A necessary and sufficient condition for solvability of the linear-quadratic control problem without stability

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by

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A NECESSARY AND SUFFICIENT CONDITION FOR SOLVABILITY OF THE LINEAR-QUADRATIC CONTROL PROBLEM WITHOUT STABILITY

ABSTRACT

In this short paper a necessary and sufficient condition is given for the existence of a minimizing control for the regular non-negative definite linear-quadratic control problem without stability.

KEYWORDS

Linear-quadratic control problem, stabilizability, optimal input, quotient space, Riccati equation.

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1. Introduction

In order to solve the regular non-negative definite linear-quadratic control problem without stability, one commonly preassumes stabilizability of the system \( \Sigma \), described by the quadruple \((A, B, C, D)\) ([1, Props. 9, 10]). The sufficiency of \((A, B)\)-stabilizability for the existence of the optimal input being known ([1]), it is clear that this condition is not necessary (take for instance a system with transfer function identically zero).

Here we will show that for every initial state the infimum of the cost criterion over the space of locally square integrable inputs is finite if and only if \((\tilde{A}_0, \tilde{B})\) is stabilizable, where \(\tilde{A}_0\) and \(\tilde{B}\) are the quotient maps of \(A_0 = A - B(D'D)^{-1}D'C\) and \(B\) w.r.t. the quotient space \(\mathbb{R}^n/V(\Sigma)\) (\(\mathbb{R}^n\) denotes the n-dimensional state space). The linear subspace \(V(\Sigma)\) is the weakly unobservable subspace (also called the output nulling subspace, see e.g. [2, Def. 3.8] and [1]). In addition, it will turn out that in case of \((\tilde{A}_0, \tilde{B})\)-stabilizability the optimal control exists and is unique.

Moreover it will follow directly from the foregoing that the Algebraic Riccati Equation has a positive semi-definite solution if and only if \((\tilde{A}_0, \tilde{B})\) is stabilizable. This result is of interest since most articles dealing with these equations start from sufficient conditions for the existence of a solution. The condition presented here indeed covers such pathological cases as the one mentioned above.
2. Preliminaries

Consider the finite-dimensional linear time-invariant system \( \Sigma \):
\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x_0, \quad (2.1a) \\
y &= Cx + Du, \quad (2.1b)
\end{align*}
\]
with \( D \) left invertible, and the quadratic cost criterion
\[
J(x_0, u) = \int_0^\infty y'y \, dt. \quad (2.2)
\]
Here \( x(t) \in \mathbb{R}^n \) and \( u(t), y(t) \) are \( m-, r- \)-dimensional real vectors, respectively. If \( L^m_{2, \text{loc}}(\mathbb{R}^+) \) denotes the space of \( m- \)-vectors whose components are locally square integrable over \( \mathbb{R}^+ \), then we state the non-negative definite linear-quadratic control problem without stability \((\text{LQCP})^-\) as follows.

Define for every \( x_0 \)
\[
J^-(x_0) := \inf \{ J(x_0, u) \mid u \in L^m_{2, \text{loc}}(\mathbb{R}^+) \}. \quad (2.3)
\]

\((\text{LQCP})^-\): Determine for every initial state \( J^-(x_0) \) such that it is finite and compute an optimal control, if it exists.

We will call \((\text{LQCP})^-\) **solvable** if (2.3) is finite for all \( x_0 \).

Indeed it is well known (see e.g. [1]) that in case of \((A, B)-\text{stabilizability}\) the optimal control actually exists and is unique. More precisely, if \((A, B)\) is stabilizable then it holds that
\[
J^-(x_0) = x_0'K^-x_0 \quad (2.4)
\]
with the real symmetric matrix \( K^- \) being the smallest positive semi-definite solution of the Algebraic Riccati Equation
\[
0 = C'C + A'K + KA - (KB + C'D)(D'D)^{-1}(B'K + D'C). \quad (2.5)
\]
Furthermore, the optimal control exists for all \( x_0 \) and it is given by the feedback law
\[
u = -(D'D)^{-1}(B'K^- + D'C)x. \quad (2.6)
\]
See also [2], [5], [6].
In this paper we will "replace" the \((A, B)\)-stabilizability by a "reduced order" stabilizability assumption that will prove to be both sufficient and necessary for solvability of \((LQCP)^-\). In addition, we will see that we only have to deal with a "reduced order" Riccati Equation instead of with \((2.5)\): our equation will be of dimension \((n - \dim(V(\Sigma))) \leq n\).

We will need the next definitions. If \(A: \mathbb{R}^n \to \mathbb{R}^n\) is a linear map and \(\mathcal{F}\) is an \(A\)-invariant subspace, then we introduce the induced map \((\bar{x} := \mathbb{R}^n/\mathcal{F})\) \(\bar{A} : \bar{x} \to \bar{x}\) defined by
\[
\bar{A} \bar{x} := \frac{A}{A_{\mathcal{F}}}
\]
with \(\bar{x} = x + \mathcal{F}\).

Analogously, if \(B\) is a linear map from input space to state space, then we define the map \(\bar{B}\) from input space to quotient space \(\bar{x}\) by \(\bar{B}u := \bar{Bu}\).

Finally, the weakly unobservable subspace \(V(\Sigma)\) turns out to play an important role. We recall that it is the subspace of points \(x_0\) for which there exists a smooth input such that the resulting output is identically zero ([2, Def. 3.8], [3, Def. 3.6]).

Our result now runs as follows. First we show that \(V(\Sigma)\) is \(A_0\)-invariant with \(A_0 = A - B(D'D)^{-1}D'C\). Thus \(\bar{A}_0\) and \(\bar{B}\) are defined w.r.t. \(\mathbb{R}^n/V(\Sigma)\) and our claim then reads:

\((LQCP)^-\) is solvable \(\iff (\bar{A}_0, \bar{B})\) is stabilizable.
3. The result

We start with computing $\mathcal{V}(\Sigma)$. We split up the output $y$ in the following way. Write $D = UG$, with $U$ left orthogonal and $G$ invertible (Gram-Schmidt). If $U_c$ is left orthogonal and such that $[U, U_c]$ is both orthogonal and invertible, then for $y_1 := U'y$ and $y_2 := U_c'y$ we find

$$y_1 = U'Cx + Gu,$$
$$y_2 = U_c'Cx.$$  \hfill (3.1a, 3.1b)

Applying the preliminary feedback

$$u = - G^{-1}U'Cx + v$$  \hfill (3.2)

in (2.1a) and (3.1a) then yields

$$x = A_0x + Bv, \quad x_0,$$
$$y_1 = Bv,$$
$$y_2 = U_c'Cx,$$  \hfill (3.3a, 3.3b, 3.3c)

with $A_0 = A - BG^{-1}U'C = A - B(D'D)^{-1}D'C$. \hfill (3.3d)

Lemma.

$$\mathcal{V}(\Sigma) = \langle \ker(C_0) | A_0 \rangle$$  \hfill (3.4)

with $C_0 = (I - D(D'D)^{-1}D')C$. \hfill (3.5)

Proof. If $y_1 = 0$ and $y_2 = 0$ then $x_0 \in \ker(U_c'C)|A_0\rangle$ and conversely. Then the claim follows from the observation that $\ker(U'_c) = \ker(U_c U'_c)$ and $U_c U'_c = (I - UU') = (I - D(D'D)^{-1}D')$.

Corollary.

$\mathcal{V}(\Sigma)$ is $A_0$-invariant.

Then define $\tilde{x} := \mathbb{R}^n/\mathcal{V}(\Sigma)$ and let $\tilde{A}_0, \tilde{B}$ be the induced maps as defined above.
THEOREM.

(LQCP)” is solvable $\iff (\overline{A_0}, \overline{B})$ is stabilizable.

Proof. Consider (3.3) and let $\mathcal{X}_2$ be a subspace such that $\mathcal{V}(\mathcal{L})\mathcal{X}_2 = \mathbb{R}^n$. Then the equations in (3.3) transform into

$$
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2
\end{bmatrix} =
\begin{bmatrix}
A_{011} & A_{012} \\
0 & A_{022}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} v,
\begin{bmatrix}
X_{01} \\
X_{02}
\end{bmatrix},
$$

(3.6a)

$$
y_1 = Gv,
$$

(3.6b)

$$
y_2 = [0 \ C_2] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},
$$

(3.6c)

with $(C_2, A_{022})$ observable. In addition,

$$
J(x_0, u) = \int_0^t \mathcal{V}'(\mathcal{L})\mathcal{V} + x_2' C_2' C_2 x_2 \, dt.
$$

(3.6d)

Hence we establish from (3.6) that for solving (LQCP)” we may confine ourselves to the subsystem

$$
\begin{align*}
\dot{X}_2 &= A_{022} X_2 + B_2 v, \quad X_{02} \\
y_1 &= Gv, \quad y_2 = C_2 X_2.
\end{align*}
$$

(3.7a)

(3.7b)

Now assume that $(A_{022}, B_2)$ is stabilizable. Then the solution of (LQCP)” runs in the same manner as in (2.4)-(2.6):

$$
J^-(x_0) = x_0' K_{22}^- x_0
$$

(3.8)

with $K_{22}^-$ the smallest positive semi-definite solution of the reduced order Riccati Equation (dimension $n - \dim(\mathcal{V}(\mathcal{L}))$

$$
0 = C_2' C_2 + A_{022}' K_{22} + K_{22} A_{022} + K_{22} B_2 (D'D)^{-1} B_2' K_{22}.
$$

(3.9)

Moreover, $x_0' K_{22}^- x_0 = x_0' K^- x_0$ and the optimal control $u^-$ is given by the feedback

$$
u^- = -(D'D)^{-1} D' C x + v^-
$$

(3.10)

with $v^-. = -(D'D)^{-1} B_2' K_{22}^- x_2 = -(D'D)^{-1} B' K^- x$, recall (3.2).

Also, due to the $(C_2, A_{022})$-observability, $K_{22}^-$ is the only positive semi-definite solution of (3.9) and the resulting closed-loop matrix for $x_2$,

$$
A_{022}^- := A_{022} - B_2 (D'D)^{-1} B_2' K_{22}^-,
$$

(3.11)

is asymptotically stable ([1], [2], [5]). Thus the optimal $x_2$, $x_2^-$, tends to zero for large $t$. Observe that even $K_{22}^- > 0$. 

Conversely, suppose that for every $x_0$ a control $u^*$ exists such that $J(x_0, u^*) < \infty$.

Then instead of the infinite horizon case consider the finite horizon problem ($T > 0$)

Find

$$J(x_0, T) := \inf \{ J(x_0, u, T) | u \in U_T(0, T) \}$$

and compute, if it exists, the optimal input.

Here, of course, $J(x_0, u, T) = \int_0^T y'y \, dt$. For every $T$ the solution of this problem is ([5], [6])

$$J(x_0, T) = x_0'K_2(T)x_0$$

with $K_2(T)$ the solution of the Riccati Differential Equation

$$\dot{K}_2(t) = C_2'C_2 + A_022'K_2(t) + K_2(t)A_022 + K_2(t)B_2(D'D)^{-1}B_2'K_2(t),$$

$$K_2(0) = 0,$$

see (3.7). Now

$$x_0'K_2(T)x_0 \leq J(x_0, u^*, T) \leq J(x_0, u^*)$$

and since $K_2(T) \geq 0$ ($t \geq 0$) and non-decreasing for increasing $t$, we may conclude that $\tilde{K}_2 := \lim_{T \rightarrow \infty} K_2(T)$ exists. This means that $\tilde{K}_2$ is a positive semi-definite solution of (3.9). Now let $K_2$ be an arbitrary solution of (3.9). Then, by completing the square, one easily establishes that (3.6)

$$J(x_0, u, T) = \int_0^T \left[ v'G'Gv + x_2'C_2'C_2x_2 \right] dt = \int_0^T \left[ v' + x_2'K_2B_2(D'D)^{-1}(D'D)[v + (D'D)^{-1}B_2'K_2x_2] dt + x_0'K_2x_0 - x_2'(T)K_2x_2(T) \right].$$

Thus, for $u = -(D'D)^{-1}D'Cx + v$ with $v = -(D'D)^{-1}B_2'K_2x_2$, $J(x_0, u, T) = x_0'K_2x_0 - x_2'(T)K_2x_2(T) \geq x_0'K_2(T)x_0$ from which we conclude:

(a) the feedback law with $K_2 = \tilde{K}_2$ is optimal for (LQCP); 
(b) if $K_2$ is an arbitrary positive semi-definite solution of (3.9), then $K_2 \geq \tilde{K}_2$ ($x_2'(T)K_2x_2(T)$ always $\geq 0$).
Hence we again conclude from the \((C_2, A_{022})\)-observability that there is a feedback law for \(v\) such that the resulting closed-loop matrix for \(x_2\) is asymptotically stable (namely \(v = -(D'D)^{-1}B_2\tilde{K}_{22}x_2\)). Or, in other words, \((A_{022}, B_2)\) is stabilizable.

This completes the proof (I am indebted to Professor M.L.J. Hautus for the last part).

**COROLLARY 1.**

Let \(\tilde{A}_0, \tilde{B}\) be stabilizable. Then

\[
J^-(x_0) = \min \{J(x_0, u) \mid u \in L^2_{\text{loc}}(\mathbb{R}^4), \ u \text{ such that } (x/\sqrt{V(x)})(\infty) = 0\}.
\]

**Proof.** See the first part of the proof for the Theorem.

**COROLLARY 2.**

There exists a positive semi-definite solution of the Algebraic Riccati Equation (2.5) if \((\tilde{A}_0, \tilde{B})\) is stabilizable.

**Proof.** To start, observe that every solution \(K_{22}\) of (3.9) corresponds to a solution \(K\) of (2.5) with matrix representation

\[
\begin{bmatrix}
0 & 0 \\
0 & K_{22}
\end{bmatrix}
\]

w.r.t. the new basis (see (3.6)).

Hence if \((\tilde{A}_0, \tilde{B})\) is stabilizable, then \(\begin{bmatrix} 0 & 0 \\ 0 & K_{22}^- \end{bmatrix}\), with \(K_{22}^-\) the unique positive definite solution of (3.9), corresponds to a positive semi-definite solution of (2.5).
Conversely, for every real symmetric $K$ satisfying (2.5) we may write for the finite horizon cost criterion (see the second part of the proof for the Theorem)
\[
J(x_0, u, T) = \int_0^T [u' + x'(C'D + KB)(D'D)^{-1}](D'D) dt + [u + (D'D)^{-1}(B'K + D'C)x] dt + x_0'Kx_0 - x'(T)Kx(T),
\]
from which we conclude that for $K \succeq 0$ there is an input $u^*$ such that $J(x_0, u^*) < \infty$ (namely $u^* = - (D'D)^{-1}(B'K + D'C)x$; note again that $x'(T)Kx(T)$ is always $\geq 0$). But then, by following the line of the second part of the proof for the Theorem, we establish that $(\tilde{A}_0, \tilde{B})$ is stabilizable.

Remarks.

1. Corollary 1 is also proven in [4, Prop. 3.3].

2. Note that the smallest positive semi-definite solution $K^-$ of (2.5) has w.r.t. the new basis the matrix representation
\[
\begin{bmatrix}
0 & 0 \\
0 & K_{22}^-
\end{bmatrix}
\]
since $V(Z) = \ker K^- (K_{22}^-)$ is the unique positive definite solution of (3.9).

3. For standard problems ($C'D = 0$) it follows that $V(Z) = < \ker(C)|A>$ and $A_0 = A$. Thus $\tilde{X} = \mathbb{R}^n < \ker(C)|A>$ and the condition in the Theorem becomes: $(\tilde{A}, \tilde{B})$ is stabilizable.

4. From Corollary 2 we establish as a by-result that (2.5) has both a positive semi-definite and a negative semi-definite solution if and only if $(\tilde{A}_0, \tilde{B})$ is controllable. Note that indeed even for a Riccati Equation of the form
\[
0 = A'K + KA - KB'B'K
\]
(which has both a solution $\succeq 0$ and $\preceq 0$, namely $K \equiv 0$) this condition is trivially satisfied.
5. The Theorem stated in this article for a regular (LQCP)\(^{-}\)
\((\ker(D) = 0)\) can be generalized for the singular (LQCP)\(^{-}\),
i.e. the case that \(\ker(D) \neq 0\). Using the terminology in
[3], we have the following.

If \(\mathcal{V}_d(\Sigma) = \mathcal{V}(\Sigma) + \mathcal{W}(\Sigma)\), where the latter subspace is
defined in [3, Def. 3.3], and \(\varpi = \mathbb{R}^n/\mathcal{V}_d(\Sigma)\), then the
necessary and sufficient condition for solvability of the
(LQCP)\(^{-}\) with \(D\) not left invertible is

\((\bar{A}_D, \bar{B}_D)\) is stabilizable.

Here the matrices \(A_D\) and \(B_D\) are yielded by the
generalized dual structure algorithm, see [3, sec. 4].
Conclusion.

A necessary and sufficient condition for solvability of the linear-quadratic control problem without stability has been derived. This condition is given in terms of the original system coefficients and is furthermore necessary and sufficient for the existence of a positive semi-definite solution of the Algebraic Riccati Equation.
References.


