \{K = K' | F(K) \geq 0\}$ and $T(s) := D + C(sI - A)^{-1}B$, the transfer function. Then for all $K \in \Gamma$, $\text{rank } (F(K)) \geq \rho := \text{rank } (T(s))$.

Proposition 2.2 ([4], [5]).

Let $\Gamma_{\text{min}} := \{K \in \Gamma | \text{rank } (F(K)) = \rho\}$. Then there exists a positive semi–definite $K_T \in \Gamma_{\text{min}}$ such that, for all $x_0$, $J^T(x_0) = x_0'K_TX_0$.

Now let $W = W(\Sigma) := \{x_0 \in \mathbb{R}^n | \exists T > 0 \forall \varepsilon > 0 \exists y, y' \in \mathbb{R}^n \int_0^\infty \|y - y\|^2 dt \leq \varepsilon \text{ and supp}(x) \subset [0, T]\}$ ([7]). Then it can be shown that $W = 0$ if and only if $D'Q > 0$. Define $W_2 := W \cap (C^{-}\text{lim}(D))$ and let $W_1$ be a left invertible matrix such that $W_1 \oplus W_2 = W$ where $\text{im}(W_1) = W_1$. Introduce $A_0 := A - B(D'D)^+D'C$ and $C_0 := (I - D(D'D)^+D'C)$, with $(D'D)^+$ denoting the Moore–Penrose inverse of $(D'D)$. Then $L_1 := W_1'C_0C_0W_1 > 0$ since $C^{-}\text{lim}(D) = \ker(C_0)$. Next, we set

$$\phi_0(K) := C_0'C_0 + A_0'K + KA_0 - KB(D'D)^+B'K \quad (2.3)$$

and

$$\phi(K) := \phi_0(K) - \phi_0(K)W_1L_1^{-1}W_1'\phi_0(K). \quad (2.4)$$

Finally, let

$$\hat{P} := W_1L_1^{-1}W_1'C_0'C_0, \quad \hat{A}_0 := A_0(I - \hat{P}) \quad (2.5)$$

and

$$\hat{\Lambda}_0(K) := \hat{A}_0 - (B(D'D)^+B' + A_0W_1L_1^{-1}W_1'A_0')K \quad (2.6)$$

for any $K \in \Gamma$.

Theorem 2.3.

It holds that $\Gamma_{\text{min}} = \{K = K' | \phi(K) = 0, W \subset \ker(K)\}$. Also, $\hat{A}_0(W) \subset W$ and, if $K \in \Gamma$, then $W \subset \ker(K)$. Any other left invertible matrix $\hat{W}_1$ such that $\text{im}(\hat{W}_1) \oplus W_2 = W$ yields the same $\phi(K)$ as defined in (2.4). Moreover, if $\hat{K}, \hat{A}_0(K)$ denote the induced maps of $\hat{A}_0, \hat{A}_0(K)$ w.r.t. $\mathbb{R}^n = \mathbb{R}^n/W$, then these maps do not depend on the choice for $W_1$ as well.
Remark 2.4.

If \( W = 0 \), then \( \phi(K) = \phi_0(K) \) and \( \phi_0(K) = 0 \) is easily seen to be the ordinary ARE. In addition, \( \hat{A}_0 = A_0 = A - B(D'D)^{-1}D'C \), \( \hat{A}_0(K) = A_0 - B(D'D)^{-1}B'K \). Observe that \( \phi(K) \) is indeed a quadratic form in \( K \) since \( KW = 0! \)

Next, let \( K \in \Gamma \). Then \( KW = 0 \) and hence we can define \( K : \mathbb{R}^n \to \mathbb{R}^n \) by \( K \tilde{x} := Kx \) (\( \tilde{x} = x + W \)). Let \( K^+ \) be the largest and let \( K^- \) be the smallest semi-definite solution of \( \Gamma_{\min} \) ([2], [5], [3]). Define \( \bar{A} := (K^+ - K^-) \) and \( V_0 := \ker(\bar{A}) \). If \( \bar{A}_0 := \bar{A}_0(K^-) \), then it can be shown that \( \bar{A}_0(V_0) \subset V_0 \) and \( \sigma(\bar{A}_0|V_0) \subset \mathbb{C} \). Moreover, there exists a unique subspace \( V^+ \subset (V + W)/W \) (where \( V = V(\Sigma) := \{x_0 | \exists u : y = 0 \} \) ([6] - [7])) such that \( \bar{A}_0(V^+) \subset V^+ \), \( \sigma(\bar{A}_0|V^+) \subset \mathbb{C}^+ \) and \( V_0 \oplus V^+ = \mathbb{R}^n \). Since, actually, \( \ker(K^-) = (V + W)/W \), we have that \( \bar{A}_0(V^+) \subset V^+ \). Also, we find that \( \bar{A}_0(V + W) \subset (V + W) \), \( \sigma(\bar{A}_0|(V + W)/W) = \sigma^*(\Sigma) \), the set of the invariant zeros ([5]), and it turns out that \( V^+ \) equals the space spanned by the generalized eigenvectors in \( (V + W)/W \) corresponding to \( \lambda \in \sigma^*(\Sigma) \) that are in \( \mathbb{C}^+ \).

Theorem 2.5

Let \( V_1 \subset V^+ \) be such that \( \bar{A}_0(V_1) \subset V_1 \). If \( P_0 : \mathbb{R}^n \to \mathbb{R}^n \) denotes the canonical projection, \( P_0^{-1}(S) := \{x | P_0x \in S \subset \mathbb{R}^n \} \) and \( \bar{A}^{-1}(H) := \{\tilde{x} | \bar{A} \tilde{x} \in H \subset \mathbb{R}^n \} \), then \( V_2 := \bar{A}^{-1}(P_0^{-1}(V_1)) \) is such that \( V_1 \oplus V_2 = \mathbb{R}^n \), \( \bar{A}_0^+(V_2) \subset V_2 \) and \( \bar{A}_0^+(V^+) \subset \mathbb{C}^+ \) (here \( \bar{A}_0^+ = \bar{A}_0(K^+) \)). Let \( \bar{F} \) denote the projection of \( \mathbb{R}^n \) onto \( V_1 \) and along \( V_2 \), then \( K = TP_0 \) with \( T = K^+\bar{F} + K^+(I - \bar{F}) = K^+(I - \bar{F}) \) in \( \Gamma_{\min} \) and positive semi-definite. Conversely, if \( K \geq 0 \) and \( K \in \Gamma_{\min} \), then there exists a unique \( V_1 \), satisfying all conditions given above, such that \( K = T \). In addition, if \( K_1, K_2 \) (both \( \geq 0 \) and in \( \Gamma_{\min} \)) are supported by \( V_{11}, V_{12} \), respectively, then \( K_1 \geq K_2 \) if and only if \( V_{11} \subset V_{12} \).
Remark 2.6.

We have established a one-to-one correspondence between the positive semi-definite rank minimizing solutions of the DI and certain subspaces in $\mathbb{R}^n/W$ in the style of [1] and [10]. Note that if $W = 0$, then the results from [8] – [9] are recovered. Finally, we state that for every $K \geq 0$ in $\Gamma_{\min}$ it holds that $J_{\ker(K)}(x_0) = x_0'Kx_0$.

This result is, in a way, the converse of Prop. 2.2.

References.


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