A unified approach to sequences, bags, and trees

by

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A unified approach to sequences, bags, and trees

Introduction

This paper contains a formal definition of constructs, a new concept of which sequences, bags, and trees may be regarded as special cases. This unification of data structures commonly regarded as unrelated leads to a simpler treatment of several operations that are usually defined by structural induction for each case separately.

Constructs

DEFINITION. An ordered set $A$ is a pair $(V_A, \leq_A)$, where $V_A$ is a set and $\leq_A$ is a relation on $V_A$ satisfying the following laws: for arbitrary $a, b, c$ in $V_A$,

(i) $a \leq_A a$ (reflexivity),
(ii) $a \leq_A b \land b \leq_A a \Rightarrow a = b$ (antisymmetry),
(iii) $a \leq_A b \land b \leq_A c \Rightarrow a \leq_A c$ (transitivity).

For simplicity's sake, we usually write $a \in A$ instead of $a \in V_A$. We shall interpret $a \geq_A b$ as $b \leq_A a$, and $a <_A b$ as $a \leq_A b \land a \neq b$.

For an ordered set $A$ and an arbitrary set $X$, the set $A \rightarrow X$ consists of the pairs $(\leq_A, f)$ with $f \in V_A \rightarrow X$. Note that this definition ensures that $A \neq B$ implies $(A \rightarrow X) \cap (B \rightarrow X) = \emptyset$. Once again for simplicity, we will write $f \in A \rightarrow X$ instead of $(\leq_A, f) \in A \rightarrow X$.

DEFINITION. Let $A$ and $B$ be ordered sets. An isomorphism of $A$ onto $B$ is a bijection $\sigma$ of $V_A$ onto $V_B$ that satisfies

$$a_0 \leq_A a_1 \Rightarrow \sigma(a_0) \leq_B \sigma(a_1)$$

for all $a_0, a_1$ in $A$. 


DEFINITION. Let $A$, and $B$ be ordered sets, $X$ and $Y$ arbitrary sets. Mappings $f \in A \rightarrow X$ and $g \in B \rightarrow Y$ are called equivalent (notation: $f \sim g$) if there exists an isomorphism $\sigma$ of $A$ onto $B$ such that $f = g \circ \sigma$. (Note that this implies $X = Y$.)

THEOREM 0. Let $F$ denote the class of all mappings that belong to a set of the form $A \rightarrow X$, where $A$ is an ordered set and $X$ any set. Then $\sim$ is reflexive, symmetric and transitive on $F$.

PROOF. It is reflexive, since for any $f$ in $F$,

$$f = f \circ id$$

where $id$ denotes the identity mapping on the domain of $f$. It is symmetric, since

$$f = g \circ \sigma = g = f \circ \sigma^{-1}$$

Finally, it is transitive, since

$$f = g \circ \sigma \land g = h \circ \tau \Rightarrow f = h \circ (\tau \circ \sigma)$$

THEOREM 1. It is possible to associate with every $f$ in $F$ a set $C(f)$ in such a way that

$$C(f) = C(g) \Rightarrow f \sim g$$

PROOF. If $F$ had been a set, one could have taken $C(f)$ to be the equivalence class of $f$ with respect to the equivalence relation $\sim$. However, $F$ is not a set. We shall circumvent this objection by constructing nonempty sets $C(f)$ that consist of some, but not all, mappings equivalent to $f$.

By transfinite induction, we define for every ordinal $\alpha$ a set $R_\alpha$ in the following way: $R_0 = \emptyset$, $R_{\alpha+1} = P(R_\alpha)$, and for every limit ordinal $\alpha$, $R_\alpha$
is the union of all sets \( R_\beta \) with \( \beta < \alpha \). For every set \( X \), the rank of \( X \) is defined as the least \( \alpha \) with \( X \subset R_\alpha \); in Zermelo-Fraenkel set theory, every set has rank \([3, 5]\).

The elements of \( F \) are themselves sets: if \( f \in A \to X \), then

\[
f \in P(V_A \times V_A) \times P(V_A \times X)
\]

Now define \( C(f) \) as the class of all \( g \) in \( F \) with \( g \approx f \) such that \( g \) is of minimal rank. If \( \alpha \) is the rank of the elements of \( C(f) \), we find that every element of \( C(f) \) belongs to \( P(P(R_\alpha)) \), so \( C(f) \) is indeed a set.

We shall call the sets \( C(f) \) constructs; as we shall see below, they are sufficiently general for such a vague term to be appropriate.

In order to escape the obligation to give every mapping a name, we introduce an alternative notation for constructs. If \( A \) is an ordered set and \( E \) an expression in the single unbound identifier \( i \), we define

\[
(\Sigma i : i \in A : E) = C(f)
\]

where \( f \) is the mapping that satisfies \( f(a) = E(i:= a) \) for every \( a \) in \( A \) and is surjective (in other words, has codomain \( \{E(i:=a) \mid a \in A\} \)).

This new notation also provides us with an easy way to restrict the domain of a construct, which is sometimes called filtering. Let \( p \) be a boolean function defined on \( A \). We define

\[
(\Sigma i : i \in A \land p(i) : E) = (\Sigma i : i \in B : E)
\]

where \( V_B \) is the subset of those \( a \) in \( V_A \) that satisfy \( p(a) \), and \( \leq \) is the restriction of \( \leq_A \) to \( V_B \times V_B \). In case \( A \) consists of the natural numbers with their usual ordering, the conjunct \( i \in A \) is usually omitted.

**Sequences**

**EXAMPLE 0.** Let \( A \) denote the set \( \{0, 1, 2\} \) with the usual ordering.
Define \( f: \mathbb{A} \to \{2, 3, 6\} \) by \( f(0) = 2, f(1) = 3, f(2) = 6 \). Now a mapping \( g: \mathbb{B} \to \mathbb{Y} \) satisfies \( g = f \) if and only if \( \mathbb{Y} = \{2, 3, 6\} \) and \( \mathbb{V}_\mathbb{B} \) consists of three elements, say \( \mathbb{V}_\mathbb{B} = \{b_0, b_1, b_2\} \) with \( b_0 \preceq b_1 \preceq b_2 \), such that \( g(b_0) = 2, g(b_1) = 3, g(b_2) = 6 \). What these different \( g \) have in common is that they take the values 2, 3 and 6 in that order. Therefore we wish to identify \( C(f) \) with the finite sequence \( \langle 2, 3, 6 \rangle \). Below we shall give a definition of finite sequences that achieves precisely this effect.

**DEFINITION.** A finite sequence is a construct, say \( C(f) \) with \( f: \mathbb{A} \to \mathbb{X} \), such that \( f \) is surjective, \( \mathbb{V}_\mathbb{A} \) is finite and \( \preceq_\mathbb{A} \) is a linear ordering (i.e., for every \( a_0 \) and \( a_1 \) in \( \mathbb{V}_\mathbb{A} \), it is true that \( a_0 \preceq_\mathbb{A} a_1 \lor a_1 \preceq_\mathbb{A} a_0 \)).

**REMARK.** The condition that \( f \) be surjective has the effect that sequences are determined by the terms and their order alone, not by the type of these terms. For instance, there is only one sequence consisting of the numbers 0 and 1 in that order, regardless of whether these are considered as elements of the set of integers or of the set \( \{0, 1, 25, 1988\} \). There is also only one empty sequence. If some application should require the introduction of typed sequences, this is achieved by the removal of the surjectivity condition. Note, however, that Theorem 2 below will then lose its validity.

In particular, if \( p \) is a boolean function on the natural numbers satisfying

\[
(N \ i:: p(i)) \iff \ldots
\]

the construct

\[
(\exists i:: p(i):: x_i)
\]

is a finite sequence, one that the present author has been in the habit of denoting by \( (\text{SEQ } i:: p(i):: x_i) \). In the literature, many other notations can be found, for instance...
from \([4]\). Note, however, that not all authors make a clear distinction between \(f\) and \(C(f)\); this makes it uncertain whether dummy transformations are permitted. In case \(p(i) = 1 \leq i \leq n\), it is common practice to denote the above sequence by \(\langle x_1, x_2, \ldots, x_n \rangle\).

**Theorem 2.** Every finite sequence can be written as \(\langle x_1, x_2, \ldots, x_n \rangle\) in precisely one way.

**Proof.** Let \(C(f)\) be a finite sequence, where \(f \in A \rightarrow X\). Let \(n\) be the number of elements of \(VA\). Define \(\sigma \in VA \rightarrow \{1, 2, \ldots, n\}\) by

\[
\sigma(a) = (\{N: i \in A: i \preceq_A a\})
\]

We shall now prove that \(\sigma\) is an isomorphism from \(A\) onto \(\{1, 2, \ldots, n\}\) with the usual ordering.

Take \(a_0\) and \(a_1\) in \(VA\) with \(a_0 \preceq_A a_1\) and \(a_0 \neq a_1\) (remember that this is abbreviated as \(a_0 \prec_A a_1\)). By the antisymmetry of \(\preceq_A\), this implies \(\neg a_1 \preceq_A a_0\). Now

\[
\sigma(a_0) \prec \sigma(a_1)
\]

= \{definition of \(\sigma\)\}

\[
(\{N: i \in A: i \preceq_A a_0\}) < (\{N: i \in A: i \preceq_A a_1\})
\]

= \{domain split\}

\[
(\{N: i \in A: i \preceq_A a_0\}) < (\{N: i \in A \land i \preceq_A a_0: i \preceq_A a_1\}) + (\{N: i \in A \land \neg i \preceq_A a_0: i \preceq_A a_1\})
\]

= \{\(a_0 \preceq_A a_1\), transitivity of \(\preceq_A\)\}

\[
0 < (\{N: i \in A \land \neg i \preceq_A a_0: i \preceq_A a_1\})
\]

= \{\(\neg a_1 \preceq_A a_0\), reflexivity of \(\preceq_A\)\}

true

This proves

\[
a_0 \prec_A a_1 \Rightarrow \sigma(a_0) < \sigma(a_1).
\]
By symmetry, the same formula holds with \( a_0 \) and \( a_1 \) interchanged. Now we can (finally!) use the linearity of \( \preceq_A \) to obtain

\[
\begin{align*}
a_0 \preceq_A a_1 & = \sigma(a_0) \preceq \sigma(a_1) \\
\end{align*}
\]

and also

\[
\begin{align*}
a_0 \neq a_1 & \Rightarrow \sigma(a_0) \neq \sigma(a_1) \\
\end{align*}
\]

in other words, \( \sigma \) is injective. Since domain and codomain of \( \sigma \) have the same finite number of elements, \( \sigma \) is in fact a bijection.

We have now proved that \( \sigma \) is an isomorphism from \( A \) onto \( \{1, 2, \ldots, n\} \) with the usual ordering. Since \( \sigma^{-1} \) must then also be an isomorphism, the definition of equivalence gives \( f \approx f \circ \sigma^{-1} \), hence,

\[
\begin{align*}
C(f) &= \{ f \approx f \circ \sigma^{-1}, \text{ definition of } C \} \\
C(f \circ \sigma^{-1}) &= \{ \text{surjectivity of } f, \text{ definition of } C \} \\
(C \ i: 1 \leq i \leq n: f(\sigma^{-1}(i)) &= \{ \text{definition of } \langle \ldots \rangle \} \\
\langle f(\sigma^{-1}(1)), f(\sigma^{-1}(2)), \ldots, f(\sigma^{-1}(n)) \rangle &= \ldots \\
\end{align*}
\]

It remains to show uniqueness. Suppose that also \( C(f) = \langle x_1, x_2, \ldots, x_n \rangle \). Then there exists an isomorphism \( \tau \) from \( \{1, 2, \ldots, n\} \) with the usual ordering onto itself, such that for every \( i \),

\[
x_i = f(\sigma^{-1}(\tau(i)))
\]

However, such a \( \tau \) can only be the identity mapping, as we shall now prove. Consider a minimal \( i \) with \( \tau(i) \neq i \). Then

\[
\tau(i) \neq i
\]
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\[ \tau(\tau(i)) \neq \tau(i) \land \tau^{-1}(\tau(i)) \neq \tau^{-1}(i) \]

so no such \( i \) exists. Hence \( \tau \) must be the identity mapping.

EXAMPLE 1. By means of dummy transformations and filtering, the same finite sequence can be written in a variety of ways. For instance,

\[ (\forall i: 0 \leq i \leq 2: i^2 + 2) \]
\[ (\forall i: 3 \leq i \leq 5: i^2 - 6i + 11) \]
\[ (\forall i: 2 \leq i \leq 6 \land i \neq 4 \land i \neq 5: i) \]
\[ (\forall i: 2 \leq i \leq 3 \lor i = 6: i) \]
\[ (\forall i: i > 1 \land 6 \mod i = 0: i) \]

all denote the sequence \(<2, 3, 6>\).

DEFINITION. An infinite sequence is a construct, say \( C(f) \) with \( f \in A \to X \), such that \( f \) is surjective and \( A \) is isomorphic to the natural numbers with their usual ordering.

EXAMPLE 2. The constructs

\[ (\forall i:: 2i + 1) \]

and
(C i: i mod 2 ≠ 0: i)

both denote the same infinite sequence, the one whose terms are the odd natural numbers in their usual order. It is not uncommon to denote this sequence by

<1, 3, 5, 7, ...>

However, the usefulness of such a notation heavily depends on how strongly the first few terms suggest the general law. From

<2, 3, 5, 7, 11, ...>

we can probably guess that

(C i: i > 1 ∧ (A j, k: j > 1 ∧ k > 1: j*k ≠ i): i)

is meant. But what about

<0, 1, 25, 1988, ...>

REMARK. It is easy to define the reverse of a construct, say C(f) with f ∈ A → X: simply replace the ordering ≤A by its inverse relation ≥A. This corresponds with the well-known reversal operation on finite sequences. If we wish to have the set of all sequences closed under reversal, we must also admit left-infinite sequences.

EXAMPLE 3. Let A denote the set {0, 1, 2}, ordered discretely: a0 ≤A a1 = a0 = a1. Define f ∈ A → {7, 10} by f(0) = 7, f(1) = 10, f(2) = 7. A mapping g ∈ B → Y satisfies g ≈ f if and only if Y = {7, 10} and Vm consists of three elements, ordered discretely, such that g maps two of these onto 7 and one onto 10. What these different g have in common is that
they take the value 7 twice and the value 10 once. Therefore we wish to identify \( C(f) \) with the bag \([7, 7, 10]\). Below we shall give a definition of bags that achieves precisely this effect.

**DEFINITION.** A bag is a construct, say \( C(f) \) with \( f \in A \rightarrow X \), such that \( f \) is surjective and \( \leq_A \) is discrete (i.e., \( a_0 \leq_A a_1 = a_0 = a_1 \)).

The \( B \)-quantifier that is sometimes used for bags relates to our \( C \)-quantifier as follows: if \( p \) is a boolean function on the set \( N \) of natural numbers,

\[
(\text{\( B \)} i: p(i): x_i) = (\text{\( C \)} i: i \in (N, =) \land p(i): x_i)
\]

In case \( p(i) = 1 \leq i \leq n \), this bag may also be denoted by \([x_1, x_2, \ldots, x_n]\).

A bag \( C(f) \), where \( f \in A \rightarrow X \), is called finite if \( V_A \) is finite.

**THEOREM 3.** Every finite bag can be written as \([x_1, x_2, \ldots, x_n]\). The elements \( x_1, x_2, \ldots, x_n \) are uniquely determined but for permutations.

**PROOF.** Let \( C(f) \) be a finite bag, where \( f \in A \rightarrow X \). Let \( n \) denote the number of elements of \( V_A \). This means that there exists a bijection \( \sigma \) from \( \{1, 2, \ldots, n\} \) onto \( V_A \); such a \( \sigma \) is also an isomorphism from \( \{1, 2, \ldots, n\} \) with the discrete order onto \( A \). By the definition of equivalence, \( f = f \circ \sigma \), hence

\[
C(f) = \begin{align*}
&= \{ f = f \circ \sigma, \ \text{definition of } C \} \\
&= \{ \text{surjectivity of } f, \ \text{definition of } C \} \\
&= \{ \text{definition of } [\ldots] \} \\
&= \{ f(\sigma(1)), f(\sigma(2)), \ldots, f(\sigma(n)) \}
\end{align*}
\]

It remains to show uniqueness but for permutations. Suppose that also \( C(f) \)

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= \{x_1, x_2, \ldots, x_n\}. Then there exists an isomorphism \( \tau \) from \( \{1, 2, \ldots, n\} \) with the discrete ordering onto itself, i.e., a permutation of these numbers, such that

\[
x_1 = f(\sigma(\tau(i)))
\]

We shall now prove that our definition of bags is equivalent to the usual one.

**DEFINITION.** For a bag \( B \), say \( B = \mathcal{C}(f) \) with \( f \in A \rightarrow X \), and for \( x \in X \), the number \( \#(B, x) \) is defined as

\[
\#(B, x) = (\{ i : i \in A : f(i) = x \})
\]

**THEOREM 4.** For any two bags \( B_0, B_1 \), say \( B_0 = \mathcal{C}(f) \) and \( B_1 = \mathcal{C}(g) \) with \( f \in A \rightarrow X \) and \( g \in B \rightarrow Y \),

\[
(0) \quad B_0 = B_1 \iff X = Y \land (\forall x \in X : \#(B_0, x) = \#(B_1, x))
\]

**PROOF.** According to the definition of \( \mathcal{C} \), the bags \( \mathcal{C}(f) \) and \( \mathcal{C}(g) \) are equal if and only if there exists an isomorphism \( \sigma \) from \( A \) onto \( B \) such that \( f = g \circ \sigma \). As \( A \) and \( B \) are discretely ordered, such an isomorphism is nothing but a bijection from \( V_A \) onto \( V_B \). From the existence of such a bijection, the first conjunct of (0) clearly follows; we now prove that the second one does too. For any \( x \) in \( X \),

\[
\#(B_0, x)
\]

= [definition of \#]

(\( \{ i : i \in A : f(i) = x \} \))

= [\( f = g \circ \sigma \)]

(\( \{ i : i \in A : g(\sigma(i)) = x \} \))

= [dummy transformation, \( j := \sigma(i) \)]

(\( \{ j : j \in B : g(j) = x \} \))
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= [definition of $\#$]

$\#(B_1, x)$.

On the other hand, assume that $\#(B_0, x) = \#(B_1, x)$ for every $x$ in $X$. Then there exists, for every $x$ in $X$, a bijection $\sigma_x$ from $f^{-1}[x]$ onto $g^{-1}[x]$. Define $\sigma \in \mathcal{V}_A \rightarrow \mathcal{V}_B$ by

$$\sigma(a) = \sigma_{f(a)}(a).$$

We prove the injectivity of $\sigma$. For all $a_0, a_1$ in $\mathcal{V}_A$,

$$\sigma(a_0) = \sigma(a_1) = [\text{definition of } \sigma] = \sigma_{f(a_0)}(a_0) = \sigma_{f(a_1)}(a_1) = [\text{application of } g] = g(\sigma_{f(a_0)}(a_0)) = g(\sigma_{f(a_1)}(a_1)) = [\text{definition of } \sigma_x \text{ implies } g(\sigma_x(y)) = x] = f(a_0) = f(a_1) \wedge \sigma_{f(a_0)}(a_0) = \sigma_{f(a_1)}(a_1) = [\text{injectivity of } \sigma_{f(a_0)}] = a_0 = a_1.\]$$

Moreover, $\sigma$ is surjective: for any $b$ in $\mathcal{V}_B$ we have $b \in g^{-1}[g(b)]$, so by the bijectivity of the $\sigma_x$, there exists an $a$ in $f^{-1}[g(b)]$ with $\sigma_{g(b)}(a) = b$. Since $a \in f^{-1}[g(b)]$, we have $f(a) = g(b)$, so $\sigma(a) = \sigma_{f(a)}(a) = \sigma_{g(b)}(a) = b$.

We have now established that $\sigma$ is a bijection from $\mathcal{V}_A$ onto $\mathcal{V}_B$. For any $a$ in $\mathcal{V}_A$, $g(\sigma(a)) = g(\sigma_{f(a)}(a)) = f(a)$. Together with $X = Y$ this yields $f = g \circ \sigma$.

Theorem 4 shows that a bag $B$ may equally well be identified with the function $\#(B, x) \in X \rightarrow N$; that is the usual definition.
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Concatenation and summation

DEFINITION. Let $A$ and $B$ be ordered sets. The concatenation of $A$ and $B$, denoted $A \circlearrowleft B$, is the ordered set $(V_{A \circlearrowleft B}, S_{A \circlearrowleft B})$, where

$$V_{A \circlearrowleft B} = \{(0, a) \mid a \in V_A\} \cup \{(1, b) \mid b \in V_B\}$$

and

$$(i_0, c_0) \leq_{A \circlearrowleft B} (i_1, c_1)$$

$$(i_0 = 0 \land i_1 = 1) \lor (i_0 = i_1 = 0 \land c_0 \leq c_1) \lor (i_0 = i_1 = 1 \land c_0 \leq c_1).$$

DEFINITION. Consider constructs $C(f)$ and $C(g)$, where $f \in A \to X$ and $g \in B \to Y$. The concatenation of $C(f)$ and $C(g)$, denoted $C(f) \circlearrowleft C(g)$, is the construct $C(h)$, where $h \in (A \circlearrowleft B) \to X \cup Y$ is defined by

$$(1) \quad h(0, a) = f(a) \land h(1, b) = g(b)$$

for $a \in A, b \in B$.

THEOREM 5 ("domain split law"). For $f$ defined on $A \circlearrowleft B$,

$$(\exists i : i \in A \circlearrowleft B : f(i))$$

$$(\exists i : i \in A : f(0, i)) \circlearrowleft (\exists i : i \in B : f(1, i))$$

EXAMPLE 4.

(a) $\langle 1, 2, 5 \rangle \circlearrowleft \langle 5, 3, 0 \rangle = \langle 1, 2, 5, 5, 3, 0 \rangle$.

(b) $\langle i : 0 \leq i < m : f(i) \rangle \circlearrowleft \langle i : 0 \leq i < n : g(i) \rangle$

$= \langle i : 0 \leq i < m+n : if i < m \rightarrow f(i) \land i \geq m \rightarrow g(i-m) fi \rangle$.

(c) $\langle i : i \in (\mathbb{N}, \geq) : f(i) \rangle \circlearrowleft \langle i : i \geq 1 : f(i) \rangle$

$= \langle i : i \in (\mathbb{Z}, \leq) : f(|i|) \rangle$.

(d) $\langle i : f(i) \rangle \circlearrowleft \langle 2 \rangle$

$= \langle i : i \in (\omega+1, \leq) : if i < \omega \rightarrow f(i) \land i = \omega \rightarrow 2 fi \rangle$. 
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\( (e) \quad <1, 2, 5> ++ [2, 2] \) is the partially ordered construct

\[
\begin{array}{c}
1 \\
2 \\
5 \\
\end{array}
\]

\[
\begin{array}{c}
2 \\
\end{array}
\]

Concatenation of constructs reduces to the normal concatenation operator when the constructs are sequences. There is also an operation on constructs that generalizes the normal operator on bags:

**DEFINITION.** Let \( A \) and \( B \) be ordered sets. The sum of \( A \) and \( B \), denoted \( A + B \), is the ordered set \((V_{A+B}, \leq_{A+B})\), where \( V_{A+B} = V_{A++B} \) and

\[
(i_0, c_0) \leq_{A+B} (i_1, c_1) \iff (i_0 = i_1 = 0 \land c_0 \leq c_1) \lor (i_0 = i_1 = 1 \land c_0 \not\leq c_1)
\]

**DEFINITION.** Consider constructs \( C(f) \) and \( C(g) \), where \( f \in A \to X \) and \( g \in B \to Y \). The sum of \( C(f) \) and \( C(g) \), denoted \( C(f) + C(g) \), is the construct \( C(h) \), where \( h \in (A + B) \to X \cup Y \) is defined by (1).

**THEOREM 6** ("domain split law"). For \( f \) defined on \( A + B \),

\[
(C \ i : i \in A + B : f(i)) = (C \ i : i \in A : f(0, i)) + (C \ i : i \in B : f(1, i))
\]

In the final part of this section, we explore the relationship between ++ and +.

**DEFINITION.** Let \( A \) be an ordered set. Consider \((\leq_A, f) \in A \to X\). Define

\[
\text{Bag}(\leq_A, f) = C(=, f)
\]

i.e., the construct that corresponds to \( f \) considered as a mapping defined on
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a discretely ordered set.

THEOREM 7. For \( f, g \) in \( F \),

\[
C(f) = C(g) \Rightarrow \text{Bag}(f) = \text{Bag}(g)
\]

PROOF. Let \( \sigma \) be an isomorphism from \( A \) onto \( B \). Then \( \sigma \) is injective, so for \( a_0, a_1 \) in \( A \),

\[
a_0 = a_1 = \sigma(a_0) = \sigma(a_1)
\]

Therefore \( \sigma \) is an isomorphism from \( (V_A, =) \) onto \( (V_B, =) \).

COROLLARY. We can define \( \text{BAG}(CO) \) for every construct \( CO \) in the following way: choose an \( f \) such that \( CO = C(f) \) and let \( \text{BAG}(CO) = \text{Bag}(f) \). The theorem ensures that the result is independent of the choice of \( f \). Every bag \( B \) can be obtained in this way, since \( \text{BAG}(B) = B \).

THEOREM 8. For constructs \( CO, CI \),

\[
\text{BAG}(CO ++ CI) = \text{BAG}(CO) + \text{BAG}(CI)
\]

PROOF. Write \( CO = C(s_A, f) \), where \( f \in V_A \to X \), and \( CI = C(s_B, g) \), where \( g \in B \to Y \). According to the definition of ++, we have \( CO ++ CI = C(s_{A+B}, h) \), where \( h \) is defined by (1). Therefore

\[
\text{BAG}(CO ++ CI) = \{ \text{definition of BAG} \}
\]

\[
\text{Bag}(s_{A+B}, h)
\]

\[
= \{ \text{definition of Bag} \}
\]

\[
C(=, h)
\]

\[
= \{ \text{definition of +} \}
\]

\[
C(=, f) + C(=, g)
\]

\[
= \{ \text{definition of Bag} \}
\]

\[
\text{Bag}(s_A, f) + \text{Bag}(s_B, g)
\]
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= {definition of BAG }

\[ \text{BAG}(C_0) + \text{BAG}(C_1) \].

 Canonical decomposition

We shall call a construct \( C(f) \), with \( f \in A \to X \), empty if \( A \) is empty. It follows that there is only one empty construct, namely \( \{\emptyset\} \).

We shall call a construct irreducible if it cannot be written as the concatenation or sum of nonempty constructs. Clearly, every finite construct may be built from irreducible constructs with the operators ++ and +. The question naturally occurs whether this decomposition is in some sense unique. Obviously, it cannot be completely unique, since both ++ and + are associative and + is commutative (as operators between constructs; they are so up to isomorphism as operators between ordered sets). Therefore, we must allow rearrangement of terms and introduction of parentheses that exploit these properties; for instance, \([2] + ([1] + [0]) = [0] + [1] + [2]\). However, apart from this the decomposition turns out to be unique. The next three theorems have the purpose of establishing this.

THEOREM 9. There are no nonempty constructs \( C_0, C_1, C_2, C_3 \) with

\[ C_0 ++ C_1 = C_2 + C_3 \].

PROOF. For elements \( a \) and \( b \) of an ordered set \( A \) we define

\[ a \leftrightarrow A b = (E x: x \in A: (x \leq A a \lor x \leq A b) \land (x \leq A b \lor x \geq A a)) \].

Now ++ has the property that for nonempty ordered sets \( A \) and \( B \),

\[ (A, a, b: a \in A ++ B \land b \in A ++ B: a \leftrightarrow A b) \].

This can be seen in the following way. If \( a \) and \( b \) are both in the part of \( A ++ B \) that corresponds to \( A \), every element \( x \) of the part that corresponds to \( B \) is comparable to both \( a \) and \( b \). Similarly if they are both
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in the other part. If they are in different parts, they are themselves comparable and we may take the \( x \) in the definition of \( \leftrightarrow_A \) to be \( a \) or \( b \).

On the other hand, \( + \) has the property that for nonempty ordered sets \( A \) and \( B \),

\[(a, b : a \in A + B \land b \in A + B : \neg a \leftrightarrow_A b) .\]

To see this, it suffices to take \( a \) in the part of \( A + B \) that corresponds to \( A \) and \( b \) in the part that corresponds to \( B \).

Therefore no ordered set can be written nontrivially as both a concatenation and a sum; it follows that no construct can either.

THEOREM 10. If \( C_0, C_1, C_2, C_3 \) are constructs such that

\[ C_0 + C_1 = C_2 + C_3 , \]

there exists a construct \( C_4 \) such that

\[ C_0 = C_2 + C_4 \lor C_2 = C_0 + C_4 . \]

PROOF. Let \( C_0 + C_1 \) be \( C(f) \) with \( f \in A \rightarrow X \). Let \( A_0, A_1, A_2, A_3 \) denote the parts of \( A \) corresponding to \( C_0, C_1, C_2, C_3 \) respectively. For \( a \in A_0 \setminus A_2 \) and \( b \in A_2 \setminus A_0 \) we have \( a \not<_A b \) since \( C(f) = C_0 + C_1 \), and \( a >_A b \) since \( C(f) = C_2 + C_3 \). These are mutually contradictory, so

\[A_0 \setminus A_2 = \emptyset \lor A_2 \setminus A_0 = \emptyset .\]

We only consider the case \( A_0 \setminus A_2 = \emptyset \); the other one is symmetric. Since in this case \( A_0 = A_0 \cap A_2 \), it follows that \( C_2 = C_0 + C(f \mid A_2 \setminus A_0) . \)

THEOREM 11. Suppose that \( C_0 \) is a nonempty construct that is not the sum of two nonempty constructs. If \( C_1, C_2, C_3 \) are constructs such that
(1) \[ C_0 + C_1 = C_2 + C_3 \]

there exists a construct \( C_4 \) such that

\[ C_2 = C_0 + C_4 \vee C_3 = C_0 + C_4 \]

**Proof.** Let \( A_0 \) through \( A_3 \) be defined as in the preceding proof. Obviously,

\[ A_0 \cap A_2 \neq \emptyset \vee A_0 \cap A_3 \neq \emptyset \]

We assume the first disjunct; the other one is treated by exchanging the roles of \( C_2 \) and \( C_3 \).

If \( A_0 \setminus A_2 \neq \emptyset \), we have \( C_0 = C(f \mid A_0 \cap A_2) + C(f \mid A_0 \setminus A_2) \), where both terms are nonempty. Since it is given that \( C_0 \) is not such a sum, it follows that \( A_0 \setminus A_2 = \emptyset \). Therefore \( A_0 = A_0 \cap A_2 \), so \( C_2 = C_0 + C(f \mid A_2 \setminus A_0) \).

**Corollary.** Decomposition of a finite construct into irreducible constructs by means of the operators ++ and + is unique except for the trivial variations that correspond to the associativity of both operators and the commutativity of +.

**Example 5.** Let \( A \) denote the set of words of length at most 2 over the alphabet \( \{0, 1, 2\} \). For words \( t \) and \( u \) we let \( t \preceq_A u \) mean that \( t \) is a prefix of \( u \). Define \( f \) on \( A \) as the sum of the numerical values of the symbols in a word. We claim that \( C(f) \) is a proper abstraction for the ternary tree

![Ternary Tree Diagram]
Note that the left-right ordering of the successors of any given node is purely arbitrary: interchanging, for instance, the leftmost 0 and 1 on the bottom row would not change $C(f)$.

It is our intention to give the name "tree" to a construct of the kind considered in Example 5. We shall now proceed to formalize the definition and explore the relationship between those constructs and the concept of a tree as it is known from graph theory.

**Definition.** Let $A$ be an ordered set. The relation $\langle A \rangle$ is defined on $A$ by

$$a \langle A \rangle b = a \langle A \rangle b \land \neg (\exists x: x \in A: a \langle A \rangle x \langle A \rangle b).$$

If $a$ and $b$ satisfy $a \langle A \rangle b$, we say that $a$ is a predecessor of $b$, and that $b$ is a successor of $a$.

**Definition.** An ordered set $A$ is called tree-like if $A$ is finite and there exists an $r$ in $A$ such that for all $a$ in $A$,

$$(\exists x: x \in A: x \langle A \rangle a) = \text{if } a = r \rightarrow 0 \| a \neq r \rightarrow 1 \text{ fi}.$$

If such an $r$ exists, it is obviously unique; we call it the root of $A$.

**Theorem 12.** For every tree-like ordered set $A$,

$$(\forall x, y: x \in A \land y \in A: x \langle A \rangle y) = |V_A| - 1.$$

**Proof.** Let $A$ be tree-like with root $r$. Then

$$(\forall x, y: x \in A \land y \in A: x \langle A \rangle y) = \{\text{domain split}\}$$

$$(\exists y: y \in A: (\forall x: x \in A: x \langle A \rangle y)).$$
THEOREM 13. In a tree-like ordered set, the root is the least element.
PROOF. Let $A$ be tree-like with root $r$. Define $d \in A \to \mathbb{N}$ by

$$d(a) = (\forall x : x \in A : x \prec_A a) .$$

The claim that every $a$ in $A$ satisfies $r \preceq_A a$ will be proved by induction on $d(a)$. If $d(a) = 0$, we deduce from $x \prec_A a$ that $a = r$ and therefore, by reflexivity, $r \preceq_A a$. If $d(a) > 0$, there is an element $b$ in $A$ with $b \prec_A a$. Now

$$d(a)$$

$$= \{\text{definition of } d\}$$

$$(\forall x : x \in A : x \prec_A a)$$

$$= \{\text{domain split}\}$$

$$(\forall x : x \in A \land x \prec_A b : x \prec_A a) + (\forall x : x \in A \land \neg x \prec_A b : x \prec_A a)$$

$$= \{ b \prec_A a \ , \ \text{so} \ x \prec_A b = x \prec_A a \}$$

$$= (\forall x : x \in A : x \prec_A b) + (\forall x : x \in A \land \neg x \prec_A b : x \prec_A a)$$

$$= \{ \neg b \prec_A b \ \text{and} \ b \prec_A a \}$$

$$= (\forall x : x \in A : x \prec_A b)$$

$$= \{\text{definition of } d\}$$

$d(b)$.

Applying the induction hypothesis to $b$, we obtain $r \preceq_A b$. Hence, by transitivity, $r \preceq_A a$. ■
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DEFINITION. For any tree-like ordered set $A$ with root $r$, the \textit{predecessor function} $p_A$ is determined by $p_A \in A \setminus \{r\} \to A$ and $p_A(a) (\preceq_A) a$ for all $a$ in $A \setminus \{r\}$.

THEOREM 14. For all $a$ and $b$ in a tree-like ordered set $A$,

$$a \preceq_A b = (\exists i : i \in N : a = p_A^i(b)) .$$

PROOF. Define $d \in A \times A \to N$ by

$$d(a, b) = (\exists x : x \in A : a \prec_A x \preceq_A b) .$$

We shall prove the implication from left to right by induction on $d(a, b)$. The theorem then follows by observing that the other implication is a direct consequence of the transitivity of $\preceq_A$.

Base: assume $a \preceq_A b$. Then

$$d(a, b) = 0 = (\exists x : x \in A : a \prec_A x \preceq_A b) = (a \preceq_A b = a = b \lor a \prec_A b) = (a = b \lor a = p_A(b)) = (\exists i : i \in N : a = p_A^i(b)) .$$

Step: suppose $d(a, b) > 0$. Then there exists an element $c$ of $A$ with $a \prec_A c \preceq_A b$. As $d(a, c) < d(a, b)$, the induction hypothesis gives an $i$ with $a = p_A^i(c)$. As also $d(c, b) < d(a, b)$, the induction hypothesis gives a $j$ with $c = p_A^j(b)$. Therefore $a = p_A^{i+j}(b)$.

Given an ordered set $A$, we may construct an undirected graph $G$ (to be referred to as "the graph of $A"$) as follows. Vertices of $G$ are the
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elements of \( V_A \); there is an edge between \( a \) and \( b \) iff \( a (\prec_A) b \lor b (\prec_A) a \). If \( A \) is tree-like, Theorems 13 and 14 together imply that \( G \) is connected (since any two points are connected by way of the root). Moreover, by Theorem 12, the number of edges of \( G \) is 1 less than the number of vertices. This establishes that \( G \) is a tree in the sense of graph theory. The next theorem shows that every graph-theoretical tree can be obtained in this way.

**Theorem 15.** Let \( G \) be a connected undirected graph of which the number of edges is 1 less than the number of vertices. Then there exists a tree-like ordered set \( A \) such that \( G \) is the graph of \( A \).

**Proof.** For \( V_A \), we take the collection of vertices of \( G \). A root \( r \) is chosen arbitrarily in \( V_A \). It is known from graph theory (e.g. [1], Theorem 4.1) that \( G \) is free of cycles; therefore, for every \( a \) in \( V_A \) there exists precisely one path from \( a \) to \( r \). Let \( a \preceq_A b \) mean that \( a \) occurs on the path from \( b \) to \( r \). Then \( A = (V_A, \preceq_A) \) is an ordered set.

Moreover, \( a (\prec_A) b \) now has the meaning that \( a \) is the first node after \( b \) on the path from \( b \) to \( r \); therefore \( a (\prec_A) b \lor b (\prec_A) a \) implies the existence of an edge between \( a \) and \( b \).

Conversely, let \( a \) and \( b \) be connected by an edge of \( G \). If \( a \preceq_A b \), the node \( a \) does not occur on the path from \( b \) to \( r \). Then the edge from \( a \) to \( b \), followed by the path from \( b \) to \( r \), constitutes a path from \( a \) to \( r \) on which \( b \) occurs as the first node after \( a \). Hence \( b (\prec_A) a \). We have now proved that

\[ a \preceq_A b \lor b (\prec_A) a \quad . \]

By symmetry, the same formula holds with \( a \) and \( b \) interchanged. Distributing \( \land \) over \( \lor \) yields

\[ a (\prec_A) b \lor b (\prec_A) a \quad . \]

The last two paragraphs together show that two nodes are connected by an edge iff one precedes the other in the ordering of \( A \). Therefore \( G \) is the
graph of $A$.  

REMARK. The difference in style between this proof and the preceding ones is one of the reasons why we have chosen to define constructs in the language of ordered sets rather than that of graphs.

The idea of a tree as we shall now define it is slightly different from the concept in graph theory. By a tree, we shall mean an arrangement of (not necessarily distinct) values; isomorphic trees that have the same values in corresponding places will be identified. The following definition is intended to reflect this point of view.

DEFINITION. A tree is a construct $C(f)$, say with $f \in A \rightarrow X$, such that $A$ is tree-like and $f$ is surjective.

EXAMPLE 6. In Example 5, a tree was introduced as the construct $C(f)$ of an explicitly given function $f$. However, it could also have been described with the aid of the operators $++$ and $+$, since the tree involved is exactly equal to

$$[0] ++ ([0] ++ ([0] + [1] + [2]))$$
$$+ ([1] ++ ([1] + [2] + [3]))$$
$$+ ([2] ++ ([2] + [3] + [4]))$$

This is no accident: in the rest of this section we shall show that every tree can be decomposed in this way.

THEOREM 16. Every tree may be built from singleton bags with the operators $++$ and $+$.

PROOF. Consider a tree $C(f)$, where $f \in A \rightarrow X$. Let $r$ be the root of $A$. We prove the statement by induction on the number of elements of $A$. If $|A| = 1$, we have $C(f) = [f(r)]$. If $|A| > 1$, consider

$$B = \{ b \in A \mid r (\langle A \rangle b) \}.$$
We shall prove that

\[(2) \quad C(f) = [f(r)]^+ \iff (\exists b : b \in B : (\forall a : a \in T(b) : f(a)))\]

where \(\exists\) denotes the quantifier corresponding to the operator \(\oplus\) and where

\[T(b) = \{a \in A \mid b \leq_A a, \leq_A\}\]

Note that

\[T(b) \text{ is tree-like with root } b\]

\[= \{\text{definition of tree-like}\}\]

\[\forall a : a \in T(b) : (N x : x \in T(b) : x \langle_A a)\]

\[= \{\text{if } a = b \rightarrow 0 \mid a \neq b \rightarrow 1 \} fi\]

\[= \{\text{definition of } p_A\}\]

\[\forall a : a \in T(b) : (N x : x \in T(b) : x = p_A(a))\]

\[= \{\text{if } a = b \rightarrow 0 \mid a \neq b \rightarrow 1 \} fi\]

\[= \{\text{one-point rule for } N\}\]

\[\forall a : a \in T(b) : p_A(a) \in T(b) \iff a \neq b\]

\[= \{\text{definition of } T\}\]

\[\forall a : b \leq_A a : b \leq_A p_A(a) \iff a \neq b\]

\[= \{\text{domain split}\}\]

\[\forall a : b \langle_A a : b \leq_A p_A(a)\} \land \neg b \leq_A p_A(b)\]

\[= \{p_A(b) \langle_A b, \text{ antisymmetry}\}\]

\[\forall a : b \langle_A a : b \leq_A p_A(a)\}

\[= \{\text{Theorem 14}\}\]

\[\forall a : (E i : i > 0 : b = p_A^i(a)) : (E i : i \geq 0 : b = p_A^i(p_A(a)))\]

\[= \{\text{true}\}\]

As \(|T(b)| < |A|\) for all \(b\) in \(B\), the theorem then follows by induction.
It remains to prove (2). Due to Theorem 13, it is sufficient to prove that for $a_0, a_1$ in $A \setminus \{r\}$ and $b_0, b_1$ in $B$,

\begin{align*}
(3) \quad & a_0 \preceq_A a_1 \Rightarrow (\exists b: b \in B: a_0 \in T(b) \land a_1 \in T(b)) \quad ,
(4) \quad & b_0 \neq b_1 \Rightarrow T(b_0) \cap T(b_1) = \emptyset .
\end{align*}

Proof of (3):

$$
a_0 \preceq_A a_1 = \{\text{Theorem 13, } a_0 \neq r \}$$
$$a_0 \preceq_A a_1 \land r \prec_A a_0$$
$$= \{\text{Theorem 14} \}$$
$$a_0 \preceq_A a_1 \land (\exists i: i > 0: r = p_A^i(a_0))$$
$$= \{\text{dummy transformation, } b:=p_A^{-1}(a_0) \}$$
$$a_0 \preceq_A a_1 \land (\exists b: b \in A: r \prec_A b \land b \preceq_A a_0)$$
$$= \{\text{transitivity} \}$$
$$= (\exists b: b \in A: r \prec_A b \land b \preceq_A a_0 \land b \preceq_A a_1)$$
$$= \{\text{definitions of } B \text{ and } T \}$$
$$= (\exists b: b \in B: a_0 \in T(b) \land a_1 \in T(b)) .$$

Proof of (4):

$$a \in T(b_0) \land a \in T(b_1)$$
$$= \{\text{definition of } T \}$$
$$b_0 \preceq_A a \land b_1 \preceq_A a$$
$$= \{\text{Theorem 14} \}$$
$$= \{k:= \lfloor |j-i| \rfloor \}$$
$$= \{p_A(b_0) = r , \quad p_A(b_1) = r , \quad r \text{ not in the domain of } p_A \}$$
$$= \{ b_0 = b_1 \lor b_0 = r \lor b_1 = r \}$$
$$= \{ b_0 \in B , \text{ so } b_0 \neq r ; b_1 \in B , \text{ so } b_1 \neq r \}$$
$$= \{ b_0 = b_1 \} \quad .$$
Theorem 16 establishes that every tree may be decomposed into singleton bags. Since these are obviously irreducible, the corollary to Theorem 15 shows that this decomposition is essentially unique.

REMARK. It is not true that every finite construct can be decomposed into singleton bags. As a counterexample, consider the ordered set \( A \) consisting of \( \{2, 3, 6, 9\} \) with the divisibility ordering. If \( f \) is the identity function on \( A \), then \( C(f) \) is the construct

\[
\begin{array}{c}
2 \\
\searrow \\
3 \\
\nearrow \\
6 \\
\searrow \\
9
\end{array}
\]

This is not a concatenation, since we have, in the notation of the proof of Theorem 9, \( \sim 2 \leadsto 9 \). Neither is it a sum, since its graph is connected.

**Structural induction**

Let \( X \) be a set and let \( \text{Co}(X) \) denote the set of all constructs that can be built from singleton bags over \( X \) by means of ++ and +. Then all finite nonempty sequences, bags and trees with values from \( X \) belong to \( \text{Co}(X) \). Due to the essential uniqueness of the canonical decomposition, we may define functions on \( \text{Co}(X) \) by describing their values on singleton bags and their behaviour under ++ and +.

**DEFINITION.** Let \( \$ \) denote a commutative, associative operator on \( X \). Then \( \$/ \) is the \( X \)-valued function on \( \text{Co}(X) \) that satisfies

\[
\begin{align*}
\$/[x] &= x \text{ for all } x \text{ in } X, \\
\$/ (C_0 ++ C_1) &= (\$/C_0) \$ (\$/C_1) \text{ for all } C_0, C_1 \text{ in } \text{Co}(X), \\
\$/ (C_0 + C_1) &= (\$/C_0) \$ (\$/C_1) \text{ for all } C_0, C_1 \text{ in } \text{Co}(X).
\end{align*}
\]

(In [0] and [2] the symbol / was introduced for sequences and bags separate-
ly. The pronunciation of / is variously reported as "reduce" [0] or "inserted in" [2].)

EXAMPLE 7. If * denotes multiplication in Z, then

(a) */[2, 3, 5] = */([2] ++ [3] ++ [5]) = 2 * 3 * 5 = 30 ;
(b) */([2, 3, 5] = */([2] + [3] + [5]) = 2 * 3 * 5 = 30 ;
(c) */([2, 3] ++ [2, 3, 5])

EXAMPLE 8. Many familiar quantifiers can be defined as the application of a function of the form $\Sigma f$ on Co(X). For instance,

\[
\begin{aligned}
+/(C i: i \in A: x_i) &= (\Sigma i: i \in A: x_i) \\
+/(C i: i \in A: [x_i]) &= (\Sigma i: i \in A: x_i) \\
+/([C i: i \in A: x_i]) &= (\Sigma i: i \in A: x_i) \\
\min/(C i: i \in A: x_i) &= (\min i: i \in A: x_i) \\
\max/(C i: i \in A: x_i) &= (\max i: i \in A: x_i) \\
\wedge/(C i: i \in A: b_i) &= (\wedge i: i \in A: b_i) \\
\vee/(C i: i \in A: b_i) &= (\vee i: i \in A: b_i)
\end{aligned}
\]

DEFINITION. For $f \in X \rightarrow Y$ the mapping $f^* \in \text{Co}(X) \rightarrow \text{Co}(Y)$ is defined by

\[
\begin{aligned}
f^*[x] &= [f(x)] \text{ for all } x \text{ in } X, \\
f^*(C_0 ++ C_1) &= (f^*C_0) ++ (f^*C_1) \text{ for all } C_0, C_1 \text{ in } \text{Co}(X), \\
(f^*(C_0 + C_1) &= (f^*C_0) + (f^*C_1) \text{ for all } C_0, C_1 \text{ in } \text{Co}(X). \\
\end{aligned}
\]

(See again [0] and [2] for sequences and bags separately. The pronunciation is "map" [0] or "applied to all" [2].)

EXAMPLE 9. Let $f \in Z \rightarrow Z$ be defined by $f(x) = x^2$. Then

(a) $f^*[2, 3, 5] = [4, 9, 25]$ ;
(b) $f^*[2, 3, 5] = [4, 9, 25]$ ;
(c) $f^*([2, 3] ++ [2, 3, 5]) = [4, 9] ++ [4, 9, 25]$.
EXAMPLE 10. Let \([\square] \in X \rightarrow \text{Co}(X)\) be defined by \([\square](x) = [x]\). Then

\[
\text{BAG } \mid \text{Co}(X) = +/ \cdot [\square]^* 
\]

Heaps and search trees

We would like to use the word "heap" to refer to trees in which the values occurring on any root path are ascending. It is very easy to phrase this in the language of constructs:

**DEFINITION.** Let \(A\) be a tree-like ordered set, \(B\) a linearly ordered set, \(f \in V_A \rightarrow V_B\). The tree \(C(\leq_A, f)\) is called a heap with respect to \(\leq_B\) if for every \(a_0\) and \(a_1\) in \(A\),

\[
a_0 \leq_A a_1 \Rightarrow f(a_0) \leq_B f(a_1) \quad .
\]

The modelling of search trees presents much more difficulties, mainly because the definition of tree is such that the successors of a node cannot be distinguished.

**DEFINITION.** Let \(A\) be a tree-like ordered set, \(B\) a linearly ordered set, \(f \in V_A \rightarrow V_B\). An element \(a\) of \(A\) is called left with respect to \((\leq_A, \leq_B, f)\) if \(a\) is not the root of \(A\) and

\[
(\forall x : a \leq_A x : f(x) <_B f(p_A(a))) 
\]

An element \(a\) is called right with respect to \((\leq_A, \leq_B, f)\) if \(a\) is left with respect to \((\leq_A, \geq_B, f)\).

**THEOREM 17.** Let \(A\) be a tree-like ordered set, \(B\) a linearly ordered set, \(f \in V_A \rightarrow V_B\). Assume that

\[
(5) \quad \text{every element of } A \text{ except the root is either left or right} .
\]
Then the following conditions are equivalent:

(6) every element of \( A \) has at most one left successor and at most one right successor;

(7) for all \( a, b, c \) in \( A \) such that \( b \) and \( c \) are different successors of \( a \),

\[
 f(b) \preceq_B f(c) \Rightarrow f(b) \preceq_B f(a) \preceq_B f(c)
\]

(Here "left" and "right" are with respect to \((\preceq_A, \preceq_B, f)\).)

PROOF. Assume (6). Take \( a, b, c \) in \( A \) such that \( b \) and \( c \) are different successors of \( a \) with \( f(b) \preceq_B f(c) \). Suppose \( f(a) \preceq_B f(b) \). Then \( b \) is not left and by transitivity of \( \preceq_B \) we have \( f(a) \preceq_B f(c) \), so \( c \) is not left either. From (5) we conclude that \( b \) and \( c \) are both right. This contradicts (6), so \( -f(a) \preceq_B f(b) \). From the linearity of \( \preceq_B \) we see that \( f(b) \preceq_B f(a) \). Similarly we get \( f(a) \preceq_B f(c) \).

Now assume (7). Let \( b \) and \( c \) be different successors of \( a \). As \( \preceq_B \) is linear, \( f(b) \) and \( f(c) \) are comparable, say \( f(b) \preceq_B f(c) \). From (7) we see that \( f(b) \preceq_B f(a) \), so \( b \) is not right, and that \( f(a) \preceq_B f(c) \), so \( c \) is not left.

If \( A, B \) and \( f \) have properties (5) through (7), while \((\preceq_A, f) \cong (\preceq_A, f_1)\), then (5) through (7) also hold with \( A \) and \( f \) replaced by \( A_1 \) and \( f_1 \). Therefore the following definition makes sense.

DEFINITION. Let \( A \) be a tree-like ordered set, \( B \) a linearly ordered set, \( f \in V_A \to V_B \). The tree \( C(\preceq_A, f) \) is called a search tree with respect to \( \preceq_B \) if (5) and (6) (or, equivalently, (5) and (7)) hold.
References


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A. Bijlsma
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