An exact solution for diffraction of a line-source field by a half-plane*

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This paper deals with the exact solution of a special electromagnetic diffraction problem, namely, diffraction of a line-source field by a half-plane. The line source is located on the surface of the half-plane, and radiates an E-polarized wave described by \( u_p = H_{11}^{(1)}(kr_1) \sin \phi_1 \), where \( n = 1, 2, 3, \ldots \), and \((r_1, \phi_1)\) are polar coordinates with the origin at the source point. A new, closed-form, exact solution for the total field on the shadow boundary is presented. This exact solution consists of \( n \) terms of order \( k^{-1/2} \) where \( n = 1, 2, \ldots, n \). Its first two terms, which are of orders \( k^{-1/2} \) and \( k^{-3/2} \) relative to the incident field, agree with the asymptotic solution derived in a companion paper by the uniform asymptotic theory of edge diffraction.

1. INTRODUCTION

The diffraction of a line-source field by a half-plane was treated by ray techniques in Ref. 1, referred to hereafter as Part I. The diffraction problem considered there has been sketched in Fig. 1. A perfectly conducting half-plane at \( x = 0, y = 0 \) is illuminated by a cylindrical wave due to an (anisotropic) line source located at \( (x = -d \times \cos \Omega, y = d \sin \Omega) \). By using the uniform asymptotic theory of edge diffraction, an asymptotic solution for the total field up to and including terms of order \( k^{-3/2} \) has been obtained in Part I. That solution was given in (1.2.3) and (1.2.4) for the general case, and in (1.3.6) and (1.4.4) for two special cases. (Equations from Part I are quoted by their numbers preceded by 1.) As the uniform asymptotic theory is a formal asymptotic method based on an unproved ansatz, the solution obtained from it in Part I, of course, may or may not check with the asymptotic expansion of the exact solution for the problem under consideration. In the present paper, we will derive the exact solutions for some test cases, and show that they are in complete agreement with the solution obtained by the uniform asymptotic theory.

An arbitrary cylindrical wave emanated from a line source may be considered as a superposition of the multipole fields

\[
\begin{align*}
u_l'(r_1, \phi_1) &= H_{11}^{(1)}(kr_1) \left[ \begin{array}{c} \cos \phi_1 \\ \sin \phi_1 \end{array} \right], \quad n = 0, 1, 2, \ldots, \quad (1.1a) \\
\end{align*}
\]

where \((r_1, \phi_1)\) are polar coordinates with the origin at the source point (Fig. 1). For the case \( n = 0 \) in (1.1a), i.e., when the line source is isotropic, exact solutions to the diffraction problem were first derived by Carslaw and Macdonald around the turn of the century. More easily accessible is the elegant solution due to Clemmow as described in Ref. 2, pp. 580-84. Clemmow's approach is first to decompose the Hankel function \( H_{11}^{(1)} \) as an angular spectrum of plane waves. For each plane of the wave spectrum, the Sommerfeld half-plane solution applies and, then, the total field solution is expressed as a superposition integral with the Sommerfeld half-plane solution weighted by the spectrum of the incident field as its integrand. The same approach can also be applied in principle for the cases \( n \neq 0 \). However, the superposition integrals in the latter cases become quite complex, and to our knowledge no explicit solution has been obtained. Since our main purpose is to check the validity of the asymptotic solution given in Part I, we will not solve the diffraction problem of Fig. 1 in its full generality. Instead, our attention will be focused on a test case. In this test case, we assume \( i = E_i \) (E-polarized wave), \( \Omega = 0 \) (line source on the upper surface of the half-plane), and \( \phi = \pi \) (observation point on the shadow boundary). This case corresponds to Case A discussed in Part I, Sec. 3. The incident field will be given by (1.1). Thus, the solution to be derived should eventually be compared with (1.3.9) and (1.3.11).

Our method of solution consists of two main steps. In the first one (Secs. 2 and 3), for incident fields in (1.1) with \( n = 1 \) and 2, the total field solutions are obtained through differentiation of Clemmow's solution for the isotropic line source, and the enforcement of the edge condition. Guided by those results, we then derive in Sec. 4 a recursion relation for the total field on the shadow boundary due to a general incident field with an index \( n \) in (1.1). The recurrence relation is subsequently solved by two methods in Secs. 4 and 5.

Several conventions used in this paper are stated below: (i) The time factor is \( \exp(-i\omega t) \) and is suppressed. (ii) Unless explicitly mentioned otherwise in Sec. 2, only the case of E-polarization is considered and \( n = E_i \). (iii) Three sets of polar coordinates are employed (Fig. 1): \((r, \phi)\) has its origin at the edge point \((x = 0, y = 0)\); \((r', \phi')\) at the source point \((x = -d \cos \Omega, y = d \sin \Omega)\); and \((r_0, \phi_0)\) at the image source point \((x = -d \cos \Omega, y = -d \sin \Omega)\). All angles take values between 0 and \( 2\pi \); \( \phi, \phi_1, \text{ and } \Omega \) are measured clockwise, and \( \phi_1 \) counterclockwise.

2. DIFFERENTIATION OF SOLUTIONS TO EDGE-DIFFRACTION PROBLEMS

In this section, we deduce a theorem on the differentiation of solutions to half-plane diffraction problems. In Sec. 3, this theorem will be applied to the diffraction
problem of Fig. 1 in the cases of an incident field \((1.1)\) with \(n = 1\) and 2. The solution to these problems will be obtained by differentiation of the known solution to the diffraction problem for the isotropic line source as presented below.

Referring to Fig. 1, we consider the diffraction of the cylindrical wave
\[
u^I(r_1, \phi_1) = H^{(1)}_0(kr_1) ,
\]
(2.1)
due to an isotropic line source, by the half-plane \(Y = 0\). The two cases of \(E\)-polarization and \(H\)-polarization are treated simultaneously, and the resulting total field is defined by
\[
u_1 (r, \phi) = H^{(1)}_0(kr) + \frac{4}{\pi l} \left( \frac{r}{l} \right)^{1/2} \exp(ikd) \cos \frac{1}{2} \phi + O(r) ,
\]
(2.7a)
which comply with the edge condition (2.3).

Now let us consider the diffraction of an \(E\)-polarized wave due to an anisotropic line source and given by
\[
u^I(r_2, \phi_2) = \frac{\partial}{\partial \phi} H^{(1)}_0(kr_2) .
\]
(2.8)
Because \(\partial \nu_1 / \partial x\) satisfies the boundary condition on the half-plane \(x \leq 0, \ y = 0\), it would appear that the total field for the present problem is simply given by \(\partial \nu_1 / \partial x\), where \(\nu_1\) is given in (2.4). However, such a result is incorrect since \(\partial \nu_1 / \partial x\) does not satisfy the edge condition: generally \(\partial \nu_1 / \partial x = O(r^{1/2})\) [compare with (2.7)], whereas the correct total field should be \(O(r^{1/8})\) near the edge \(r = 0\). Therefore, \(\partial \nu_1 / \partial x\) must be supplemented with an additional term that should satisfy the reduced wave equation, the radiation condition, and the boundary condition on the half-plane, and should compensate the edge singularity of \(\partial \nu_1 / \partial x\). It is easily found that the additional term is a multiple of
\[
H^{(1)}_0(kr) \sin \frac{1}{2} \phi = -i \left( \frac{2}{\pi k} \right)^{1/2} \exp(ikr) \frac{r^{1/2}}{\sqrt{2\pi}} \sin \frac{1}{2} \phi .
\]
(2.9)
The total field \(\nu_1\) due to the incident field in (2.8) is now given by
\[
\nu_1 (r, \phi) = \frac{\partial \nu_1}{\partial x} + A_1 \exp(ikr) \frac{r^{1/2}}{\sqrt{2\pi}} \sin \frac{1}{2} \phi ,
\]
(2.10)

where the constant \(A_1\) is to be determined by the requirement that at the edge \(r = 0\), the \(r^{1/2}\)-singularities in the two terms in (2.10) should cancel. It can easily be shown that (2.10) satisfies all conditions for the present diffraction problem. Hence, by relying on uniqueness, (2.10) does represent the exact total field due to diffraction of the incident field in (2.8).

Next consider the diffraction of a \(H\)-polarized wave due to an anisotropic line source and given by
\[
u^I(r_2, \phi_2) = \frac{\partial}{\partial \phi} H^{(1)}_0(kr_2) .
\]
(2.11)

By employing a similar argument as before, it is found that the total field \(\nu_2\) in this case is given by
\[
\nu_2 (r, \phi) = \frac{\partial \nu_2}{\partial y} + B_2 \exp(ikr) \frac{r^{1/2}}{\sqrt{2\pi}} \sin \frac{1}{2} \phi ,
\]
(2.12)

where \(\nu_2\) is given in (2.4). Because \(\nu_2 = 0\) on the half-plane, the tangential total electrical field at \(x \leq 0, \ y = 0\), which is proportional to
\[
\frac{\partial \nu_2}{\partial y} = \frac{\partial \nu_2}{\partial y} - B_2 \exp(ikr) \frac{r^{1/2}}{\sqrt{2\pi}} \sin \frac{1}{2} \phi ,
\]
also vanishes on the half-plane. The constant \(B_2\) in (2.12) can be determined by enforcing the edge condition \(\nu_2 (r, \phi) = O(1)\) as \(r \to 0\).

Guided by the two results in (2.10) and (2.12), we can state the following theorem for the differentiation of solutions to the half-plane diffraction problem sketched in Fig. 1.
According to the theorem in Sec. 2, the resulting total field $u(r, \phi)$ is found to be

$$u(r, \phi) = \frac{1}{k} \frac{\partial u_2}{\partial y} + B_1 \frac{\exp(ikr)}{r^{1/2}} \sin \frac{1}{2} \phi,$$  

(3.4)

where $u_2$ is given in (2.4). To determine $B_1$, the behavior of $\partial u_2/\partial y$ near the edge should be examined. With the help of (2.7b), we have

$$\frac{\partial u_2}{\partial y} = \frac{2}{\pi} \frac{\exp(ikd)}{r^{1/2}} \cos \Omega \ r^{-1/2} \sin \frac{1}{2} \phi + O(1), \quad r - 0.$$  

(3.5)

The edge condition requires that $u$ in (3.4) must be free from the $r^{-1/2}$-singularity. In view of (3.5), this requirement is satisfied if $B_1$ assumes the value

$$B_1 = \frac{2i}{\pi} \frac{\exp(ikd)}{r^{1/2}} \cos \Omega.$$

(3.6)

Thus, (3.4) and (3.6) give the exact total field (valid for all $\Omega$ and $\phi$ between 0 and $2\pi$) due to diffraction of the incident field (3.1). Specializing the solution in (3.4) for the case $\Omega = 0^+$, we have

$$u(r, \phi) = \frac{4}{i \pi k} \int_{\xi = 0} \left[ \frac{\exp(ikr)}{r^{1/2}} \sin \frac{1}{2} \phi \right] \, d\mu,$$

(3.7)

where $\xi = (r + d - r_1)^{1/2} \text{sgn}(\cos \frac{1}{2} \phi)$. Along the shadow boundary $\phi = \pi$, it is easily shown that

$$\frac{\partial u}{\partial y} \mid_{\phi = \pi} = 0,$$

(3.8)

Using (3.8) in (3.7), we obtain

$$u(r, \phi = \pi) = \frac{\exp[i(kr + kd + \frac{1}{2} \eta)]}{\pi k(r + d)^{1/2}} \cdot \frac{2^1}{2 \pi r^{1/2}} \, d\xi.$$  

(3.9)

This is the exact total field on the shadow boundary due to the incidence of (3.1) with $\Omega = 0^+$. When (3.9) is compared with the asymptotic solution in (1.3.9), they coincide.

Next consider the diffraction of the $E$-polarized wave given by

$$u'(r_1, \phi_1) = H^{11}_n(kr_1) \sin \phi_1.$$  

(3.10)

If one uses (3.2) and the relation

$$\frac{\partial}{\partial x} = - \cos \phi_1 \frac{\partial}{\partial r_1} + \sin \phi_1 \frac{1}{r_1} \frac{\partial}{\partial \phi_1},$$

(3.11)

(3.10) may be rewritten as

$$u'(r_1, \phi_1) = \frac{2}{k^2} \frac{\partial^2}{\partial x \partial y} H^{11}_n(kr_1).$$

(3.12)

On extending the theorem in Section 2, it is found that the total field $u$ in the present diffraction problem may be expressed as

$$u(r, \phi) = \frac{2}{k^2} \left[ \frac{\partial^2 u_2}{\partial x \partial y} + A_3 \frac{\exp(ikr)}{r^{1/2}} \sin \frac{1}{2} \phi \right] + B_3 \frac{\exp(ikr)}{r^{1/2}} \left[ 1 - \frac{1}{ikr} \right] \sin \frac{1}{2} \phi,$$

(3.13)

where $u_2$ is given by (2.4), and the constants $A_3$ and $B_3$ are to be determined by enforcing the edge condition. The second and third terms in (3.13) are multiples of $H^{11}_n(kr) \sin \frac{1}{2} \phi$ and $H^{11}_n(kr) \sin \frac{1}{2} \phi$, respectively; these terms do satisfy the wave equation, the radiation condition, and the boundary condition on the half-plane. Near the edge $r = 0$, it can be shown that
\[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{\pi^2} \frac{\exp(ikd)}{d^{3/2}} \cos^2 \frac{\Omega}{2} r^{1/2} \sin \frac{\phi}{2} + \frac{k \exp(ikd)}{\pi} \]
\[ \times \left(1 - \frac{1}{ikd}\right) \cos^2 \frac{\Omega}{2} r^{-1/2} \sin \frac{\phi}{2} + O(1), \quad r \to 0. \quad (3.15) \]

The edge condition requires that \( u \) in (3.13) must be free from the \( r^{-1/2} \)-singularity and the \( r^{-1/2} \)-singularity near the edge. These requirements determine \( A_3 \) and \( B_3 \) with the results
\[ A_3 = \frac{\pi}{\pi} \exp(ikd) \left(1 - \frac{1}{ikd}\right) \cos^2 \frac{\Omega}{2}, \quad (3.16a) \]
\[ B_3 = \frac{\pi}{\pi} \exp(ikd) \cos^2 \frac{\Omega}{2}. \quad (3.16b) \]

Thus, (3.13) and (3.16) give the exact total field (valid for all \( \Omega \) and \( \phi \) between \( 0 \) and \( 2\pi \)) due to diffraction of the incident field (3.10). Specializing this solution for \( \Omega = 0^+ \) and \( \phi = \pi \), we obtain
\[ u(r, \phi = \pi) = \exp[i(kr + kd + \pi)] \]
\[ \times \frac{4r^{1/2}}{\pi 2d^{3/2}} \left[1 + \frac{r + 3d}{2kd}(r + d)\right]. \quad (3.17) \]

This exact solution again verifies the asymptotic solution derived by the uniform asymptotic theory and given in (3.9).

The above procedure can be continued to derive the total field due to diffraction of a higher-order multipole field in (1.1b). However, this is not necessary. In the next two sections, we will derive a recurrence relation for the total field on the shadow boundary, due to the incidence of a general multipole field, and obtain the desired field solution by solving the recurrence relation.

### 4. Diffraction of a General Multipole Field Due to a Line Source

This section deals with the diffraction of the line-source field (1.1) with general \( n \) by the half-plane \( x \leq 0 \), \( y = 0 \) (Fig. 1). The line source is located on the upper surface of the half-plane (\( \Omega = 0^+ \)) and the incident field (1.1) is an \( E \)-polarized wave. We will determine the resulting total field on the shadow boundary \( \phi = \pi \).

Consider first the case of an incident field (1.1a) which is symmetric with respect to the plane \( y = 0 \). Then the fields produced by the source at \( (x = -d, y = 0^+) \) and its image at \( (x = d, y = 0^-) \) cancel exactly. Hence, the total field is identically zero everywhere. This result verifies the asymptotic solution derived by the uniform asymptotic theory and given in (3.11).

Next consider the diffraction of the asymmetric \( E \)-polarized wave as given in (1.1b), viz.,
\[ u_0(x_1, \phi_1) = H_{0}^{(1)}(kr_1) \sin \phi_1, \quad n = 0, 1, 2, \ldots. \quad (4.1) \]

Let the resulting total field on the shadow boundary \( \phi = \pi \) be denoted by
\[ u(r, \phi = \pi) = g_\alpha(r) \quad \text{for} \quad \Omega = 0^+, \quad (4.2) \]
then obviously
\[ g_\alpha(r) = 0 \quad (4.3) \]
and, according to (3.9) and (3.17),
\[ g_\alpha(r) = \exp[i(kr + kd + \pi/2)] \frac{2}{\pi r^{1/2}} \left[4r^{1/2} + 2d \right]. \quad (4.4) \]

We will derive a recurrence relation for the functions \( g_\alpha \). For this purpose, we observe that
\[ \frac{\partial^2 u}{\partial x^2} = -\frac{1}{2} k H_{0}^{(1)}(kr_1) \sin(n - 1)\phi_1 + \frac{1}{2} k H_{0}^{(1)}(kr_1) \sin(n + 1)\phi_1 \]
\[ = -\frac{1}{2} k u_{0n} + \frac{1}{2} k u_{0n+1}, \quad n = 1, 2, 3, \ldots, \]
where (3.11) and some well-known recurrence relations for the Hankel function have been used. In view of (4.6), the total field on the shadow boundary, due to the incident field \( \partial u_0/\partial x \), is then equal to
\[ -\frac{1}{2} k u_{0n+1} + \frac{1}{2} k u_{0n}, \quad (4.7) \]

On the other hand, referring to the theorem in Sec. 2, the total field is also given by
\[ \frac{\partial}{\partial x} g_\alpha(r) = C_n \exp[ikr/\sqrt{2}] = g'_\alpha(r) - C_n \exp[ikr/\sqrt{2}], \quad (4.8) \]
due to the constants \( C_n \) are determined by the requirement that the \( r^{1/2} \)-singularity in the total field at the edge should vanish (edge condition); hence,
\[ C_n = \lim_{r \to 0^+} r^{1/2} g'_\alpha(r). \quad (4.9) \]

By equating (4.6) and (4.8), a recurrence relation is obtained,
\[ g_{0n}(r) = g_{0n}(r) + \frac{2}{k} g'_\alpha(r) - \frac{2C_n}{k} \exp[ikr/\sqrt{2}], \quad n = 1, 2, 3, \ldots. \quad (4.10) \]

It has been verified that \( g_{00}, g_1, \) and \( g_2 \) in (4.3)—(4.5) do satisfy (4.10). The field \( g_\alpha \) are now completely specified by (4.10) and the “initial values” \( g_0 \) and \( g_1 \) in (4.3) and (4.4).

We will now solve the recurrence relation in (4.10). Because \( g_\alpha \) obviously consists of \( n \) terms of order \( k^n \), \( p = 1, 2, \ldots, n \), we can introduce the ansatz
\[ g_\alpha(r) = \frac{2i}{\pi} \exp[i(kr + kd + \frac{1}{2} j \pi)] \sum_{n}^n (ik)^n A_n g_n(r), \quad (4.11) \]
where the coefficients \( A_n \) and the functions \( g_n \) are to be determined. The ansatz is rather special in that \( g_n \) do not depend on \( n \), i.e., all \( g_n \) are expressed in terms of the same set of \( g_n \). This choice is suggested by (4.4) and (4.5) where the leading terms contain the same function of \( r \). Without loss of generality we may assume
\[ A_{00} = 1. \quad (4.12) \]

### A. Determination of \( G_n \)

A comparison of (4.4) and (4.5) with (4.11) yields immediately
\[ G_1(r) = \frac{\pi^{1/2}}{d^{1/2} (r + d)} \quad G_2(r) = \frac{\pi^{1/2} (y + 3d)}{d^{1/2} (r + d)}. \quad (4.13) \]
By introducing the notation
\[
\Gamma_\rho = \lim_{t \to \pm 1/2} \rho^{t/2} G_p(r),
\tag{4.14}
\]
(4.15) becomes
\[
\gamma = 2^{2\rho} \frac{\pi}{\rho} \exp(\rho^2) \sum_{k=1}^{\infty} \frac{\rho^k}{2^k} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_\rho A_{n,p}.
\tag{4.16}
\]

Inserting (4.11) and (4.15) into (4.10) and equating corresponding terms containing the same power of \( k^1 \), we obtain
\[
A_{n+1,p} G_p(r) = -A_{n+1,0} G_p(r) + 2 A_{n,p} G_p(r) + 2 A_{n-1,p} [G_{n-1}(r) - \Gamma_{n-1} r^{-1/2}].
\tag{4.17}
\]
In (4.16) we set \( p = n + 1 \); then, in view of \( A_{0,0} = A_m \), \( A_{n-1,0} = A_{n+1,0} = 0 \), we have
\[
G_{n+1}(r) = 2 G_n(r) - 2 r^n r^{-1/2}, \quad n = 1, 2, 3, \ldots.
\tag{4.18}
\]

This equation recursively defines the functions \( G_p \). It has been verified that \( G_2 \) in (4.13) does satisfy (4.17).

Then, by use of a well-known integral representation for the hypergeometric function (see Ref. 7, Chapter 15),
\[
G_p(r) = \frac{1}{2^2} \sum_{q=1}^{\infty} \frac{(q + \frac{1}{2}) (q + \frac{3}{2}) \cdots (q + \frac{1}{2}) (q + \frac{1}{2})}{(q + \frac{1}{2}) (q + \frac{1}{2})} \cdot \frac{1}{d^q} (r/d)^q d^q,
\tag{4.19}
\]
which is valid for \( 0 < r < d \). Starting from (4.19), \( G_p(r) \) may be written as a hypergeometric function (see Ref. 7, Eq. (15.3.1)),
\[
G_p(r) = \frac{(-1)^q}{2^q} (\frac{2^q - 1}{2^q - 1}) \frac{1}{d^q} (r/d)^q F(p + \frac{1}{2}, 1; \frac{3}{2}; -r/d),
\tag{4.20}
\]
where \( F \) is a hypergeometric function, (Ref. 7, Eq. (15.3.1)).

Then, by use of a well-known integral representation for the hypergeometric function [Ref. 7, Eq. (15.3.1)],
\[
G_p(r) = \frac{(2^q - 1)!}{2^q (q!)^2} \frac{1}{d^q} (r/d)^q F(p + \frac{1}{2}, 1; \frac{3}{2}; -r/d),
\tag{4.21}
\]
The latter representation for \( G_p \), which is an analytic continuation of the one in (4.19), is valid for all \( r \) between 0 and \( \infty \). Alternatively, by use of a linear transformation formula for the hypergeometric function (Ref. 7, Eq. (15.3.12)), \( G_p \) can be reduced to
\[
G_p(r) = (1)^{2q} (p - 1)^{1/2} \frac{1}{d^q} (r/d)^q F(p + \frac{1}{2}, 1; \frac{3}{2}; -r/d), \quad p = 1, 2, 3, \ldots.
\tag{4.22}
\]
The result in (4.22) is the desired final expression for the functions \( G_p \).

B. Determination of \( A_{n,p} \)

The use of (4.17) in (4.16) leads to
\[
A_{n+1,p} = -A_{n+1,0} + 2 A_{n,p} + A_{n-1,p} - 1,
\tag{4.23}
\]
which can be solved in a standard manner subject to the side condition (4.12). The result is
\[
A_{n} = \left( \frac{n + p - 1}{2p - 1} \right),
\tag{4.24}
\]
where \( (\cdot) \) denotes the binomial coefficient.

In summary, the exact total field on the shadow boundary \( \phi = \pi \), due to diffraction of an incident \( E \)-polarized wave in (4.1) with \( \omega = 0 + \), is given by an \( n \)-term sum, namely,
\[
g_n(r) = \frac{2i}{\pi} \exp[i(kr + kd + \frac{1}{2} n)]
\times \sum_{n=1}^{\infty} \frac{n + p - 1}{2p - 1} (ik)^n G_n(r), \quad n = 1, 2, 3, \ldots,
\tag{4.25}
\]
where \( \{G_n\} \) are given in (4.22). The first two terms of (4.25) are
\[
g_n(r) = \exp[i(kr + kd + \frac{1}{2} n)] \frac{2n r^1}{\pi k(r + d) d^1} \left[ 1 + i (n^2 - 1) (r + 3d) \right] + O(k^2),
\tag{4.26}
\]
which agrees with the asymptotic solution in (1.3.9), derived from the uniform asymptotic theory. For the case \( n = 1 (n = 2) \), there is only one term (two terms) in (4.25); thus, the asymptotic solution in (1.3.9) becomes exact in these cases.

5. ALTERNATIVE SOLUTION OF THE RECURRENCE RELATION

For the diffraction of the \( E \)-polarized wave given in (4.1) with \( \Omega = 0 + \), the resulting total field on the shadow boundary is denoted by \( g_n(r) \) as indicated in (4.2). We will now present an alternative method for solving the recurrence relation for \( \{G_p\} \) in (4.10), or
\[
\delta_{m1} = \frac{2i}{\pi} G_n(r) - \frac{2 C_n \exp ik r}{k} \frac{n + p - 1}{2p - 1}, \quad n = 1, 2, 3, \ldots.
\tag{5.1}
\]
In this method, the integral in (4.21) is obtained in a more natural manner.

Consider first the constants \( \{C_n\} \) as defined by (4.9). From (4.4) and (4.5), it may be shown that \( C_1 \) and \( C_2 \) may be expressed in terms of Hankel functions of half-integral order and of argument \( kd \):
\[
C_1 = \frac{i}{\pi k} \exp(ikd) \frac{1}{d^{1/2}} \left[ H_{1/2}^{(1)}(kd) + i H_{1/2}^{(2)}(kd) \right],
\tag{5.2a}
\]
\[
C_2 = \frac{2}{\pi k} \exp(ikd) \frac{1}{d^{1/2}} \left[ H_{1/2}^{(1)}(kd) + i H_{1/2}^{(2)}(kd) \right].
\tag{5.2b}
\]
Guided by these results, it is conjectured that
\[
C_n = \left( \frac{-1}{2} \right)^{n} \left( \frac{k}{2 \pi} \right)^{1/2} \left[ H_{1/2}^{(1)}(kd) + i H_{1/2}^{(2)}(kd) \right].
\tag{5.3}
\]
It has been verified that this conjecture also holds true for $C_0$ and $C_1$.

Furthermore, observe that (5.1) and (5.3) remain valid when $n$ is a negative integer. As a matter of fact, for negative $n$, the incident field in (4.1) becomes

$$u(r_1, \phi_1) = H_{p+1}^{-1}(kr_1) \sin(-n\phi_1) = (-1)^{m+n} H_{p+1}(kr_1, \phi_1).$$

Thus, the associated total field $g_n$ and constant $C_n$ satisfy

$$g_n(r) = (-1)^{n+1} g_n(r), \quad C_n = (-1)^m C_n.$$  

It is easily seen that (5.1) and (5.3) are consistent with the symmetry relation in (5.5).

The recurrence relation (5.1), valid now for all positive and negative integer $n$, is solved next by a formal generating-function technique. Introduce the generating function

$$F(r, \theta) = \sum_{n=1}^{\infty} g_n(r) \exp(in\theta),$$

then (5.1) implies the following differential equation for $F$:

$$\frac{\partial}{\partial r} F(r, \theta) + ik \sin \theta F(r, \theta) = C(\theta) \frac{\exp(ikr)}{r^{1/2}},$$

where

$$C(\theta) = \sum_{n=1}^{\infty} C_n \exp(in\theta).$$

The differential equation (5.7) is solved by variation of parameters. Since $F(r, \theta), r = 0$ at $r = 0$ (edge condition), we obtain the solution

$$F(r, \theta) = C(\theta) \int_0^r \exp(ik[t-(r-t)\sin \theta]) t^{1/2} dt.$$

Using the well-known generating function for Bessel functions [Ref. 7, Eqs. (9.1.42) and (9.1.43)], one has

$$\exp(-ik(r-t)\sin \theta) = \sum_{p=0}^{\infty} J_p(k(r-t)) \exp(-ip\theta)$$

and (5.8) can be rewritten as

$$F(r, \theta) = \sum_{n=1}^{\infty} C_n \exp(i\theta \sum_{p=0}^{\infty} \exp(-ip\theta) \int_0^r J_p(k(r-t)) \exp(ikt) t^{1/2} dt.$$  

Comparing (5.10) and (5.6), we immediately deduce that

$$g_n(r) = \sum_{p=0}^{\infty} C_{pn} \int_0^r J_p(k(r-t)) \exp(ikt) t^{1/2} dt.$$  

Substitute the conjectured values (5.3) for $C_{pn}$ into (5.11), then by use of the following identities [See Ref. 7, Eq. (9.1.79)]:

$$\sum_{p=0}^{\infty} (-1)^p H_{p+1}^{(1)}(kr) J_p(k(r-t)) = H_{p+1}^{(1)}(kr),$$

$$\sum_{p=0}^{\infty} (-1)^p H_{p+1}^{(1)}(kr) J_p(k(r+t)) = H_{p+1}^{(1)}(kr),$$

$$t^{1/2} \exp(ikt) = i \left( \frac{\pi k}{2} \right)^{1/2} H_{1/2}^{(1)}(kt),$$

we obtain the desired solution for $g_n(r)$, namely,

$$g_n(r) = \frac{1}{2}(-1)^n i \int_0^r H_{n+1}^{(1)}(k(r-t)) dt + i H_{n+1}^{(1)}(k(r+t)) H_{1/2}^{(1)}(kt) dt, \quad n = 1, 2, 3, \cdots.$$  

The result in (5.15) is an exact representation of the total field on the shadow boundary. The derivation of (5.15) is based on the conjectured values (5.3) for $C_{n}$ and a formal generating-function technique. The convergence of the series involved and the interchange of the order of summation and integration were not seriously studied. Thus, (5.15) requires the following additional verification:

(i) Determine $C_n$ from (4.9) and (5.15), then the conjectured value (5.3) is precisely recovered.

(ii) By direct substitution, the solution in (5.15) has been shown to satisfy the recurrence relation (5.1).

This verification shows that $g_n$ is given by the exact representation (5.15).

To derive a more explicit solution from (5.15), we may express the Hankel function in terms of elementary functions,

$$H_{n+1}^{(1)}(z) = \left( \frac{2}{\pi} \right)^{1/2} \exp[i(z - \frac{1}{2} \pi)]$$

$$\times \sum_{p=0}^{\infty} \frac{1}{(n+p)!} (-1)^p \frac{1}{2p!} (2iz)^p.$$  

As a result, (5.15) is reduced to

$$g_n(r) = \frac{2i}{\pi} \exp[i(kr + kd + \frac{1}{2} \pi n)] \sum_{p=0}^{\infty} \frac{(n+p)!}{(n-p)!} \frac{1}{(2p)!}$$

$$\times (-1)^p z^p \int_0^r (r + d - t)^{n+1} t^{1/2} dt.$$  

With the help of the representation (4.21) of $G_p(r)$, (5.17) may be rewritten as

$$g_n(r) = \frac{2i}{\pi} \exp[i(kr + kd + \frac{1}{2} \pi n)]$$

$$\times \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} (2i)^p G_p(r)$$  

which agrees with (4.25), the solution obtained by the first method.

6. DISCUSSION AND NUMERICAL RESULTS

In the present paper, the exact solution to the diffraction of a line-source field by a half-plane is studied by analytical methods. When the incident field given in (4.1) is an $E$-polarized wave and is due to a line source located on the upper surface of the half-plane, the exact total field on the shadow boundary is given in (4.25), which is an $n$-term sum ($n$ is the index of the incident field), or in (5.15), which is a finite integral. The first two terms of (4.25) agree with the asymptotic solution determined by the uniform asymptotic theory in Part I.

For a given incident field (fixed $n$), the total field in (4.25) or (5.15) depends on two parameters $d$ and $r$, which are the distances from the edge to the source,
and to the observation point, respectively. For the extreme case \((r/d) = 0\) (near field or faraway source), it is found from (5.15) that

\[
g_s(r) = (-1)^n \left( \frac{2kr}{\pi} \right)^{1/2}
\times \exp[ikr]H_{1/2}^{(1)}(kd) + iH_{1/2}^{(2)}(kd) \left[ 1 + O\left( \frac{r}{d} \right) \right]
\]

\[
= \frac{4i}{\pi} \exp[i(kr + kd + \frac{1}{2}n\pi)] \left( \frac{r}{d} \right)^{1/2}
\times \sum_{p=1}^{n} \frac{(n+p-1)!}{(n-p)!(p-1)!} \frac{(-1)^{p+1}}{(2ikd)^p}
\times \left[ 1 + O\left( \frac{r}{d} \right) \right], \quad \left( \frac{r}{d} \right) < 0.
\]

\[
(6.1)
\]

For the other extreme case \((d/r) = 0\) (far field or nearby source), it can be shown from (4.22) and (4.25) that

\[
\overline{g}_s(v) = \exp[-i(kr + kd + \frac{1}{2}n\pi)]g_s(r)
\]

\[
(6.3)
\]
as a function of $kr$ for two incident fields $n = 2$ and $n = 4$; $\overline{g}_n$ is calculated from (4.23) and (4.22) with one, two, \ldots, or $n$ terms in the sum, where the one with $n$ terms is the exact solution. Since $kd = 2\pi$ is relatively small and $n = 4$ corresponds to a rapidly varying incident field, the curves calculated with one or two terms in the sum in Fig. 3 do not converge well to the exact solution. In particular, we note in Fig. 3 that the curves calculated with one term show a reasonable magnitude but the phase is far off.

The poor convergence mentioned above becomes less serious as $kd$ is increased, as indicated in (6.1) and (6.2). In Fig. 4, we reconsider the case presented in Fig. 3 but with $kd = 6\pi$ (triple the previous value). The curves calculated with two terms already give good results in both magnitude and phase.

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