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THE CURVATURE OF CONJUGATE PROFILES IN POINTS OF CONTACT

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The curvature of conjugate profiles in points of contact

0. Introduction and summary
Consider a point of contact on conjugate tooth profiles of the pinion and the gear in a gear pair. The following quantities with respect to the curvature of the conjugate profiles in the point of contact are of practical interest:

- the principal curvatures of the conjugate profiles in points of contact;
- the principal directions of the conjugate profiles in points of contact, in particular the angle between corresponding principal directions on the conjugate profiles;
- the direction of the curve of contact in points of contact.

The principal curvatures at a point on a surface are the real eigenvalues of a symmetric (linear) operator on the tangent space of the surface at the point under consideration. The principal directions at a point on a surface are the mutually perpendicular eigenvectors of this symmetric operator. In Section 1, we give a survey of the theory of the curvature of surfaces, using elementary linear algebra. The results can be found in books on differential geometry, e.g. [1], [2], [5]. The conjugate profiles in a gear pair are in contact along a curve of contact. In Section 1 we discuss the curvature of two bodies in the points on their curve of contact. We present formulas for the angle between the corresponding principal directions of the boundary surfaces of the bodies on the curve of contact and for the angle between the direction of the curve of contact and the principal direction of one of the boundary surfaces corresponding to its smaller principal curvature.

Consider a point on the curve of contact of two conjugate tooth profiles in a gear pair. The tangent spaces at this point of contact on the conjugate profiles coincide. Hence, there are two symmetric operators $B_1$ and $B_2$, both acting on the common tangent space of the conjugate profiles at the point of contact. The real eigenvalues of $B_1$ ($B_2$) are the principal curvatures of profile 1(2) and the corresponding eigenvectors are in the direction of the principal directions of profile 1(2). The symmetric operator $B = B_1 + B_2$ is non-negative and has an eigenvalue zero. The eigenvector of $B$ with eigenvalue zero is in the direction of the curve of contact. The positive eigenvalue of $B$ is the relative curvature of the conjugate profiles in the point of contact under consideration. The above considerations constitute the starting point of a new and efficient algorithm for the calculation of the curvature of conjugate profiles in points of contact for the general gear pair. We describe this algorithm in Section 2.
1. The curvature of surfaces

Let $S$ be a surface in three-dimensional space. The linear space of vectors from a point $P$ on $S$ parallel to the tangent plane of $S$ at $P$ is referred to as the tangent space $T(P)$ of $S$ at $P$. The unit normal vector on $S$ at the point $P$ is denoted by $\mathbf{n}(P)$. In every point on $S$ there exist two unit normal vectors, and therefore we make a choice such that the vector $\mathbf{n}(P)$ depends continuously on the point $P$ on $S$.

Let $\mathbf{t} \in T(P)$ be a unit tangent vector of $S$ at $P$ and let $C$ be a curve on $S$ through $P$ such that $\mathbf{t}$ is a tangent vector of $C$ at $P$. Take $s$ for the arc length parameter on $C$ such that $P$ corresponds to $s = 0$ and such that $s$ is increasing in the direction of $\mathbf{t}$. We consider five vector fields on the curve $C$:

\begin{align*}
\mathbf{t}(s) &= \text{unit tangent vector of } C \text{ at the point corresponding to } s \text{ and in the direction of increasing } s; \\
\mathbf{t}_s(s) &= \frac{d\mathbf{t}(s)}{ds}; \\
\mathbf{n}(s) &= \text{unit normal vector on } S \text{ at the point on } C \text{ corresponding to } s; \\
\mathbf{n}_s(s) &= \frac{d\mathbf{n}(s)}{ds}; \\
\mathbf{n}^*(s) &= \text{principal normal on } C \text{ at the point corresponding to } s.
\end{align*}

The curvature $\kappa(s)$ of $C$ at the point on $C$ corresponding to $s$, satisfies the Frenet formula

$$
\mathbf{t}_s(s) = \kappa(s) \mathbf{n}^*(s).
$$

Later in this section we shall show, that there exists a symmetric (linear) operator $B$ on the tangent space $T(P)$ of $S$ at $P$, such that

$$
\mathbf{n}_s(0) = -B \mathbf{t}.
$$

Differentiation of the relation $(\mathbf{t}(s), \mathbf{n}(s)) = 0$ yields

$$
(\mathbf{t}_s(s), \mathbf{n}(s)) + (\mathbf{t}(s), \mathbf{n}_s(s)) = 0.
$$

Use the Frenet formula (1.2) to obtain

$$
\kappa(s) (\mathbf{n}^*(s), \mathbf{n}(s)) = -(\mathbf{t}(s), \mathbf{n}_s(s)).
$$

Substitution of $s = 0$ in (1.4) yields
\( (1.5) \quad \kappa(P; C) = \frac{\langle t, B t \rangle}{\cos(\Omega(P; C))}, \)

where \( B \) is the symmetric operator on \( T(P) \) in \( (1.3) \) and

\( \kappa(P; C) = \text{curvature of } C \text{ at } P , \)

\( n^*(P; C) = \text{principal normal on } C \text{ at } P , \)

\( n(P) = \text{normal on } S \text{ at } P . \)

Let \( \Omega(P; C) \) be the angle between \( n^*(P; C) \) and \( n(P) \). From \( (1.5) \) we get

\( (1.6) \quad \kappa(P; C) = \frac{\langle t, B t \rangle}{\cos(\Omega(P; C))} . \)

The plane through \( P \) parallel to the tangent vector \( t \) and the normal vector \( n(P) \) is denoted by \( N(P; t) \). The curve of intersection of the plane \( N(P; t) \) and the surface \( S \) is referred to as the normal section through \( P \) in the direction \( t \). The curvature of the normal section at \( P \) in the direction \( t \) is called the normal curvature \( \kappa(P; t) \) of the surface \( S \) at the point \( P \) in the direction \( t \). If the normal vector \( n(P) \) is in the direction of the center of curvature of the normal section, then the normal curvature \( \kappa(P; t) \) is positive, otherwise negative. If the normal curvature \( \kappa(P; t) \) is positive, then \( n(P) \) and the principal normal on the normal section at \( P \) coincide, otherwise they have opposite directions. Hence, by use of \( (1.6) \), we get

\( (1.7) \quad \kappa(P; t) = \frac{\langle t, B t \rangle}{\cos(\Omega(P; C))} . \)

The linear operator \( B \) on \( T(P) \) is symmetric. Hence, there exist two mutually perpendicular unit vectors \( t_1(P) \in T(P) \) and \( t_2(P) \in T(P) \) such that

\( (1.8) \quad \kappa_1(P) := \kappa(P; t) \leq \kappa(P; t_1) \leq \kappa(P; t_2) =: \kappa_2(P) . \)

The normal curvatures \( \kappa_1(P) \) and \( \kappa_2(P) \) in \( (1.8) \) are said to be the principal curvatures of the surface \( S \) at the point \( P \), and the corresponding tangent vectors \( t_1(P) \) and \( t_2(P) \) the principal directions of the surface \( S \) at the point \( P \). The principal curvatures \( \kappa_1(P) \) and \( \kappa_2(P) \) are the real eigenvalues of the symmetric operator \( B \) on \( T(P) \). The principal directions \( t_1(P) \) and \( t_2(P) \) are the corresponding mutually perpendicular eigenvectors of the symmetric operator \( B \) on \( T(P) \).

Let \( 0 \leq \Theta(t) < 2\pi \) be such that
Substitution of (1.9) in (1.7) yields

\[ \kappa(P; t) = \kappa_1(P) \cos^2(\Theta(t)) + \kappa_2(P) \sin^2(\Theta(t)) . \]

Equation (1.10) is known as the equation of Euler. The curvature of the surface \( S \) at the point \( P \) is often described in terms of

\[ \bar{\kappa}(P) := \kappa_1(P) + \kappa_2(P), \]

the mean curvature of \( S \) at \( P \).

\[ \hat{\kappa}(P) := \kappa_1(P) \kappa_2(P), \]

the total (Gaussian) curvature of \( S \) at \( P \).

Hence,

\[ \bar{\kappa} = \kappa_1 + \kappa_2 = \text{trace}(B), \]

\[ \hat{\kappa} = \kappa_1 \kappa_2 = \det(B), \]

\[ \kappa_1 = \frac{(\bar{\kappa} - (\bar{\kappa}^2 - 4\hat{\kappa})^{\frac{1}{2}})}{2}, \]

\[ \kappa_2 = \frac{(\bar{\kappa} + (\bar{\kappa}^2 - 4\hat{\kappa})^{\frac{1}{2}})}{2}. \]

We now discuss the symmetric operator \( B \) on \( T(P) \) referred to in (1.3). For ease of calculation we introduce a right-handed cartesian coordinate system. The corresponding coordinates are denoted by \((x, y, z) = \bar{x}\). The surface \( S \) is specified by the parametric equations, with parameters \( \lambda \) and \( \mu \),

\[ \bar{x} = \bar{x}(\lambda, \mu) = (x(\lambda, \mu), y(\lambda, \mu), z(\lambda, \mu)) . \]

We suppose that the tangent vectors \( \bar{x}_\lambda := \partial \bar{x}/\partial \lambda \) and \( \bar{x}_\mu := \partial \bar{x}/\partial \mu \) are linearly independent at every point on \( S \). Hence, the vectors \( \bar{x}_\lambda \) and \( \bar{x}_\mu \) are the elements of a basis in the tangent space \( T(P) \) of \( S \) at \( P \). For \( \bar{u} \in T(P) \) we write \( \bar{u} = \alpha \bar{x}_\lambda + \beta \bar{x}_\mu \), and we introduce the \((2 \times 1)-\)matrix \( \hat{\bar{u}} \) corresponding to the vector \( \bar{u} \in T(P) \)

\[ \hat{\bar{u}} := [\alpha, \beta]^T , \]

where \([\alpha, \beta]^T\) is the transpose of \([\alpha, \beta]\). For \( \bar{u} \in T(P) \), the matrix \( \hat{\bar{u}} \) depends on the parameterization \((\lambda, \mu)\) of the surface \( S \). Let \((\lambda', \mu')\) be another parameterization of the surface \( S \). The parameters \( \lambda \) and \( \mu \) are invertible functions of the parameters \( \lambda' \) and \( \mu' \), say

\[ \lambda = \lambda(\lambda', \mu'), \]

\[ \mu = \mu(\lambda', \mu') . \]

The \((2 \times 1)-\)matrices corresponding to the vector \( \bar{u} \in T(P) \) with respect to the parameterizations \((\lambda, \mu)\) and \((\lambda', \mu')\), are denoted by \( \hat{\bar{u}} \) and \( \hat{\bar{u}}' \) respectively. One easily verifies
where the (2×2)-matrix $S$ is given by
\[
S = \begin{bmatrix}
\frac{\partial \lambda}{\partial \lambda'} & \frac{\partial \lambda}{\partial \mu'} \\
\frac{\partial \mu}{\partial \lambda'} & \frac{\partial \mu}{\partial \mu'}
\end{bmatrix}.
\]

Let $A$ be a linear operator on the tangent space $T(P)$ at $P$ on $S$. We introduce the (2×2)-matrix $\hat{A}$ corresponding to the linear operator $A$ on $T(P)$, such that
\[
(Au) = \hat{A}u, \quad u \in T(P).
\]

The (2×2)-matrices corresponding to the linear operator $A$ on $T(P)$ with respect to the parameterizations $(\lambda, \mu)$ and $(\lambda', \mu')$ of $S$, are denoted by $\hat{A}$ and $\hat{A}'$, respectively. One easily verifies
\[
\hat{A}' = S^{-1}A S,
\]
where the (2×2)-matrix $S$ is given in (1.16).

Consider the (2×2)-matrix $Q$ at $P$ on $S$
\[
Q = \begin{bmatrix}
E & F \\
F & G
\end{bmatrix},
\]
where
\[
E = E(\lambda, \mu) := (x_\lambda, x_\lambda),
\]
\[
F = F(\lambda, \mu) := (x_\lambda, x_\mu),
\]
\[
G = G(\lambda, \mu) := (x_\mu, x_\mu).
\]

The quantities $E, F, G$ in (1.20) are the fundamental coefficients of the first order at the point $P(\lambda, \mu)$ on $S$. It is easily verified, that for $u, v \in T(P)$
\[
(u, v) = (\hat{u})^T Q \hat{v}.
\]

Let $(\lambda', \mu')$ be another parameterization of the surface $S$. The corresponding fundamental coefficients of the first order are denoted by $E', F', G'$, and
\[
Q' = \begin{bmatrix}
E' & F' \\
F' & G'
\end{bmatrix}.
\]

We have
where the \((2 \times 2)\)-matrix \(S\) is given in (1.16).

The components of the unit normal vector \(\mathbf{n}(P)\) on \(S\) at \(P(\lambda, \mu)\) are \(\mathbf{n}(\lambda, \mu)\). Consider the \((2 \times 2)\)-matrix \(R\) at \(P\) on \(S\)

\[(1.23) \quad R = \begin{bmatrix} L & M \\ M & N \end{bmatrix},\]

where

\[(1.24) \quad L = L(\lambda, \mu) := (n_\lambda, x_{\lambda \lambda}) = -(n_\lambda, x_\lambda), \]

\[(1.24) \quad M = M(\lambda, \mu) := (n_\lambda, x_{\lambda \mu}) = -(n_\lambda, x_\mu), \]

\[(1.24) \quad N = N(\lambda, \mu) := (n_\mu, x_{\mu \mu}) = -(n_\mu, x_\mu).\]

The quantities \(L, M, N\) in (1.24) are the fundamental coefficients of the second order at the point \(P(\lambda, \mu)\) on \(S\). Let \((\lambda', \mu')\) be another parameterization of the surface \(S\). The corresponding coefficients of the second order are denoted by \(L', M', N'\), and

\[(1.25) \quad R' = \begin{bmatrix} L' & M' \\ M' & N' \end{bmatrix}.\]

We have

\[(1.25) \quad R' = S^T R S,\]

where the \((2 \times 2)\)-matrix \(S\) is given in (1.16).

Since \((n, n) = 1\), we have \(n_\lambda \in T(P)\) and \(n_\mu \in T(P)\).

Write

\[(1.26) \quad n_\lambda = \alpha x_\lambda + \beta x_\mu, \]

\[(1.26) \quad n_\mu = \gamma x_\lambda + \delta x_\mu.\]

It follows from (1.26) and the definitions in (1.20) and (1.24), that

\[(1.27) \quad \alpha = (FM - GL)/(EG - F^2), \]

\[(1.27) \quad \beta = (FL - EN)/(EG - F^2), \]

\[(1.27) \quad \gamma = (FN - GM)/(EG - F^2), \]

\[(1.27) \quad \delta = (FM - EN)/(EG - F^2).\]

Now introduce that \((2 \times 2)\)-matrix
From (1.27, 28) and (1.19), (1.23) we conclude, that

\[ B = Q^{-1} R. \]

Let \((\lambda', \mu')\) be another parameterization of the surface \(S\). The matrix corresponding to \(\hat{B}\) in (1.28) is denoted by \(\hat{B}'\). From (1.22), (1.25) and (1.29) we get

\[ \hat{B}' = S^{-1} \hat{B} S, \]

where the \((2 \times 2)\)-matrix \(S\) is given in (1.16). Hence, the matrix \(\hat{B}\) in (1.28) corresponds to a linear operator \(B\) on the tangent space \(T(P)\) at \(P\) on \(S\); compare (1.30) and (1.18). We shall show, that \(B\) is the symmetric operator on \(T(P)\) referred to in (1.3).

For \(\mathbf{u}, \mathbf{v} \in T(P)\) we have, by use of (1.21), (1.17), (1.29),

\[ (\mathbf{u}, B\mathbf{v}) = (\hat{\mathbf{u}})^T Q Q^{-1} R \hat{\mathbf{v}} = (\hat{\mathbf{u}})^T R \hat{\mathbf{v}} = (\hat{\mathbf{v}})^T R \hat{\mathbf{u}} = (\mathbf{v}, B\mathbf{u}). \]

Hence, the operator \(B\) is symmetric.

Let \(C\) be a curve on \(S\) through \(P\) with parametric equations

\[ \mathbf{x} = \mathbf{x}(s) = \mathbf{x}(\lambda(s), \mu(s)). \]

The parameter \(s\) in (1.31) is the arc length parameter on \(C\) such that \(P\) corresponds to \(s = 0\). The components of the vectors \(\mathbf{n}(s) = \mathbf{n}(\lambda(s), \mu(s))\) and \(\mathbf{n}_s(s) = d\mathbf{n}(s)/ds\), compare (1.1), are \(\mathbf{n}(s)\) and \(\mathbf{n}_s(s)\) respectively. The vector

\[ \mathbf{t} = \mathbf{x}_s(0) \frac{d\lambda}{ds}(0) + \mathbf{x}_\mu(0) \frac{d\mu}{ds}(0) \]

is a unit tangent vector of \(C\) at \(P\), and clearly

\[ \mathbf{t} = [\frac{d\lambda}{ds}(0), \frac{d\mu}{ds}(0)]^T. \]

From (1.26)

\[ \mathbf{n}_s(0) = \mathbf{n}_x(0) \frac{d\lambda}{ds}(0) + \mathbf{n}_\mu(0) \frac{d\mu}{ds}(0) = \]

\[ = (\alpha \frac{d\lambda}{ds}(0) + \gamma \frac{d\mu}{ds}(0)) \mathbf{x}_x + (\gamma \frac{d\lambda}{ds}(0) + \delta \frac{d\mu}{ds}(0)) \mathbf{x}_\mu. \]

Hence, by use of (1.28) and (1.33),
In view of (1.17) we conclude, that

\begin{equation}
(1.36) \quad n_3(0) = -Bt
\end{equation}

which proves that $B$ is the symmetric operator on $T(P)$ referred to in (1.3).

Using (1.12), (1.19), (1.23) and (1.27, 28, 29), we conclude for the mean curvature $\bar{\kappa}(P)$ and the total curvature $\hat{\kappa}(P)$ of $S$ at $P (\lambda, \mu)$

\begin{equation}
(1.37) \quad \bar{\kappa}(\lambda, \mu) = \text{trace}(B) = -\alpha - \delta = (EN + GL - 2FM)/(EG - F^2),
\end{equation}

\begin{equation}
(1.37) \quad \hat{\kappa}(\lambda, \mu) = \det(B) = \det(R)/\det(Q) = (LN - M^2)/(EG - F^2).
\end{equation}

Consider two rigid bodies $L_1$ and $L_2$ with boundary surfaces $S_1$ and $S_2$. The bodies $L_1$ and $L_2$ are in contact along a curve $C$ of contact. The inner unit normal vector on $S_1$ and $S_2$ is denoted by $n_1$ and $n_2$ respectively. Hence, the normal vectors $n_1$ and $n_2$ have opposite directions on $C$, i.e., $n_1 = -n_2$ on $C$. We consider a point on the curve $C$ of contact. As a point on $S_1$ we denote it by $P$, as a point on $S_2$ by $Q$. The tangent spaces $T_1(P)$ and $T_2(Q)$ of $S_1$ and $S_2$ coincide and we write $T(P, Q) := T_1(P) = T_2(Q)$. The symmetric operators $B_1(P)$ and $B_2(Q)$ referred to in (1.3) both act on the common tangent space $T(P, Q)$. Let $t_c \in T(P, Q)$ be a unit tangent vector of the curve $C$ of contact at $P = Q$. Since $n_1(P) = -n_2(Q)$, we conclude from (1.3) that

\begin{equation}
(1.38) \quad B_1(P) \cdot t_c = -B_2(Q) \cdot t_c.
\end{equation}

Suppose, there exists a unit vector $t \in T(P, Q)$ such that the normal curvature $\kappa_1(P; t)$ of $S_1$ at $P$ in the direction $t$ is negative. Then the normal curvature $\kappa_2(Q; t)$ of $S_2$ at $Q$ in the direction $t$ must be positive and since the rigid bodies $L_1$ and $L_2$ do not penetrate each other, we must have

$$0 < 1/\kappa_2(Q; t) \leq -1/\kappa_1(P; t),$$

and therefore (compare (1.7))

$$\kappa_1(P; t) + \kappa_2(Q; t) = (t, (B_1(P) + B_2(Q)) t) \geq 0.$$

Hence

$$(t, B_1(P) t) < 0 \Rightarrow (t, (B_1(P) + B_2(Q)) t) \geq 0,$$

and similarly
From the above considerations we may infer, that for \( \vec{t} \in T(P, Q) \) we have

\[(1.39) \quad (\vec{t}, (B_1(P) + B_2(Q)) \vec{t}) \geq 0.\]

Consider the symmetric operator

\[(1.40) \quad B(P, Q) := B_1(P) + B_2(Q)\]

on the common tangent space \( T(P, Q) \). By virtue of (1.38) and (1.39) we may conclude that \( B(P, Q) \) is non-negative and has an eigenvalue zero. For the unit vector \( \vec{t} \in T(P, Q) \) the relative curvature \( \kappa_r(P, Q; \vec{t}) \) of the bodies \( L_1 \) and \( L_2 \) at \( P = Q \) on the curve \( C \) of contact in the direction \( \vec{t} \) is defined by

\[(1.41) \quad \kappa_r(P, Q; \vec{t}) = (\vec{t}, B(P, Q) \vec{t}).\]

The relative curvature \( \kappa_r(P, Q; \vec{t}) \) reaches a maximum in the direction perpendicular to the direction of the curve \( C \) of contact. This maximum is referred to as the relative curvature \( \kappa_r(P, Q) \) of the bodies \( L_1 \) and \( L_2 \) at \( P = Q \) on the curve \( C \) of contact. The principal curvatures \( \kappa_{11}(P) \leq \kappa_{12}(P) \) of \( S_1 \) at \( P \) are the real eigenvalues of the symmetric operator \( B_1(P) \) and the principal curvatures \( \kappa_{21}(Q) \leq \kappa_{22}(Q) \) of \( S_2 \) at \( Q \) are the real eigenvalues of the symmetric operator \( B_2(Q) \). Hence, the relative curvature \( \kappa_r(P, Q) \) of the bodies \( L_1 \) and \( L_2 \) at \( P = Q \) on \( C \) equals

\[(1.42) \quad \kappa_r(P, Q) = \kappa_{11}(P) + \kappa_{12}(P) + \kappa_{21}(Q) + \kappa_{22}(Q) = \bar{\kappa}_1(P) + \bar{\kappa}_2(Q),\]

where \( \bar{\kappa}_1(P) \) and \( \bar{\kappa}_2(Q) \) are the mean curvatures of \( S_1 \) and \( S_2 \) at \( P \) and \( Q \) respectively.

Suppose that \( P \) is an umbilical point of \( S_1 \), i.e.

\[(1.43) \quad \kappa_{11}(P) = \kappa_{12}(P) =: \kappa_1(P).\]

Let \( \vec{t}_{21}(Q) \) and \( \vec{t}_{22}(Q) \) be unit tangent vectors in \( T(P, Q) \) in the principal directions of \( S_2 \) at \( Q \) corresponding to the principal curvatures \( \kappa_{21}(Q) \) and \( \kappa_{22}(Q) \) respectively. We take \( \vec{t}_{21}(Q) \) and \( \vec{t}_{22}(Q) \) as the elements of an orthonormal basis in the common tangent space \( T(P, Q) \). The matrix representations \( \hat{B}_1(P) \) and \( \hat{B}_2(Q) \) of the operators \( B_1(P) \) and \( B_2(Q) \) have the form

\[(1.44) \quad \hat{B}_1(P) = \kappa_1(P) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{B}_2(Q) = \begin{bmatrix} \kappa_{21}(Q) & 0 \\ 0 & \kappa_{22}(Q) \end{bmatrix}.\]
Since $B(P, Q)$ is non-negative and singular, we have

\begin{equation}
\text{trace}(\hat{B}_1(P) + \hat{B}_2(Q)) \geq 0. 
\end{equation}

(1.45)

\[ \det(\hat{B}_1(P) + \hat{B}_2(Q)) = 0. \]

Hence,

\begin{equation}
\kappa_{21}(Q) = -\kappa_1(P), 
\end{equation}

(1.46)

\[ \kappa_{21}(Q) < \kappa_{22}(Q) \Rightarrow t_e = t_{21}(Q), \]

where $t_e \in T(P, Q)$ is a unit tangent vector of $C$ at $P = Q$.

Suppose that $Q$ is an umbilical point of $S_2$, i.e.

\begin{equation}
\kappa_{21}(Q) = \kappa_{22}(Q) := \kappa_2(Q). 
\end{equation}

(1.47)

Let $t_{11}(P)$ and $t_{12}(P)$ be unit tangent vectors in $T(P, Q)$ in the principal directions of $S_1$ at $P$ corresponding to the principal curvatures $\kappa_{11}(P)$ and $\kappa_{12}(P)$ respectively.

We have, compare (1.46),

\begin{equation}
\kappa_{11}(P) = -\kappa_2(Q) 
\end{equation}

(1.48)

\[ \kappa_{11}(P) < \kappa_{12}(P) \Rightarrow t_e = t_{11}(P). \]

Now suppose that $P$ and $Q$ are not umbilical points of $S_1$ and $S_2$ respectively, i.e.

\begin{equation}
\kappa_{11}(P) < \kappa_{12}(P) \text{ and } \kappa_{21}(Q) < \kappa_{22}(Q). 
\end{equation}

(1.49)

We take $t_{11}(P)$ and $t_{12}(P)$ as the elements of an orthonormal basis in the common tangent space $T(P, Q)$. Choose $-\pi/2 < \Theta \leq \pi/2$ such that the vector $t_{11}(\Theta) \cos(\Theta) + t_{12}(\Theta) \sin(\Theta)$ is in the principal direction of $S_2$ at $Q$ corresponding to the principal curvature $\kappa_{21}(Q)$ of $S_2$ at $Q$. The matrix representations $\hat{B}_1(P)$ and $\hat{B}_2(Q)$ of the operators $B_1(P)$ and $B_2(Q)$ have the form

\[ \hat{B}_1(P) = \begin{bmatrix} \kappa_{11} & 0 \\ 0 & \kappa_{12} \end{bmatrix}, \]

(1.50)

\[ \hat{B}_2(Q) = \begin{bmatrix} \kappa_{21} \cos^2(\Theta) + \kappa_{22} \sin^2(\Theta), \cos(\Theta) \sin(\Theta)(\kappa_{21} - \kappa_{22}) \\ \cos(\Theta) \sin(\Theta)(\kappa_{21} - \kappa_{22}), \kappa_{21} \sin^2(\Theta) + \kappa_{22} \cos^2(\Theta) \end{bmatrix}. \]

Since $\det(\hat{B}_1(P) + B_2(Q)) = 0$, we get
From (1.51) and (1.42) we conclude
\[ l_{C1}(P) + l_{C2}(Q) \geq 0, \]
\[ l_{C2}(Q) + l_{C1}(P) \geq 0, \]
\[ l_{C1}(P) + l_{C2}(Q) \leq 0, \]
\[ \Theta = 0 \iff l_{C1}(P) + l_{C2}(Q) = 0, \]
\[ \Theta = \pi/2 \iff l_{C1}(P) + l_{C2}(Q) = 0 \text{ or } l_{C2}(Q) + l_{C1}(P) = 0. \]

Choose \(-\pi/2 < \eta \leq \pi/2\) such that the vector \( \vec{t}_{11} \cos(\eta) + \vec{t}_{12} \sin(\eta) \) is in the direction of the curve \( C \) of contact at \( P = Q \). The vector \( \vec{t}_{11} \cos(\eta) + \vec{t}_{12} \sin(\eta) \) is an eigenvector of \( B(P, Q) \) having eigenvalue zero. Hence, use (1.50),
\[ \Theta = 0 \Rightarrow \eta = 0, \]
\[ \Theta = \pi/2 \text{ and } l_{C1}(P) + l_{C2}(Q) = 0 \Rightarrow \eta = 0, \]
and if \( \Theta \neq 0 \) and \( \Theta \neq \pi/2 \), then
\[ \eta = \arctan \left( \frac{l_{C1}(P) \cos(\Theta) + l_{C2}(Q) \sin^2(\Theta) + l_{C1}(P)}{\cos(\Theta) \sin(\Theta) (l_{C2}(Q) - l_{C2}(P))} \right). \]

The sign of \( \Theta \) depends on the orientation of the vectors \( \vec{t}_{11}(P) \) and \( \vec{t}_{12}(P) \). We can choose the orientation of the vectors \( \vec{t}_{11}(P) \) and \( \vec{t}_{12}(P) \) such that \( 0 \leq \Theta \leq \pi/2 \).

2. The curvature of conjugate profiles in points of contact

We consider the general gear pair. The smaller of the two gears is called the pinion and the larger the gear. The axes of the pinion and the gear are fixed in space. The angular velocity vector of the pinion and the gear is denoted by \( \omega_1 \) and \( \omega_2 \) respectively; see Figure 2.1. The angular velocity ratio \( U \) of the gear pair is constant and equals
\[ U = \omega_1 / \omega_2, \]
where \( \omega_1 = |\omega_1| \) and \( \omega_2 = |\omega_2| \).

Choose the point \( 0_1 \) on the pinion axis and the point \( 0_2 \) on the gear axis such that the line \( 0_10_2 \) is perpendicular to both axes. If the pinion axis and the gear axis intersect in a point \( 0 \), then
We introduce a right-handed cartesian coordinate system which is fixed in space. The corresponding coordinates are denoted by \((x, y, z) = \mathbf{x}\). The point \(0_1\) on the pinion axis is the origin of the coordinate system. The coordinate axes of the coordinate system are shown in Figure 2.1. The distance of the pinion axis and the gear axis is denoted by \(a\). The angle between the angular velocity vectors \(\omega_1\) and \(\omega_2\) of the pinion and the gear is \(\delta\), \(-\pi < \delta \leq \pi\). The sign of \(\delta\) is chosen such that the components \(\omega_2\) of the angular velocity vector \(\omega_2\) of the gear are given by \(\omega_2 = \omega_2 \left(\cos(\delta), 0, \sin(\delta)\right)\); see Figure 2.1.

![Figure 2.1. The pinion axis and the gear axis in the space-fixed coordinate system.](image)

The tooth profiles on the mating gear teeth are conjugate, i.e. they are shaped such as to produce a constant angular velocity ratio during meshing. Let the tooth profile on a tooth of the pinion in the reference position be specified by the parametric equations, with parameters \(\lambda\) and \(\mu\),

\[
\mathbf{x} = \mathbf{x}_1(\lambda, \mu) = (x_1(\lambda, \mu), y_1(\lambda, \mu), z_1(\lambda, \mu))
\]

with respect to the coordinate system in Figure 2.1. Furthermore, let \(\phi(\lambda, \mu)\) be the angle of rotation of the pinion around its axis from the reference position \(\phi = 0\) in the orientation of \(\omega_1\), such that the point \(P(\lambda, \mu)\) on the pinion profile (2.2) is in contact with the conjugate gear profile. The angle of rotation \(\phi(\lambda, \mu)\) can be calculated from the equation

\[
a(\lambda, \mu) \sin(\phi(\lambda, \mu)) + b(\lambda, \mu) \cos(\phi(\lambda, \mu)) = c(\lambda, \mu)
\]
where

\[
\begin{align*}
\alpha(\lambda, \mu) &= \sin(\delta) \frac{\partial (z_1^2 + x_1^2, y_1)}{\partial (\lambda, \mu)} + 2a \cos(\delta) \frac{\partial (z_1, x_1)}{\partial (\lambda, \mu)}, \\
b(\lambda, \mu) &= \sin(\delta) \frac{\partial (x_1^2 + y_1^2, z_1)}{\partial (\lambda, \mu)} + 2a \cos(\delta) \frac{\partial (x_1, y_1)}{\partial (\lambda, \mu)}, \\
c(\lambda, \mu) &= (U - \cos(\delta)) \frac{\partial (y_1^2 + z_1^2, x_1)}{\partial (\lambda, \mu)} + 2a \sin(\delta) \frac{\partial (y_1, z_1)}{\partial (\lambda, \mu)},
\end{align*}
\]

with

\[
\frac{\partial (p, q)}{\partial (\lambda, \mu)} = \frac{\partial p}{\partial \lambda} \frac{\partial q}{\partial \mu} - \frac{\partial p}{\partial \mu} \frac{\partial q}{\partial \lambda},
\]

and \(x_1, y_1, z_1\) as specified in (2.2).

For a proof of equation (2.3) the reader is referred to [3, § 1.4] or [4].

Let \(P\) be a point on the pinion profile with parametric equations (2.2) in the reference position \(\phi = 0\). The position in space of the pinion point \(P\) at the moment of contact in \(P\) with the conjugate gear profile, is called the contact position \(IP\) of \(P\). Let \(P\) be the pinion point corresponding to the parameters \((\lambda, \mu)\) in (2.2). The coordinates of the pinion point \(P = P(\lambda, \mu)\) in the reference position \(\phi = 0\) are \(x(\lambda, \mu)\) in (2.2). The coordinates \(x(\lambda, \mu)\) of the contact position \(IP = IP(\lambda, \mu)\) can be written in the form

\[
x(\lambda, \mu) = \exp(\phi(\lambda, \mu)\Omega_1) x(\lambda, \mu),
\]

where

\[
\phi(\lambda, \mu) \text{ as calculated from equation (2.3)},
\]

the antisymmetric matrix

\[
\Omega_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

For a description of rotations by means of the exponential function with matrix-argument, we refer to [3, § 1.3] or [4]. The coordinates \(x(\lambda, \mu)\) in (2.4) of the contact position \(IP(\lambda, \mu)\) are independent of the choise of the reference position \(\phi = 0\). The equations (2.4) are the parametric equations, with parameters \(\lambda\) and \(\mu\), of the so called surface of action of the gear pair, i.e. the surface containing all contact positions.

Let \(Q\) be the point of contact on the conjugate gear profile corresponding to the point \(P\) on the pinion profile. The image of the contact position \(IP\) under the rotation over the angle
\(-k\phi(\lambda, \mu), k = 1/U,\) around the gear axis in the orientation of \(\vec{\omega}_2,\) is the position of the gear point \(Q\) in the reference position \(\phi = 0\). Hence, the coordinates \(x_2(\lambda, \mu)\) of the gear point \(Q = Q(\lambda, \mu)\) in the reference position \(\phi = 0\) can be written in the form
\[
(2.5) \quad x_2(\lambda, \mu) = \exp(-k\phi(\lambda, \mu)\Omega_2)(\vec{x}(\lambda, \mu) - \vec{a}) + \vec{a},
\]
where
\[
\vec{x}(\lambda, \mu) \text{ see (2.4), } k = 1/U,
\]
\(\vec{a} = (0, a, 0)\) are the coordinates of the point \(O_2\) on the gear axis; compare Figure 2.1,

the antisymmetric matrix
\[
\Omega_2 = \begin{bmatrix}
0 & -\sin(\delta) & 0 \\
\sin(\delta) & 0 & -\cos(\delta) \\
0 & \cos(\delta) & 0
\end{bmatrix}
\]

The equations (2.5) are the parametric equations, with parameters \(\lambda\) and \(\mu,\) of the conjugate gear profile in the reference position \(\phi = 0.\)

Let \(n_1(P)\) be the unit inner normal vector on the pinion profile at the pinion point \(P,\) i.e. the vector \(n_1(P)\) is in the direction of the pinion tooth at \(P.\) The components \(n_1(\lambda, \mu)\) of the normal vector \(n_1(P)\), \(P = P(\lambda, \mu),\) in the reference position \(\phi = 0\) can be calculated by use of the parametric equations (2.2). The components \(\tilde{n}(\lambda, \mu)\) of the common unit normal vector on the pinion profile and the gear profile at the moment of contact in the pinion point \(P = P(\lambda, \mu),\) can be written in the form
\[
(2.6) \quad \tilde{n}(\lambda, \mu) = \exp(\phi(\lambda, \mu)\Omega_1) n_1(\lambda, \mu).
\]

The vector \(\tilde{n}(\lambda, \mu)\) is independent of the choice of the reference position \(\phi = 0.\)

Let \(n_2(Q)\) be the unit inner normal vector on the gear profile at the point \(Q\) of contact corresponding to the pinion point \(P.\) The components \(n_2(\lambda, \mu)\) of the normal vector \(n_2(Q),\) \(Q = Q(\lambda, \mu),\) in the reference position \(\phi = 0\) can be calculated from (2.6). We get
\[
(2.7) \quad n_2(\lambda, \mu) = -\exp(-k\phi(\lambda, \mu)\Omega_2) \tilde{n}(\lambda, \mu).
\]

We give formulas for the fundamental coefficients of the conjugate profiles in points of contact. By use of (2.4-7) we get
\[ x_{1,\lambda} = \exp(-\phi \Omega_1)(\bar{x}_\lambda - \phi_\lambda \Omega_1 \bar{x}) , 
\]
\[ x_{1,\mu} = \exp(-\phi \Omega_1)(\bar{x}_\mu - \phi_\mu \Omega_1 \bar{x}) , 
\]
\[ n_{1,\lambda} = \exp(-\phi \Omega_1)(\bar{n}_\lambda - \phi_\lambda \Omega_1 \bar{n}) , 
\]
\[ n_{1,\mu} = \exp(-\phi \Omega_1)(\bar{n}_\mu - \phi_\mu \Omega_1 \bar{n}) , 
\]
\[ x_{2,\lambda} = \exp(-k \phi \Omega_2)(\bar{x}_\lambda - k \phi_\lambda \Omega_2 (\bar{x} - \bar{a})) , 
\]
\[ x_{2,\mu} = \exp(-k \phi \Omega_2)(\bar{x}_\mu - k \phi_\mu \Omega_2 (\bar{x} - \bar{a})) , 
\]
\[ n_{2,\lambda} = -\exp(-k \phi \Omega_2)(\bar{n}_\lambda - k \phi_\lambda \Omega_2 \bar{n}) , 
\]
\[ n_{2,\mu} = -\exp(-k \phi \Omega_2)(\bar{n}_\mu - k \phi_\mu \Omega_2 \bar{n}) . 
\]

The fundamental coefficients are calculated by use of the definitions in (1.20) and (1.24) and the relations (2.8). Furthermore, we use the orthogonality of the matrix \( \exp(A) \), provided that the matrix \( A \) is antisymmetric. The fundamental coefficients \( E_1, \ldots, N_1 \) of the pinion profile at the point \( P = P(\lambda, \mu) \) can be calculated by use of the formulas

\[ E_1 = (x_{1,\lambda}, x_{1,\lambda}) = (\bar{x}_\lambda - \phi_\lambda \Omega_1 \bar{x}, \bar{x}_\lambda - \phi_\lambda \Omega_1 \bar{x}) , \]
\[ F_1 = (x_{1,\lambda}, x_{1,\mu}) = (\bar{x}_\lambda - \phi_\lambda \Omega_1 \bar{x}, \bar{x}_\mu - \phi_\mu \Omega_1 \bar{x}) , \]
\[ G_1 = (x_{1,\mu}, x_{1,\mu}) = (\bar{x}_\mu - \phi_\mu \Omega_1 \bar{x}, \bar{x}_\mu - \phi_\mu \Omega_1 \bar{x}) , \]
\[ L_1 = -(n_{1,\lambda}, x_{1,\lambda}) = -(\bar{n}_\lambda - \phi_\lambda \Omega_1 \bar{n}, \bar{x}_\lambda - \phi_\lambda \Omega_1 \bar{x}) , \]
\[ M_1 = -(n_{1,\lambda}, x_{1,\mu}) = -(\bar{n}_\lambda - \phi_\lambda \Omega_1 \bar{n}, \bar{x}_\mu - \phi_\mu \Omega_1 \bar{x}) , \]
\[ N_1 = -(n_{1,\mu}, x_{1,\mu}) = -(\bar{n}_\mu - \phi_\mu \Omega_1 \bar{n}, \bar{x}_\mu - \phi_\mu \Omega_1 \bar{x}) . \]

The fundamental coefficients \( E_2, \ldots, N_2 \) of the gear profile at the point \( Q = Q(\lambda, \mu) \) of contact corresponding to the pinion point \( P = P(\lambda, \mu) \), can be calculated by use of the formulas

\[ E_2 = (x_{2,\lambda}, x_{2,\lambda}) = (\bar{x}_\lambda - k \phi_\lambda \Omega_2 (\bar{x} - \bar{a}), \bar{x}_\lambda - k \phi_\lambda \Omega_2 (\bar{x} - \bar{a})) , \]
\[ F_2 = (x_{2,\lambda}, x_{2,\mu}) = (\bar{x}_\lambda - k \phi_\lambda \Omega_2 (\bar{x} - \bar{a}), \bar{x}_\mu - k \phi_\mu \Omega_2 (\bar{x} - \bar{a})) , \]
\[ G_2 = (x_{2,\mu}, x_{2,\mu}) = (\bar{x}_\mu - k \phi_\mu \Omega_2 (\bar{x} - \bar{a}), \bar{x}_\mu - k \phi_\mu \Omega_2 (\bar{x} - \bar{a})) , \]
\[ L_2 = -(n_{2,\lambda}, x_{2,\lambda}) = -(\bar{n}_\lambda - k \phi_\lambda \Omega_2 \bar{n}, \bar{x}_\lambda - k \phi_\lambda \Omega_2 (\bar{x} - \bar{a})) , \]
\[ M_2 = -(n_{2,\lambda}, x_{2,\mu}) = -(\bar{n}_\lambda - k \phi_\lambda \Omega_2 \bar{n}, \bar{x}_\mu - k \phi_\mu \Omega_2 (\bar{x} - \bar{a})) , \]
\[ N_2 = -(n_{2,\mu}, x_{2,\mu}) = -(\bar{n}_\mu - k \phi_\mu \Omega_2 \bar{n}, \bar{x}_\mu - k \phi_\mu \Omega_2 (\bar{x} - \bar{a})) . \]

Finally, we summarize the steps in the algorithm for the calculation of the curvature of conjugate profiles in points of contact:
1. Introduce the constants
   \[ a = (0, a, 0) \], where \( a \) is the distance of the pinion axis and the gear axis; compare Figure 2.1;
   - the antisymmetric matrices
     \[
     \Omega_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}
     \quad \text{and} \quad
     \Omega_2 = \begin{bmatrix} 0 & -\sin(\delta) & 0 \\ \sin(\delta) & 0 & -\cos(\delta) \\ 0 & \cos(\delta) & 0 \end{bmatrix},
     \]
   where \( \delta \) is the angle between the pinion axis and the gear axis; compare Figure 2.1;
   - \( k = 1/U \), where \( U \) is the angular velocity ratio of the gear pair; compare (2.1).

2. Specify the tooth profile on a tooth of the pinion by means of its parametric equations
   \[ \bar{x} = \bar{x}_1(\lambda, \mu) \] in the reference position \( \phi = 0 \) and with respect to the cartesian coordinate system in Figure 2.1; compare (2.2)

3. Determine the angle of rotation \( \phi(\lambda, \mu) \) by use of equation (2.3).

4. Determine the coordinates \( \bar{z}(\lambda, \mu) \) of the contact position \( \bar{IP}(\lambda, \mu) \) of the pinion point \( P = P(\lambda, \mu) \) by use of (2.4).

5. Determine the components \( \bar{n}_1(\lambda, \mu) \) of the unit inner normal vector on the pinion profile at the point \( P = P(\lambda, \mu) \) in the reference position \( \phi = 0 \) by use of the parametric equations (2.2).

6. Determine the components \( \bar{n}(\lambda, \mu) \) of the common unit normal on the pinion profile and the gear profile at the moment of contact in the pinion point \( P = P(\lambda, \mu) \) by use of (2.6).

7. Determine the fundamental coefficients \( E_1, \cdots, N_1 \) of the pinion profile at the pinion point \( P = P(\lambda, \mu) \) by use of (2.9).

8. Determine the fundamental coefficients \( E_2, \cdots, N_2 \) of the gear profile at the gear point \( Q = Q(\lambda, \mu) \) of contact corresponding to the pinion point \( P = P(\lambda, \mu) \) by use of (2.10).

9. Determine the mean curvatures \( \bar{k}_1(P) \) and \( \bar{k}_2(Q) \) of the pinion profile and the gear profile at the points \( P \) and \( Q \) of contact by use of (1.37).
10. Determine the total curvatures $\tilde{k}_1(P)$ and $\tilde{k}_2(Q)$ of the pinion profile and the gear profile at the points $P$ and $Q$ of contact by use of (1.37).

11. Determine the relative curvature $k_r(P, Q) = \tilde{k}_1(P) + \tilde{k}_2(Q)$ of the conjugate profiles in the points $P$ and $Q$ of contact; compare (1.42). If $k_r(P, Q)$ turns out to be negative, then $P$ and $Q$ are not real points of contact since undercutting has occurred; compare (1.39).

12. Determine the principal curvatures $k_{11}(P) \leq k_{12}(P)$ of the pinion profile at the pinion point $P = P(\lambda, \mu)$ by use of (1.12).

13. Determine the principal curvatures $k_{21}(Q) \leq k_{22}(Q)$ of the gear profile at the gear point $Q = Q(\lambda, \mu)$ of contact corresponding to the pinion point $P = P(\lambda, \mu)$ by use of (1.12).

14. Determine the angle $\Theta$ between the corresponding principal directions of the conjugate profiles at the points $P$ and $Q$ of contact by use of (1.51).

15. Determine the angle $\eta$ between the curve of contact and the principal direction of the pinion profile corresponding to $k_{11}$ at the points $P$ and $Q$ of contact by use of (1.53, 54).

References


