Feedback decoupling and stabilization for linear systems with multiple exogenous variables

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FEEDBACK DECOUPLING AND STABILIZATION
FOR LINEAR SYSTEMS
WITH MULTIPLE EXOGENOUS VARIABLES

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In the present monograph we study the existence and design of automatic controllers for dynamical systems for which the interaction with the outside world plays an important role.

A (simplified) example of such a system is a blending process. In a blending process one mixes two or more gasses (or liquids) with different concentrations of some substance in order to obtain a blend with a prescribed concentration of that substance. In such a process, the concentrations in the incoming gasses are quantities that are brought into the process from the outside, whereas the concentration in the blend may be relevant in further processing outside the process. In view of such later processing we assume that it is desired to have the concentration in the blend constant. Then, if by some means the concentrations in the incoming gasses are known and are constant, we can calculate the amounts of flow of the different incoming gasses necessary for a proper blending. If, in addition, flow control valves are present in the pipes that take care of the transportation of the incoming gasses, we can actually accomplish that the proper amounts of the flow of the different gasses meet in the blending point.

However, if for any reason the concentrations in the incoming gasses change, also the amounts of flow of the different gasses needed for a proper blending may change. In turn, this may imply that also the valves have to be adjusted. In order to find out how we have to adjust these valves, we have to know what the concentrations (or the changes of the concentrations) in the incoming gasses are. Therefore, it is necessary to have equipment that can measure the concentrations in the incoming gasses and perhaps also in the blend itself.

As announced above, in the present work we are interested in automatic controllers. For our blending process, such automatic controllers will be mechanisms (software or hardware) that based on signals of the measurement
devices, representing the measurements, produce signals, representing valve adjustments, that can be sent to the valve engines. Such control devices are feedback mechanisms because they feed information about the process back to the controls of the process.

The automatic controller that we would like to have for our blending process has to be such that in the controlled process the changes in the concentration of the substance in the blend are zero, no matter what the changes in the concentration of the substance in the incoming gasses are. If we can construct such a controller, we say that we have been able to achieve decoupling (between the changes in concentration of the incoming gasses and the outgoing blended gas) by means of (measurement) feedback.

We stress the fact that the above example is just a very simplified presentation of reality. Nevertheless, the example should make clear that the systems we are interested in may have two types of input and two types of output.

The first type of input, called control input, represents the actions that one can undertake to regulate the system. The second type of input, called exogenous input, stands for (unknown) influences entering the system from outside. The first type of output, called measurement output, reflects the state of part of the variables involved in the system. The second type of output, called exogenous output, represents the outcome of the system relevant to the outside world.

In contrast to the above example, all systems in this monograph will be formulated mathematically only and will have a finite-dimensional linear time-invariant state space representation.

If, for a given system, we have an undesired transfer from the exogenous input(s) to the exogenous output(s), this feature may be reflected in the measurement output. Then, noticing this, we can try to improve and compensate this undesired behavior by applying an appropriate control input. Since we are interested in automatic control, we assume that these control inputs are generated by a dynamical system that is driven by the measurement outputs. We call this type of controller dynamic (measurement) feedback and we assume that these feedback systems also have a finite-dimensional linear time-invariant state space representation.
The control problems that we study for a given system will be formulated in terms of finding a dynamic feedback such that the 'controlled' or 'closed loop' system satisfies certain, in advance given properties. We say that a given control problem is solvable if we can actually find and construct a dynamic feedback that meets the requirements of the problem.

In order to tackle the control problems in an elegant way, we shall adopt the so-called 'geometric approach' towards linear control theory. For a fundamental treatment of this approach, we refer to Wonham (1979). The generalizations and modifications of the theory presented in this reference that we shall need, can be found in Schumacher (1981) and Willems (1981), (1982), respectively.

The main features of the control problems successfully solved by this 'geometric approach' involve decoupling (see example above), almost decoupling, internal stabilization and input/output stabilization.

In Chapter 1 of this monograph we introduce all the system theoretic concepts needed for the development of later chapters. Furthermore, in Chapter 1 we introduce most of the notation.

In Chapter 2 we study our first actual control problem. The basic motivation for the study of the problem lies in that fact that it includes the extension towards measurement feedback of control problems considered in Houtus (1980) and Trentelman (1986).

In Chapters 3 and 4 we consider the extension towards measurement feedback of problems of 'noninteracting control' and 'almost noninteracting control' as formulated in Willems (1980). Also in Chapter 3, we extend and give alternative frequency domain oriented proofs of some of the results in the latter reference.

Finally, in the spirit of Chapters 3 and 4, and inspired by Wonham (1979, Section 9.8), in Chapter 5 we formulate and study a number of new problems concerning 'triangular decoupling' and 'almost triangular decoupling' by measurement feedback.
In this first chapter we shall introduce all the systems theoretic concepts that will be important in the development of later chapters where we study actual control problems. Furthermore, in the present chapter we shall introduce most of the notation that we use throughout this work.

The systems theoretic concepts to be introduced all have their origin in the geometric approach towards linear control theory (cf. Wonham (1979)). Originally, those concepts were defined in the time domain using state space descriptions of linear systems. In this monograph we shall introduce the concepts in a uniform way by means of frequency domain descriptions, also starting from state space representations of linear systems. The connection between our definitions and the original description in the time domain, and various properties of the concepts will be given in the form of propositions and theorems.

The linear system that we shall use for the introduction of the various concepts, together with its dual, will be described in Section 1.1. Also in Section 1.1, we review some elementary facts from linear algebra and matrix theory. In Section 1.2 we describe what we mean by stability and stabilizability of linear systems. In Section 1.3 we discuss the notion of invariance and the ability to achieve invariance for linear systems. Section 1.4 will consist of a combination of some of the concepts introduced in Sections 1.2 and 1.3. Finally, in Section 1.5 we give an introduction to the notion of dynamic feedback (cf. Schumacher (1981)). This notion will play an important role in this monograph.
1.1. Basic concepts and notation

In this section we start with giving the description of the type of system that plays an important role throughout this monograph. These systems consist of a combination of a linear inhomogeneous first order ordinary differential equation and a linear algebraic equation:

\[
\begin{align*}
(1.1a) \quad & x(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad (x(t) = \frac{d}{dt} x(t)) , \\
(1.1b) \quad & y(t) = Cx(t) .
\end{align*}
\]

Here, the variables \(x, u\) and \(y\) are real-valued vectors and \(A, B\) and \(C\) are real constant matrices of suitable dimensions. The independent variable \(t\) will be interpreted as time.

As announced, systems of type (1.1) play an important role in the present work. On the one hand, this type of system will be used in the present chapter for the introduction of some systems theoretic concepts needed in the development of later chapters. On the other hand, systems of type (1.1) can be considered to be the fundament of all the systems that we shall study in those later chapters. In fact, there the systems always consist of a differential equation, obtained from (1.1a) by adding extra terms to the right-hand side of (1.1a), the algebraic equation (1.1b) and a number of additional equations of type (1.1b). Consequently, the variables \(x, u\) and \(y\) will be present in every system that we shall study. Throughout this monograph we call the variable \(x\) the state, the variable \(u\) the (control) input and the variable \(y\) the (measurement) output of the system.

Furthermore, we always assume that \(x\) is a real \(n\)-vector, \(u\) is a real \(m\)-vector and \(y\) is a real \(p\)-vector. Correspondingly, \(A\) is a real \(n \times n\)-matrix, \(B\) is a real \(n \times m\)-matrix and \(C\) is a real \(p \times n\)-matrix.

In the present work we denote the field of real numbers by \(\mathbb{R}\) and the field of complex numbers by \(\mathbb{C}\). Furthermore, we denote the real linear space of \(a\)-vectors with real components by \(\mathbb{R}^a\) and the real linear space of \(a \times b\)-matrices with real entries by \(\mathbb{R}^{a \times b}\). Hence, formulated in this notation, we assume throughout that \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}^m\) and \(y(t) \in \mathbb{R}^p\) for all \(t \geq 0\), and \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\) and \(C \in \mathbb{R}^{p \times n}\).
For obvious reasons the equation (1.1a) will be called the state evolution equation and the equation (1.1b) will be called the (measurement) output equation.

Throughout the present work we assume that any control input \( u: [0, \infty) \to \mathbb{R}^m \) is a piecewise continuous function. Given such a control input and the state at time \( t = 0 \), \( x(0) = x_0 \), the solution of (1.1a) is given by the well-known variation of constants formula

\[
(1.2) \quad x(t) = e^{tA} x_0 + \int_0^t e^{(t-\tau)A} Bu(\tau) d\tau.
\]

Note that the linear system described by (1.1) is completely determined by the matrices \( A, B \) and \( C \).

By letting \( T \) denote matrix transposition we can also associate another system of type (1.1) to the matrices \( A, B \) and \( C \). This system is called the dual system of (1.1) and is described by the following equations:

\[
(1.3a) \quad \dot{x}'(t) = A^T x'(t) + C^T u'(t),
\]
\[
(1.3b) \quad y'(t) = B^T x'(t).
\]

We want to point out that our notion of dual system is slightly different from the usual notion of dual system (cf. Kalman (1969), Brockett (1970)).

We conclude the present section by reviewing some basic facts from matrix theory and linear algebra. First of all, we shall use the symbol \( 0 \) for anything that is zero (as a number, vector, matrix, linear subspace, etc.). If \( M \in \mathbb{R}^{a \times b} \) is a given matrix, then we denote its image by \( \text{im} M \) and its nullspace by \( \ker M \), i.e.

\[
\text{im} M = \{ w \in \mathbb{R}^a \mid \exists v \in \mathbb{R}^b : w = Mv \} \quad \text{and} \quad \ker M = \{ v \in \mathbb{R}^b \mid Mv = 0 \}.
\]

Furthermore, we say that the matrix \( M \) is injective (or: has full column rank) if \( \ker M = 0 \), surjective (or: has full row rank) if \( \text{im} M = \mathbb{R}^a \) and regular (or: nonsingular or invertible) if the matrix \( M \) is injective as well as surjective. The latter implies that a regular matrix is square and has full rank. We denote the inverse of a regular matrix \( M \) by \( M^{-1} \), i.e.

\[
M M^{-1} = M^{-1} M = I,
\]

where \( I \) denotes the identity matrix of appropriate
dimensions. All the linear subspaces that we work with will be real vector spaces, where sometimes we use standard complexifications (cf. Arnold (1973)).

If \( V_1 \) and \( V_2 \) are two linear subspaces in \( \mathbb{R}^b \), then we denote their sum by \( V_1 + V_2 \) and their intersection by \( V_1 \cap V_2 \). \( V_1^\perp \) will denote the orthogonal complement of \( V_1 \) in \( \mathbb{R}^b \) with respect to the standard euclidean inner product in \( \mathbb{R}^b \). If \( V_1 \) is a linear subspace in \( \mathbb{R}^b_1 \) and \( V_2 \) is a linear subspace in \( \mathbb{R}^b_2 \), then \( V_1 \oplus V_2 \) denotes the linear subspace in \( \mathbb{R}^{b_1+b_2} \) defined as

\[
V_1 \oplus V_2 := \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mid v_1 \in V_1, v_2 \in V_2 \right\}.
\]

If \( M \in \mathbb{R}^{b \times b} \) is a given square matrix then \( \sigma(M) \) denotes the set of \( b \) complex zeros of the characteristic polynomial of \( M \). Furthermore, if \( V \) is a linear subspace in \( \mathbb{R}^b \), then we say that \( V \) is an \( M \)-invariant subspace if \( MV \subseteq V \).

If \( V_1 \) and \( V_2 \) are two linear subspaces in \( \mathbb{R}^b \) such that \( V_1 \subseteq V_2 \), then there exists a (nonunique) regular matrix \( S \in \mathbb{R}^{b \times b} \) that can be partitioned as \( S = [S_1, S_2, S_3] \) such that \( \text{im } S_1 = V_1 \) and \( \text{im } [S_1, S_2] = V_2 \). If, in addition, \( V_1 \) and \( V_2 \) are \( M \)-invariant subspaces, then with respect to the basis in \( \mathbb{R}^b \) formed by the columns of \( S \), the matrix \( M \) has the following form:

\[
M = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
0 & M_{22} & M_{23} \\
0 & 0 & M_{33}
\end{bmatrix},
\]

where \( M_{11}, M_{22} \) and \( M_{33} \) are square matrices.

Clearly, \( \sigma(M) = \sigma(M_{11}) \cup \sigma(M_{22}) \cup \sigma(M_{33}) \) where \( \cup \) denotes the union with any common elements repeated (cf. Wonham (1979)).

We define \( \sigma(M|V_1) := \sigma(M_{11}) \) and \( \sigma(M|V_2/V_1) := \sigma(M_{22}) \). It can be proved that this definition is independent of the particular choice of the matrix \( S = [S_1, S_2, S_3] \).
1.2. Stability issues

In the present section we introduce the concepts of stability, stabilizability and detectability for the class of linear systems described by (1.1). To this end, we consider a linear system of type (1.1) and for a start we assume that $B = 0$. Furthermore, for the time being, we do not pay any attention to the measurement output equation (1.1b). Hence, we are dealing with the linear autonomous system described by the homogeneous differential equation:

$$
(1.4) \quad \dot{x}(t) = Ax(t).
$$

From (1.2) we know that for a given initial state $x_0$ the solution of (1.4) reads $x(t) = e^{tA}x_0$. As an introduction to the general definition of stability of system (1.4) we now define:

**Definition 1.1:**

The linear system (1.4) is called *asymptotically stable* if

$$
\lim_{t \to \infty} x(t) = 0
$$

for every initial state $x_0 \in \mathbb{R}^n$.

The following theorem is well known (cf. Arnold (1973)).

**Theorem 1.2:**

The following statements are equivalent:

1. The linear system (1.4) is asymptotically stable.
2. $\sigma(A) \subseteq \mathbb{C}^-$, where $\mathbb{C}^- := \{ z \in \mathbb{C} \mid \text{Re } z < 0 \}$.
3. The rational matrix $(sI - A)^{-1}$ has all its poles in $\mathbb{C}^-$.

Throughout this monograph we denote the set of rational functions with real coefficients by $\mathbb{R}(s)$. $\mathbb{R}_0(s)$ and $\mathbb{R}_+^a(s)$ will denote the set of proper rational functions with real coefficients and the set of strictly proper rational functions with real coefficients, respectively. $\mathbb{R}_-^{a\times b}(s)$, $\mathbb{R}_0^{a\times b}(s)$ and $\mathbb{R}_+^{a\times b}(s)$ will denote the set of $a \times b$-vectors with entries in $\mathbb{R}(s)$, $\mathbb{R}_0(s)$ and $\mathbb{R}_+(s)$, respectively, while $\mathbb{R}_-^{a\times b}(s)$, $\mathbb{R}_0^{a\times b}(s)$ and $\mathbb{R}_+^{a\times b}(s)$ will denote the set of $a \times b$-matrices with entries in $\mathbb{R}(s)$, $\mathbb{R}_0(s)$ and $\mathbb{R}_+(s)$, respectively.
If $\sum_{i=1}^{2N} \xi_i s^{-i}$ with $\xi_i \in \mathbb{R}^b$ for all $i \geq N$ is the power series of a rational vector $\xi \in \mathbb{R}^b(s)$, i.e. $\xi(s) = \xi_i s^{-i}$, and if $V$ is a linear subspace in $\mathbb{R}^b$, then we say that $\xi \in V$ if $\xi_i \in V$ for all $i \geq N$. In particular, we say that $\xi = 0$ if $\xi_i = 0$ for all $i \geq N$.

By Theorem 1.2, asymptotic stability of the system (1.4) is related to the particular subset $\mathbb{C}^-$ of the complex plane. In the present work we shall relate stability of the system (1.4) to more general subsets of the complex plane, so-called 'stability regions'. We shall restrict ourselves to 'symmetric' stability regions. That is, we shall consider the set of stability regions $\emptyset$, defined by $\Theta := \{\Gamma \subseteq \mathbb{C} | \lambda \in \Gamma \Rightarrow \overline{\lambda} \in \Gamma \text{ and } \mathbb{R} \cap \Gamma \neq \emptyset\}$. Here, the overbar denotes complex conjugation. It is clear that $\mathbb{C}^- \subseteq \Theta$.

Throughout this first chapter we assume that a stability region $\mathbb{C}_g \subseteq \Theta$ is given ('g' stands for 'good'), and we call a rational function, vector or matrix stable if the latter has all its poles in $\mathbb{C}_g$.

(In later chapters we shall often be dealing with pairs of stability regions, $\mathbb{C}_f, \mathbb{C}_s \subseteq \Theta$ with $\mathbb{C}_f \equiv \mathbb{C}_s$ ('f' stands for 'fast' and 's' stands for 'slow'). There, we shall call a rational function, vector or matrix $f$-stable, $s$-stable if the latter has all its poles in $\mathbb{C}_f, \mathbb{C}_s$, respectively.) Thus, given $\mathbb{C}_g \subseteq \Theta$, inspired by the third statement of Theorem 1.2, we define:

**Definition 1.3:**
The linear system (1.4) is called stable if the rational matrix $(sI - A)^{-1}$ is stable.

Then we have:

**Theorem 1.4:**
The linear system (1.4) is stable if and only if $\sigma(A) \subseteq \mathbb{C}_g$.

As an alternative for Definition 1.3 we could have defined:
The linear system (1.4) is called stable if for every initial state $x_0 \in \mathbb{R}^n$ the rational vector $(sI - A)^{-1}x_0$ is stable, or:
The linear system (1.4) is called stable if for every initial state $x_0 \in \mathbb{R}^n$ there exists a stable rational vector $\xi \in \mathbb{R}^n(s)$ such that $x_0 = (sI - A)\xi(s)$. 
It is clear that in case the linear system (1.4) is not stable, there may still exist initial states \( x_0 \in \mathbb{R}^n \) such that the rational vector \((sI-A)^{-1}x_0\) is stable. Therefore the following definition makes sense.

**Definition 1.5:**

\[ \chi_g(A) := \{ x_0 \in \mathbb{R}^n \mid \text{the rational vector } (sI-A)^{-1}x_0 \text{ is stable} \}. \]

It can be shown that \( \chi_g(A) \) is the sum of the generalized eigenspaces of \( A \) corresponding to the eigenvalues of \( A \) in \( \mathcal{G} \). Furthermore, if

\[
p(s) = \det(sI-A) = p_g(s)p_d(s),
\]

where \( p_g \) is a real polynomial with all its zeros in \( \mathcal{G} \) and \( p_d \) is a real polynomial with all its zeros in \( \mathcal{G}' := \{ z \in \mathcal{G} \mid z \notin \mathcal{G} \} \) then \( \chi_g(A) = \ker p_g(A) \) (cf. Wonham (1979)). Likewise, \( \chi_b(A) \), the sum of the generalized eigenspaces of \( A \) corresponding to the eigenvalues of \( A \) in \( \mathcal{G}' \), is equal to \( \ker p_b(A) \).

We now return to the linear system described by (1.1a) and we drop the assumption that \( B = 0 \). As an introduction to the concept of stabilizability for systems described by (1.1a) we define:

**Definition 1.6:**

The linear system (1.1a) is called asymptotically stabilizable if for every initial state \( x_0 \in \mathbb{R}^n \) there exists a piecewise continuous control input \( u: [0, \infty) \to \mathbb{R}^m \) such that \( \lim_{t \to \infty} x(t) = 0 \).

Then we have (cf. Hautus (1970), Hautus (1980)):

**Theorem 1.7:**

The following statements are equivalent:

1. The linear system (1.1a) is asymptotically stabilizable.
2. There exists a matrix \( F \in \mathbb{R}^{m \times n} \) such that \( s(A + BF) < 0 \).
3. For every initial state \( x_0 \in \mathbb{R}^n \) there exists rational vectors \( \xi \in \mathbb{R}^n(s) \) and \( \omega \in \mathbb{R}^m(s) \), both with only poles in \( \mathcal{G}^- \), such that

\[
x_0 = (sI-A)\xi(s) - Bu(s).
\]
In view of the third statement of the above theorem we define (see also Hautus (1980) for a slightly different definition):

Definition 1.8:
If for a given initial state \( x_0 \in \mathbb{R}^n \) going with the linear system (1.1a) there exist rational vectors \( \xi \in \mathbb{R}^n(s) \) and \( \omega \in \mathbb{R}^n(s) \) such that
\[ x_0 = (sI - A)\xi(s) - B\omega(s), \]
then the latter expression for \( x_0 \) is called a \((\xi, \omega)\)-representation of \( x_0 \) (\( x_0 \) is said to have a \((\xi, \omega)\)-representation).

We shall call a \((\xi, \omega)\)-representation regular if both \( \xi \) and \( \omega \) are strictly proper rational vectors and we shall call a \((\xi, \omega)\)-representation stable if both \( \xi \) and \( \omega \) are stable rational vectors.

(In later chapters when \( \xi, \xi, \in \Theta \) are two stability regions, we shall call a \((\xi, \omega)\)-representation \( f \)-stable, \( s \)-stable if both rational vectors \( \xi \) and \( \omega \) are \( f \)-stable, \( s \)-stable, respectively.)

We claim that a regular \((\xi, \omega)\)-representation of \( x_0 \) can be considered as the Laplace transform of (1.1a) with initial state \( x(0) = x_0 \) and
\( u: [0, \infty) \to \mathbb{R}^m \) a piecewise continuous control input, provided certain conditions are satisfied. Indeed, if both functions \( x \) and \( u \) associated with (1.1a) are such that their Laplace transforms exist and are strictly proper rational vectors, then the expression
\[ x_0 = (sI - A)\xi(s) - B\omega(s) \]
can be considered to be the Laplace transform of (1.1a) with \( x(0) = x_0 \).

Functions that have a strictly proper rational Laplace transform are called Bohl functions (cf. Trentelman (1986)). If \( h \) is a Bohl function then it can be shown that there exist real matrices \( F, G \) and \( H \) of suitable dimensions such that \( h(t) = Fe^{Gt}H \). If in this representation of \( h \) the dimension of \( G \) is taken minimal, then we define \( \nu(h) := \nu(G) \).

Inspired by the third statement of Theorem 1.7 we now define the concept of stabilizability of the linear system (1.1a) as follows. To this end, let \( \xi, \xi, \in \Theta \) be a given stability region.

Definition 1.9:
The linear system (1.1a) is called stabilizable if every initial state \( x_0 \in \mathbb{R}^n \) has a stable regular \((\xi, \omega)\)-representation.
Theorem 1.10:
The following statements are equivalent:

1. The linear system (1.1a) is stabilizable.
2. There exists a matrix \( F \in \mathbb{R}^{m \times n} \) such that \( (A + BF) \subseteq \xi_g. \)
3. For every initial state \( x_0 \in \mathbb{R}^n \) there exists a piecewise continuous control input \( u: [0,\infty) \rightarrow \mathbb{R}^m \) such that \( x: [0,\infty) \rightarrow \mathbb{R}^m \) is a Bochi function and \( o(x) \subseteq \xi_g. \)
4. \( \text{im} (A - \lambda I - A) + \text{im} B = \mathbb{R}^n \) for all \( \lambda \in \xi \setminus \xi_g. \)
5. There exist stable rational matrices \( X \in \mathbb{R}^{n \times n}(s) \) and \( U \in \mathbb{R}^{m \times n}(s) \) such that \( I = (sI - A)X(s) - BU(s). \)

Proof:

Note that the stabilizability of (1.1a) only depends on the matrices \( A \) and \( B. \) Therefore, instead of the stating that the linear system (1.1a) is stabilizable we shall often say that the pair \( (A,B) \) is stabilizable.

If the linear system is not stabilizable, there may still exist initial states \( x_0 \in \mathbb{R}^n \) that do have a stable regular \((\xi,\omega)\)-representation. Therefore, we define (cf. Hautus (1980)):

Definition 1.11:
\[ X_{\text{stab}}(A,B) := \{ x_0 \in \mathbb{R}^n \mid x_0 \text{ has a stable regular } (\xi,\omega)-\text{representation} \}. \]

\( X_{\text{stab}}(A,B) \) is called the stabilizable subspace associated with the linear system (1.1a) (cf. Hautus (1980)). It can be shown (cf. Schumacher (1981)) that
\[ X_{\text{stab}}(A,B) = \xi_g(A) + \langle A \mid \text{im } B \rangle, \]
where we have denoted
\[ \langle A \mid \text{im } B \rangle := \text{im } B + A \text{ im } B + \ldots + A^{n-1} \text{ im } B = \sum_{i=0}^{n-1} A^i \text{ im } B. \]

Here, the last equality is based on the Cayley-Hamilton theorem (cf. Gantmacher (1959)). \( \langle A \mid \text{im } B \rangle \) is known as the controllable subspace (cf.}
Wonham (1979)) and is equal to the smallest $A$-invariant subspace that contains in $B$.

Until now we have ignored the measurement output equation (1.1b) and we have been concerned with the state evolution equation (1.1a) only, where, in first instance, we have assumed that $B = 0$. In the remainder of the present section we shall again assume that $B = 0$, but now we do include equation (1.1b). Hence, in the remainder of this section we are dealing with the linear system described by:

\[
\begin{align*}
(1.4) & \quad \dot{x}(t) = Ax(t), \\
(1.1b) & \quad y(t) = Cx(t).
\end{align*}
\]

In Section 1.1 we have introduced the linear system (1.5), the dual of system (1.1). It is clear that the linear system dual to the linear system (1.4), (1.1b) is described by:

\[
\begin{align*}
(1.3a) & \quad \dot{x}'(t) = A^T x'(t) + C^T u'(t).
\end{align*}
\]

The latter is a linear system of the kind with respect to which we have introduced the notion of stabilizability earlier this section. Now we define:

**Definition 1.12:**

The linear system (1.4), (1.1b) (or the pair $(C,A)$) is called detectable if the linear system (1.3a) is stabilizable.

By dualization of Theorem 1.10 we obtain:

**Theorem 1.13:**

The following statements are equivalent:

1. The linear system (1.4), (1.1b) is detectable.
2. There exists a matrix $J \in \mathbb{R}^{n \times p}$ such that $\sigma(J + C) \subseteq \sigma_g(B)$.
3. $\ker (\lambda I - A) \cap \ker C = 0$ for all $\lambda \notin \sigma_g(B)$.
4. There exist stable rational matrices $Z \in \mathbb{R}_+^{n \times n}(s)$ and $Y \in \mathbb{R}_+^{n \times p}(s)$ such that $1 = Z(s)(sI - A) - Y(s)C$. 
Definition 1.14:

\[ X_{\text{det}}(A, C) := \left(X_{\text{stab}}(A^T, C^T)\right)^\perp. \]

\( X_{\text{det}}(A, C) \) is called the undetectable subspace associated with the linear system (1.4), (1.1b). We have \( X_{\text{det}}(A, C) = X_0(A) \cap \langle \ker C \rangle \) (cf. Schumacher (1981)) where we have denoted

\[ \langle \ker C \rangle := \ker C \cap \ker A \cap \ldots \cap \ker C A^{n-1} = \bigcap_{i=0}^{\infty} \ker C A^i. \]

Once more, the last equality is based on the Cayley-Hamilton theorem. \( \langle \ker C \rangle \) is known as the unobservable subspace associated with the linear system (1.4), (1.1b) and is equal to the largest \( A \)-invariant subspace contained in \( \ker C \).

1.3. Invariance issues

In this section we consider linear systems of type (1.1) in connection with linear subspaces that are invariant or that can be made invariant. For this purpose, we again start with the linear autonomous system described by (1.4) and we define:

Definition 1.15:

A linear subspace \( V \) in \( \mathbb{R}^n \) is called dynamically invariant (with respect to the linear system (1.4)) if \( x(t) \in V \) for all \( t \geq 0 \) and for every initial state \( x_0 \in V \).

Then we have (cf. Hautus (1980)):

Theorem 1.16:
The following statements are equivalent:

1. The linear subspace \( V \) in \( \mathbb{R}^n \) is dynamically invariant.
2. \( AV \subseteq V \).
3. For every initial state \( x_0 \in V \) there exists a rational vector \( \xi \in \mathbb{R}^n(s) \) such that \( x_0 = (sI - A)\xi(s) \) and \( \xi \in V \).
Observe that from the second statement of Theorem 1.16 it follows that notions of A-invariance, as introduced in Section 1.1, and dynamic invariance, as defined above, are in fact the same. Therefore, in the sequel we shall refer to dynamic invariance as A-invariance.

If \( V \) is not an A-invariant subspace in \( \mathbb{R}^n \) there may still exist initial states \( x_0 \in V \) for which there exists a rational vector \( \xi \in \mathbb{R}_+^n(s) \) such that \( x_0 = (sI - A)\xi(s) \) and \( \xi \in V \).

Assume that \( V = \ker H \) for some appropriate matrix \( H \). Then we can define:

**Definition 1.17:**

\[
I^*(\ker H; A) := \{ x_0 \in \ker H \mid \text{there exists a rational vector } \xi \in \mathbb{R}_+^n(s) \text{ such that } x_0 = (sI - A)\xi(s) \text{ and } H\xi = 0 \}.
\]

Observe that, if the rational vector \( \xi \in \mathbb{R}_+^n(s) \) is such that \( x_0 = (sI - A)\xi(s) \), then \( \xi(s) = \sum_{i=0}^{\infty} A^i x_0 s^{-i-1} \). If, in addition, \( H\xi(s) = 0 \) then \( HA^i x_0 = 0 \) for all \( i \geq 0 \). So, we could just as well have defined:

\[
I^*(\ker H; A) := \{ x_0 \in \mathbb{R}^n \mid \text{there exists a rational vector } \xi \in \mathbb{R}_+^n(s) \text{ such that } x_0 = (sI - A)\xi(s) \text{ and } H\xi = 0 \},
\]

or even:

\[
I^*(\ker H; A) := \{ x_0 \in \mathbb{R}^n \mid HA^i x_0 = 0 \text{ for all } i \geq 0 \}.
\]

It follows that \( I^*(\ker H; A) = \ker H \mid A \). Hence, \( I^*(\ker H; A) \) is the largest A-invariant subspace contained in \( \ker H \).

Again, we return to the linear system described by (1.1a) and we define (cf. Wonham (1979), Hautus (1980), Basile and Marro (1969)):

**Definition 1.18:**

A linear subspace \( V \) in \( \mathbb{R}^n \) is called a controlled invariant (or: \( (A,B) \)-invariant) subspace (with respect to the linear system (1.1a)) if every initial state \( x_0 \in V \) has a regular \( (\xi,\psi) \)-representation such that \( \xi \in V \).
If we omit the assumption that the \((x,w)\)-representation should be regular, we obtain (cf. Willems (1981), Trentelman (1986)):

**Definition 1.19:**
A linear subspace \(V\) in \(\mathbb{R}^n\) is called an almost controlled invariant (or: almost \((\Lambda,B)\)-invariant) subspace (with respect to the linear system (1.1a)) if every initial state \(x_0 \in V\) has a \((x,w)\)-representation such that \(x \in V\).

The names attached to the two types of subspaces defined above will become more clear after we have stated the following results (cf. Wonham (1979), Hautus (1980), Basile and Marro (1969)):

**Theorem 1.20:**
The following statements are equivalent:

1. The linear subspace \(V\) is controlled invariant.
2. \(AV \subseteq V + \text{im } B\).
3. There exists a matrix \(F \in \mathbb{R}^{m \times n}\) such that \((A + BF)V \subseteq V\).
4. For every initial state \(x_0 \in V\) there exists a piecewise continuous control \(u: [0,\infty) \rightarrow \mathbb{R}^m\) such that \(x(t) \in V\) for all \(t \geq 0\).

From Theorem 1.20 it is clear that starting in a controlled invariant subspace one can stay in that subspace by using an appropriate control input. Moreover, it follows that the latter can be done by means of a feedback control law.

**Theorem 1.21:**
The following statements are equivalent:

1. The linear subspace \(V\) is an almost controlled invariant subspace.
2. For all \(\varepsilon > 0\) there exists a matrix \(F \in \mathbb{R}^{m \times n}\) such that
   \[
   \inf \|e^{t(A+BF)}x_0\|_V \leq \varepsilon \text{ for all } t \geq 0 \text{ and for every } x_0 \in V \text{ with } \|x_0\|_V \leq 1.
   \]
3. For all \(\varepsilon > 0\) and for every \(x_0 \in V\) there exists a piecewise continuous control \(u: [0,\infty) \rightarrow \mathbb{R}^m\) such that \(\inf \|x(t) - v\|_V \leq \varepsilon \text{ for all } t \geq 0\).

**Proof:**
From Theorem 1.21 it follows that starting in an almost controlled invariant subspace one can stay arbitrarily close to that subspace by using appropriate control inputs. Moreover, it follows that the latter can be also achieved by appropriate feedback control laws.

Again, assume that \( U = \ker H \) for some appropriate matrix \( H \). Then, much in the spirit of Definition 1.17, we can define (cf. Hausus (1980)):

**Definition 1.22:**

\[
U^\ast(\ker H) \ (= U^\ast(\ker H; A, B)) := \{ x_0 \in \ker H \mid x_0 \text{ has a regular } (\xi, \omega)\text{-representation such that } H\xi = 0 \}.
\]

Note that we just as well could have defined:

\[
U^\ast(\ker H) := \{ x_0 \in \mathbb{R}^n \mid x_0 \text{ has a regular } (\xi, \omega)\text{-representation such that } H\xi = 0 \}.
\]

Indeed, if the rational vectors \( \xi \in \mathbb{R}_+^n(s) \) and \( \omega \in \mathbb{R}_+^n(s) \) are such that for \( x_0 \in \mathbb{R}^n \) we have \( x_0 = (sI - A)\xi(s) - B\omega(s) \), then it follows that

\[ s\xi(s) = x_0 + \xi'(s) \text{ where } \xi'(s) = A\xi(s) + B\omega(s) \in \mathbb{R}_+^n(s). \]

If, in addition, \( H\xi = 0 \), then \( Hx_0 = 0 \) and \( H\xi' = 0 \). Hence, \( x_0 \in \ker H \).

Furthermore, we can define (cf. Schumacher (1983)):

**Definition 1.23:**

\[
U^\ast_a(\ker H) \ (= U^\ast_a(\ker H; A, B)) := \{ x_0 \in \ker H \mid x_0 \text{ has a } (\xi, \omega)\text{-representation such that } H\xi = 0 \}.
\]

We can now not replace \( \ker H \) by \( \mathbb{R}^n \). This is caused by the fact that the \( (\xi, \omega)\)-representation is not necessarily regular. Therefore, we define separately (cf. Schumacher (1983)):

**Definition 1.24:**

\[
U^\ast_b(\ker H) \ (= U^\ast_b(\ker H; A, B)) := \{ x_0 \in \mathbb{R}^n \mid x_0 \text{ has a } (\xi, \omega)\text{-representation such that } H\xi = 0 \}.
\]
Of course, \( V^*_d(\ker H) = V^*_b(\ker H) \cap \ker H \). Furthermore, we have (cf. Hautus (1980), Schumacher (1983)):

**Proposition 1.25:**

\( V^*(\ker H) \) is the largest controlled invariant subspace in \( \ker H \).

\( V^*_a(\ker H) \) is the largest almost controlled invariant subspace in \( \ker H \).

In the sequel we shall not need the subspace \( V^*_a(\ker H) \). The subspaces \( V^*(\ker H) \) and \( V^*_b(\ker H) \) will be used quite frequently in later chapters. Therefore, it will be necessary to give schemes that can be used for the calculation of these subspaces. To that end, we state the following algorithms where we assume that the matrices \( A, B \) and \( H \) are given (cf. Willems (1981), Wonham (1979)).

**Algorithms 1.26:**

Consider the following iteration processes:

\[
\begin{align*}
V_0 &= \ker H, & V_{k+1} &= \ker H \cap \{x \in \mathbb{R}^n \mid Ax \in (V_k + \text{im } B)\} \\
&\quad \text{for all } k \geq 0, \\
R_0 &= \text{im } B, & R_{k+1} &= \text{im } B + \{x \in \mathbb{R}^n \mid x \in A(R_k \cap \ker H)\} \\
&\quad \text{for all } k \geq 0.
\end{align*}
\]

By induction it can be proved that \( 0 \subseteq V_{k+1} \subseteq V_k \subseteq \mathbb{R}^n \) and \( 0 \subseteq R_k \subseteq R_{k+1} \subseteq \mathbb{R}^n \) for all \( k \geq 0 \). So it is clear that the limits \( V_\infty \) and \( R_\infty \) exist and are determined within a finite number (\( \leq n \)) of iterations.

Then we have \( V^*(\ker H) = V_\infty \) and \( V^*_b(\ker H) = V_\infty + R_\infty \).

Of course, \( V^*_d(\ker H) = \ker H \cap (V_\infty + R_\infty) \).

We now consider the linear system described by (1.4), (1.1b) together with the corresponding dual system described by (1.3a).

We define (cf. Schumacher (1979), Basile and Marro (1969), Willems (1982)):

**Definition 1.27:**

A linear subspace \( S \) in \( \mathbb{R}^n \) is called a conditioned invariant (or: \((C,A)\)-invariant) subspace (with respect to the linear system (1.4), (1.1b)) if the linear subspace \( S^\perp \) in \( \mathbb{R}^n \) is a controlled invariant subspace (with respect to the dual system (1.3a)).
Definition 1.28:
A linear subspace $S$ in $\mathbb{R}^n$ is called an almost conditioned invariant (or: almost $(C,A)$-invariant) subspace (with respect to the linear system (1.4), (1.1b) if the linear subspace $S^\perp$ in $\mathbb{R}^n$ is an almost controlled invariant subspace with respect to the dual system (1.3a)).

Then, dualizing Theorems 1.20 and 1.21 yields:

Theorem 1.29:
The following statements are equivalent:

1. The linear subspace $S$ is conditioned invariant.
2. $A(S \cap \ker C) \subseteq S$.
3. There exists a matrix $J \in \mathbb{R}^{n \times p}$ such that $(A + JC)S \subseteq S$.

Theorem 1.30:
The linear subspace $S$ in an almost conditioned invariant subspace if and only if for all $c > 0$ there exists a matrix $J \in \mathbb{R}^{n \times p}$ such that
\[
\inf_{c \in \mathbb{C}} \|(A + JC)\| < c \quad \text{for all } \|x_0\| = 1.
\]

The dual versions of the fourth statement of Theorem 1.20 and the third statement of Theorem 1.21 do not have a straightforward interpretation in the context of the linear system (1.4), (1.1b) and are therefore left out of Theorems 1.29 and 1.30.

If, in addition to the matrices $A$ and $C$, a matrix $G$ of suitable dimensions is given, then by dualization we can obtain the following (cf. Willems and Commault (1981), Schumacher (1981), Willems (1982)):

Definition 1.31:
\[
S^* (\text{im } G) := S^* (\text{im } G; A, C) := \left( V^* (\ker G^T; A^T, C^T) \right)^\perp, \\
S^*_0 (\text{im } G) := S^*_0 (\text{im } G; A, C) := \left( V^*_0 (\ker G^T; A^T, C^T) \right)^\perp, \\
S^*_b (\text{im } G) := S^*_b (\text{im } G; A, C) := \left( V^*_b (\ker G^T; A^T, C^T) \right)^\perp.
\]
Proposition 1.32:
$S^*(\text{im } G)$ is the smallest conditioned invariant subspace containing $\text{im } G$.
$S^a(\text{im } G)$ is the smallest almost conditioned invariant subspace containing $\text{im } G$.

Algorithm 1.33:
Consider the iteration processes (compare (1.5) and (1.6))

(1.7) \[ S_0 = \text{im } G, \quad S_{k+1} = \text{im } G + \{x \in \mathbb{R}^n \mid x \in A(S_k \cap \ker C)\} \]
for all $k \geq 0$,

(1.8) \[ N_0 = \ker C, \quad N_{k+1} = \ker C \cap \{x \in \mathbb{R}^n \mid Ax \in (N_k + \text{im } G)\} \]
for all $k \geq 0$.

Then $0 \subseteq S_k \subseteq S_{k+1} \subseteq \mathbb{R}^n$ and $0 \subseteq N_{k+1} \subseteq N_k \subseteq \mathbb{R}^n$ for all $k \geq 0$.

Hence, $S_\infty$ and $N_\infty$ exist and are determined within a finite number of iterations. Now, $S^*(\text{im } G) = S_\infty$, $S^b(\text{im } G) = S_\infty \cap N_\infty$ and $S^a(\text{im } G) = \text{im } G + (S_\infty \cap N_\infty)$.

1.4. Combined stability and invariance issues

In the present section we consider the combination of some of the concepts that we introduced in the previous two sections. To this end we consider the linear system (1.1) and we assume that $\Theta \subseteq \Theta$ is a given stability region.

Definition 1.34:
A linear subspace $V$ in $\mathbb{R}^n$ is called a stabilizability subspace (with respect to the linear system (1.1a)) if every initial state $x_0 \in V$ has a stable regular $(t, u)$-representation with $x \in V$.

From this definition it is clear that every stabilizability subspace is a controlled invariant subspace. Furthermore, it will turn out that the linear system obtained by restricting the linear system (1.1a) controlled by a suitable state feedback to a stabilizability subspace is a stabilizable system. In fact, we have the following (cf. Hautus (1980), Schumacher (1982)).
Theorem 1.35:
The following statements are equivalent:

1. $V$ is a stabilizability subspace.
2. $(tI - A)V + \text{im } B = V + \text{im } B$ for all $t \in \mathbb{C} \setminus \{0\}$.
3. There exists a matrix $F \in \mathbb{R}^{m \times n}$ such that $(A + BF)V \subseteq V$ and $a(A + BF|V) \subseteq \mathbb{C}_+$.
4. For every initial state $x_0 \in V$ there exists a piecewise continuous control $u: [0,\infty) \to \mathbb{R}^m$ such that $x(t) \in V$ for all $t > 0$ and $a(x) \subseteq \mathbb{C}_+$.

Observe that, if $V = \mathbb{R}^n$, then the statement that $V$ is a stabilizability subspace implies that the pair $(A,B)$ is stabilizable. Furthermore, if $\mathbb{C}_+ = \mathbb{C}$, then every controlled invariant subspace is also a stabilizability subspace. If the linear subspace $V$ in $\mathbb{R}^n$ is a stabilizability subspace and the pair $(A,B)$ is stabilizable, then there exists a matrix $F \in \mathbb{R}^{m \times n}$ such that $(A + BF)V \subseteq V$ and $a(A + BF) \subseteq \mathbb{C}_+$ (cf. Wonham (1979)).

In the spirit of the previous sections we define:

Definition 1.36:
$$
V^g_{\ker H}(= V^g_{\ker H}(A,B)) := \{x_0 \in \ker H \mid x_0 \text{ has a stable regular } (t,u)-\text{representation with } Ht = 0\}.
$$

Then we have (cf. Hautus (1980)):

Proposition 1.37:
$$
V^g_{\ker H} \text{ is the largest stabilizability subspace in } \ker H.
$$

Also, the linear subspace $V^g_{\ker H}$ will be used frequently in future chapters. Therefore, we present the following algorithm by which this subspace can be computed:

Algorithm 1.38:
Let $F \in \mathbb{R}^{m \times n}$ be a matrix such that $(A + BF)V^g_{\ker H} \subseteq V^g_{\ker H}$, where $V^g_{\ker H}$ can be calculated by means of the iteration process described by (1.5). For the computation of the matrix $F$ we refer to Wonham (1979), exercise 5.6. Then it can be shown that
\[ \mathcal{V}_g^*(\ker H) = \left( \mathcal{X}_g(A + BF) \cap \mathcal{V}_g^*(\ker H) \right) + \left( \mathcal{R}_\infty \cap \mathcal{V}_g^*(\ker H) \right), \]

where \( \mathcal{R}_\infty \) is the limit of the iteration process described by (1.6).

The validity of the above expression for \( \mathcal{V}_g^*(\ker H) \) can be proved starting from the expression for \( \mathcal{V}_g^*(\ker \Pi) \) given in Wonham (1979), section 5.5.

Note that, if \( \mathcal{C}_g = \emptyset \), then \( \mathcal{V}_g^*(\ker H) = \mathcal{V}_g^*(\ker H) \), and note that

\[ \mathcal{V}_g^*(\ker H) \cap \text{im } B \subseteq \mathcal{V}_g^*(\ker H) \cap \mathcal{R}_\infty \subseteq \mathcal{V}_g^*(\ker H). \]

Furthermore, note that, if \( H = 0 \) (\( \ker H = \mathbb{R}^n \)), then \( \mathcal{V} = \mathbb{R}^n \), implying that \( \Delta \mathcal{V}_\infty \subseteq \mathcal{V}_\infty \), and \( \mathcal{R}_\infty = \sum_{i=0}^{\infty} A^i \text{im } B = \langle A \mid \text{im } B \rangle \). Hence,

\[ \mathcal{V}_g^*(\mathbb{R}^n) = \mathcal{X}_g(A) + \langle A \mid \text{im } B \rangle = \mathcal{X}_{\text{stab}}(A, B). \]

We consider now the linear system described by (1.4), (1.1b) together with its dual (1.3a).

**Definition 1.39:**
A linear subspace \( S \) in \( \mathbb{R}^n \) is called a detectability subspace (with respect to the linear system (1.4), (1.1b)) if the linear subspace \( S^\perp \) in \( \mathbb{R}^n \) is a stabilizability subspace (with respect to the dual system (1.3a)).

**Theorem 1.40:**
The following statements are equivalent:

1. \( S \) is a detectability subspace.
2. \((A - \lambda I)^{-1} S \cap \ker \mathcal{C} = S \cap \ker \mathcal{C} \) for all \( \lambda \in \mathbb{C} \setminus \mathcal{C}_g \).
3. There exists a matrix \( J \in \mathbb{R}^{n \times p} \) such that \((A + JC)S \subseteq S \) and \( \sigma(A + JC \mid \mathbb{R}^n/S) \subseteq \mathcal{C}_g \).

Note that, if \( S = 0 \), then the statement that \( S \) is a detectability subspace implies that the pair \((C, A)\) is detectable. If \( \mathcal{C}_g = \emptyset \), then any conditioned invariant subspace is also a detectability subspace. Finally, if the pair \((C, A)\) is detectable and \( S \) is a detectability subspace, then there exists a matrix \( J \in \mathbb{R}^{n \times p} \) such that \((A + JC)S \subseteq S \) and \( \sigma(A + JC) \subseteq \mathcal{C}_g \).
Definition 1.41:

\[ S_g^*(\text{im} \, G) = S_g^*(\text{im} \, C; A, C) := (V_g^*(\ker C^T; A^T, C^T))^\perp. \]

Then we have:

Proposition 1.42:

\( S_g^*(\text{im} \, C) \) is the smallest detectability subspace containing \( \text{im} \, G \).

Algorithm 1.43:

Let \( J \in \mathbb{R}^{n \times p} \) be a matrix such that \((A + J C)S_m \subseteq S_m^*\), then

\[ S_g^*(\text{im} \, C) = (X_b(A + J C) + S_m^*) \cap (N_m + S_m), \]

where \( S_m \) and \( N_m \) are the limits of the iteration processes described by (1.7) and (1.8), respectively.

Note that, if \( C = C \), then \( S_g^*(\text{im} \, C) = S_m = S_g^*(\text{im} \, G) \). Furthermore, \( S_g^*(\text{im} \, G) + \ker C \subseteq S_g^*(\text{im} \, C) \) and \( S_g^*(0) = X_b(A) \cap \ker C|A^o = X_{det}(A, C) \).

1.5. Dynamic feedback

Again, we consider the linear system described by (1.1).

In the present section we shall be dealing with 'dynamic' feedback around this system. This being in contrast with the previous sections where the system theoretic concepts introduced were related to 'static' feedback in the form of a state feedback \( u(t) = Fx(t) \) or an output 'injection' \( Jy(t) \).

As announced, in the present section we shall be dealing with linear dynamic feedback and it will turn out that this kind of feedback can be considered a generalization of static output feedback of the form \( u(t) = M y(t) \). The adjective 'dynamic' reflects the fact that the feedback can be considered a dynamic system in the usual sense with a state evolution equation and an output equation.

Dynamic feedback, also called dynamic compensation, will play an important role in the sequel of this monograph. The dynamic feedback (laws), also
called (dynamic) compensators, that we shall use in the sequel are described as follows:

\begin{align*}
(1.9a) \quad \dot{w}(t) &= Kw(t) + Ly(t), \\
(1.9b) \quad u(t) &= Mw(t) + Ny(t).
\end{align*}

Here, \( w(t) \in \mathbb{R}^k \) denotes the state of the compensator and \( K \in \mathbb{R}^{k \times k} \), \( L \in \mathbb{R}^{k \times p} \), \( M \in \mathbb{R}^{m \times k} \) and \( N \in \mathbb{R}^{m \times p} \). The integer \( k \geq 0 \) is called the order of the compensator. In the case that \( k = 0 \) we are dealing with static output feedback (zero order compensator) of the form \( u(t) = Ny(t) \), while for \( k > 0 \) dynamics will be involved in the compensator.

The closed loop (or: controlled) linear system obtained by the interconnection of the linear system (1.1) and the compensator (1.9) is described by:

\begin{equation}
(1.10) \quad \dot{x}(t) = Ax(t) + e,
\end{equation}

where we have denoted

\begin{align*}
(1.11a) \quad x_c(t) &:= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \\
(1.11b) \quad A_c &:= \begin{bmatrix} \Delta + BC & BM \\ LC & K \end{bmatrix}.
\end{align*}

Note that \( x_c(t) \in \mathbb{R}^{n+k} \), the state space of the closed loop system.

It should be clear that, when designing a compensator of type (1.9) for the linear system (1.1), one always has certain control objectives in mind (for instance: stabilization, decoupling, etc.). In many cases these control objectives require the closed loop system (1.10) to satisfy certain conditions (for instance: the closed loop system should be stable, in the state space of the closed loop system there should exist linear subspaces with certain properties, etc.). However, when starting the design of any compensator of type (1.9) one only has the to-be-controlled system available. Therefore, it is important to establish relations between system (1.1) and the ultimate closed loop system (1.10). Then, one one hand,
these relations have to make clear how the desired control objectives
have to be translated into necessary properties of system (1.1).
While, on the other hand, these relations have to explain which control
objectives can be achieved when system (1.1) satisfies certain properties.
It turns out that, for establishing many of these relations, the following
two linear subspaces depending on a linear subspace \( \mathcal{W}_e \) in \( \mathbb{R}^{n+k} \) are crucial
(cf. Schumacher (1981); in fact all material presented in this section can
be found in this reference).

\[
\begin{align*}
(1.12a) \quad \mathbf{p}(\mathcal{W}_e) &= \{ x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^k : \left[ \begin{array}{c} x \\ w \end{array} \right] \in \mathcal{W}_e \}, \\
(1.12b) \quad \mathbf{i}(\mathcal{W}_e) &= \{ x \in \mathbb{R}^n \mid \left[ \begin{array}{c} x \\ 0 \end{array} \right] \in \mathcal{W}_e \}. 
\end{align*}
\]

\( \mathbf{p}(\mathcal{W}_e) \) denotes the projection of \( \mathcal{W}_e \) onto \( \mathbb{R}^n \) along \( \mathbb{R}^k \) and \( \mathbf{i}(\mathcal{W}_e) \) denotes the
intersection of \( \mathcal{W}_e \) with \( \mathbb{R}^n \). Of course, \( \mathbf{i}(\mathcal{W}_e) \subseteq \mathbf{p}(\mathcal{W}_e) \) for all \( \mathcal{W}_e \subseteq \mathbb{R}^{n+k} \).
Now we can establish the following relations between the closed loop sys-
tem (1.10) and the 'to-be-controlled' system (1.1). To this end, let
\( \mathcal{C}_e \subseteq \mathcal{C} \) be a given stability region and let \( A_c \) be given by (1.11b).

**Theorem 1.44:**

1. If \( \sigma(A_c) \subseteq \mathcal{C}_e \), then the pair \( (A, \mathcal{N}) \) is stabilizable and the pair \( (C, A) \)
is detectable.

2. If \( \mathcal{W}_e \) is an \( A_c \)-invariant subspace in \( \mathbb{R}^{n+k} \), then \( \mathbf{p}(\mathcal{W}_e) \) is a controlled
   invariant subspace and \( \mathbf{i}(\mathcal{W}_e) \) is a conditioned invariant subspace in
   \( \mathbb{R}^n \).

3. If \( \sigma(A_c) \subseteq \mathcal{C}_e \) and \( \mathcal{W}_e \) is an \( A_c \)-invariant subspace in \( \mathbb{R}^{n+k} \), then \( \mathbf{p}(\mathcal{W}_e) \)
is a stabilizability subspace and \( \mathbf{i}(\mathcal{W}_e) \) is a detectability subspace in
   \( \mathbb{R}^n \).

**Proof:**

1. Follows from 3) by successively specifying \( \mathcal{W}_e = \mathbb{R}^{n+k} \) and \( \mathcal{W}_e = 0 \).

2. Follows from 3) by specifying \( \mathcal{C}_e = \mathcal{C} \).

3. Let \( x \in \mathbf{p}(\mathcal{W}_e) \). Then there exists \( w \in \mathbb{R}^k \) such that \( x_e = \left[ \begin{array}{c} x \\ w \end{array} \right] \in \mathcal{W}_e \).
   Define
\[ \xi_e(s) = \begin{bmatrix} \xi(s) \\ \omega(s) \end{bmatrix} := (sI - A_e)^{-1} x_e, \quad \text{where } \xi \in \mathbb{R}_+^n(s) \text{ and } \omega \in \mathbb{R}_+^m(s). \]

From Theorems 1.14 and 1.16 it follows that \( \xi_e \in \mathcal{W}_e \) and \( \xi_e \) is stable. Since \( \xi_e \in \mathcal{W}_e \) it is clear that \( \xi \in \mathcal{P}(\mathcal{W}_e) \). Furthermore, it follows that \( x = (sI - A)\xi(s) - B(NC\xi(s) + M\omega(s)) \), where \( \xi, \omega \) and consequently \( NC\xi + M\omega \) are stable and strictly proper. Hence, \( \mathcal{P}(\mathcal{W}_e) \) is a stabilizability subspace.

Dually, we can prove that \( \mathcal{P}(\mathcal{W}_e) \) is a detectability subspace.

Corollary 1.45:

4) If \( \sigma(A_e) \subseteq \mathcal{G}_e \) and \( \mathcal{W}_e \) is an \( A_e \)-invariant subspace in \( \mathbb{R}^{n+k} \), then the pair \( (A,B) \) is stabilizable, the pair \( (C,A) \) is detectable, \( \mathcal{P}(\mathcal{W}_e) \) is a stabilizability subspace and \( \mathcal{P}(\mathcal{W}_e) \) is a detectability subspace.

Theorem 1.44 states which properties will be satisfied with respect to system (1.1) if the closed loop system (1.10) satisfies certain control objectives. However, we can also give results for the converse. To this end, we consider system (1.1) and we let a stability region \( \mathcal{G}_e \subseteq \mathcal{G} \) be given.

Theorem 1.46:

Let \( S, V \) be linear subspaces in \( \mathbb{R}^n \) such that \( S \subseteq V, S \) is a detectability subspace and \( V \) is a stabilizability subspace, and let the pair \( (A,B) \) be stabilizable and the pair \( (C,A) \) be detectable. Then there exists a compensator (1.9) and a linear subspace \( \mathcal{W}_e \) in \( \mathbb{R}^{n+k} \), the state space of the resulting closed loop system (1.10), such that \( \sigma(A_e) \subseteq \mathcal{G}_e, A_e \mathcal{W}_e \subseteq \mathcal{W}_e \) and \( S \oplus 0 \subseteq \mathcal{W}_e \subseteq V \oplus \mathbb{R}^k \).

The proof of Theorem 1.46 will be given later in this section in the form of a construction. We proceed with two special cases of Theorem 1.46.

Corollary 1.47:

1) If the pair \( (A,B) \) is stabilizable and the pair \( (C,A) \) is detectable, then there exists a compensator (1.9) such that the closed loop system (1.10) is stable.
2) Let $S$ be a conditioned invariant subspace and let $V$ be a controlled invariant subspace such that $S \subseteq V$. Then there exists a compensator (1.9) and a linear subspace $W_c$ in $\mathbb{R}^{n+k}$, the state space of the resulting closed loop system (1.10), such that $A_c W_c \subseteq W_c$ and $(S \oplus 0) \subseteq W_c \subseteq (V \oplus \mathbb{R}^k)$.

Proof:
1) Follows from Theorem 1.46 by specifying $S = 0$ and $V = \mathbb{R}^n$.

2) Follows from Theorem 1.46 by specifying $E = 0$.

As announced, we shall provide a proof of Theorem 1.46 by means of a construction. We give this construction in full detail here, because it will be used frequently in later chapters.

Construction 1.48:
Consider the linear system (1.1) and let $E \in \mathbb{R}^n$ be a given stability region. Suppose that $S$ and $V$ are linear subspaces in $\mathbb{R}^n$ such that $S \subseteq V$. Furthermore, suppose that $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$ and $C \in \mathbb{R}^{m \times p}$ are matrices such that $(A + BF)V \subseteq V$, $(A + JC)S \subseteq S$, $(A + ENC)S \subseteq V$, $a(A + BF) \subseteq E_c$ and $a(A + JC) \subseteq E$.

(Clearly, this implies that the pair $(A, B)$ is stabilizable, the pair $(C, A)$ is detectable, $S$ is a detectability subspace and $V$ is a stabilizability subspace.)

Now, let $k = n$ and let $K \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{n \times p}$ and $M \in \mathbb{R}^{m \times n}$ be matrices defined by $K := A + BF + JC - ENC$, $L := EN - J$ and $M := V - NC$. Furthermore, let $W_c$ be the linear subspace in $\mathbb{R}^{2n}$ defined by

(1.12) $W_c := \left\{ \begin{bmatrix} s \\ 0 \\ v \end{bmatrix} \mid s \in S, v \in V \right\}.$

Then we have $a(A_c) \subseteq E_c$, $A_c W_c \subseteq W_c$ and $(S \oplus 0) \subseteq W_c \subseteq (V \oplus \mathbb{R}^k)$.

Proof:
It is clear that with these specific choices of the matrices $K$, $L$ and $M$ we have that $A_c = A_c$, where
Ae \equiv [A+BNC & B(F-NC) \\
(BN-J)C & A+BF+JC-BNC].

Now let \( S_e \in \mathbb{R}^{2n \times 2n} \) be defined by \( S_e := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). Then
\[
\sigma(\hat{A}_e) = \sigma(A+BF) \cup \sigma(A+JC),
\]
since
\[
S_e \hat{A}_e S_e^{-1} = \begin{bmatrix} A+BF & BF-BNC \\
0 & A+JC \end{bmatrix}.
\]

Next, let \( x_e \in W_e \), i.e. \( x_e = [s] + [v] \) with \( s \in S \) and \( v \in V \). Then
\[
A_e x_e = \hat{A}_e \begin{bmatrix} s \\ 0 \end{bmatrix} + \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} (A+JC)s \\ 0 \end{bmatrix} + \begin{bmatrix} (A+BNC)s \\ (A+JC)s \end{bmatrix} + \begin{bmatrix} (A+BF)v \end{bmatrix},
\]
from which it is clear that \( A_e x_e \in W_e \). Hence, \( A_e W_e \subseteq W_e \).

Finally, by (1.12) it is clear that \((S \oplus 0) \subseteq W_e \subseteq (V \oplus \mathbb{R}^n)\).

Special cases 1.49:

1) Let \( F \in \mathbb{R}^{m \times n} \) and \( J \in \mathbb{R}^{p \times p} \) be matrices such that \( \sigma(A+BF) \subseteq \mathcal{G}_e \) and \( \sigma(A+JC) \subseteq \mathcal{G}_e \), and let \( N \in \mathbb{R}^{m \times p} \) be an arbitrary matrix.

Then \( \sigma(\hat{A}_e) \subseteq \mathcal{G}_e \), where \( \hat{A}_e \) is given by (1.13).

2) Let \( S, V \) be linear subspaces in \( \mathbb{R}^n \) such that \( S \subseteq V \), and let \( F \in \mathbb{R}^{m \times n} \), \( J \in \mathbb{R}^{p \times p} \) and \( N \in \mathbb{R}^{m \times p} \) be matrices such that \( (A+BF)V \subseteq V \), \( (A+JC)S \subseteq S \) and \( (A+BNC)S \subseteq V \). Then \( \hat{A}_e W_e \subseteq W_e \), where \( \hat{A}_e \) is given by (1.13) and \( W_e \) is given by (1.12).

Proof of Theorem 1.46:

From Section 1.4 the existence of matrices \( F \in \mathbb{R}^{m \times n} \) and \( J \in \mathbb{R}^{p \times p} \) such that \( (A+BF)V \subseteq V \), \( (A+JC)S \subseteq S \), \( \sigma(A+BF) \subseteq \mathcal{G}_e \) and \( \sigma(A+JC) \subseteq \mathcal{G}_e \) is clear.

From the next lemma the existence follows of a matrix \( N \in \mathbb{R}^{m \times p} \) such that \( (A+BNC)S \subseteq V \), whereupon application of construction 1.48 completes the proof of Theorem 1.46.
Lemma 1.50:
Let $S$ be a conditioned invariant subspace and let $V$ be a controlled invariant subspace such that $S \subseteq V$. Then there exists a matrix $X \in \mathbb{R}^{nxp}$ such that $(A + BNC)S \subseteq V$.

Proof:
See Lemma 5.5. See also Schumacher (1981), Lemma 2.3.
CHAPTER 2
DISTURBANCE DECOUPLING AND OUTPUT STABILIZATION
BY MEASUREMENT FEEDBACK WITH INTERNAL STABILITY

In this chapter we shall study our first actual control problem. The main reason for studying the problem will be the fact that it includes the extension towards measurement feedback of two important control problems of which up to now only the solution with (static) state feedback was known.

In order to describe the first control problem for which we shall establish this extension, we assume that we have a linear system that, apart from a control input and a measurement output, compare (1.1), has an additional exogenous input and an additional exogenous output. Furthermore, we assume that a stability region $\mathcal{C} \in \Theta$ is given, where $\Theta$ is defined as in Section 1.2.

![Figure 1](image.png)

Then our first problem consists of finding a control law (compensator) that feeds the measurement output back to the control input such that in the 'controlled' system the exogenous output depends on the exogenous input in a stable fashion.

For the case that the measurement output is assumed to be equal to the state of the system and the control law is assumed to be static state feedback, this problem has been studied and solved by Hautus (1980).
For a description of the second control problem for which we shall establish the extension mentioned before, we assume that we have a linear system as described above, now with two exogenous outputs instead of one.

![Diagram](image)

Figure 2.

Then our second problem will consist of finding a control law, similar to the one described above, such that in the 'controlled' system one exogenous output is independent of the exogenous input, while the other exogenous output depends on the exogenous input in a stable fashion.

A special case of this problem, namely the case that the measurement output is assumed to be equal to the state of the system and the control law is assumed to be static state feedback, has been introduced and solved in Trentelman (1986).

As announced, the control problems described above will be special cases of the main control problem of the present chapter. In particular, this will imply that the systems that we shall consider, in addition to a control input and a measurement output, will have two exogenous outputs. For reasons of symmetry we assume that the systems under consideration also have two exogenous inputs.
Then, if we control such a system by means of a measurement feedback compensator as described in Section 1.5, we obtain a closed loop system (= controlled system) that has two exogenous inputs and two exogenous outputs. Hence, the closed loop system is described by four transfer matrices. Now, the main feature of our control problem will be to find a measurement feedback compensator such that three of the four transfer matrices that describe the closed loop system are identically equal to zero, while the remaining fourth transfer matrix is stable.

In fact, we shall study an even more complex control problem, because in our problem formulation we shall also require closed loop internal stabilization.

A nice accidental circumstance will be the fact that our control problem, in addition to the two control problems mentioned above, also includes a number of control problems that are very well known in the literature.

The outline of the present chapter is as follows.

In Section 2.1 we shall introduce the linear system for which we shall formulate our main problem. This problem formulation will also be included in Section 2.1. In Section 2.2 we shall consider an important special case of our main problem. The problem treated in Section 2.2 consists of the control problem mentioned before, studied in Trentelman (1986), extended with the requirement of internal stabilization. Section 2.3 consists of a pure dualization of the results of Section 2.2. In Section 2.4, after some preliminary results, we shall state and prove computable conditions that are necessary and sufficient for the solvability of our main control
problem. We shall conclude the present chapter with Section 2.5. There we shall consider some special cases of our main problem. As announced previously, these special cases will include the extension towards (dynamic) measurement feedback of the control problems studied in Hautus (1980) and Trentelman (1986).

2.1. Problem formulation

In this section we shall introduce the linear system that plays an important role in the present chapter and that we shall use for the formulation of our main problem.

As an introduction to this linear system, we shall first consider the following extension of system (1.1) (see also Fig. 1):

\begin{align}
(2.1a) \quad \dot{x}(t) &= Ax(t) + Bu(t) + Gv(t), \\
(2.1b) \quad y(t) &= Cx(t), \\
(2.1c) \quad z(t) &= Hx(t).
\end{align}

Here, \(x(t), u(t), y(t), A, B, C\) are as described in Section 1.1, \(v(t) \in \mathbb{R}^q\) denotes the exogenous input and \(z(t) \in \mathbb{R}^r\) denotes the exogenous output of the system. Furthermore, \(C \in \mathbb{R}^{n \times q}\) and \(H \in \mathbb{R}^{r \times n}\).

We may think of the exogenous input as a disturbance input that is entering the system, taking arbitrary values and that is not available for control purposes. The exogenous output may be thought of as an output imposed upon the system by the control objective that we have in mind.

As in Section 1.5, we let the system (2.1) be controlled by means of the dynamic measurement feedback compensator:

\begin{align}
(1.9a) \quad \dot{w}(t) &= Kw(t) + Ly(t), \\
(1.9b) \quad u(t) &= Mw(t) + Ny(t),
\end{align}

where \(w(t), K, L, M, N\) are as described in Section 1.5. The resulting closed loop system is given by:
where \( x_e(t) \) and \( A_e \) are described by (1.11), and \( G_e := [0, H_e : [H, 0]]. \)

One of the control problems that is very well known in the literature is the following (cf. Schumacher (1979), Akashi and Tani (1979)).

Given the linear system (2.1), find a measurement feedback compensator (1.9) such that in the closed loop system (2.2): \( H_e(sI - A_e)^{-1} G_e = 0. \)

This control problem is known as the disturbance decoupling problem by measurement feedback, abbreviated as (DDPM). It is clear that whenever (DDPM) is solvable, there exists a measurement feedback compensator (1.9) such that in the closed loop system (2.2) the exogenous output is independent of (is decoupled from) the exogenous input. In that case the exogenous output is only influenced by the initial state of the closed loop system (see also (1.2)).

If, in addition to being independent of the exogenous input, we require the exogenous output to depend on any initial state of the closed loop system in a stable way, we obtain the following special case of the so-called regulator problem (cf. Francis (1977), Schumacher (1981), Wonham (1979)).

Given the linear system (2.1), find a measurement feedback compensator (1.9) such that in the closed loop system (2.2): \( H_e(sI - A_e)^{-1} G_e = 0 \) and \( H_e(sI - A_e)^{-1} \) is stable.

We shall abbreviate this problem as (RP) (regulator problem), where we note that the actual regulator problem as studied in the cited references is formulated more generally.

Furthermore, as announced in Section 1.2, if we use the adverb 'stable', this should always be understood to be taken with respect to a given stability region \( G \in G \). Given such a stability region we now return to (DDPM).
If (DDPM) is solvable and a measurement feedback compensator (1.9) solving the problem is available, it may happen that the state of the closed loop system exhibits unstable behavior although the feature of decoupling is maintained. Therefore, the following extension of (DDPM) makes sense (cf. Schumacher (1981), Imai and Akashi (1981), Willems and Commault (1981)).

Given the linear system (2.1), find a measurement feedback compensator (1.9), such that in the closed loop system (2.2): \( H_c(sI - A_g)^{-1} G_a = 0 \) and \( \sigma(A_e) \leq \xi_f \).

This control problem is known as the disturbance decoupling problem with measurement feedback and internal stabilization. We abbreviate this problem as (DDPM)' . Note that if \( \xi_f \leq \xi_f \) the requirement \( \sigma(A_e) \leq \xi_f \) guarantees that the state of the closed loop system remains bounded whenever it is driven through (2.1a) by a bounded exogenous input.

The feature of internal stabilization can also be imposed in (RP). In order to formulate this extension of (RP) we need two stability regions \( \xi_f, \xi_g \geq 0 \) with \( \xi_f \leq \xi_g \). Then, this extended problem can be described as follows.

Given the linear system (2.1), find a measurement feedback compensator (1.9) such that in the closed loop system (2.2): \( H_c(sI - A_e)^{-1} G_e = 0 \), \( H_c(sI - A_e)^{-1} \) is f-stable and \( \sigma(A_e) \leq \xi_g \).

We shall denote this problem by (RP)' .

It will be seen at the end of the present section that the four problems described previously are all special cases of the main control problem of this chapter. In order to formulate our main control problem we assume that both the exogenous input and the exogenous output can be decomposed into two parts.

In particular, we assume that we can write

\[
\begin{align*}
v &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad z &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.
\end{align*}
\]

Furthermore, we assume the matrices \( G \) and \( H \) to be partitioned as
such that $Gv(t) = G_1 v_1(t) + G_2 v_2(t)$, $z_1(t) = H_1 x(t)$ and $z_2(t) = H_2 x(t)$. Then the linear system (2.1) is described by (see also Fig. 3):

$$(2.3a) \quad x(t) = Ax(t) + Bu(t) + G_1 v_1(t) + G_2 v_2(t),$$

$$(2.3b) \quad y(t) = Cx(t),$$

$$(2.3c) \quad z_1(t) = H_1 x(t), \quad z_2(t) = H_2 x(t),$$

where $x(t), u(t), y(t), A, B$ and $C$ are as described in Section 1.1, and where we assume that $v_1(t) \in \mathbb{R}^{q_1}, v_2(t) \in \mathbb{R}^{q_2}$ are two exogenous inputs and $z_1(t) \in \mathbb{R}^{r_1}, z_2(t) \in \mathbb{R}^{r_2}$ are two exogenous outputs. Furthermore, $G_1 \in \mathbb{R}^{n \times q_1}, G_2 \in \mathbb{R}^{n \times q_2}, H_1 \in \mathbb{R}^{r_1 \times n}$ and $H_2 \in \mathbb{R}^{r_2 \times n}$.

The closed loop system obtained by the interconnection of the system (2.3) and the measurement feedback compensator (1.9), is described by:

$$(2.4a) \quad \dot{x}_e(t) = A_e x_e(t) + G_{1,e} v_1(t) + G_{2,e} v_2(t),$$

$$(2.4b) \quad z_1(t) = H_{1,e} x_e(t), \quad z_2(t) = H_{2,e} x_e(t),$$

where $x_e(t)$ and $A_e$ are as described by (1.11), and

$$G_{1,e} := \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \quad G_{2,e} := \begin{bmatrix} G_2 \\ 0 \end{bmatrix}, \quad H_{1,e} := [H_1, 0] \quad \text{and} \quad H_{2,e} := [H_2, 0].$$

Let $T$ denote the transfer matrix of the closed loop system, i.e.

$$T(s) = H_e (sI - A_e)^{-1} G_e \quad \text{with} \quad H_e = \begin{bmatrix} H_{1,e} \\ H_{2,e} \end{bmatrix} \quad \text{and} \quad G_e = [G_{1,e}, G_{2,e}].$$

Clearly, $T$ can be partitioned as $T = (T_{ij}) \quad (i,j = 1,2)$, where

$$T_{ij}(s) = H_{i,e} (sI - A_e)^{-1} G_j,e.$$

Now, we can define our main problem.
Definition 2.1:

Given the linear system (2.3) and stability regions \( \mathcal{F}, \mathcal{S} \subseteq \mathbb{C} \) with \( \mathcal{F} \subseteq \mathcal{S} \), find a measurement feedback compensator (1.9) such that in the closed loop system (2.4) we have \( T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} \) is an \( \mathcal{F} \)-stable rational matrix and \( \sigma(A_c) \subseteq \mathcal{F} \).

In the sequel we shall refer to this problem as problem 2.1.

Note that if \( q_2 = 0, r_2 = 0 \), that is, the matrices \( G_2 \) and \( H_2 \) do not appear, then problem 2.1 is equal to (DDPM)' with \( \mathcal{R}_g = \mathcal{R}_s \). If, in addition, \( \mathcal{R}_s = \mathcal{R} \) then problem 2.1 coincides with (DDPM). Furthermore, if \( G_1 = G' \), \( G_2 = I \) and \( r_1 = 0 \), then problem 2.1 coincides with (RP)' where \( \mathcal{R} = G' \). Finally, if in addition \( \mathcal{R}_s = \mathcal{R} \) then problem 2.1 is equal to (RP) with \( \mathcal{R}_g = \mathcal{R} \) and \( G' = G \).

Other cases of interest for the study of problem 2.1 are the cases that \( q_1 = 0 = r_1 \) with \( \mathcal{R}_s = \mathcal{R} \) or \( \mathcal{R}_s \neq \mathcal{R} \), and \( q_1 = 0 \) with \( \mathcal{R}_s = \mathcal{R} \) or \( \mathcal{R}_s \neq \mathcal{R} \). In Section 2.6 it will be shown that these special cases are the extensions towards (dynamic) measurement feedback of the problems treated in Hautus (1980) and Trentelman (1986).

2.2. Disturbance decoupling with output stabilization by state feedback with internal stability

In the present section we are concerned with a special case of problem 2.1 as defined in the previous section. The results of this section are important because they will be used in the derivation of necessary and sufficient conditions for the solvability of problem 2.1. Furthermore, the results of the present section establish a solution for the generalization towards internal stabilization of a problem that has been studied in Trentelman (1986). Many of the results of the present section show resemblance to the results in the latter reference (see also Hautus (1980)).

In order to give a mathematical formulation of the problem of this section, we consider the following linear system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Gv(t), \\
z_1(t) &= H_1 x(t), \quad z_2(t) = H_2 x(t).
\end{align*}
\]
As in the previous section, we denote \( H = \begin{bmatrix} H_1 \\ H_0 \end{bmatrix} \). Furthermore, we assume that two stability regions \( \mathcal{C}_f, \mathcal{C}_g \in \mathbb{B} \) with \( \mathcal{C}_f \subseteq \mathcal{C}_g \) are given. We can now consider the following problem.

Find a static state feedback such that the closed loop system is \( s \)-stable and such that the first exogenous output is decoupled and the second exogenous output is \( f \)-stabilized with respect to the exogenous input.

Following Trentelman (1986) we shall refer to this problem as (DDPOS)', where the ' stands for the requirement of internal stabilization.

Formulated mathematically (DDPOS)' reads

**Definition 2.2:**

Find a matrix \( F \in \mathbb{R}^{m \times n} \), representing a static state feedback \( u(t) = Fx(t) \) for the system (2.1a), (2.3c), such that \( H_1(sI - (A + BF))^{-1} G = 0 \), \( H_2(sI - (A + BF))^{-1} G \) is \( f \)-stable and \( \sigma(A + BF) \subseteq \mathcal{C}_g \).

In order to derive necessary and sufficient conditions for the solvability of (DDPOS)', we introduce the following linear subspace.

**Definition 2.3:**

\[
V_{f,s}^f(\ker H_1, \ker H) = V_{f,s}^f(\ker H_1, \ker H; A, B)
\]

\[
\{ x_0 \in \mathbb{R}^n \mid x_0 \text{ has an } s \text{-stable regular } (\xi, \omega) \text{-representation with } H_1 \xi = 0 \text{ and } H_2 \xi \text{ } f \text{-stable} \}.
\]

Note that we just as well could have defined

\[
V_{f,s}^f(\ker H_1, \ker H) : = \{ x_0 \in \ker H \mid x_0 \text{ has an } s \text{-stable regular } (\xi, \omega) \text{ representation with } H_1 \xi = 0 \text{ and } H_2 \xi \text{ } f \text{-stable} \}.
\]

Now the following results are much in the spirit of Hautus (1980) and Trentelman (1986).
Theorem 2.4:

\[ V^*_f(s)(\ker H_1, \ker H) = V^*_f(\ker H_1) + V^*_s(\ker H) \]

Proof:
From Definition 1.36 it is clear that \( V^*_s(\ker H) \subseteq V^*_f,s(\ker H_1, \ker H) \) and \( V^*_f(\ker H_1) \subseteq V^*_f,s(\ker H_1, \ker H) \). Hence,

\[ V^*_f(\ker H_1) + V^*_s(\ker H) \subseteq V^*_f,s(\ker H_1, \ker H) \]

and it remains to be shown that the converse inclusion also holds.
To this end, let \( x \in V^*_f,s(\ker H_1, \ker H) \) have an \( s \)-stable regular \( (\xi, \omega) \)-representation with \( H_1 \xi = 0 \) and \( H \xi \) \( f \)-stable. Then we can decompose \( \xi = \xi_1 + \xi_2 \) and \( \omega = \omega_1 + \omega_2 \) where \( \xi_1, \omega_1 \) are \( f \)-stable strictly proper rational vectors and \( \xi_2, \omega_2 \) are strictly proper rational vectors with only poles in \( \mathcal{F} \sim \mathcal{F}_f \). So, we can write

\[ x = (sI - \Lambda) \xi_1(s) + H \omega_1(s) = (sI - \Lambda) \xi_2(s) - H \omega_2(s) \]

Now observe that both sides of this expression are proper rational and that the left-hand side has only poles in \( \mathcal{F}_f \), whereas the right-hand side has only poles in \( \mathcal{F}_s \sim \mathcal{F}_f \). This implies that both sides are equal and constant; meaning that the static parts of both sides are equal while the strictly proper parts of both sides are equal to zero.

Hence, \( x - \xi_1 = \xi_2 \), \( \xi_1 = (sI - \Lambda) \xi_1(s) - H \omega_1(s) \) and \( \xi_2 = (sI - \Lambda) \xi_2(s) - H \omega_2(s) \)

where \( \xi_{1,2} := \lim_{s \to \infty} s \xi_{1,2}(s) \).

Since \( H \xi_1 \) and \( \xi_1 \) are \( f \)-stable and \( \xi_2 \) has only poles in \( \mathcal{F}_s \sim \mathcal{F}_f \), it follows that \( H \xi_2 = H(\xi - \xi_1) = 0 \), from which it is immediate that \( \xi_{2,1} \in V^*_s(\ker H) \).

Furthermore, since \( H_1 \xi = H_1(\xi_1 + \xi_2) = 0 \) and \( H_1 \xi_2 = 0 \) because \( H \xi_2 = 0 \), it follows that \( H_1 \xi_1 = 0 \). The latter implies that \( \xi_{1,1} \in V^*_f(\ker H_1) \). Hence,

\[ x = \xi_{1,1} + \xi_{2,1} \in V^*_f(\ker H_1) + V^*_s(\ker H) \]

So,

\[ V^*_f,s(\ker H_1, \ker H) \subseteq V^*_f(\ker H_1) + V^*_s(\ker H) \]

which completes the proof. \( \blacksquare \)
Theorem 2.5:
Let the pair $(A, B)$ be $s$-stabilisable. Then there exists a matrix

$$F \in \mathbb{R}^{m \times n}$$

such that $(A + BF)V_{f, s}^*(\ker H_1, \ker H) \subseteq V_{f, s}^*(\ker H_1, \ker H)$,

$$A + BF)(A + BF)^{-1} x \in \mathbb{R}^n$$

is $f$-stable for all $x \in V_{f, s}^*(\ker H_1, \ker H)$.

Proof:
Note that $V_{s}^*(\ker H) \subseteq V_{f, s}^*(\ker H_1, \ker H)$. There exists a matrix $P_0 \in \mathbb{R}^{m \times n}$ such that $(A + BF_0)V_{s}^*(\ker H) \subseteq V_{s}^*(\ker H_1, \ker H)$ and $A + BF_0 \subseteq \mathbb{R}^n$ (cf. Schumacher (1981; Lemma 1.10), Wonham (1979; Exercise 9.1)). See also Lemma 5.5.

We claim that now also $(A + BF_0)V_{f}^*(\ker H_1) \subseteq V_{f}^*(\ker H_1)$. Indeed, observe that on the one hand

$$(A + BF_0)V_{f}^*(\ker H_1) \subseteq V_{f, s}^*(\ker H_1, \ker H) = V_{f}^*(\ker H_1) + V_{s}^*(\ker H),$$

while on the other hand

$$(A + BF_0)V_{f}^*(\ker H_1) \subseteq V_{f}^*(\ker H_1) + \ker H.$$

Then, using the modular distributive rule we can conclude that

$$(A + BF_0)V_{f}^*(\ker H_1) \subseteq (V_{f}^*(\ker H_1) + \ker H) \cap V_{f, s}^*(\ker H_1, \ker H) = V_{f}^*(\ker H_1) \cap (\ker H \cap V_{f, s}^*(\ker H_1, \ker H)).$$

Now note that $V_{f, s}^*(\ker H_1, \ker H) \subseteq V_{f}^*(\ker H_1)$ and $V_{f}^*(\ker H_1) \cap \ker H \subseteq V_{f}^*(\ker H_1)$ (see Section 1.4). It is then immediate that

$$(A + BF_0)V_{f}^*(\ker H_1) \subseteq V_{f}^*(\ker H_1).$$

Let $[X_1, X_2, X_3, X_4] \in \mathbb{R}^{4 \times 4}$ be an invertible matrix such that

$\ker H_1 = V_{f}^*(\ker H_1) \cap V_{s}^*(\ker H),$ $\ker H_1 = V_{f}^*(\ker H)$ and $\ker H_1 = V_{s}^*(\ker H_1).$
Furthermore, let \([U_1, U_2] \in \mathbb{R}^{m \times m}\) be an invertible matrix such that

\[ B \in U_1 = \nu^*_s(\ker H_1) \cap \ker B \cdot \]

Note that the columns of \([X_1, X_2, X_3, X_4]\) and \([U_1, U_2]\) form a basis of \(\mathbb{R}^n\) and \(\mathbb{R}^m\), respectively. With respect to these bases the matrices \(A + BF_0, B, H_1\) and \(H_2\) have the following forms:

\[
(A + BF_0) = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
0 & A_{22} & 0 & A_{23} \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22} \\
B_{31} & B_{32} \\
0 & B_{42}
\end{bmatrix},
\]

\[ H_1 = [0, 0, 0, H_{14}] \quad \text{and} \quad H_2 = [0, 0, H_{23}, H_{24}] \cdot \]

Observe that \(\sigma(A + BF_0) \mid \nu^*_s(\ker H_1) = \sigma(A_{11}) \cup \sigma(A_{22}) \subseteq \mathcal{C}_s\).

Because \(\nu^*_s(\ker H_1)\) is an \(s\)-stabilizability subspace it follows that the pair \(A_{11}, A_{13}, B_{11}\) and consequently the pair \((A_{33}, B_{31})\) is \(s\)-stabilizable (cf. Hautus (1970), Hautus (1980)). Furthermore, because the pair \((A, B)\) is \(s\)-stabilizable, also the pair \((A + BF_0, B)\) and consequently the pair \((A_{44}, H_{42})\) is \(s\)-stabilizable. Hence, there exist matrices \(F_{13}, F_{24}\) of suitable dimensions such that \(\sigma(A_{33} + B_{31} F_{13}) \subseteq \mathcal{C}_s\) and \(\sigma(A_{44} + B_{42} F_{24}) \subseteq \mathcal{C}_s\).

Now, let \(F, F_1 \in \mathbb{R}^{m \times n}\) be matrices adapted to the chosen bases in \(\mathbb{R}^n\) and \(\mathbb{R}^m\) defined as

\[ F := \begin{bmatrix}
0 & F_{13} & 0 \\
0 & 0 & F_{24}
\end{bmatrix} \quad \text{and} \quad F := Y_0 + F_1 \cdot \]

Then the matrix \(F\) is such that it meets the requirements mentioned in the theorem. \(\blacksquare\)

Now we have:

**Theorem 2.6:**

The following statements are equivalent:

(1) (MMRIG) is solvable.
(2) The pair \((A, B)\) is s-stabilizable and \(\text{im } G \subseteq V_{f,s}(\ker H_1, \ker H)\).

(3) There exist s-stable rational matrices \(X \in \mathbb{R}^{n \times n}_+(s)\) and \(U \in \mathbb{R}^{m \times n}_+(s)\) such that \((sI - A)X(s) - BU(s) = I, H_1XG = 0\) and \(HXG\) is an s-stable rational matrix.

**Proof:**

(1) \(\Rightarrow\) (3) Take \(X(s) = (sI - (A + BF))^{-1}\) and \(U(s) = FX(s)\).

(3) \(\Rightarrow\) (2) From Theorem 1.10 it follows that the pair \((A, B)\) is s-stabilizable. Furthermore, let \(v \in \mathbb{R}^q\) and define \(g := Gv \in \mathbb{R}^n, \xi(s) := X(s)g \in \mathbb{R}^p(s)\) and \(\omega(s) := U(s)g \in \mathbb{R}^m(s)\). Observe that \(\xi\) and \(\omega\) are s-stable rational vectors that satisfy \(g = (sI - A)\xi(s) - B\omega(s),\) \(H_1\xi = 0\) and \(H\xi\) is s-stable. Hence, \(G \in V_{f,s}(\ker H_1, \ker H)\) and therefore \(\text{im } G \subseteq V_{f,s}(\ker H_1, \ker H)\).

(2) \(\Rightarrow\) (1) Take the matrix \(F \in \mathbb{R}^{m \times n}\) as indicated in Theorem 2.5. Then, since \(V_{f,s}(\ker H_1, \ker H) \subseteq \ker H_1\), it is immediate that \(H_1(sI - (A + BF))^{-1}G = 0, H(sI - (A + BF))^{-1}G\) is s-stable and \(\sigma(A + BF) \subseteq \mathbb{C}_s\).

**Remark 2.7:**

Recall that \(V_{s}(\ker H) \subseteq \mathbb{R}^{n \times n}_+(s)\) and \(V_{s}(\ker H) \subseteq \ker H_1\). Let \(F \in \mathbb{R}^{m \times n}\) be a matrix as constructed in Theorem 2.5 and let \([X_1, X_2, X_3] \in \mathbb{R}^{n \times n}\) be an invertible matrix such that \(X_1 = V_{s}(\ker H)\) and \(X_2 = V_{s}(\ker H_1, \ker H)\). Then, with respect to the basis in \(\mathbb{R}^n\) formed by the columns of \([X_1, X_2, X_3]\), the matrices \(A + BF, H_1, H_2\) have the following form:

\[
(A + BF) = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
0 & A_{22} & A_{23} \\
0 & 0 & A_{33}
\end{bmatrix}, \quad H_1 = [0, 0, H_{13}] \text{ and } H_2 = [0, H_{22}, H_{23}].
\]

It follows from Theorem 2.5 that \(\sigma(A_{11}) \cup \sigma(A_{33}) \subseteq \mathbb{C}_s\) and \(\sigma(A_{22}) \subseteq \mathbb{C}_f\). Therefore, with respect to the basis mentioned above we have

\[
H_1(sI - (A + BF))^{-1} = [0, 0, P_3(s)]
\]

and

\[
H_2(sI - (A + BF))^{-1} = [0, Q_2(s), Q_3(s)].
\]
where $P_3$, $Q_3$ are s-stable strictly proper rational matrices and $Q_2$ is an f-stable strictly proper rational matrix.

In Section 2.4 it will be seen that the special forms of $H_1(sI - (A + BF))^{-1}$ and $H_2(sI - (A + BF))^{-1}$ given above are useful in the derivation of conditions for the solvability of problem 2.1.

### 2.3. Disturbance decoupled and stabilized estimation with internal stability

The objective of this section is to solve a problem that can be considered the dual of (DDPOS)' as defined in the previous section. The results of the present section can be obtained simply by dualizing the results of Section 2.2, and will be useful in the derivation of necessary and sufficient conditions for the solvability of problem 2.1. The linear system with respect to which the results of this section will be formulated is described as follows.

\[(2.5a) \quad \dot{x}(t) = Ax(t) + C_1 v_1(t) + C_2 v_2(t),\]
\[(2.1b) \quad y(t) = Cx(t),\]
\[(2.1c) \quad z(t) = Hx(t).\]

It is clear that the structure of this system is equal to the structure of the dual of the system described by (2.1a), (2.3c).

Again, we denote $G = [C_1, C_2]$ and we assume that $C_f, C_g \in \Phi$ with $C_f \subseteq C_g$ are two given stability regions. Then the problem of this section, abbreviated (DDPOS)'1, completely dual to the formulation of (DDPOS)', is formulated as follows.

**Definition 2.8:**

(DDPOS)'1 consists of finding a matrix $J \in \mathbb{R}^{nxp}$, representing an output injection $Jy(t)$ for the system (2.5a), (2.1b), (2.1c), such that

\[H(sI - (A + JC))^{-1} C_1 = 0, \quad H(sI - (A + JC))^{-1} C_2 \text{ is f-stable and } \sigma(A + JC) \subseteq C_g.\]

Apart from being the dual of (DDPOS)', it is possible to give an interpretation of (DDPOS)'1 in terms of estimators, explaining the title of the present section. Indeed, let $J \in \mathbb{R}^{nxp}$ be a matrix that satisfies the
requirements of (DDPOS)$^\dagger$. Now, consider the following system that produces an estimate $\hat{z}$ of the exogenous output $z$:

$$\begin{align*}
\dot{x}(t) &= (A + JC)x(t) - Jy(t), \\
\hat{z}(t) &= Hx(t).
\end{align*}$$

Denote the error between $x$ and $\hat{x}$ by $e$, i.e., $e = x - \hat{x}$. Then

$$\begin{align*}
\dot{e}(t) &= (A + JC)e(t) + G_1v_1(t) + G_2v_2(t), \\
z(t) &= Hx(t),
\end{align*}$$

It is then clear that the above estimation scheme for $z$ produces an estimate $\hat{z}$ such that the error $z - \hat{z}$ is independent of the exogenous input $v_1$ and depends on the exogenous input $v_2$ in a $s$-stable way, while the error equation, as a dynamical system, is $s$-stable.

The next definition is the result of dualizing Definition 2.3.

**Definition 2.9:**

$$S_{f,s}(\text{im } G_1, \text{im } G) = S^s_{f,s}(\text{im } G_1, \text{im } G; A, C) :=$$

$$[V_{f,s}(\text{ker } C_1^T, \ker C_1^T; A^T, C^T)]^\perp$$

The following results may be obtained from Theorems 2.4, 2.5, and 2.6 by pure dualization.

**Theorem 2.10:**

$$S_{f,s}(\text{im } G_1, \text{im } G) = S^s_{f}(\text{im } G_1) \cap S^s_{g}(\text{im } G).$$

**Theorem 2.11:**

Let the pair $(C, A)$ be $s$-detectable. Then there exists a matrix $J \in \mathbb{R}^{nxp}$ such that $(A + JC)S^s_{f,s}(\text{im } G_1, \text{im } G) \subseteq S^s_{f,s}(\text{im } G_1, \text{im } G)$, $(A + JC)S^s_{s}(\text{im } G) \subseteq S^s_{s}(\text{im } G)$, $\sigma(A + JC) \equiv \mathcal{C}$, and $x^T(sI - (A + JC))^{-1}c$ is $s$-stable for all $x \in (S_{f,s}(\text{im } G_1, \text{im } G))^\perp$. 

Theorem 2.12:
The following statements are equivalent:

1. \((DDTOS)^{-1}\) is solvable.
2. \(S_{\theta}(\text{im } G_1, \text{im } G) \subseteq \ker H\) and the pair \((C,A)\) is \(s\)-detectable.
3. There exist \(s\)-stable rational matrices \(X \in \mathbb{R}^{n \times m}(s)\) and \(Y \in \mathbb{R}^{n \times p}(s)\) such that \(X(s)(sI-A) - Y(s)C = I\), \(HXG_1 = 0\) and \(HXG_2\) is \(s\)-stable.

Remark 2.13:
Note that \(\text{im } G_1 \subseteq S_{\theta}(\text{im } G_1, \text{im } G) \subseteq S^*_s(\text{im } G)\) and \(\text{im } C_1 \subseteq \text{im } G \subseteq S^*_s(\text{im } G)\). Let \(J \in \mathbb{R}^{nxp}\) be a matrix as indicated in Theorem 2.11 and let \([X_1, X_2, X_3] \in \mathbb{R}^{nxn}\) be an invertible matrix such that \(\text{im } \hat{X}_1 = S_{\theta}(\text{im } G_1, \text{im } G)\) and \(\text{im } [X_1, X_2] = S^*_s(\text{im } G)\). Analogously as in Remark 2.7 it follows that, with respect to the basis in \(\mathbb{R}^n\) formed by the columns \([X_1, X_2, X_3]\) we have

\[
(sI - (A + JC))^{-1} C_1 = \begin{bmatrix} K_1(s) \\ 0 \end{bmatrix} \quad \text{and} \quad (sI - (A + JC))^{-1} C_2 = \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix},
\]

where \(K_1\) and \(U_1\) are \(s\)-stable strictly proper rational matrices and \(U_2\) is an \(s\)-stable strictly proper rational matrix.

### 2.4. Main result

In this section we shall derive the main result of the present chapter. The result consists of conditions in state space terms that are necessary and sufficient for the solvability of problem 2.1. In the derivation of these conditions the two lemmas formulated below will be useful. For the formulation of these lemmas we go back to the linear system described by (1.1) and we relate stability to a given stability region \(\mathfrak{g} \subseteq \mathfrak{g}\).

Lemma 2.14:

1. \(C(sI-A)^{-1}B = 0\) if and only if \(C(sI-A)^{-1}b = 0\) for every vector \(b \in \langle A | \text{im } B \rangle\).
2. \(C(sI-A)^{-1}B\) is stable if and only if \(C(sI-A)^{-1}b\) is stable for every vector \(b \in \langle A | \text{im } B \rangle\).
Proof:
Since \( \text{im } B \subseteq \langle A \mid \text{im } B \rangle \), the (if)-part of both statements is obvious.

In order to prove the (only if)-part of each of the two statements, recall that \( C(sI - A)^{-1} B = \sum_{k=0}^{\infty} C A^k B s^{-k+1} \). Then, it is immediate that
\[
C(sI - A)^{-1} B = 0 \text{ if and only if } C A^k B = 0 \text{ for all } k \geq 0.
\]
Also it follows immediately that \( s C(sI - A)^{-1} B - C B = C(sI - A)^{-1} AB \). Therefore, if \( C(sI - A)^{-1} B = 0 \), then \( CB = 0 \) and \( C(sI - A)^{-1} AB = 0 \) and if \( C(sI - A)^{-1} B \) is stable, then \( C(sI - A)^{-1} AB \) is stable.

Repeated application of these arguments leads to the statements that, if \( C(sI - A)^{-1} B = 0 \), then \( C(sI - A)^{-1} A^k B = 0 \) for all \( k \geq 0 \) and if \( C(sI - A)^{-1} B \) is stable, then \( C(sI - A)^{-1} A^k B \) is stable for all \( k \geq 0 \).

The (only if)-part of each of the two statements of Lemma 2.14 can now be completed using the definition of \( \langle A \mid \text{im } B \rangle \) and the fact that the sum and the product of stable rational matrices remain stable.

**Lemma 2.15:**

Let \( S_1, S_2 \) be \((C,A)\)-invariant subspaces in \( \mathbb{R}^n \) and let \( V_1, V_2 \) be \((A,B)\)-invariant subspaces in \( \mathbb{R}^n \) such that \( S_1 \subseteq S_2 \subseteq V_2 \) and \( S_1 \subseteq V_1 \subseteq V_2 \). Then there exists a matrix \( N \in \mathbb{R}^{m \times p} \) such that \((A + BNC)S_1 \subseteq V_1 \) and \((A + BNC)S_2 \subseteq V_2 \).

For a proof of this lemma we refer to Lemma 5.5, where a more general result is formulated and proved. See also Schumacher (1981), Lemma 3.6.

We are now in the position to state and prove the main result of this chapter. To this end, let the system (2.3) be given and let \( \zeta_f, \zeta_s \in 0 \) be two stability regions with \( \zeta_f \subseteq \zeta_s \). Then, our main result reads as follows (see also Van der Woude (1986)).

**Theorem 2.16:**

Problem 2.1 is solvable if and only if \( S^*_f(\text{im } G) \subseteq V^*_f, (\ker H, \text{ker } H), \)
\( S^*_s(\text{im } G_s, \text{im } G) \subseteq V^*_s(\ker H) \). the pair \((A,B)\) is \( s\)-stabilisable and the pair \((C,A)\) is \( s\)-detectable.
Proof:
(only if) Assume the problem 2.1 is solvable. That is, assume that we have a measurement feedback compensator (1.9) such that in the closed loop system (2.4) \( T_{11} = 0, T_{21} = 0, T_{12} = 0, T_{22} = 0 \) is \( f \)-stable and \( \sigma(A_e) \subseteq \mathcal{F}_e^r \).

Then, by Theorem 1.44, it follows that the pair \((A,B)\) is \( s \)-stabilizable and the pair \((C,A)\) is \( s \)-detectable.

Let \( W_e \) be the linear subspace in \( \mathbb{R}^{n+k} \), the state space of the closed loop system, defined by

\[
W_e = \left\{ A_e \right\} \text{im} \ G_e = \sum_{l=0}^m A_e^l \text{im} \ G_e \quad (= \text{im} \ G_e + A_e \text{im} \ G_e + \ldots + A_e^{n+k-1} \text{im} \ G_e).
\]

Here, we have denoted \( G_e = \begin{bmatrix} C \\ 0 \end{bmatrix} \).

From this definition it is clear that \( \text{im} \ G_e \subseteq W_e \) and \( A_e \text{im} \ G_e \subseteq W_e \). By Lemma 2.14 and the facts that \( H_{1,e}(sI-A_e)^{-1} G_e = H_{1,e}(sI-A_e)^{-1}[G_{1,e}^c, G_{2,e}] = [T_{11}(s), T_{12}(s)] = 0 \) and \( H_{2,e}(sI-A_e)^{-1} G_e = [T_{21}(s), T_{22}(s)] = [0, T_{22}(s)] \) is \( f \)-stable, it follows that \( H_{1,e}(sI-A_e)^{-1} x_e = 0 \) and \( H_{2,e}(sI-A_e)^{-1} x_e = 0 \) is \( f \)-stable for all \( x_e \in W_e \). Hence, \( \text{im} \ G_e \subseteq W_e \subseteq \ker H_{1,e} \).

Let \( S \) and \( V \) be the linear subspaces in \( \mathbb{R}^n \) defined by

\[
S = \{ x \in \mathbb{R}^n \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in W_e \} \quad \text{and} \quad V = \{ x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^k : \begin{bmatrix} x \\ w \end{bmatrix} \in W_e \}.
\]

Observe that \( S \) is the intersection of \( W_e \) with \( \mathbb{R}^n \), while \( V \) is the projection of \( W_e \) onto \( \mathbb{R}^n \) along \( \mathbb{R}^k \). It is then clear that \( \text{im} \ G \subseteq S \subseteq V \subseteq \ker H_1 \).

By Theorem 1.44, it follows that \( S \) is an \( s \)-detectability subspace. So, by Proposition 1.42, we may conclude \( S^s = (\text{im} \ G) \subseteq V \subseteq \ker H_1 \).

Now, take \( x \in V \). By the definition of \( V \) there exists a vector \( w \in \mathbb{R}^k \) such that \( \begin{bmatrix} x \\ w \end{bmatrix} \in W_e \). Let \( x \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^m \) be rational vectors defined as

\[
\begin{bmatrix} \xi(s) \\ \lambda(s) \end{bmatrix} = (sI-A_e)^{-1} \begin{bmatrix} x \\ w \end{bmatrix}.
\]

Because \( \sigma(A_e) \subseteq \mathcal{F}_e \), it is clear that \( \xi \) and \( \lambda \) are \( s \)-stable. Furthermore, using the fact that \( H_{1,e}(sI-A_e)^{-1} x_e = 0 \) and \( H_{2,e}(sI-A_e)^{-1} x_e \) is \( f \)-stable.
for all $x_e \in W_e$, it follows that $H_1 x_e = 0$ and $H_2 x_e$ is $f$-stable. Finally, note that

$$
\begin{bmatrix}
x \\
v
\end{bmatrix} = (sI - A_e) \begin{bmatrix}
\xi(s) \\
\lambda(s)
\end{bmatrix} = \begin{bmatrix}
sI - (A + BNC) & BN \\
LC & K
\end{bmatrix} \begin{bmatrix}
\xi(s) \\
\lambda(s)
\end{bmatrix},
$$

from which it is immediate that $x = (sI - A)\xi(s) - B\omega(s)$, where $\omega = Ne\xi + M\lambda$.

Note that $\omega$ is also an $s$-stable strictly proper rational vector.

Now, by Definition 2.3, it follows that $x \in V_{f,s}^e(\ker H_1, \ker H)$. Hence, we have $S_a^e(\im G) \subseteq V_{f,s}^e(\ker H_1, \ker H)$. By dual reasoning, we may derive $S_{f,s}(\im G, \im G) \subseteq V_{f,s}^e(\ker H)$.

(If) During this part of the proof we use the notation:

$V_{f,s}^e = V_{f,s}^e(\ker H_1, \ker H)$, $V_s^e = V_s^e(\ker H_1, \ker H)$, $S_{f,s} = S_{f,s}(\im G_1, \im G)$ and $S_s = S_s(\im G)$. So, we assume that $S_{f,s} \subseteq V_{f,s}^e$, $S_s \subseteq V_{f,s}$, the pair $(A,B)$ is $s$-stabilizable and the pair $(C,A)$ is $s$-detectable.

Since $S_{f,s}^e$, $S_s$ are $(C,A)$-invariant subspaces in $\mathbb{R}^n$ and $V_{f,s}^e$, $V_{f,s}$ are $(A,B)$-invariant subspaces in $\mathbb{R}^n$ satisfying $S_{f,s}^e \subseteq S_s \subseteq V_{f,s}^e$ and $S_{f,s} \subseteq V_{f,s}$, Lemma 2.15 implies the existence of a matrix $N \in \mathbb{R}^{m \times p}$ such that $(A + BNC)S_{f,s} \subseteq V_{f,s}^e$ and $(A + BNC)S_s \subseteq V_{f,s}^e$. Furthermore, let $F \in \mathbb{R}^{m \times n}$ be a matrix as indicated in Theorem 2.5 and let $J \in \mathbb{R}^{n \times p}$ be a matrix as indicated in Theorem 2.11.

As in Section 1.5, we set $k = n$ and we let $K \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{n \times p}$ and $M \in \mathbb{R}^{m \times m}$ be matrices defined as $K = A + BF + JC - BNC$, $L = BN - J$ and $M = F - NC$. The matrices $K$, $L$, $M$ and $N$, in the obvious way, constitute a measurement feedback compensator (1.9) which, we claim, solves problem 2.1. In order to prove validity of this claim, we remark the following.

(1) With the measurement feedback compensator made up of the matrices $K$, $L$, $M$ and $N$ as described above, the closed loop system is given by

$$
x_{e}(t) = \hat{A}_e x_e(t) + G_{1,e} v_1(t) + G_{2,e} v_2(t),
$$

$$
z_1(t) = H_{1,e} x_e(t), \quad z_2(t) = H_{2,e} x_e(t).
$$

Here, $G_{1,e}$, $G_{2,e}$, $H_{1,e}$ and $H_{2,e}$ are as described in Section 2.1, and $\hat{A}_e$ is given by (1.13).
In Construction 5.48 we have proved that \( \delta(\xi) = \sigma(A + BF) \cup \sigma(A + JC) \). So, with the matrices \( F \) and \( J \) as described above, we have \( \sigma(\xi) \subseteq \xi \). Next, we claim that

\[
H_{1,e} (sI - \xi)^{-1} C_{j,e} = H_{1,e} (sI - (A + BF))^{-1} C_{j,e} + H_{1,e} (sI - (A + JC))^{-1} C_{j,e} + H_{1,e} (sI - (A + BF))^{-1}(sI - (A + BNC))(sI - (A + JC))^{-1} C_{j,e}.
\]

Indeed, with

\[
s^{-1}_{e} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}
\]

it follows that

\[
H_{1,e} (sI - \xi)^{-1} C_{j,e} = H_{1,e} s^{-1}_{e} (sI - \xi) C_{j,e}^{-1} s_{e} = [H_{1,e}, C] (sI - \begin{bmatrix} A + BF & BF - BNC \\ 0 & A + JC \end{bmatrix})^{-1} [G_{j}] = H_{1,e} (sI - (A + BF))^{-1} C_{j,e} + H_{1,e} (sI - (A + BF))^{-1}(sI - (A + BNC))(sI - (A + JC))^{-1} C_{j,e}.
\]

Then, using \((BF - BNC) = (sI - (A + BNC)) - (sI - (A + BF))\), the proof of the desired expression for \( H_{1,e} (sI - \xi)^{-1} C_{j,e} \) can be completed.

(2) Note that we have the following subspace inclusions:

\( V_s \subseteq V_{s,\theta} \subseteq \text{ker } H, V_s \subseteq \text{ker } H \subseteq \text{ker } H_1, \text{im } G_1 \subseteq S_{f_\theta} \subseteq S_\theta \) and \( \text{im } G_1 \subseteq S_\theta \).

Now, let \( [X_1', X_2', X_3', X_4', X_5', X_6'] \in \mathbb{R}^{n \times n} \) be an invertible matrix such that \( \text{im } X_1 = S_{f_\theta}, \text{im } [X_1', X_2', X_3'] = V_s, \text{im } [X_1', X_2', X_4', X_5'] = S_\theta \) and \( \text{im } [X_1', X_2', X_3', X_4', X_5', X_6'] = V_{f_\theta} \).

By Remark 2.7 it follows that, with respect to the basis of \( \mathbb{R}^n \) formed by the columns of \( [X_1', X_2', X_3', X_4', X_5', X_6'] \) we have the following:

\[
H_1 (sI - (A + BF))^{-1} = [0, 0, 0, 0, 0, P_6 (s)]
\]

and

\[
H_2 (sI - (A + BF))^{-1} = [0, 0, 0, Q_4 (s), Q_3 (s), Q_6 (s)].
\]
where \( P_6, Q_6 \) are \( s \)-stable strictly proper rational matrices and \( Q_4, Q_5 \) are \( f \)-stable strictly proper rational matrices.

Analogously, by Remark 2.13 it follows that with respect to the same basis

\[
(sI - (A + JC))^{-1} G_1 = \begin{bmatrix}
R_1(s) \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

and

\[
(sI - (A + JC))^{-1} G_2 = \begin{bmatrix}
U_1(s) \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where \( R_1, U_1 \) are \( s \)-stable strictly proper rational matrices and \( U_2, U_4 \) are \( f \)-stable strictly proper rational matrices.

Furthermore, with respect to the basis mentioned above, the matrices \( G_1, G_2, H_1, H_2 \) and \( A + \text{BNC} \) can be written as

\[
G_1 = \begin{bmatrix}
G_{11} \\
0 \\
U \\
0 \\
0 \\
0
\end{bmatrix}, \quad G_2 = \begin{bmatrix}
G_{12} \\
G_{22} \\
0 \\
G_{42} \\
0 \\
0
\end{bmatrix}
\]

\[
H_1 = [0, 0, 0, 0, 0, H_{16}], \quad H_2 = [0, 0, 0, H_{24}, H_{25}, H_{26}]
\]

and

\[
A + \text{BNC} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
0 & 0 & a_{63} & 0 & a_{65} & a_{66}
\end{bmatrix}
\]

(3) It is now easy to see that

\[
H_i (sI - (A + BF))^{-1} G_j = 0, \quad H_i (sI - (A + JC))^{-1} G_j = 0
\]

and

\[
H_i (sI - (A + BF))^{-1} (sI - (A + \text{BNC})) (sI - (A + JC))^{-1} G_j = 0
\]
for all \((i,j) = (1,1), (2,1)\) and \((1,2)\). Furthermore, it follows that

\[
H_2(sI - (A + BF))^{-1} G_2 = Q_4(s) G_{42},
\]

\[
H_2(sI - (A + JC))^{-1} G_2 = H_{24} U_4(s)
\]

and

\[
H_2(sI - (A + BF))^{-1} G_2 = Q_4(s)(-A_{42} U_2(s) + (sI - A_{44}) U_4(s)) - Q_5(s)(A_{52} U_2(s) + A_{54} U_4(s)).
\]

By the previous it is now clear that \(H_{2e}(sI - \hat{A}_e)^{-1} G_2, e = 0\) for all \((i,j) = (1,1), (2,1)\) and \((1,2)\), and \(H_{2e}(sI - \hat{A}_e)^{-1} G_2, e\) is f-stable. This completes the proof of the (if)-part.

We can now formulate the following corollary of Theorem 2.16. To this end, we assume stability to be taken with respect to a single stability region \(C \subseteq \mathbb{R}\) and we define

\[
V_{g}(\ker H_1, \ker H) = V_{g}(\ker H_1, \ker H; A, B) := \{x_0 \in \mathbb{R}^n | x_0 \text{ has a regular } (\ell, \omega)\text{-representation with } H_1(= 0 \text{ and } H \text{ stable})\}
\]

and

\[
S_{g}(\text{im } G_1, \text{im } G) = S_{g}(\text{im } G_1, \text{im } G; A, C) :=
\]

\[
= (V_{g}(\ker G_1^T, \ker G^T; A^T, C^T))^T.
\]

Let the linear system (2.3) be given. Then we can formulate

**Corollary 2.17:**

There exists a measurement feedback compensator (1.9) such that in the closed loop system (2.4) \(T_{11} = 0, T_{21} = 0, T_{12} = 0\) and \(T_{22}\) is stable (with respect to \(C\)) if and only if \(S^*(\text{im } G) \subseteq V_{g}(\ker H_1, \ker H)\) and \(S_{g}(\text{im } G_1, \text{im } G) \subseteq V^*(\ker H_1, \ker H)\).

**Proof:**

In Theorem 2.16, set \(\xi_1 = \mathcal{C}\) and \(\xi_2 = \mathcal{C}\). Then \(S^*(\text{im } G) = S_{g}^*(\text{im } G), V^*(\ker H) = V_{g}^*(\ker H), S_{g}(\text{im } G_1, \text{im } G) = S_{g}(\text{im } G_1, \text{im } G)\) and \(V_{g}(\ker H_1, \ker H) = V_{g}(\ker H_1, \ker H)\).
2.5. Special cases

In this last section of the present chapter we consider some special cases of problem 2.1.

In fact, it are these special cases that make the study of problem 2.1 worthwhile. As announced before, these special cases are the nontrivial extensions towards measurement feedback of problems of which, up to now, only the version using state feedback was solved.

Also in the present section, we examine the well-known control problems (DDPM), (DDPM)', (RF) and (RF)' mentioned in the beginning of this chapter.

The first control problem that we consider will be the problem of stabilizing the exogenous output of a system with respect to the exogenous input using measurement feedback. Furthermore, we shall treat this problem with and without the additional requirement of closed loop internal stability. For the problem with closed loop internal stability we assume that two stability regions $C_f, C_s \in \Theta$ with $C_f \subseteq C_s$ are given. For the problem without closed loop internal stability we assume that only one stability region $C_g \in \Theta$ is given.

Let the linear system (2.1) be given. Then the problem of stabilizing the exogenous output with respect to the exogenous input, abbreviated (OSDPM) (cf. Hautus (1980)), consists of finding a measurement feedback compensator (1.9) such that in the closed loop system (2.2) $H_e(sI-A_e)^{-1} C_e$ is stable (with respect to $C_g$).

The problem of stabilizing the exogenous output with respect to the exogenous input and with internal stabilization, (OSDPM)', consists of finding a measurement feedback compensator (1.9) such that in the closed loop system (2.2) $H_e(sI-A_e)^{-1} C_e$ is $f$-stable and $o(A_e) \subseteq C_s$.

It should be clear that (OSDPM) is the extension towards measurement feedback of the problem of stabilizing the exogenous output with respect to the exogenous input using static state feedback. The latter problem, abbreviated (OSDP), was introduced and solved in Hautus (1980) and is formulated as follows.

Given the linear system described by (2.1a) and (2.1c), find a static state feedback $u(t) = Fx(t)$ such that in the closed loop system
\( H(sl - (A + BF))^{-1} G \) is stable (with respect to an a priori given stability region \( \mathcal{C} \subseteq \mathcal{R} \)).

If, instead of one stability region \( \mathcal{C} \), we are given two stability regions \( \mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{R} \) and it is required that \( H(sl - (A + BF))^{-1} G \) be \( f \)-stable while \( \sigma(A + BF) \subseteq \mathcal{S}_g \), we arrive at the following problem: The problem of stabilizing the exogenous output with respect to the exogenous input with internal stabilization using state feedback. This problem we shall abbreviate \((\text{OSDP})'\).

Returning to \((\text{OSDP})\) and \((\text{OSDP})'\), it is clear that these problems may be obtained from problem 2.1 by specifying that \( q_1 = 0, r_1 = 0 \), so the matrices \( C_1 \) and \( \mathcal{H}_1 \) do not appear, and \( G = G_2, H = H_2 \).

Then, by Theorem 2.16 and Corollary 2.17 we can conclude:

**Corollary 2.18:**

1. \((\text{OSDP})\) is solvable if and only if \( S^*(\text{im } G) \subseteq V_g(\mathcal{R}^n, \ker H) \) and \( S_g(0, \text{im } G) \subseteq V^*(\ker H) \).

2. \((\text{OSDP})'\) is solvable if and only if \( S^*_g(\text{im } G) \subseteq V^*_g(\mathcal{R}^n, \ker H) \), \( S^*_g(0, \text{im } G) \subseteq V^*(\ker H) \), the pair \((A, B)\) is \( s \)-stabilizable and the pair \((C, A)\) is \( s \)-detectable.

It follows from Theorem 2.4 that \( V_g(\mathcal{R}^n, \ker H) = V^*_g(\mathcal{R}^n) + V^*(\ker H) \). Now recall that \( V^*_g(\mathcal{R}^n) = \langle A | \text{im } B \rangle + X^*_g(A) \) (see Section 1.4). Thus, \( \text{im } B = V^*_g(\mathcal{R}^n) \) and \( A V^*_g(\mathcal{R}^n) \subseteq V^*_g(\mathcal{R}^n) \). Therefore,

\[
A V^*_g(\mathcal{R}^n, \ker H) \subseteq A V^*_g(\mathcal{R}^n) + A V^*(\ker H) \subseteq V^*_g(\mathcal{R}^n) + V^*(\ker H) + \text{im } B = V^*_g(\mathcal{R}^n) + V^*(\ker H) = V_g(\mathcal{R}^n, \ker H).
\]

Hence, \( V_g(\mathcal{R}^n, \ker H) \) is an \( A \)-invariant subspace.

Using the fact that \( \text{im } \mathcal{C} \subseteq S^*(\text{im } \mathcal{C}) \subseteq \langle A | \text{im } \mathcal{C} \rangle \), we obtain that \( S^*(\text{im } \mathcal{C}) \subseteq V_g(\mathcal{R}^n, \ker H) \) if and only if \( \text{im } \mathcal{C} \subseteq V_g(\mathcal{R}^n, \ker H) \). By dual reasoning, we can derive that \( S^*_g(0, \text{im } G) \subseteq V^*_g(\ker H) \) if and only if \( S^*_g(0, \text{im } G) \subseteq \ker H \). Therefore, we may conclude.
Corollary 2.19:
(OSDPM) is solvable if and only if $\text{im } G \subseteq V_g(\mathbb{R}^n, \ker H)$ and $S_g(O, \text{im } G) \subseteq \ker H$.

At this point we want to recall that $\text{im } G \subseteq V_g(\mathbb{R}^n, \ker H)$ is a necessary and sufficient condition for the solvability of (OSDP) (cf. Hautus (1980)). So, the remarkable fact shows up that due to the $A$-invariance of both $V_g(\mathbb{R}^n, \ker H)$ and $S_g(O, \text{im } G)$, and the facts that $\text{im } G \subseteq S^*(\text{im } G) \subseteq \langle A \mid \text{im } G \rangle$ and $\langle \ker H \mid A \rangle \subseteq V^*(\ker H) \subseteq \ker H$, the solvability of (OSDPM) is equivalent to the solvability of both (OSDP) and (OSDP)$^\perp$, the dual of (OSDP). Here, (OSDP)$^\perp$ is formulated as follows.

Given the linear system described by (2.1), find a matrix $J \in \mathbb{R}^{n \times p}$ defining an output injection $Jy(t)$ for (2.18) such that in the resulting system the transfer matrix $H(sI - (A + J\xi))^{-1}C$ is stable.

Another remarkable point is that due to the $A$-invariance of $S_g(O, \text{im } G)$ and $V_g(\mathbb{R}^n, \ker H)$, in the construction as described in the (ii)-part of Theorem 2.16, the matrix $M \in \mathbb{R}^{m \times n}$ can be taken equal to the zero matrix. This implies that we do not need a static measurement feedback part in the compensator that solves (OSDPM). Consequently, the transfer matrix of such a compensator will be a strictly proper rational matrix.

From the previous it is clear that also $V_{g, s}(\mathbb{R}^n, \ker H)$ and $S_s(O, \text{im } G)$ are $A$-invariant subspaces.

However, since in general when $\mathcal{C}_g \neq \mathcal{C}$, it is not true that $S^*(\text{im } G) \subseteq \langle A \mid \text{im } G \rangle$ or $\langle \ker H \mid A \rangle \subseteq V^*(\ker H)$, a simplification, as in the above, of the conditions that are equivalent to the solvability of (OSDPM)' is not possible! Nevertheless, analogously to the previous, we can conclude that the compensator (1.9) that solves (OSDPM)' can be constructed to have a strictly proper rational transfer matrix.

The problem we shall treat in the second part of the present section can be considered a generalization of the problems (OSDPM) and (OSDPM)', as introduced above, but also as an extension of (DDPOS)' as described in Section 2.2. To that end, we consider the linear system described by (2.3a), (2.1b) and (2.1c). Furthermore, we assume that either a single stability region $\mathcal{C}_g \in \Theta$ is given, or that two stability regions $\mathcal{C}_s, \mathcal{C}_g \in \Theta$ with $\mathcal{C}_g \subseteq \mathcal{C}_s$ are given. Then we can formulate:
The problem of decoupling the first exogenous output and stabilizing the second exogenous output with respect to the exogenous input using measurement feedback, abbreviated (DDPOSM), consists of finding a measurement feedback compensator (1.9) such that in the resulting closed loop system $H_1 e(sI - A_e)^{-1} G_e = 0$ and $H_2 e(sI - A_e)^{-1} G_e$ is stable (relative $F_g$).

The problem of decoupling the first exogenous output and stabilizing the second exogenous output with respect to the exogenous input with internal stabilization using measurement feedback, abbreviated as (DDPOSM)', consists of finding a measurement feedback compensator (1.9) such that in the resulting closed loop systems $H_1 e(sI - A_e)^{-1} G_e = 0$, $H_2 e(sI - A_e)^{-1} G_e$ is I-stable and $c(A_e) \subseteq C_s$.

By specifying that $q_1 = 0$ and $G_2 = G$ in problem 2.1 we obtain the following:

Corollary 2.20:

1) (DDPOSM) is solvable if and only if $S^*(im G) \subseteq V_g(\ker H_1, \ker H)$ and $S^*_g(0, im G) \subseteq V^*(\ker H)$.

2) (DDPOSM)' is solvable if and only if $S^*_g(im G) \subseteq V^*_g(\ker H_1, \ker H)$, $S^*_g(0, im G) \subseteq V^*_g(\ker H)$, the pair $(A, B)$ is s-stabilizable and the pair $(C, A)$ is s-detectable.

As in the previous, it can be shown that (DDPOSM) is solvable if and only if $S^*(im G) \subseteq V_g(\ker H_1, \ker H)$ and $S^*_g(0, im G) \subseteq \ker H$. No further simplification of the first of these two conditions is possible. Moreover, it can be shown that, in general, a measurement feedback compensator that solves (DDPOSM) or (DDPOSM)' needs to have a nonzero static measurement feedback part.

Finally, we return to the four well-known control problems mentioned in Section 2.1. As explained in that section, these four problems can be considered special cases of problem 2.1. Then, from Theorem 2.16 and Corollary 2.17, we can derive:
Corollary 2.21:

1) (DDPM) is solvable if and only if $S^*(\text{im } G) \subseteq V^*(\ker H)$.

2) (RP) is solvable if and only if $S^*_g(\text{im } G) \subseteq V^*_g(\ker H)$ and $\mathbb{R}^n = V^*_g(\mathbb{R}^n, \ker H)$.

3) (DDPM)' is solvable if and only if $S^*_g(\text{im } G) \subseteq V^*_g(\ker H)$, the pair $(A, B)$ is stabilizable and the pair $(C, A)$ is detectable (with respect to $\xi_g \in \mathcal{G}$).

4) (RP)' is solvable if and only if $S^*_g(\text{im } G) \subseteq V^*_g(\ker H)$, $\mathbb{R}^n = V^*_g(\mathbb{R}^n, \ker H)$, the pair $(A, B)$ is $s$-stabilizable and the pair $(C, A)$ is $s$-detectable.

Proof:

1) In Corollary 2.17 specify $\alpha_2 = 0, \gamma_2 = 0, C_1 = G$ and $H_1 = H$.

2) In Corollary 2.17 specify $\alpha_1 = 0, C_1 = G, C_2 = I$ and $H_2 = H$.

3) In Theorem 2.16 specify $\alpha_2 = 0, \gamma_2 = 0, C_1 = G, H_1 = H$ and $\xi_g = \Phi_g$.

4) In Theorem 2.16 specify $\alpha_1 = 0, C_1 = G, C_2 = I$ and $H_2 = H$. 

$\blacksquare$
In this chapter we consider a further extension of the type of linear system that we considered in Chapter 2. To be more specific, in the present chapter we shall be dealing with linear systems that, apart from a control input and a measurement output, have \( \nu \) exogenous inputs and \( \nu \) exogenous outputs, where \( \nu \) is an integer larger than one.

If a system of this kind is controlled by means of a measurement feedback compensator as described in Section 1.5, we obtain a closed loop system with \( \nu \) exogenous inputs and \( \nu \) exogenous outputs. So, the transfer matrix of the closed loop system can be partitioned according to the dimensions of the exogenous inputs and outputs as a \( \nu \times \nu \) block matrix.

It is clear that the blocks of this transfer matrix depend on the parameters (matrices) of both the system and the compensator applied.

![Diagram](image)

**Figure 4**

Now, the problem of noninteracting control (by measurement feedback) consists of the following. Find a compensator such that the off-diagonal blocks of the transfer matrix of the closed loop system are identically equal to zero.
The problem of almost noninteracting control (by measurement feedback) can be formulated as follows. For each $\varepsilon > 0$, find a compensator such that the off-diagonal blocks of the transfer matrix of the closed loop system, in some appropriate norm, are smaller than $\varepsilon$.

In these formulations no requirements are made with respect to the diagonal blocks of the transfer matrix of the closed loop system.

In the present chapter we show that for linear systems as described above, it is always possible to check whether the two control problems introduced above are solvable or not. However, for $\mu > 2$ we are not able to derive simple, intuitively appealing conditions in state space - or frequency domain terms that are necessary and sufficient for the solvability of these problems. In fact, for $\mu = 2$, the derivation of such conditions is involved to such an extent that we have made it the major subject of Chapter 4.

The importance of conditions in state space - or frequency domain terms is that they can serve as a starting point in the study of control problems that are more realistic and more complicated than the two problems introduced before. For instance, such more complicated control problems could be the two control problems mentioned above extended with some stability requirements.
At this point, we want to make clear that the point of view towards non-interacting control as exposed in the present chapter is completely different from the 'classical' point of view on noninteracting control (cf. Morse and Wonham (1971), Hautus and Heyman (1980) and the references cited therein). Indeed, in the 'classical' context, it is assumed that the linear systems, in addition to a control input, only have exogenous outputs. The problem of noninteracting control in the 'classical' context then consists of finding a state feedback compensator that has \( n + 1 \) inputs, one of these being the state of the system, and one output, serving as the control input to the system, such that the transfer matrix of the closed loop system has a block diagonal structure. Furthermore, in contrast with our approach, in the 'classical' noninteracting control problem it is required that the diagonal blocks of the transfer matrix of the closed loop system satisfy some output controllability condition or a condition of rank preservation.

The outline of the present chapter is as follows. In Section 3.1 we give a mathematical formulation of the two main problems of this chapter. In addition, we describe a method by which it is possible to decide whether each of the two problems is solvable or not. The method in principle provides, if it exists, the corresponding compensator. In Section 3.2 we formulate a modified version of our two main problems. More concretely, in Section 3.2, we allow for state feedback instead of measurement feedback in the compensation scheme. So, at that point we are dealing with problems of noninteracting and almost noninteracting control by (dynamic) state feedback. Problems of this kind were formulated for the first time in Willems (1980), and were developed in Trentelman and Van der Woode (1986).

In the first reference, results were derived using state space methods. In the second reference, as we shall do in Sections 3.3 and 3.4, these derivations were carried out using the concept of \((\xi, \omega)\)-representation. In fact, Sections 3.2, 3.3 and 3.4 are based largely on this last reference. The approach using \((\xi, \omega)\)-representations turns out to be very powerful and enables us to provide elegant and rigorous proofs for our results. Due to the fact that we allow state feedback, we are in the position to derive necessary and sufficient conditions in state space terms for the solvability of the noninteracting and almost noninteracting control problem by
(dynamic) state feedback. In fact, we are able to do more. We shall also derive necessary and sufficient conditions for the solvability of the non-interacting control problem by state feedback extended with some additional stability requirements.

3.1. First principles

In this chapter we consider the linear system described by

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) + \sum_{i \in \mu} G_i v_i(t), \\
y(t) &= Cx(t), \\
z_i(t) &= H_i x(t), \quad (i \in \mu),
\end{align}

where \( \mu > 1 \) and \( \mu := \{1, 2, \ldots, \mu\} \). The meaning of \( x(t), u(t), y(t), A, B \) and \( C \) is as described in Section 1.1, \( v_i(t) \in \mathbb{R}^{q_i} \) denotes the \( i \)-th exogenous input and \( z_i(t) \in \mathbb{R}^{r_i} \) the \( i \)-th exogenous output \( (i \in \mu) \). \( G_i \) and \( H_i \) \( (i \in \mu) \) are real matrices of appropriate dimensions.

Throughout the present chapter we assume that the linear system (3.1) is controlled by means of the type of feedback compensator that we introduced in Section 1.5

\begin{align}
(1.9a) \quad \hat{w}(t) &= Kw(t) + Ly(t), \\
(1.9b) \quad u(t) &= Mw(t) + Ny(t).
\end{align}

The interconnection of this type of compensator with the linear system (3.1) yields a closed loop system with \( \nu \) exogenous inputs and \( \nu \) exogenous outputs. This closed loop system is described by

\begin{align}
\dot{x}_e(t) &= A_e x_e(t) + \sum_{i \in \mu} G_{i,e} v_i(t), \\
z_i(t) &= H_{i,e} x_e(t), \quad (i \in \mu),
\end{align}

where \( x_e(t) \) and \( A_e \) are given by (1.11), \( G_{i,e} := \begin{bmatrix} G_i \\ 0 \end{bmatrix} \) and \( H_{i,e} := [H_i, 0] \) \( (i \in \mu) \).
Let $T$ denote the transfer matrix of the closed loop system (3.2). Then $T$ can be partitioned according to the dimensions of the exogenous inputs and outputs as $T = \begin{pmatrix} T_{ij} \end{pmatrix}$ \((i, j \in \mathbb{N})\), where $T_{ij}(s) := H_{ij} e^{(s\mathbf{I} - A \mathbf{e})^{-1}} G_{ij}$.

$T_{ij}$ denotes the transfer matrix between the $j$-th exogenous input and the $i$-th exogenous output in the closed loop system (3.2).

We denote the transfer matrices in the 'open loop' system (3.1) by

$$
P(s) := C(s\mathbf{I} - A)^{-1} B, \quad M_i(s) := C(s\mathbf{I} - A)^{-1} G_i,$$

$$
L_i(s) := H_i(s\mathbf{I} - A)^{-1} B, \quad X_{ij}(s) := H_i(s\mathbf{I} - A)^{-1} G_j,$$

and the transfer matrix of the compensator (1.9) by

$$
P(s) := N + M(s\mathbf{I} - K)^{-1} L.
$$

An easy calculation shows that in the closed loop system (3.2)

$$
T_{ij} = K_{ij} + L_i X M_j \quad (i, j \in \mathbb{N}),
$$

where $X := (1 - FP)^{-1} F$. Note that the inverse in the latter expression exists as a rational matrix because $T - FP$ is a bicausal rational matrix (cf. Hautus and Heyman (1979)). A bicausal rational matrix is a proper rational matrix with a proper rational inverse. A proper rational matrix is bicausal if and only if its determinant does not vanish at infinity. It is clear that $X$ is a proper rational matrix and that $F = X(I + PX)^{-1}$.

In the spirit of Willems (1980) we formulate the following two control problems, where we assume that the linear system (3.1) is given.

**Definition 3.1:**

The noninteracting control problem by measurement feedback, (NICPM), consists of finding a measurement feedback compensator (1.9) such that in the closed loop system (3.2) $T_{ij} = 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$.

If, instead of being zero, we require the off-diagonal blocks of the transfer matrix of the closed loop system to be arbitrarily small with respect to an appropriate norm, we arrive at the almost version of (NICPM).
Therefore, in order to be able to state the formulation of this almost version, we have to specify the norm with respect to which we shall measure the magnitudes of the off-diagonal blocks of the transfer matrix. To this extent, recall that $\mathbb{C}^-$ is the set of complex numbers with negative real part, and let $W \in \mathbb{C}^{mxn}(s)$ be a given $\mathbb{C}^-$-stable rational matrix. Then we define the $H^\infty$-norm of $W$, denoted by $\|W\|_\infty$, as

$$
\|W\|_\infty := \sup_{\Re s \geq 0} \sigma(W(s)),
$$

where, for all $s \in \mathbb{C}$, $\sigma(W(s))$ denotes the largest singular value of the complex matrix $W(s)$.

Note that, due to the $\mathbb{C}^-$-stability and the strict properness of $W$ we have that

$$
\|W\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma(W(i\omega)).
$$

In the sequel, whenever we use $\|W\|_\infty$, we implicitly assume that $W$ is a $\mathbb{C}^-$-stable strictly proper rational matrix.

For more details concerning the $H^\infty$-norm, we refer to Vidyasagar (1985) or Desoer and Vidyasagar (1975).

We can now state the formulation of the almost version of (NICPM), where we again assume that the linear system (3.1) is given.

**Definition 3.2:**

The almost noninteracting control problem by measurement feedback, (ANICPM), consists of finding, for all $\epsilon > 0$, a measurement feedback compensator (1.9) such that in the closed loop system (3.2) $\|T_{ij}\|_\infty \leq \epsilon$ for all $i,j \in \mu$ with $i \neq j$.

The following corollary is now immediate.

**Corollary 3.3:**

1) (NICPM) is solvable if and only if there exists a rational matrix $X \in \mathbb{C}^{mxn}(s)$ such that $K_{ij} + X_{ij}X_{mj} = 0$ for all $i,j \in \mu$ with $i \neq j$. 


2) (ANICPMu) is solvable if and only if for all $c > 0$ there exists a rational matrix $X \in \mathbb{R}_0^{m \times p}(s)$ such that $\|K_{ij} + L_i X M_j\|_\infty \leq c$ for all $i,j \in \Pi$ with $i \neq j$.

In order to develop a method by which we can check the solvability of (NICPMu) and (ANICPMu) we consider the rational matrix equation

$$(RME) \quad UX = W,$$

where $U \in \mathbb{R}_+^{a \times b}(s)$, $W \in \mathbb{R}_+^{b \times c}(s)$ are given rational matrices and $X \in \mathbb{R}_+^{b \times c}(s)$ is the unknown rational matrix. We say that (RME) is solvable over $\mathbb{R}(s)$, respectively over $\mathbb{R}_0(s)$, if there exists a rational matrix $X \in \mathbb{R}_+^{b \times c}(s)$, respectively $X \in \mathbb{R}_0^{b \times c}(s)$, such that (RME) is satisfied.

The next theorem is due to Willems (1981) and plays an important role in the present as well as in future chapters.

**Theorem 3.4:**

For every $c > 0$ there exists a rational matrix $X \in \mathbb{R}_0^{b \times c}(s)$ such that $\|UX - W\|_\infty \leq c$ if and only if (RME) is solvable over $\mathbb{R}(s)$.

The proof of this theorem, given in Willems (1981), is quite involved. It makes a detour via a state space realization of the 'transfer matrix' $[U,W]$. Then, with respect to the obtained realization, various nontrivial subspaces and state feedback matrices are computed, from which eventually a proof of Theorem 3.4 is deduced.

We shall present here a more direct proof of Theorem 3.4. In order to do so, we need two preliminary lemmas. Throughout the proofs of these lemmas and of Theorem 3.4, we call a rational function, vector or matrix stable if the latter has all its poles in $\mathbb{C}^-$. Furthermore, we denote the set of polynomials with real coefficients by $\mathbb{R}[s]$ and the set of a $a \times b$-matrices with entries in $\mathbb{R}[s]$ by $\mathbb{R}_+^{a \times b}(s)$. Finally, if $V \in \mathbb{R}_+^{a \times b}(s)$ is a given nonzero rational matrix, then we define the degree of $V$, denoted by $\mathcal{D}_V$, as

$$\mathcal{D}_V := \min \{d \in \mathbb{Z} \mid s^{-d} V(s) \in \mathbb{R}_0^{a \times b}(s)\}.$$

Note that for polynomial matrices this notion of degree coincides with the
usual notion of the degree of a polynomial matrix. Now we can state:

Lemma 3.5:
Consider the rational matrix equation (RME). If (RME) is solvable over \( \mathbb{R}(s) \), then there exists a rational matrix \( X' \in \mathbb{R}^{b \times c}_0(s) \) such that \( UX' - W \) is a stable strictly proper rational matrix.

Proof:
Let \( X \in \mathbb{R}^{b \times c}_0(s) \) be a rational matrix such that \( UX = V \) and let \( r := 3X \).
If \( r \leq 0 \), then \( X \in \mathbb{R}^{b \times c}_0(s) \), set \( X' := X \) and we have that \( UX' - V = 0 \) is a stable strictly proper rational matrix. Therefore, assume \( r > 0 \). Determine a polynomial matrix \( N \in \mathbb{R}^{a \times c}(s) \) and a polynomial \( d \in \mathbb{R}(s) \) such that \( W = N/d \) and denote \( p := 3d \). Because \( W \in \mathbb{R}^{a \times c}(s) \), it is clear that \( 3N < p \).

Let \( q \in \mathbb{R}(s) \) be a polynomial of degree \( n + p \) with all its zeros in \( \mathbb{C}^\times \) and determine polynomials \( p, q \in \mathbb{R}(s) \) such that \( q = ud + p \) and \( 2p < p \). Define the rational function \( r \in \mathbb{R}(s) \) by \( r := p/q \) and observe that \( s^r(s) \in \mathbb{R}^{b \times c}_0(s) \). Next, define the rational matrix \( X' \in \mathbb{R}^{b \times c}_0(s) \) by \( X' := rX \). Then it follows that \( X' \in \mathbb{R}^{b \times c}_0(s) \), because \( X'(s) = s^r(s) s^{-1} X(s) \) where both \( s^r(s) \) and \( s^{-1} X(s) \) are proper rational. Furthermore,

\[
UX' - W = rUX - W = (r-1)W = \left( \frac{p-q}{q} \right) \frac{N}{d} = \frac{N}{q} \frac{N}{d} = \frac{N}{d} \cdot N,
\]
from which it is clear that \( UX' - W \) is a stable strictly proper rational matrix.

Lemma 3.6:
Consider the rational matrix equation (RME) and assume that the rational matrix \( W \in \mathbb{R}^{a \times c}_+(s) \) is stable. Let \( \kappa \) be a real positive number such that \( \kappa > c \) where \( c := \|W\|_\infty \). Then we have the following.
If \( X \in \mathbb{R}^{b \times c}_0(s) \) is a rational matrix such that \( \|UX - W\|_\infty \leq \kappa \) then there exists a rational matrix \( X' \in \mathbb{R}^{b \times c}_0(s) \) with \( \delta X' = 3X - 1 \) such that \( \|UX' - W\|_\infty \leq 2\kappa \).

Proof:
We recall that \( W \) is a strictly proper rational matrix, i.e. \( \lim_{|\omega| \to \infty} W(i\omega) = 0 \).
Therefore, there exists a real number \( R > 0 \) such that \( \sup_{|\omega| \geq R} \|W(i\omega)\| < \kappa \).
Let \( \lambda \) be a real number such that \( 0 < \lambda < \frac{\varepsilon}{n} \) and define 
\[ f(s) = (s\lambda + 1)^{-1} . \]
Note that \( f \) is a stable strictly proper rational function. Hence, \( \| f \|_\infty \) exists and 
\[ \| f \|_\infty = \sup_{\Re s \geq 0} |f(s)| = \sup_{\omega \in \mathbb{R}} |f(i\omega)| = \sup_{\omega \in \mathbb{R}} (\lambda^2 \omega^2 + 1)^{-\frac{1}{2}} = 1 . \]

Let \( X' \in \mathbb{R}^{b \times c}(s) \) be the rational matrix defined by \( X' := fX \). Then 
\[ 3X' - 3X - 1 . \]
Furthermore, 
\[ UX' - W = fUX' - W = f(UX' - W) + (f-1)W , \]
where both \( f(UX' - W) \) and \((f-1)W\) are stable strictly proper rational matrices. So, \( \| UX' - W \|_\infty \) is well-defined and 
\[ \| UX' - W \|_\infty \leq \| f(UX' - W) \|_\infty + \| (f-1)W \|_\infty \leq 2c . \]

Now observe that 
\[ \| (f-1)W \|_\infty = \sup_{\Re s \geq 0} |f(s)-1|W(s) = \sup_{\omega \in \mathbb{R}} \left( |f(i\omega)-1|W(i\omega) \right) = \sup_{|\omega| = \infty} \left( |f(i\omega)-1|W(i\omega) \right) \]
\[ = \max \left[ \sup_{|\omega| = \infty} \left( |f(i\omega)-1|W(i\omega) \right) , \sup_{|\omega| > \infty} \left( |f(i\omega)-1|W(i\omega) \right) \right] \]
\[ \leq \max \left[ \sup_{|\omega| = \infty} \left( \sqrt{\frac{\lambda^2 \omega^2}{\lambda^2 \omega^2 + 1} \cdot \varepsilon} \right) , \sup_{|\omega| > \infty} \left( \sqrt{\frac{\lambda^2 \omega^2}{\lambda^2 \omega^2 + 1} \cdot \varepsilon} \right) \right] \]
\[ \leq \max \left[ \frac{\varepsilon}{\lambda} \cdot \sigma , \frac{\varepsilon}{\lambda} \cdot \sigma \right] = \varepsilon , \]

Consequently, \( \| UX' - W \|_\infty < 2c . \)

\textbf{Proof of Theorem 3.4:}

\text{(only if)} In order to show the existence of a rational matrix \( X \in \mathbb{R}^{b \times c}(s) \) satisfying (RME) it suffices to show that for every rational vector \( v \in \mathbb{R}^b(s) \) satisfying \( v^T U = 0 \), we have \( v^T W = 0 \). It even suffices to show that for every rational vector \( v \in \mathbb{R}^b(s) \) satisfying \( v^T U = 0 \) and \( v, v^T W \)
are stable, we have $v^T W = 0$. Therefore, assume that $v \in \mathbb{R}_+^b(s)$ is such that $v^T U = 0$, and $v, v^T W$ are stable. Then we have the following. For every $\varepsilon > 0$ there exists a rational matrix $X' \in \mathbb{R}_0^{b \times c}(s)$ such that

$$\|v^T W\|_{\infty} = \|v^T (UX' - W)\|_{\infty} \leq \|v^T\|_{\infty} c.$$ 

Hence, for every $\varepsilon > 0$, $\|v^T W\|_{\infty} \leq \varepsilon \|v\|_{\infty}$. Thus, $\|v^T W\|_{\infty} = 0$, or equivalently $v^T W = 0$.

(i) Assume that $X \in \mathbb{R}_0^{b \times c}(s)$ is a rational matrix that satisfies (RME). Let $\varepsilon > 0$ be arbitrary. Then we construct a rational matrix $X' \in \mathbb{R}_0^{b \times c}(s)$ such that $UX' - W$ is a stable strictly proper rational matrix. Hence, $\|UX' - W\|_{\infty}$ is well-defined.

Let $\sigma := \|UX' - W\|_{\infty}$. If $\sigma \leq \varepsilon$, then set $X' = X$ and we are done. Therefore, assume $\sigma > \varepsilon$. From Lemma 3.5, the existence follows of a rational matrix $\tilde{X} \in \mathbb{R}_0^{b \times c}(s)$ such that $UX - W$ is a stable strictly proper rational matrix. Hence, $\|UX - W\|_{\infty}$ is well-defined.

By putting $X' := \tilde{X} + \tilde{X}$, we have proved the existence of a proper rational matrix $X' \in \mathbb{R}_0^{b \times c}(s)$ such that $\|UX' - W\|_{\infty} \leq \varepsilon$. Thus the proof of Theorem 3.4 is completed.

From the proofs of Theorem 3.4 and Lemmas 3.5, 3.6 an algorithm can be extracted that for a given rational matrix $X \in \mathbb{R}_0^{b \times c}(s)$ satisfying (RME) and for a given $\varepsilon > 0$ results in a proper rational matrix $X' \in \mathbb{R}_0^{b \times c}(s)$ such that $\|UX' - W\|_{\infty} \leq \varepsilon$.

Also from the proofs of Theorem 3.4 and Lemmas 3.5, 3.6 it is clear that the following corollary is true.
Corollary 3.7:
For every \( k > 0 \) there exists a strictly proper rational matrix \( X \in \mathbb{R}_+^{b \times c}(s) \) such that \( \| UX - W \|_{\infty} < c \) if and only if (RME) is solvable over \( \mathbb{R}(s) \).

Consider the rational matrix equation

\[
(U' \cdot X' \cdot V') = W',
\]

where \( W \in \mathbb{R}^{b \times d}_+(s) \), \( U \in \mathbb{R}^{a \times b}_+(s) \) and \( V \in \mathbb{R}^{c \times d}_+(s) \) are given rational matrices and \( X \in \mathbb{R}^{b \times c}_+(s) \) is the unknown rational matrix.

It is well known that rational matrix equations of type (RME)', can be reformulated as rational matrix equations of type (RME) by means of Kronecker products.

Indeed, let \( U = (v_{ij}) \) with \( v_{ij} \in \mathbb{R}_+(s) \) for all \( i \in \mathbb{I}, j \in \mathbb{J} \), and let \( W = [w_1, w_2, \ldots, w_d] \) with \( w_i \in \mathbb{R}_+(s) \) for all \( i \in \mathbb{I} \). Now, let \( U' \in \mathbb{R}^{a \times d}_+(s) \) and \( W' \in \mathbb{R}^{d \times c}_+(s) \) be rational matrices defined by

\[
U' = (u_{ij}') := (v_{ij} u), \quad (i \in \mathbb{I}, j \in \mathbb{J}) \quad \text{and} \quad W' := \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}.
\]

Then the following holds (cf. Lancaster and Tismenetsky (1985)).
The rational matrix \( X \in \mathbb{R}_+^{b \times c}(s) \) (\( X \in \mathbb{R}_0^{b \times c}(s) \)) with \( X := [x_1, x_2, \ldots, x_c] \),
where \( x_i \in \mathbb{R}_+^{b}(s) \) (\( x_i \in \mathbb{R}_0^{b}(s) \)), satisfies \( UXV = W \) if and only if the rational vector \( X' \in \mathbb{R}_+^{b \times 1}(s) \) (\( X' \in \mathbb{R}_0^{b \times 1}(s) \)) with \( X' := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} \) satisfies \( U'X' = W' \).

By this remark and by Theorem 3.4 the next result follows readily.
Corollary 3.8:

\( (ANICPM_u) \) is solvable if and only if there exists a rational matrix 
\( X \in \mathbb{R}^{m \times p}(s) \) such that 
\( K_{ij} + L_i X M_j = 0 \) for all \( i, j \in \mathbb{U} \) with \( i \neq j \).

**Proof:**

By the previous remark it is easy to see that the \( u^2 - u \) rational matrix 
equations of type \( (RME) \) can be reformulated into \( u^2 - u \) rational matrix 
equations of type \( (RME) \) that all have the same unknown, and therefore can 
be rearranged into one large rational matrix equation of type \( (RME) \). Then 
application of Theorem 3.4, using some general properties of norms, 
completes the proof of this corollary.

By now, it should be clear that the solvability of \( (ANICPM_u) \) is equivalent 
to the solvability of a certain large rational matrix equation of type 
\( (RME) \) over \( \mathbb{R}(s) \), whereas the solvability of \( (NICPM_u) \) is equivalent to the 
solvability of the same rational matrix equation over \( \mathbb{R}_0\mathbb{Q}(s) \). Furthermore, 
it is clear that any proper rational solution of this equation provides a 
proper rational matrix 
\( X \in \mathbb{R}^{m \times p}_0(s) \) such that 
\( K_{ij} + L_i X M_j = 0 \) for all \( i, j \in \mathbb{U} \) with \( i \neq j \). The transfer matrix of a compensator \( (1.9) \) that solves 
\( (NICPM_u) \) can then be calculated as 
\( F = X(I + PX)^{-1} \).

Starting from a rational solution of this equation and a positive \( \epsilon \), a 
proper rational matrix 
\( X_\epsilon \in \mathbb{R}^{m \times p}_0(s) \) such that 
\( K_{ij} + L_i X_\epsilon M_j = 0 \) for all \( i, j \in \mathbb{U} \) with \( i \neq j \), can be calculated by means of an algorithm extracted 
from the proofs of Theorem 3.4 and Lemmas 3.5 and 3.6.

Then, 
\( F_\epsilon := X_\epsilon (I + PX_\epsilon)^{-1} \) will be the transfer matrix of a compensator 
\( (1.9) \) that achieves almost noninteraction with an accuracy less than \( \epsilon \).

Here, \( \eta \) is a constant depending on the dimensions of all the matrices involved.

We conclude the present section by showing that it is indeed possible to 
check the solvability of a rational matrix equation of type \( (RME) \) over 
\( \mathbb{R}(s) \) or over \( \mathbb{R}_0\mathbb{Q}(s) \). To this extent, let 
\( V \in \mathbb{R}^{a \times b}_0(s) \) be a given rational 
matrix. We say that rank \( V = q \) if there exists a \( q \)-th order minor of \( V \) not 
equal to 0 \( (\in \mathbb{R}(s)) \), while every \( (q+1) \)-th order minor of \( V \) is equal to 0 
\( (\in \mathbb{R}(s)) \). Furthermore, if \( V \in \mathbb{R}^{a \times b}_0(s) \) is a proper rational matrix and 
rank \( V = q \), then there exist bicausal rational matrices 
\( S \in \mathbb{R}^{a \times a}_0(s) \),
Theorem 3.9:

1) The rational matrix equation (RME) is solvable over \( \mathbb{R}(s) \) if and only if \( \text{rank } U = \text{rank } [U, W] \).

2) The rational matrix equation (RME) is solvable over \( \mathbb{R}_0(s) \) if and only if the Smith form of \([U, 0]\) is equal to the Smith form of \([U, W]\), where \( 0 \) denotes the zero matrix in \( \mathbb{R}^{x \times c} \).

Note that, once the solvability of the rational matrix equation (RME) over \( \mathbb{R}(s) \) or over \( \mathbb{R}_0(s) \) is established, it requires standard techniques to compute an actual solution \( X \in \mathbb{R}^{x \times c} \) or \( X \in \mathbb{R}_0^{x \times c} \), respectively, that satisfies (RME).

3.2. (Almost) noninteracting control by state feedback: problem formulations

As announced in the introduction, in the remainder of the present chapter, we shall be dealing with (dynamic) state feedback. This means that in equation (3.1b) we shall assume that \( C = I_1 \). Then, the interconnection of the feedback compensator (1.9) with the linear system (3.1) results in a closed loop system described by

\[
\begin{align*}
\dot{x}_e(t) &= A'_e x_e(t) + \sum_{i \in \mathcal{U}} C_{i,e} v_i(t), \\
\dot{z}_i(t) &= H_{i,e} x_e(t), & (i \in \mathcal{Y}),
\end{align*}
\]

where \( x_e(t), C_{i,e} \) and \( H_{i,e} \) (\( i \in \mathcal{Y} \)) are as in Section 3.1 and

\[
A'_e := \begin{bmatrix} A + \sum_{i \in \mathcal{Y}} E_{i,e} \end{bmatrix}.
\]
Let $T'$ denote the transfer matrix of the closed loop system (3.3) and partition $T' = (T'_{ij})$ $(i, j \in \mathcal{U})$, where $T'_{ij}(s) := H_{ij}(sI - A')^{-1} G_{ij}$ $e$ denotes the transfer matrix between the $j$-th exogenous input and the $i$-th exogenous output in (3.3).

Denote $P'(s) := (sI - A)^{-1} B$ and $M_j'(s) := (sI - A)^{-1} G_j$ $(j \in \mathcal{U})$. In terms of transfer matrices we then have

$$ T'_{ij} = X_{ij} + L_i X' M_j', $$

where $X' := (I - FP')^{-1} F$ and $X_{ij}$, $L_i$ and $F$ are as described in the previous section.

We can now formulate the following control problems where we assume that the system (3.1) with $C = I$ is given.

**Definition 3.10:**
The noninteracting control problem by state feedback, abbreviated (NICPSu), consists of finding a feedback compensator (1.9) such that in the closed loop system (3.3) $T'_{ij} = 0$ for all $i, j \in \mathcal{U}$ with $i \neq j$.

**Definition 3.11:**
The almost noninteracting control problem by state feedback, abbreviated (ANICPSu), consists of finding, for all $\epsilon > 0$, a feedback compensator (1.9) such that in the closed loop system (3.3) $\|T'_{ij}\|_\infty \leq \epsilon$ for all $i, j \in \mathcal{U}$ with $i \neq j$.

As announced previously, we shall be able to derive necessary and sufficient conditions in state space terms for the solvability of the two control problems (NICPSu) and (ANICPSu). In fact, we shall be able to derive these kind of conditions for a problem even more general than (NICPSu).

Let $\mathcal{C}_f, \mathcal{C}_s \in \Theta$ be given stability regions such that $\mathcal{C}_f \subseteq \mathcal{C}_s$. Then, much in the spirit of Chapter 2, we can define:

**Definition 3.12:**
The noninteracting control problem by state feedback with internal s-stabilization and input/output l-stabilization, abbreviated (NICPSu)'.
consists of finding a feedback compensator \((1.9)\) such that in the closed loop system \((3.3)\) \(T^i_{ij} = 0\) for all \(i, j \in \mu\) with \(i \neq j\), \(T^i_{ii}\) is \(\bar{f}\)-stable for all \(i \in \mu\) and \(v(A^v_i) \in \mathcal{S}_s\).

In order to derive necessary and sufficient conditions in state space terms for the solvability of the control problems defined above, we need the following:

Let \(\{L_i \mid i \in \mu\}\) be a family of linear subspaces in \(\mathbb{R}^n\) and denote

\[
L^v := \sum_{j \in \mu, j \neq i} L_j \quad \text{for all } i \in \mu.
\]

We say that the family is independent if \(L_i \cap L^v_i = 0\) for all \(i \in \mu\). Define the linear subspace \(L^v\) in \(\mathbb{R}^n\) as

\[
L^v := \sum_{i \in \mu} (L_i \cap L^v_i).
\]

The linear subspace \(L^v\) is called the radical of the family \(\{L_i \mid i \in \mu\}\) (cf. Wonham (1979)).

It is clear that a family of linear subspaces is independent if and only if its radical is equal to the zero subspace.

**Proposition 3.13:**

Let \(\{L_i \mid i \in \mu\}\) be a family of linear subspaces in \(\mathbb{R}^n\) and let \(\{\overline{L}_i \mid i \in \mu\}\) be a family of linear subspaces such that \(\overline{L}_i \subseteq L_i\), \(\overline{L}_i \cap L^v = 0\) and \(L_i + L^v = \overline{L}_i + L^v\) for all \(i \in \mu\). Then the family \(\{L^v, \overline{L}_1, \overline{L}_2, \ldots, \overline{L}_\mu\}\) is independent.

**Proof:**

In order to prove this proposition it has to be shown that \(\overline{L}_i \cap (L^v + \overline{L}_i) = 0\) for all \(i \in \mu\), and \(L^v \cap (\bigcup_{i \in \mu} \overline{L}_i) = 0\), where we have denoted

\[
\overline{L}^v := \sum_{j \in \mu, j \neq i} \overline{L}_j \quad \text{for all } i \in \mu.
\]

Now, on one hand we have \(\overline{L}_i \cap (L^v + \overline{L}_i) \subseteq \overline{L}_i\), while on the other hand
\[ \mathcal{I}_i \cap (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \subseteq (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \cap (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \subseteq \mathcal{L}^\mathcal{V}, \] because \( \mathcal{L}^\mathcal{V} \) is also the radical of the family \((\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \mid i \in \mathcal{Y}\) (cf. Wonham (1979), Lemma 10.1). Hence, 
\[ \mathcal{E}_i \cap (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) = \mathcal{E}_i \cap \mathcal{L}^\mathcal{V} = 0 \] for all \( i \in \mathcal{Y} \). It remains to be shown that 
\( \mathcal{L}^\mathcal{V} \cap (\mathcal{L}_i^\mathcal{V} \setminus \mathcal{I}_i^\mathcal{V}) = 0 \). To this extent, let \( x \in \mathcal{L}^\mathcal{V} \cap (\mathcal{L}_i^\mathcal{V} \setminus \mathcal{I}_i^\mathcal{V}) \). So, 
\[ x = \sum_{i \in \mathcal{Y}} \xi_i \] with \( \xi_i \in \mathcal{L}_i^\mathcal{V} \) for all \( i \in \mathcal{Y} \). It follows that 
\[ x - \sum_{i \in \mathcal{Y}} \xi_i \in (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \cap (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \subseteq (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \cap (\mathcal{L}^\mathcal{V} + \mathcal{I}_i^\mathcal{V}) \subseteq \mathcal{L}^\mathcal{V} \).

Hence, \( \xi_i = (x_i - x) + x \in \mathcal{L}^\mathcal{V} \). Because \( \mathcal{L}^\mathcal{V} \cap \mathcal{I}_i^\mathcal{V} = 0 \), it follows that \( \xi_i = 0 \). Similarly, it can be shown that \( \xi_i = 0 \) for all \( i \in \mathcal{Y} \). So, \( x = 0 \) and consequently \( \mathcal{L}^\mathcal{V} \cap (\mathcal{L}_i^\mathcal{V} \setminus \mathcal{I}_i^\mathcal{V}) = 0 \).

### 3.3. Noninteracting control by state feedback with input/output and internal stability

In the present section we shall derive necessary and sufficient conditions in state space terms for the solvability of (NICPS)\( \mathcal{Y} \) and (NICPS)\( \mathcal{Y} \)'s. Let the linear system (3.1) with \( \mathcal{C} = I \) be given and let \( \mathcal{C}_f, \mathcal{C}_s \subseteq 0 \) with \( \mathcal{C}_f \subseteq \mathcal{C}_s \) be two given stability regions. Denote

\[ \mathcal{L}_i := \text{im} \mathcal{C}_i (i \in \mathcal{Y}), \quad \mathcal{L}^\mathcal{V}_i := \sum_{j \in \mathcal{Y}, j \neq i} \mathcal{L}_j (i \in \mathcal{Y}), \]

\[ \mathcal{L}^\mathcal{V} := \sum_{i \in \mathcal{Y}} (\mathcal{L}_i \cap \mathcal{L}^\mathcal{V}_i), \]

\[ K_i := \cap \ker \mathcal{H}_j (i \in \mathcal{Y}) \] and \( K := \cap \ker \mathcal{H}_j \).

The following theorem is the main result of this section.

**Theorem 3.14:**

(NICPS)\( \mathcal{Y} \)' is solvable if and only if the pair \((\mathcal{A}, \mathcal{B})\) is \( s \)-stabilizable, 
\[ \mathcal{L}_i \subseteq \mathcal{V}_{f,s}(K_i, K) \] for all \( i \in \mathcal{Y} \) and \( \mathcal{L}^\mathcal{V} \subseteq \mathcal{V}^s_s(K) \).
Proof: (obly if) Assume that \((\text{NCP} \psi)\)' is solvable. Then there exists a feedback compensator \((1.9)\) such that in the closed loop system \((3.3)\)

\[
H_{i,e}(sI - \Lambda_c)^{-1} G_{j,e} = 0 \quad \text{for all } i,j \in \mathcal{U} \text{ with } i \neq j,
\]

\[
H_{i,e}(sI - \Lambda_c)^{-1} G_{j,e} \text{ is } \mathcal{F}\text{-stable for all } i \in \mathcal{U} \text{ and } \sigma(\Lambda_c^j) \subseteq \sigma_c.
\]

Because \(\sigma(\Lambda_c^j) \subseteq \sigma_c\), it follows from Theorem 1.44 with \(C = 1\) that the pair \((A_c, \mathcal{E})\) is \(\mathcal{F}\)-stabilizable. Next, let \(j \in \mathcal{U}\) be fixed and let \(\xi\) be a vector in \(L_j = \mathcal{F} G_{j,e}\). So, there exists a vector \(\xi \in \mathbb{R}^q\) such that \(G_j \xi = \xi\). Now, let \(G_c = [\xi]\) \(\in \mathbb{R}^n\) with \(\xi \in \mathbb{R}^n\) and \(\psi \in \mathbb{R}^m\) be the rational vector defined as

\[
\xi_e(s) := (sI - \Lambda_c^j)^{-1} G_{j,e} \psi.
\]

Then it is clear that \(H_{i,e} \xi_e = 0\) for all \(i \in \mathcal{U}\) with \(i \neq j\) and \(H_{j,e} \xi_e = 0\) for \(j \in \mathcal{U}\) with \(i \neq j\) and \(H_{j,e} \xi_e = H_j \xi\) is \(\mathcal{F}\)-stable. Furthermore, it follows that

\[
\xi = G_j \psi = (sI - \Lambda_c) \xi_e(s) = B(\mathcal{N}(s) + M\psi(s)).
\]

Because \(\sigma(\Lambda_c^j) \subseteq \sigma_c\), it is clear that \(\psi\) and consequently \(\mathcal{N} + M\psi\) are \(\mathcal{F}\)-stabilizable. Hence, the vector \(\xi \in \mathcal{F}_j\) has an \(\mathcal{F}\)-stable regular \((\xi, \psi)\)-representation with \(H_j \xi = 0\) and \(H_{j,e} \xi = 0\) for all \(i \in \mathcal{U}\) with \(i \neq j\). So, \(\xi = V_{f,s}(K_j, K)\) (see Section 2.2) and we may conclude that \(L_j \subseteq V_{f,s}(K_j, K)\) for all \(j \in \mathcal{U}\).

Now again, let \(j \in \mathcal{U}\) be fixed and let \(\xi\) be a vector in \(L_j \cap \mathcal{F}_j\). Then there exist vectors \(v \in \mathbb{R}^q\) and \(w \in \mathbb{R}^q\) where \(i \in \mathcal{U}\), \(i \neq j\), such that

\[
\xi = G_j \psi = \sum_{i \in \mathcal{U}, i \neq j} G_j w_i.
\]

We claim that \(H_{i,e}(sI - \Lambda_c)^{-1} G_{j,e} \psi = 0\) for all \(i \in \mathcal{U}\). Indeed, for \(i \in \mathcal{U}\) with \(i \neq j\), this is immediate and for \(i = j\), this is clear from the fact that

\[
H_{i,e}(sI - \Lambda_c)^{-1} G_{j,e} \psi = \sum_{i \in \mathcal{U}, i \neq j} H_{j,e}(sI - \Lambda_c)^{-1} G_{j,e} w_i = 0.
\]

As before, let \(i_c = [\psi]\) \(\in \mathbb{R}^n\) be defined as \(\xi_e(s) := (sI - \Lambda_c^j)^{-1} G_{j,e} \psi\). Then, \(H_{i,e} \xi_c = H_{i,e} \xi = 0\) for all \(i \in \mathcal{U}\). Furthermore,
\[ \xi = G_j v = (sI - A)\xi(s) - B(N\xi(s) + Mv(s)), \]

where \( \xi \) and \( N\xi + Mv \) are s-stable. Hence, the vector \( \xi \in \mathcal{L}_j \cap \mathcal{L}_j^V \) has an s-stable regular \((\xi, \omega)\)-representation with \( H_i \xi = 0 \) for all \( i \in \mu \). So, \( \xi \in \mathcal{V}_s^*(K) \) and consequently \( \mathcal{L}_j \cap \mathcal{L}_j^V \subseteq \mathcal{V}_s^*(K) \) for all \( j \in \mu \). The latter implies that

\[ \mathcal{L}_j^V = \sum_{j \in \mu} (\mathcal{L}_j \cap \mathcal{L}_j^V) \subseteq \mathcal{V}_s^*(K). \]

(if) Assume that the pair \((A, B)\) is s-stabilizable, \( \text{im} \ G_i = \mathcal{L}_i \subseteq \mathcal{V}_s(K_i, K) \) for all \( i \in \mu \) and \( \mathcal{L}_j^V \subseteq \mathcal{V}_s^*(K) \).

By Theorem 2.6 the existence follows of s-stable rational matrices \( X_j \in \mathbb{R}_s^{nxn}(s) \) and \( U_j \in \mathbb{R}_s^{nxn}(s) \) (\( j \in \mu \)), such that \((sI - A)X_j(s) - BU_j(s) = 1 \) and \( H_j X_j G_j \) is s-stable for all \( j \in \mu \), and \( H_j X_j G_j = 0 \) for all \( i, j \in \mu \) with \( i \neq j \).

Let \( \overline{G}_0 \) be an injective matrix such that \( \text{im} \overline{G}_0 = \mathcal{L}_j^V \). Then, analogously to Theorem 2.6, the existence can be proved of s-stable rational matrices \( X_0 \in \mathbb{R}_s^{nxn}(s) \) and \( U_0 \in \mathbb{R}_s^{nxn}(s) \) such that \((sI - A)X_0(s) - BU_0(s) = 1 \) and \( H_i X_0 \overline{G}_0 = 0 \) for all \( i \in \mu \).

By Theorem 1.10 we can find s-stable rational matrices \( X_{\mu + 1} \in \mathbb{R}_s^{nxn}(s) \) and \( U_{\mu + 1} \in \mathbb{R}_s^{nxn}(s) \) such that \((sI - A)X_{\mu + 1}(s) - BU_{\mu + 1}(s) = 1 \).

Now, consider the family of linear subspaces \( \{ \mathcal{L}_i \mid i \in \mu \} \) together with its radical \( \mathcal{L}_i^V \). For all \( i \in \mu \), let \( \overline{E}_i \) be a linear subspace in \( \mathcal{L}_i \) such that \( \mathcal{L}_i^V + \overline{E}_i = \mathcal{L}_i^V + \mathcal{L}_i \) and \( \mathcal{L}_i^V \cap \overline{E}_i = 0 \). In the previous section we proved that the family \( \{ \mathcal{L}_i, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_\mu \} \) is independent. For all \( i \in \mu \), let \( \overline{E}_i \) be an injective matrix such that \( \text{im} \overline{E}_i = \mathcal{L}_i^V \). It can now be shown that \([\overline{G}_0, \overline{G}_1, \ldots, \overline{G}_\mu] \) is an injective matrix (cf. Wonham (1979)).

Since for all \( i \in \mu \) we have \( \text{im} \overline{E}_i \subseteq \text{im} \overline{G}_i \subseteq \text{im} \overline{G}_i \), there exist matrices \( P_i, Q_i, R_i \) (\( i \in \mu \)) such that for all \( i \in \mu \) \( \overline{G}_i = \overline{G}_i P_i \) and \( \overline{G}_i = \overline{G}_i Q_i + \overline{G}_i R_i \).

Let \( \overline{E}_{\mu + 1} \) be an injective matrix such that \([\overline{G}_0, \overline{G}_1, \ldots, \overline{G}_\mu, \overline{G}_{\mu + 1}] \) is a square regular matrix.

Define rational matrices \( X \in \mathbb{R}_s^{nxn}(s) \) and \( U \in \mathbb{R}_s^{nxn}(s) \) such that

\[ X[\overline{G}_0, \overline{G}_1, \ldots, \overline{G}_\mu, \overline{G}_{\mu + 1}] = [X_0 \overline{G}_0, X_1 \overline{G}_1, \ldots, X_\mu \overline{G}_\mu, X_{\mu + 1} \overline{G}_{\mu + 1}] \]

and

\[ U[\overline{G}_0, \overline{G}_1, \ldots, \overline{G}_\mu, \overline{G}_{\mu + 1}] = [U_0 \overline{G}_0, U_1 \overline{G}_1, \ldots, U_\mu \overline{G}_\mu, U_{\mu + 1} \overline{G}_{\mu + 1}] \].
It is clear that both X and U are s-stable strictly proper rational matrices. Furthermore, we have \((sI - A)X(s) - BU(s) = I\). Indeed, observe that

\[
((sI - A)X(s) - BU(s))[\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_\mu, \bar{G}_{\mu+1}] =
\]

\[
[((sI - A)X(s) - BU(s))\bar{G}_0, ((sI - A)X(s) - BU(s))\bar{G}_1, \ldots, ((sI - A)X(s) - BU(s))\bar{G}_{\mu+1}] =
\]

\[
[((sI - A)X_0(s) - BU_0(s))\bar{G}_0, ((sI - A)X_1(s) - BU_1(s))\bar{G}_1, \ldots, ((sI - A)X_{\mu+1}(s) - BU_{\mu+1}(s))\bar{G}_{\mu+1}] =
\]

\[
[\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_\mu, \bar{G}_{\mu+1}].
\]

Then, for all \(i, j \in \mu\),

\[
H_i X G_j = H_i X (\bar{G}_0 Q_j + \bar{C}_i R_j) = H_i (X_0 \bar{G}_0 Q_j + X \bar{G}_j R_j) =
\]

\[
= H_i X \bar{C}_j R_j = H_i X \bar{G}_j P_j R_j .
\]

Therefore, \(H_i X G_j = 0\) for all \(i, j \in \mu\) with \(i \neq j\) and \(H_i X G_i\) is \(\mu\)-stable for all \(i \in \mu\).

Observe that \(sX(s) = I + AX(s) + BU(s)\). This implies that \(sX(s)\) is a bi-causal rational matrix. Let \(\Gamma(s) \in \mathbb{M}_0^{n \times n}(s)\) be the inverse of \(sX(s)\). Then the inverse of \(X(s)\) exists as a rational matrix. In fact, \(X^{-1}(s) = s\Gamma(s)\).

Define \(F = UX^{-1}\) and note that \(F(s) = sU(s)\Gamma(s)\) with both \(sU(s)\) and \(\Gamma(s)\) proper rational matrices. Hence, \(F\) is a proper rational matrix.

Furthermore, we have \(X(s) = (sI - A)^{-1} + (sI - A)^{-1}BU(s)\) and consequently \(U(s) = F(s)X(s) = F(s)(sI - A)^{-1} + F(s)(sI - A)^{-1}BU(s)\). The latter implies that \(U(s) = (I - F(s)P'(s))^{-1}F(s)(sI - A)^{-1}\), where \(P'\) is as defined in Section 3.2, and \(X(s) = (sI - A)^{-1} + (sI - A)^{-1}B(I - F(s)P'(s))^{-1}F(s)(sI - A)^{-1}\).

Therefore, \(0 = H_i X G_j = K_{ij} + L_i (I - FP')^{-1}P'M'\) for all \(i, j \in \mu\) with \(i \neq j\), and \(H_i X G_i = K_{ii} + L_i (I - FP')P'M'\) is \(\mu\)-stable for all \(i \in \mu\). Here, \(M'_i\) (\(i \in \mu\)) is as defined in Section 3.2, and \(L_i\) (\(i \in \mu\)), \(K_{ij}\) (\(i, j \in \mu\)) are as defined in Section 3.1.

From the latter it is clear that \(F\) may serve as the transfer matrix of the feedback compensator (1.9). Let the quadruple \(\{K, L, M, N\}\) be a minimal state...
space realization of \( F \), i.e. \( F(s) = N + M(sI - K)^{-1}L \). Then, apart from the internal stabilization, all the control objectives in the formulation of \((\text{NICPS}_u)^t\) are fulfilled.

Because the quadruple \( \{K, L, M, N\} \) represents a minimal realization of \( F \), it follows that the pair \( (K, L) \) is s-stabilizable and the pair \( (M, K) \) is s-detectable. Recall that the pair \( (A, B) \) is s-stabilizable and note that the pair \( (I, A) \) is s-detectable.

Then we can apply the following result (cf. Vidyasagar (1985)): the closed loop system is s-stable \( (\sigma(A_c) \subseteq \mathcal{G}_g) \) if and only if the following four rational matrices are s-stable: \((I - P'F)^{-1}P', (I - P'F)^{-1}P'F, (I - FP')^{-1}F \) and \((I - FP')^{-1}FP'\).

Straightforward calculation shows that
\[
(I - P'(s)F(s))^{-1}P'(s) = X(s)B,
\]
\[
(I - P'(s)F(s))^{-1}P'(s)F(s) = X(s)(sI - A) - I,
\]
\[
(I - F(s)P'(s))^{-1}F(s) = U(s)(sI - A)
\]
and
\[
(I - F(s)P'(s))^{-1}F(s)P'(s) = U(s)B,
\]
which are all s-stable rational matrices. Hence, the closed loop system is s-stable.

This completes the proof of Theorem 3.14.

Let the linear system (3.1) with \( C = I \) be given and let \( \mathcal{G}_g \subseteq \mathcal{G} \) be a stability region. The following corollary is an easy consequence of Theorem 3.14.

**Corollary 3.15:**

1) \((\text{NICPS}_u)\) is solvable if and only if \( L_i \subseteq \mathcal{V}_g^u(K_i) \) for all \( i \in \mathcal{U} \), and \( \mathcal{L}^u \subseteq \mathcal{V}_g^u(K) \).

2) There exists a feedback compensator (1.9) such that in the closed loop system (3.3) \( T_{ij}^f = 0 \) for all \( i, j \in \mathcal{U} \) with \( i \neq j \) and \( \sigma(A_i^f) \subseteq \mathcal{G}_g \) if and only if \( L_i \subseteq \mathcal{V}_g^u(K_i) \) for all \( i \in \mathcal{U} \), \( \mathcal{L}^u \subseteq \mathcal{V}_g^u(K) \) and the pair \( (A, B) \) is stabilizable (with respect to \( \mathcal{G}_g \)).
3) There exists a feedback compensator (1.9) such that in the closed loop system (3.3) \( T_{ij}' = 0 \) for all \( i, j \in \mathcal{U} \) with \( i \neq j \) and \( T_{ii}' \) is stable for all \( i \in \mathcal{U} \) if and only if \( \ell_i \leq v^*(K_i) + v^*(K) \) for all \( i \in \mathcal{U} \) and \( \ell' \leq v^*(K) \).

Proof:
In Theorem 3.4 specify: 1) \( \ell_f = \ell_s = \ell \), 2) \( \ell_f = \ell_g = \ell_g \), 3) \( \ell_f = \ell_g \). \[ \text{ } \]

3.4. Almost noninteracting control by state feedback

We conclude the present chapter by establishing necessary and sufficient conditions in state space terms for the solvability of (ANICPSU) as described in Section 3.2. Let \( \ell_i \) (\( i \in \mathcal{U} \)), \( \ell' \), \( K_i \) (\( i \in \mathcal{U} \)) and \( K \) be linear subspaces as defined in the previous section. Then the main result of the present section reads as follows.

Theorem 3.16:
(ANICPSU) is solvable if and only if \( \ell_i \leq v^*_b(K_i) \) for all \( i \in \mathcal{U} \) and \( \ell' \leq v^*_{b}(K) \).

The proof of this theorem will be omitted. It can be proved completely analogously to Theorem 3.14, using the following two propositions. In the first proposition we derive a characterization of the linear subspace \( v^*_{b}(\ker \Pi) \) (cf. Trumpelman (1986), Cor. 3.33).

Proposition 3.17:
\[ v^*_{b}(\ker \Pi) = \left\{ x_0 \in \mathbb{R}^n \mid \text{for all } \varepsilon > 0 \text{ there exists a regular } (\ell, \omega) \text{-representation with } \|H_\ell\|_{\infty} < \varepsilon \right\}. \]

Proof:
By Definition 1.24 we have that \( x_0 \in v^*_{b}(\ker \Pi) \) if and only if there exist rational vectors \( : \in \mathbb{R}^n(s) \) and \( \omega \in \mathbb{R}^n(s) \) such that \( x_0 = (sI - A)\xi(s) - B\omega(s) \) and \( HC = 0 \). If we eliminate \( \xi \) in the latter description we obtain that
$x_0 \in V_b^*(\ker H)$ if and only if there exists a rational vector $w \in \mathbb{R}^m(s)$ such that $H(sI - A)^{-1} x_0 + H(sI - A)^{-1} B w(s) = 0$. Then by application of Corollary 3.7, it follows that $x_0 \in V_b^*(\ker H)$ if and only if for all $\varepsilon > 0$ there exists a strictly proper rational vector $w_\varepsilon(s) \in \mathbb{R}^m(s)$ such that $\|H(sI - A)^{-1} x_0 + H(sI - A)^{-1} B w_\varepsilon(s)\| \leq \varepsilon$. In the latter expression, set $\varepsilon_\varepsilon(s) := (sI - A)^{-1}(x_0 + B w_\varepsilon(s))$. Then we obtain that $x_0 \in V_b^*(\ker H)$ if and only if for all $\varepsilon > 0$ there exist strictly proper rational vectors $\varepsilon_\varepsilon \in \mathbb{R}^m(s)$ and $w_\varepsilon \in \mathbb{R}^m(s)$ such that $x_0 = (sI - A) \varepsilon_\varepsilon(s) - B w_\varepsilon(s)$ and $\|H \varepsilon \varepsilon\| \leq \varepsilon$.

Proposition 3.18:

$\text{im } G \subseteq V_b^*(\ker H)$ if and only if for all $\varepsilon > 0$ there exist strictly proper rational matrices $X_\varepsilon \in \mathbb{R}^{n \times n}(s)$ and $U_\varepsilon \in \mathbb{R}^{m \times n}(s)$ such that

$I = (sI - A)X_\varepsilon(s) - B U_\varepsilon(s)$ and $\|H X_\varepsilon\| \leq \varepsilon$.

Proof:

Let $[G_1, G_2]$ be a square invertible matrix such that $\text{im } G_1 = \text{im } G$. Furthermore, let $X_2$ and $U_2$ be strictly proper rational matrices such that $G_2 = (sI - A)X_2(s) - B U_2(s)$. From Proposition 3.17 it follows that for all $\varepsilon > 0$ there exist strictly proper rational matrices $X_1, \varepsilon$ and $U_1, \varepsilon$ such that $G_1 = (sI - A)X_1, \varepsilon(s) - B U_1, \varepsilon(s)$ and $\|H X_1, \varepsilon\| \leq \varepsilon$. For all $\varepsilon > 0$, define the rational matrices $X_\varepsilon \in \mathbb{R}^{n \times n}(s)$ and $U_\varepsilon \in \mathbb{R}^{m \times n}(s)$ as

$$X_\varepsilon [G_1, G_2] = [X_1, \varepsilon, X_2] \quad \text{and} \quad U_\varepsilon [G_1, G_2] = [U_1, \varepsilon, U_2].$$

Then

$$(sI - A)X_\varepsilon(s) - B U_\varepsilon(s) = I \quad \text{and} \quad \|H X_\varepsilon\| \leq \varepsilon.$$
extension it is assumed that the linear system (3.1) with \( C = 1 \) is given and that \( \mathcal{C}_f, \mathcal{C}_g \subset 0 \) with \( \mathcal{C}_f \subseteq \mathcal{C}_g \) are given stability regions. In the above reference necessary and sufficient conditions are derived for the solvability of the extended problem. The treatment of this problem gives rise to an analysis that is considerably more involved than its 'exact' counterpart.
In this chapter we consider linear systems that, in addition to a control input and a measurement output, have two exogenous inputs and two exogenous outputs (see also Chapter 2). For systems of this kind we study two control problems that we introduced in the previous chapter. These control problems are the noninteracting control problem by measurement feedback and the almost noninteracting control problem by measurement feedback.

As announced in the introduction to the previous chapter, in the present chapter we are concerned with the derivation of necessary and sufficient conditions for the solvability of the latter two control problems, formulated in state space or frequency domain terms. These conditions should, in principle, be verifiable by means of computations.

The outline of the present section is as follows.

In Section 4.1 we briefly introduce the linear system and the two control problems that we are interested in. Section 4.2 will deal with pairs of linear matrix equations over a field. There we present a new result that provides simple necessary and sufficient conditions for the existence of a common solution to the pair of linear matrix equations $U_1XV_1 = W_1$ and $U_2XV_2 = W_2$. From this result, simple necessary and sufficient conditions in frequency domain terms for the solvability of the almost noninteracting control problem by measurement feedback will be derived.

In Sections 4.3 and 4.4 we are concerned with the derivation of necessary and sufficient conditions in state space terms for the solvability of the noninteracting control problem by measurement feedback. In these sections also the results of Section 4.2 will play an important role. In Section 4.5 we consider some special cases of the noninteracting control problem by measurement feedback. We conclude the present chapter by Section 4.6 in
which we study two control problems that, in addition to the feature of 
(almost) noninteraction, also require (almost) diagonal transfer preservation.

4.1. Introduction

As mentioned in the beginning of this chapter, we are concerned with linear systems that, in addition to a control input and a measurement output, have two exogenous inputs and two exogenous outputs (see also (2.3) or (3.1) with \( \mu = 2 \)).

\[
\begin{align*}
4.1a) & \quad \dot{x}(t) = Ax(t) + Bu(t) + C_1v_1(t) + C_2v_2(t) \\
4.1b) & \quad y(t) = Cx(t) \\
4.1c) & \quad z_1(t) = H_1x(t), \quad z_2(t) = H_2x(t).
\end{align*}
\]

As usual, we assume that the linear system (4.1) is controlled by

\[
\begin{align*}
1.9a) & \quad \dot{\omega}(t) = Kw(t) + Ly(t), \\
1.9b) & \quad u(t) = M\omega(t) + Ny(t),
\end{align*}
\]

resulting in the closed loop system described by

\[
\begin{align*}
4.2a) & \quad \dot{x}_c(t) = A_c x_c(t) + C_1,c v_1(t) + C_2,c v_2(t), \\
4.2b) & \quad z_1(t) = H_1,c x_c(t), \quad z_2(t) = H_2,c x_c(t).
\end{align*}
\]

For the meaning of all the variables involved we refer to Chapter 2 or to Chapter 3.

Again, let \( T \) denote the transfer matrix of the closed loop system (4.2) and partition \( T = (T_{ij}) \) \((i,j = 1,2)\), where \( T_{ij}(s) = \tilde{H}_{ij,c}(sI - A_c)^{-1}C_{j,c} \). We recall that in the closed loop system (4.2) the following relations hold:

\[
T_{ij} = K_{ij} + L_{ij} X M_j, \quad (i,j = 1,2),
\]
where $X = (I - FP)^{-1} P$. For the definitions of $K_{12}, L_1, M_1, P$ and $F$ we refer to Section 3.1.

Assume that the linear system (4.1) is given. In connection with this system we now recall Definitions 3.1 and 3.2, and Corollaries 3.3 and 3.8.

**Definition 4.1.**
The noninteracting control problem by measurement feedback, (NICPM2), consists of finding a measurement feedback compensator (1.9) such that in the closed loop system (4.2) $T_{12} = 0$ and $T_{21} = 0$.

**Definition 4.2:**
The almost noninteracting control problem by measurement feedback, (ANICPM2), consists of finding, for all $c > 0$, a measurement feedback compensator (1.9) such that in the closed loop system (4.2) $\|T_{12}\|_{\infty} \leq c$ and $\|T_{21}\|_{\infty} \leq c$.

**Corollary 4.3:**
1) (NICPM2) is solvable if and only if there exists a proper rational matrix $X \in \mathbb{R}_0^{m \times p}(s)$ such that $K_{12} + L_1 X M_2 = 0$ and $K_{21} + L_2 X M_1 = 0$.
2) (ANICPM2) is solvable if and only if there exists a rational matrix $X \in \mathbb{R}^{m \times p}(s)$ such that $K_{12} + L_1 X M_2 = 0$ and $K_{21} + L_2 X M_1 = 0$.

Recall that $K_{12}, K_{21}, L_1, L_2, M_1$ and $M_2$ are (strictly) proper rational matrices. Furthermore, observe that $\mathbb{R}_0(s)$ is a principal ideal domain, whereas $\mathbb{R}(s)$ is a field (cf. Vidyasagar (1985)).

Therefore, the solvability of (NICPM2) is equivalent to the existence of a common solution to a pair of linear matrix equations over the principal ideal domain $\mathbb{R}_0(s)$, whereas the solvability of (ANICPM2) is equivalent to existence of a common solution to the same pair of linear matrix equations over the field $\mathbb{R}(s)$. 
4.2. On a common solution to a pair of linear matrix equations

In this section we derive some results that will be of great importance in the remainder of the present chapter. The results of the present section will be stated in terms of an arbitrary field \( \mathbb{F} \) and deal with the existence of a common solution to a pair of linear matrix equations over \( \mathbb{F} \). The relevance of the results of the present section will become apparent when \( \mathbb{F} \) is replaced by \( \mathbb{R}(s) \). Throughout this section, \( \mathbb{F}^{a} \) denotes the set of \( a \)-vectors with entries in \( \mathbb{F} \) and \( \mathbb{F}^{a \times b} \) denotes the set of \( a \times b \)-vectors with entries in \( \mathbb{F} \). For a given matrix \( W \in \mathbb{F}^{a \times b} \), we say that the rank of \( W \) is \( q \), i.e. \( \text{rank } W = q \), if there exists a \( q \)-th order minor of \( W \) not equal to 0 (\( \in \mathbb{F} \)), while every \( (q+1) \)-th order minor of \( W \) is equal to 0 (\( \in \mathbb{F} \)).

In order to establish the main result of the present section we need the following two lemmas. In these lemmas, let \( W \in \mathbb{F}^{a \times d} \), \( U \in \mathbb{F}^{a \times b} \) and \( V \in \mathbb{F}^{b \times d} \) be given matrices.

**Lemma 4.4:**
The following statements are equivalent:

1) There exists a matrix \( X \in \mathbb{F}^{b \times c} \) such that \( UXV = W \).
2) \( \text{Im } W \subseteq \text{Im } U \) and \( \text{ker } V \subseteq \text{ker } W \).
3) \( \text{Rank } U = \text{rank } [U,W] \) and \( \text{rank } V = \text{rank } [V,W] \).

**Proof:**

1) \( \Rightarrow 2) \) See Willems (1982).

2) \( \Rightarrow 3) \) See any textbook on matrix theory, for instance Mardueff (1960), Lancaster and Tismenetsky (1985).

**Lemma 4.5:**
The following statements are equivalent:

1) There exist matrices \( X \in \mathbb{F}^{b \times d} \) and \( Y \in \mathbb{F}^{a \times c} \) such that \( UX + VY = W \).
2) \( W \) ker \( V \subseteq \text{Im } U \).
3) \( \text{Rank } \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \text{rank } \begin{bmatrix} U & W \\ 0 & V \end{bmatrix} \).
Proof:

1) $\Leftrightarrow$ 3) See Roth (1952).

1) $\Rightarrow$ 2) Take a vector $v \in \ker V$. Then $Wv = (UX + YV)v = UXv$, from which it is clear that $W \ker V \subseteq \text{im} U$.

2) $\Rightarrow$ 1) Let $[S_1, S_2] \in \mathbb{F}^{a \times d}$ be an invertible matrix such that $\text{im} S_1 = \ker V$. Then $VS_2$ is an injective matrix. Because $W \ker V \subseteq U$, there exists a matrix $T_1$ of appropriate dimensions such that $WS_1 = UT_1$. Let $T_2$ be an arbitrary matrix such that $[T_1, T_2]$ is a square matrix, let $Y \in \mathbb{F}^{a \times c}$ be a matrix such that $YVS_2 = WS_2 - UT_2$ and let $X \in \mathbb{F}^{b \times d}$ be a matrix such that $X[S_1, S_2] = [T_1, T_2]$. Then we have that

$$(UX + YV)[S_1, S_2] = U[T_1, T_2] + Y[V_S_1, VS_2] = [UT_1, UT_2 + YVS_2] = W[S_1, S_2].$$

Hence, $UX + YV = W$.

We are now able to state the main result of this section. The result provides new necessary and sufficient conditions for the existence of a common solution to a pair of linear matrix equations over a field. In our opinion, the conditions presented are simpler than the conditions derived in Mitra (1973), since the latter involve rather complicated expressions with generalized inverses of matrices.

Let $U_i \in \mathbb{F}^{a_i \times b} \quad (i = 1, 2)$, $V_i \in \mathbb{F}^{c \times d_i} \quad (i = 1, 2)$ and $W_i \in \mathbb{F}^{a_i \times d_i} \quad (i = 1, 2)$ be given matrices.

**Theorem 4.6:**

The following statements are equivalent:

1) There exists a matrix $X \in \mathbb{F}^{b \times c}$ such that $U_1XV_1 = W_1$ and $U_2XV_2 = W_2$.

2) For $i = 1, 2$, rank $U_i = \text{rank } [U_i, W_i]$ and rank $V_i = \text{rank } \begin{bmatrix} V_i \\ W_i \end{bmatrix}$, and

\[
\begin{bmatrix}
U_1 & 0 & 0 \\
U_2 & 0 & 0 \\
0 & V_1 & V_2
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
U_1 & W_1 & 0 \\
U_2 & 0 & -W_2
\end{bmatrix}.
\]
3) For $i = 1, 2$, let $U_i \subseteq \text{im} \, \overline{V}_i$ and $\ker V_i \subseteq \ker \overline{W}_i$, and
\[
\begin{bmatrix}
W_1 & 0 \\
0 & -W_2
\end{bmatrix}
\text{ker} \, \begin{bmatrix} V_1, V_2 \end{bmatrix} \subseteq \text{im} \begin{bmatrix} U_1 \\
U_2
\end{bmatrix}.
\]

Proof

2) $\Rightarrow$ 3) Follows from Lemmas 4.4 and 4.5.

1) $\Rightarrow$ 3) From Lemma 4.4 it follows that, for $i = 1, 2$, let $W_i \subseteq \text{im} \, V_i$ and $\ker V_i \subseteq \ker W_i$.

Take a vector
\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} \in \ker \begin{bmatrix} V_1, V_2 \end{bmatrix} \text{ with } v_1 \in \mathbb{F}^d_1 \text{ and } v_2 \in \mathbb{F}^d_2.
\]

Then $V_1 v_1 = -V_2 v_2 =: w$. Observe that
\[
\begin{bmatrix}
W_1 & 0 \\
0 & -W_2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix} W_1 v_1 \\
-W_2 v_2
\end{bmatrix} = \begin{bmatrix} U_1 xW_1 v_1 \\
-U_2 xW_2 v_2
\end{bmatrix} = \begin{bmatrix} U_1 \\
U_2
\end{bmatrix} xw.
\]

It follows that
\[
\begin{bmatrix}
W_1 & 0 \\
0 & -W_2
\end{bmatrix}
\text{ker} \begin{bmatrix} V_1, V_2 \end{bmatrix} \subseteq \text{im} \begin{bmatrix} U_1 \\
U_2
\end{bmatrix}.
\]

3) $\Rightarrow$ 1) Let $[\overline{V}_0, \overline{V}_1, \overline{V}_2]$ be an injective matrix such that $\text{im} \overline{V}_0 = \text{im} \overline{V}_1 \cap \text{im} \overline{V}_2$, $\text{im} \begin{bmatrix} \overline{V}_0, \overline{V}_1 \end{bmatrix} = \text{im} \overline{V}_1$ and $\text{im} \begin{bmatrix} \overline{V}_0, \overline{V}_2 \end{bmatrix} = \text{im} \overline{V}_2$.

Then there exist square invertible matrices $S_1 \in \mathbb{F}^{d_1 \times d_1}$ and $S_2 \in \mathbb{F}^{d_2 \times d_2}$ such that $V_1 S_1 = [\overline{V}_0, \overline{V}_1, 0]$ and $V_2 S_2 = [\overline{V}_0, \overline{V}_2, 0]$. Partition correspondingly $W_1 S_1 = [W'_1, \tilde{W}_1, \tilde{W}_1]$ and $W_2 S_2 = [W'_2, \tilde{W}_2, \tilde{W}_2]$.

Because, for $i = 1, 2$, $\ker W_i \supseteq \ker V_i$ we also have $\ker W_i S_i \supseteq \ker V_i S_i$ for $i = 1, 2$, from which it is clear that $\tilde{W}_1 = 0$ and $\tilde{W}_2 = 0$.

Now, observe that
Here, the last equality is due to the fact that \([\bar{V}_0,\bar{V}_1,\bar{V}_2]\) is injective. Because
\[
\begin{bmatrix}
\bar{W}_1 & 0 \\
0 & -\bar{W}_2
\end{bmatrix}
\ker [V_1, V_2] \subseteq \text{im} \begin{bmatrix}
U_1 \\
U_2
\end{bmatrix}
\]
we now have
\[
\text{im} \begin{bmatrix}
W_1' \\
W_2'
\end{bmatrix} \subseteq \text{im} \begin{bmatrix}
U_1 \\
U_2
\end{bmatrix}.
\]
Moreover, since \(\bar{V}_0\) is injective, we also have
\[
\ker \begin{bmatrix}
W_1' \\
W_2'
\end{bmatrix} \cong \ker \bar{V}_0.
\]
Thus, by Lemma 4.4 there exists a matrix \(X_0 \in \mathbb{F}^{b \times c}\) such that
\[
\begin{bmatrix}
U_1 \\
U_2
\end{bmatrix}
X_0 \bar{V}_0 = \begin{bmatrix}
W_1' \\
W_2'
\end{bmatrix}.
\]
Because, for \(i = 1,2\), \(\text{im} W_i \subseteq \text{im} U_i\) and \(\ker V_i \subseteq \ker W_i\), again by Lemma 4.4 there exists matrices \(X_1, X_2 \in \mathbb{F}^{b \times c}\) such that \(U_1 X_1 V_1 = W_1\) and \(U_2 X_2 V_2 = W_2\).
It is clear that the matrices \( X_1 \) and \( X_2 \) also satisfy \( U_1 X_1 \tilde{\nu}_1 = \tilde{w}_1 \) and \( U_2 X_2 \tilde{\nu}_2 = \tilde{w}_2 \).

Now, let \( X \in \mathbb{R}^{b \times c} \) be a matrix such that

\[
x[\tilde{\nu}_0, \tilde{\nu}_1, \tilde{\nu}_2] = [X_0 \tilde{\nu}_0, X_1 \tilde{\nu}_1, X_2 \tilde{\nu}_2].
\]

Note that

\[
U_1 X V_1 S_1 = U_1 [X_0 \tilde{\nu}_0, X_1 \tilde{\nu}_1, 0] = U_1 [X_0 \tilde{\nu}_0, X_1 \tilde{\nu}_1, 0] = [w_1^t \tilde{\nu}_1, 0] = w_1 \tilde{\nu}_1,
\]

where \( i = 1, 2 \).

Hence, the matrix \( X \in \mathbb{R}^{b \times c} \) satisfies \( U_1 X V_1 = w_1 \) and \( U_2 X V_2 = w_2 \).

We now return to the linear system (4.1) and we state verifiable necessary and sufficient conditions for the solvability of (ANICFM2) (see also Van der Woude (1987a).

**Theorem 4.7:**

The following statements are equivalent:

1) \( \text{(ANICFM2)} \) is solvable.

2) For \((i, j) = (1, 2), (2, 1)\), rank \( L_i = \text{rank} \{ L_1, K_{ij} \} \) and rank \( M_j = \text{rank} \{ K_{ij} \} \), and

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
L_1 & K_{12} & 0 \\
L_2 & 0 & -K_{21} \\
0 & M_2 & M_1
\end{bmatrix}
\]

3) For \((i, j) = (1, 2), (2, 1)\), \( S_0^* \langle \im C_i \rangle \subseteq \ker H_j \) and \( \im C_i \subseteq V_b^* (\ker H_j) \), and

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
L_1 & K_{12} & 0 \\
L_2 & 0 & -K_{21} \\
0 & M_2 & M_1
\end{bmatrix}
\]

**Proof:**

1) \( \Rightarrow \) 2) Follows from Corollary 4.3 and Theorem 4.6.

2) \( \Rightarrow \) 3) From Theorem 5.8 we know that the condition rank \( L_i = \text{rank} \{ L_1, K_{12} \} \) is equivalent to the existence of a rational matrix \( U \in \mathbb{R}^{m \times q_2(s)} \) such that
\( L_1 U + K_{12} = 0 \). Recall that \( L_1(s) = H_1(sI - A)^{-1} B \) and \( K_{12}(s) = H_1(sI - A)^{-1} G_2 \), and define \( X(s) := (sI - A)^{-1} B U(s) + (sI - A)^{-1} G_2 \). Then it is clear that \( \text{rank } L_1 = \text{rank } [L_1, K_{12}] \) is equivalent to the existence of rational matrices \( X \in \mathbb{R}^{n \times d_2}(s) \) and \( U \in \mathbb{R}^{m \times q_2}(s) \) such that \((sI - A)X(s) - BU(s) = G_2 \) and \( HX(s) = 0 \). From Definition 1.24 it follows now that the condition \( \text{rank } L_1 = \text{rank } [L_1, K_{12}] \) is equivalent to \( G_2 \leq V^*_b(ker H_1) \).

Analogously, we can prove that \( \text{rank } L_2 = \text{rank } [L_2, K_{21}] \) if and only if \( \text{im } G_1 \leq V^*_b(ker H_2) \), and \( \text{rank } M_j = \text{rank } [K_{i,j}] \) if and only if \( S_b^*(\text{im } G_j) \leq ker H_1 \), where \((i,j) = (1,2), (2,1) \).

The necessity of the last (rank) condition in both the second and the third statement of Theorem 4.7 can be argued more intuitively as follows. Suppose that \((\text{ANICPM2})\) is solvable and suppose that \( \bar{\varphi}_1 \in \mathbb{R}^{q_1}(s) \), \( \bar{\varphi}_2 \in \mathbb{R}^{q_2}(s) \) are the Laplace transforms of the two exogenous inputs \( v_1, v_2 \). Let the contribution of the \( j \)-th \((j = 1, 2) \) exogenous input to the measurement output be given by \( m_j := M_j \bar{\varphi}_j \), and its contribution to the \( i \)-th \((i = 1, 2) \) exogenous output by \( z_{ij} := K_{ij} \bar{\varphi}_j \). Now suppose that

\[
\begin{bmatrix}
    m_1 \\
    m_2
\end{bmatrix} = 0,
\]

i.e., \( \begin{bmatrix} \bar{\varphi}_1 \\ \bar{\varphi}_2 \end{bmatrix} \in \ker [M_1, M_2] \).

Hence, the measurement output based on the exogenous inputs \( \bar{\varphi}_1 \) and \( \bar{\varphi}_2 \) is zero. Then no measurement feedback compensator can reconstruct the values of \( \bar{\varphi}_1 \) and \( \bar{\varphi}_2 \). Therefore, in order to achieve 'almost noninteraction', the compensator with the control input has to simultaneously 'almost compensate' \( v_{12} = K_{12} \bar{\varphi}_2 \) and \( z_{21} = K_{21} \bar{\varphi}_1 \). This requirement is expressed by the following subspace inclusion:

\[
\begin{bmatrix}
    0 & K_{12} \\
    0 & -K_{21}
\end{bmatrix} \ker [M_2, M_1] \subseteq \text{im } \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.
\]

By Lemma 4.5, the latter subspace inclusion is equivalent to the last rank condition in both the second and the third statement of Theorem 4.7.

The successful derivation of simple necessary and sufficient conditions for the solvability of \((\text{ANICPM2})\) is for a great deal based on the fact
that we were able to establish simple necessary and sufficient conditions for the existence of a common solution to a pair of linear matrix equations over a field. Unfortunately, we are not (yet) able to derive analogous conditions for the existence of a common solution to a pair of linear matrix equations over a principal ideal domain. Consequently, we are also not able to derive simple frequency domain oriented conditions for the solvability of (NICPM2). However, in the following two sections we shall derive verifiable necessary and sufficient conditions in state space terms for the solvability of (NICPM2).

4.3. Noninteracting control by measurement feedback: sufficient conditions

As announced, in this section and also in the following section we are concerned with the noninteracting control problem by measurement feedback, (NICPM2).

In the present section, we shall derive sufficient conditions in state space terms for the solvability of the latter problem. Furthermore, we shall show that these sufficient conditions in fact give rise to computationally verifiable necessary and sufficient conditions for the solvability of (NICPM2), provided the linear system involved has certain properties.

At the same time, we shall derive necessary and sufficient conditions for the solvability of the noninteracting control problem by pure static measurement feedback.

The next result will be useful in the derivation of the sufficient conditions mentioned above.

Consider the linear system (4.1).

Lemma 4.8:

Let $S_1$, $S_2$ be $(\mathcal{G},\mathcal{A})$-invariant subspaces in $\mathbb{R}^n$ and let $V_1, V_2$ be $(\mathcal{A},\mathcal{B})$-invariant subspaces in $\mathbb{R}^n$ such that $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$. Then there exists a matrix $N \in \mathbb{R}^{m \times n}$ such that $(\mathcal{A}+\mathcal{B})S_1 \subseteq V_1$ and $(\mathcal{A}+\mathcal{B})S_2 \subseteq V_2$ if and only if

$$
\begin{bmatrix}
A & 0 \\
0 & -A
\end{bmatrix}
((S_1 \oplus S_2) \cap \ker \left[ C, D \right]) \subseteq (V_1 \oplus V_2) + \text{im } \begin{bmatrix} B \end{bmatrix}.
$$
Proof:
Let $X_1, X_2, T_1$ and $T_2$ be matrices such that $\text{im } X_i = S_i$ ($i = 1, 2$) and $\ker T_i = V_i$ ($i = 1, 2$). Then there exists a matrix $N \in \mathbb{R}^{m \times p}$ such that $(A+BNC)S_i \subseteq V_i$ ($i = 1, 2$) if and only if there exists a matrix $N \in \mathbb{R}^{m \times p}$ such that $T_iAX_i + T_iBN CX_i = 0$ ($i = 1, 2$).

By Theorem 4.6, the latter is equivalent to

\[ \text{im } T_iB \supseteq \text{im } T_i AX_i \quad (i = 1, 2), \quad \ker CX_i \subseteq \ker T_i AX_i \quad (i = 1, 2) \]

and

\[
\begin{bmatrix}
T_i AX_i & 0 \\
0 & -T_2 AX_2
\end{bmatrix}
\begin{bmatrix}
\ker [CX_i, CX_2] \subseteq \text{im }
\begin{bmatrix}
T_i B \\
T_2 B
\end{bmatrix}
\end{bmatrix}.
\]

In turn, this is equivalent to

\[ AS_i \subseteq V_i + \text{ im } B \quad (i = 1, 2), \quad A(S_i \cap \ker C) \subseteq V_i \quad (i = 1, 2) \]

and

\[
\begin{bmatrix}
A & 0 \\
0 & -A
\end{bmatrix}
\begin{bmatrix}
(S_1 \oplus S_2) \cap \ker [C, C] \subseteq (V_1 \oplus V_2) + \text{ im }
\begin{bmatrix}
B \\
B
\end{bmatrix}
\end{bmatrix}.
\]

The proof can now be completed using the observation that the conditions $AS_i \subseteq V_i + \text{ im } B$ ($i = 1, 2$) and $A(S_i \cap \ker C) \subseteq V_i$ ($i = 1, 2$) are fulfilled trivially since $S_1, S_2$ are $(C,A)$-invariant subspaces, $V_1, V_2$ are $(A,B)$-invariant subspaces and $S_1 \subseteq V_1, S_2 \subseteq V_2$.

Using Lemma 4.8 we can derive the following necessary and sufficient conditions for the solvability of the noninteracting control problem by pure static measurement feedback.

Corollary 4.9:
Let the system (4.1) be given. There exists a matrix $N \in \mathbb{R}^{m \times p}$ such that $H_1 (\omega_1 - (A+BNC))^{-1} C_2 = 0$ and $H_2 (\omega_1 - (A+BNC))^{-1} C_1 = 0$ if and only if there exist subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ in $\mathbb{R}^n$, both $(A,B)$-invariant as well as $(C,A)$-invariant, such that $\text{im } C_1 \subseteq \mathcal{W}_1 \subseteq \ker H_2$ and $\text{im } C_2 \subseteq \mathcal{W}_2 \subseteq \ker H_1$.


\[
\begin{bmatrix}
\lambda & 0 \\
0 & -\lambda
\end{bmatrix} ((\Omega_1 \Theta \Omega_2) \cap \ker [\Omega, \Theta]) \subseteq (\Omega_1 \Theta \Omega_2) + \text{im } [B].
\]

Proof:
It is easy to see that \( \Pi_1 (s I - (A+\text{BNC})^{-1} G_2 = 0 \) and \( \Pi_2 (s I - (A+\text{BNC})^{-1} G_1 = 0 \) if and only if there exist linear subspaces \( \Omega_1 \) and \( \Omega_2 \) in \( \mathbb{R}^n \) such that
\( (A+\text{BNC}) \Omega_1 = \Omega_1, \quad (A+\text{BNC}) \Omega_2 = \Omega_2, \)
\( \text{im } G_1 \subseteq \Omega_1 \subseteq \ker H_2 \) and \( \text{im } G_2 \subseteq \Omega_2 \subseteq \ker H_1 \).
In addition, observe that the linear subspaces \( \Omega_1 \) and \( \Omega_2 \) are \((A,B)\)-invariant as well as \((C,A)\)-invariant. Then the proof of this corollary can be completed by the application of Lemma 4.8 with \( \Omega_1 = S_1 - V_1 \) and \( \Omega_2 = S_2 = V_2 \).

The following theorem is the main result of this section and states sufficient conditions for the solvability of \((\text{NTCPM2})\).

Theorem 4.10:
Consider the linear system (4.1). Let \( S_2, S_2 \) be \((C,A)\)-invariant subspaces and let \( V_1, V_2 \) be \((A,B)\)-invariant subspaces in \( \mathbb{R}^n \) such that the following conditions are satisfied:

a) \( \text{im } G_1 \subseteq S_1 \subseteq \ker H_2 \), \( \text{im } G_2 \subseteq S_2 \subseteq \ker H_1 \),
b) \( S_1 + S_2 \) is a \((C,A)\)-invariant subspace,
c) \( V_1 \cap V_2 \) is an \((A,B)\)-invariant subspace, and
d) \( \begin{bmatrix}
\lambda & 0 \\
0 & -\lambda
\end{bmatrix} ((S_1 \Theta S_2) \cap \ker [C, \Theta]) \subseteq (S_1 \Theta S_2) + \text{im } [B].
\]
Then \((\text{NTCPM2})\) is solvable.

Proof:
Because of a), d) and Lemma 4.8, there exists a matrix \( N \in \mathbb{R}^{m \times n} \) such that
\( (A+\text{BNC}) S_i \subseteq V_i \quad (i = 1, 2) \). By b) and c) there exist matrices \( P \in \mathbb{R}^{m \times n} \) and \( J \in \mathbb{R}^{n \times m} \) such that \( (A+\text{BNC}) V_i \subseteq V_i \quad (i = 1, 2) \) and \( (A+\text{BNC}) S_i \subseteq S_i \quad (i = 1, 2) \)
(cf. Wohlm (1979), exercise 9.1).

Let \( \Omega_{1, \epsilon} \) and \( \Omega_{2, \epsilon} \) be linear subspaces in \( \mathbb{R}^{2n} \) defined as
\[
\Omega_{i, \epsilon} := \left\{ \begin{bmatrix} s \\ \theta_s \\ v \\ \theta_v \end{bmatrix} \mid s \in S_i, \ v \in V_i \right\} \quad (i = 1, 2)
\]
and let $A_e \in \mathbb{R}^{2n \times 2n}$ be a matrix defined as $A_e := \hat{\hat{A}}_e$, where

$$
(1.13) \quad \hat{\hat{A}}_e = \begin{bmatrix} A + BF & B(F - NC) \\ \text{B} & I_{2n} - J \end{bmatrix},
$$

The matrix $A_e$ can be considered to be obtained by the interconnection of system (4.1) and compensator (1.9) with $K = A + BF + JC - BNC$, $L = BN - J$ and $M = F - NC$ (see also Section 1.5).

By the construction described in Section 1.5 it now follows that $A_{e,1}, e \subseteq W_{1,e}$, $A_{e,2}, e \subseteq W_{2,e}$, $\text{im} G_{1,e} \subseteq W_{1,e}$, $\text{ker} H_{2,e}$ and $\text{im} G_{2,e} \subseteq W_{2,e}$. Hence, it follows that $H_{1,e} A_{e,1} e = 0$ and $H_{2,e} A_{e,2} e = 0$ for all $k \geq 0$, which implies that $T_{12}(s) = (sI - A_{e})^{-1} G_{2,e} = 0$ and $T_{21}(s) = (H_{2,e} (sI - A_{e})^{-1} G_{2,e} = 0$.

**Remark 4.11:**

Note that the various conditions appearing in Corollary 4.9 and Theorem 4.10 are rather 'abstract' in the sense that it is not clear how they should be verified computationally.

In the remainder of the present section and also in the following section, we derive verifiable conditions in connection with the solvability of (NCPM2).

For reasons of notational convenience we shall denote

$$
S_{1}^* = S^* (\text{im} G_{1}) , \quad S_{2}^* = S^* (\text{im} G_{2}) , \quad S^* = S^* ([G_{1}, G_{2}]) ,
$$

$$
V_{1}^* = V^* (\text{ker} H_{2}) , \quad V_{2}^* = V^* (\text{ker} H_{1}) \quad \text{and} \quad V^* = V^* (\text{ker} H_{1}) .
$$

Since $S_{1}^* \subseteq S^*$, $S_{2}^* \subseteq S^*$, $V_{1}^* \subseteq V_{1}^*$ and $V_{2}^* \subseteq V_{2}^*$, we have

$$
S_{1}^* + S_{2}^* \subseteq S^* \quad \text{and} \quad V_{X} \subseteq V_{1}^* \cap V_{2}^* .
$$

Moreover, $S_{1}^* + S_{2}^*$ is a $(C, A)$-invariant subspace if and only if $S_{1}^* + S_{2}^* = S^*$ and $V_{1}^* \cap V_{2}^*$ is an $(A, B)$-invariant subspace if and only if $V_{1}^* \cap V_{2}^* = V^*$.

Indeed, note that $\text{im} [G_{1}, G_{2}] \subseteq S_{1}^* + S_{2}^*$. So, if $S_{1}^* + S_{2}^*$ is a $(C, A)$-invariant subspace, then $S^* \subseteq S_{1}^* + S_{2}^*$. Hence, $S^* = S_{1}^* + S_{2}^*$. Therefore, if $S^* = S_{1}^* + S_{2}^*$,
and \( V^* - V_1^* \cap V_2^* \), then in Theorem 4.10 we may replace the pair \((S_1, S_2)\) by the pair \((S_1^*, S_2^*)\) and the pair \((V_1, V_2)\) by the pair \((V_1^*, V_2^*)\).

In the following we shall state sufficient conditions for \( V^* = V_1^* \cap V_2^* \) and \( S^* = S_1^* + S_2^* \):

To this extent, we consider a rational matrix \( W \in \mathbb{R}^{axb}(s) \) and we assume that \( W = [w_1, w_2, \ldots, w_b] \) with \( w_i \in \mathbb{R}^a(s) \) for all \( i \in b \). For all \( i \in b \), if \( w_i \neq 0 \), let \( \alpha_i := \ominus w_i \) (see Section 3.1) and define \( \tilde{V}_i(s) := s^{-\alpha_i} w_i(s) \).

Let \( \tilde{W} \in \mathbb{R}^{axb}(s) \) be the proper rational matrix obtained from \( W \) by replacing every nonzero column \( w_i \) of \( W \) by \( \tilde{V}_i \) and define \( \tilde{W} := \lim_{s \to \infty} \tilde{W}(s) \).

We say that the rational matrix \( W \in \mathbb{R}^{axb}(s) \) is column proper if the matrix \( \tilde{W} \in \mathbb{R}^{nxm}(s) \) is injective. Dually, we say that the rational matrix \( W \in \mathbb{R}^{axb}(s) \) is row proper if \( W^T \in \mathbb{R}^{bxa}(s) \) is a column proper rational matrix.

Consider the linear system described by (1,1) and let

\[
C = \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_p
\end{bmatrix}, \quad \text{where } c_i \in \mathbb{R}^{1xn} \quad \text{for all } i \in p.
\]

The following result is due to Bhattacharyya (1975).

**Proposition 4.12:**

If \( C(sI-A)^{-1} B \) is a row proper rational matrix then

\[
(V^*(\ker C; A, B)) = \bigcap_{i \in p} V^*(\ker c_i) = \bigcap_{i \in p} (V^*(\ker c_i; A, B))
\]

**Corollary 4.13:**

If \( \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} (sI - A)^{-1} B \) is a row proper rational matrix then \( V_1^* \cap V_2^* = V^* \).

**Theorem 4.14:**

Let the system (4.1) be given and assume that \( \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} (sI - A)^{-1} B \) is a row proper rational matrix and \( C(sI - A)^{-1} [G_1, G_2] \) is a column proper rational matrix.
Then (NICPM2) is solvable if and only if \( S_1^\ast \subseteq \nu_1^\ast, S_2^\ast \subseteq \nu_2^\ast \) and

\[
\begin{bmatrix}
A & 0 \\
0 & -A
\end{bmatrix}((S_1^\ast \oplus S_2^\ast) \cap \ker [C, C]) \subseteq (\nu_1^\ast \oplus \nu_2^\ast) + \text{im } [B].
\]

Proof:

(if) Follows immediately from Theorem 4.10, Remark 4.11 and Corollary 4.13.

(only if) Assume that (NICPM2) is solvable. That is, assume there exists a measurement feedback compensator (1.9) such that in the closed loop system (4.2) \( H_1,e(sI-A_e)^{-1}G_2,e = 0 \) and \( H_2,e(sI-A_e)^{-1}G_1,e = 0 \).

Let \( \omega_{1,e} \) and \( \omega_{2,e} \) be linear subspaces in \( \mathbb{R}^{n+k} \), the state space of the closed loop system, defined by

\[
\omega_{1,e} := \langle A_e | \text{im } G_1,e \rangle \quad \text{and} \quad \omega_{2,e} := \langle A_e | \text{im } G_2,e \rangle.
\]

Then it follows that \( A_e \omega_{1,e} \subseteq \omega_{1,e}, A_e \omega_{2,e} \subseteq \omega_{2,e}, \text{im } G_1,e \subseteq \omega_{1,e} \subseteq \ker H_2,e \) and \( \text{im } G_2,e \subseteq \omega_{2,e} \subseteq \ker H_1,e \). Also, it follows that

\[
A_e (\omega_{1,e} \cap \omega_{2,e}) \subseteq (\omega_{1,e} \cap \omega_{2,e}) \quad \text{and} \quad A_e (\omega_{1,e} + \omega_{2,e}) \subseteq (\omega_{1,e} + \omega_{2,e}).
\]

Let \( S_1, S_2, S, V_1, V_2, \) and \( V \) be linear subspaces in \( \mathbb{R}^n \) defined as

\[
S_i := \{ x \in \mathbb{R}^n | \begin{bmatrix} x \\ 0 \end{bmatrix} \in \omega_{i,e} \} \quad (i = 1,2),
\]

\[
S := \{ x \in \mathbb{R}^n | \begin{bmatrix} x \\ 0 \end{bmatrix} \in (\omega_{1,e} + \omega_{2,e}) \},
\]

\[
V_i := \{ x \in \mathbb{R}^n | \exists y \in \mathbb{R}^k: \begin{bmatrix} x \\ y \end{bmatrix} \in \omega_{i,e} \} \quad (i = 1,2),
\]

\[
V := \{ x \in \mathbb{R}^n | \exists y \in \mathbb{R}^k: \begin{bmatrix} x \\ y \end{bmatrix} \in (\omega_{1,e} \cap \omega_{2,e}) \}.
\]

Note that \( S_i \) is the intersection of \( \omega_{i,e} \) with \( \mathbb{R}^n \) \((i = 1,2)\), \( S \) is the intersection of \( \omega_{1,e} + \omega_{2,e} \) with \( \mathbb{R}^n \), \( V_i \) is the projection of \( \omega_{i,e} \) onto \( \mathbb{R}^n \) \((i = 1,2)\) and \( V \) is the projection of \( \omega_{1,e} \cap \omega_{2,e} \) onto \( \mathbb{R}^n \) (see also Section 1.5). By the above it is clear that
\[(S_1 \oplus 0) \subseteq \omega_1 \subseteq (V_1 \oplus \mathbb{R}^k) \quad (i = 1, 2),
\]
\[(S \oplus 0) \subseteq (\omega_1 + \omega_2, e) \subseteq ((V_1 + V_2) \oplus \mathbb{R}^k),
\]
and
\[((S_1 \cap S_2) \oplus 0) \subseteq (\omega_1, e \cap \omega_2, e) \subseteq (V \oplus \mathbb{R}^k).
\]

Also, it is clear that
\[
im G_1 \subseteq S_1 \subseteq V_1 \subseteq \ker H_2, \quad \im G_2 \subseteq S_2 \subseteq V_2 \subseteq \ker H_1,
\]
\[
im [G_1, G_2] \subseteq S \subseteq (V_1 + V_2) \quad \text{and} \quad (S_1 \cap S_2) \subseteq V \subseteq \ker \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.
\]

From Theorem 1.4.4 we have that, due to the \(A_e\)-invariance of the subspaces \(\omega_1, e, \omega_2, e\) and \((\omega_1, e \cap \omega_2, e)\), the subspaces \(S_1, S_2, S\) and \(S\) are \((C, A)\)-invariant subspaces in \(\mathbb{R}^n\). Also, due to the \(A_e\)-invariance of the subspaces \(\omega_1, e, \omega_2, e\) and \((\omega_1, e \cap \omega_2, e)\), the subspaces \(V_1, V_2\) and \(V\) are \((A, B)\)-invariant subspaces in \(\mathbb{R}^n\). Hence, it follows that \(S_1^* \subseteq V_1^*, S_2^* \subseteq V_2^*, S^* \subseteq V_1^* + V_2^*\) and \(S_1^* \cap S_2^* \subseteq V^*\).

Because \(\omega_1, e\) and \(\omega_2, e\) are \(A_e\)-invariant subspaces it follows that
\[
A_e(S_1 \oplus 0) \subseteq (V_1 \oplus \mathbb{R}^k) \quad \text{and} \quad A_e(S_2 \oplus 0) \subseteq (V_2 \oplus \mathbb{R}^k).
\]

The latter implies that \((A + BNC)S_1^* \subseteq V_1^*\) and \((A + BNC)S_2^* \subseteq V_2^*\), which, by Lemma 4.8, is equivalent to
\[
\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \left( (S_1^* \oplus S_2^*) \cap \ker [U, C] \right) \subseteq (V_1^* \oplus V_2^*) + \im \begin{bmatrix} B \\ 0 \end{bmatrix}.
\]

\[\blacksquare\]

Remark 4.15:

Note that in the proof of the (only if)-part of the previous theorem we did not use that \(\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}(s1 - A)^{-1} B\) is a row proper rational matrix and \(C(s1 - A)^{-1}[G_1, G_2]\) is a column proper rational matrix. In fact, from the proof it follows that the solvability of (NIEPM2) is equivalent to the existence of \(A_e\)-invariant subspaces \(\omega_1, e\) and \(\omega_2, e\) in \(\mathbb{R}^{n+k}\) such that
4.4. Noninteracting control by measurement feedback: necessary and sufficient conditions

In the previous section we obtained nonconstructive sufficient conditions for the solvability of (NICPM2). Furthermore, we were able to derive verifiable necessary and sufficient conditions for the solvability of (NICPM2) for systems (4.1) that satisfy certain assumptions. It is the purpose of the present section to drop these assumptions and to derive verifiable necessary and sufficient conditions for the solvability of (NICPM2).

Before stating these conditions we introduce the following.

If $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ are linear subspaces in $\mathbb{R}^n$ such that $\mathcal{L}_1 + \mathcal{L}_2 \subseteq \mathcal{L}$, then we define the set $\Phi(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L})$ as follows:

$$\Phi(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}) := \{(M_1, M_2) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} | (M_1 - I)\mathcal{L}_1 = 0, (M_2 - I)\mathcal{L}_2 = 0 \text{ and } (M_1 + M_2 - I)\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2\}.$$ 

Note that for every triple $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$ of linear subspaces in $\mathbb{R}^n$ such that $\mathcal{L}_1 + \mathcal{L}_2 \subseteq \mathcal{L}$, the set $\Phi(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L})$ is not empty.

Indeed, let $[L_0, L_1, L_2, L_3, L_4] \in \mathbb{R}^{n \times n}$ be an invertible matrix such that

- $\text{im } L_0 = \mathcal{L}_1 \cap \mathcal{L}_2$,
- $\text{im } [L_0, L_1] = \mathcal{L}_1$,
- $\text{im } [L_0, L_2] = \mathcal{L}_2$ and
- $\text{im } [L_0, L_1, L_2, L_3] = \mathcal{L}$.

Then, every pair of matrices $(M_1, M_2) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ such that

$$M_1 [L_0, L_1, L_2, L_3, L_4] = [L_0, L_1, 0, M_{13}, M_{14}]$$

and

$$M_2 [L_0, L_1, L_2, L_3, L_4] = [L_0, 0, L_2, L_3 - M_{13}, M_{24}]$$

satisfy

$$(S^*_1 \oplus 0) \subseteq \omega_{i,e} \subseteq (v^*_i \oplus \mathbb{R}^k) \quad (i = 1, 2),$$

$$(S^* \oplus 0) \subseteq (\omega_{1,e} + \omega_{2,e}) \subseteq ((v^*_1 + v^*_2) \oplus \mathbb{R}^k)$$

and

$$(S^*_1 \cap S^*_2 \oplus 0) \subseteq (\omega_{1,e} \cap \omega_{2,e}) \subseteq (v^* \oplus \mathbb{R}^k).$$
is an element of $\Phi(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L})$. In the latter, $K_{12}$, $M_{14}$ and $M_{23}$ are arbitrary matrices of suitable dimensions.

Now, the main result of the present section reads as follows (see also Van der Waerden (1987b)).

**Theorem 4.16:**

Consider the system (4.1).

$(\text{NCPM2})$ is solvable if and only if there exist pairs of matrices

$$
(D_1, D_2) \in \Phi(S_1^*, S_2^*, S^*) \quad \text{and} \quad (E_1^T, E_2^T) \in \Phi(U_1^*, U_2^*, U^*)
$$

such that $D_1 S^* \subseteq U_1^*$, $D_2 S^* \subseteq U_2^*$, $E_1 S_1^* \subseteq V^*$, $E_2 S_2^* \subseteq V^*$, $(D_1 + E_1 - I) S^* \subseteq V^*$

and $(AD_1 + E_1 A - A)(S^* \cap \ker C) \subseteq V^* + im B$.

**Remark 4.17**

In the proof of the above theorem we shall make use of the following (non-unique) representation of the subspaces $S_1^*$, $S_2^*$, $S^*$, $U_1^*$, $U_2^*$ and $V^*$. Throughout the remainder of the present section we let $[X_0, X_1, X_2, X_3]$ be an injective matrix such that $im X_0 = S_1^* \subseteq S_2^*$, $im [X_0, X_1] = S_1^*$, $im [X_0, X_2] = S_2^*$ and $im [X_0, X_1, X_2, X_3] = S^*$.

Dually, we let

$$
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
$$

be a surjective matrix such that

$$
\ker T_0 = U_1^* + U_2^*, \quad \ker \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = U_1^*, \quad \ker \begin{bmatrix} T_0 \\ T_2 \end{bmatrix} = U_2^* \quad \text{and} \quad \ker \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{bmatrix} = V^*.
$$

In the sequel, we shall use the notation
Furthermore, we let $Q_X$ and $Q_T$ be matrices of suitable dimensions such that $XQ_X = \hat{X}$ and $Q_T \hat{T} = \hat{T}$. Then, using these specific representations, the subspace inclusions in the 'unknown' matrices $D_1$, $D_2$, $E_1$, and $E_2$ appearing in the above theorem, can be reformulated as a number of linear matrix equations in the 'unknown' matrices $D_1$, $D_2$, $E_1$ and $E_2$. In turn, the linear matrix equations obtained can be transformed, by means of Kronecker products, into linear equations whose solvability can be checked using standard techniques. Hence, the conditions of Theorem 4.16 can be verified computationally.

Proof of Theorem 4.16:

(only if) From Remark 4.15 the existence follows of $A_e$-invariant subspaces $\mathcal{W}_{i,e}$ and $\mathcal{W}_{2,e}$ in $\mathbb{R}^{n+k}$ such that

\[(S_1^* \otimes 0) \subseteq \mathcal{W}_{i,e} \subseteq (V_i^* \otimes \mathbb{R}^k) \quad (i = 1, 2),\]

\[(S_2^* \otimes 0) \subseteq (\mathcal{W}_{1,e} + \mathcal{W}_{2,e}) \subseteq ((V_1^* + V_2^*) \otimes \mathbb{R}^k)\]

and

\[(S_1^* \cap S_2^* \otimes 0) \subseteq (\mathcal{W}_{1,e} \cap \mathcal{W}_{2,e}) \subseteq (V^* \otimes \mathbb{R}^k).\]

Let $S_1^*$, $S_2^*$, $S^*$, $V_1^*$, $V_2^*$ and $V^*$ have representations as indicated in Remark 4.17, then we have

\[\text{im} \begin{bmatrix} X_0 & X_1 \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{W}_{i,e} \subseteq \ker \begin{bmatrix} T_0 \\ T_i \end{bmatrix} \quad (i = 1, 2),\]

\[\text{im} \begin{bmatrix} X_0 & X_1 & X_2 & X_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subseteq (\mathcal{W}_{1,e} + \mathcal{W}_{2,e}) \subseteq \ker \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{bmatrix} \]

and
\[
\begin{bmatrix}
X_0 \\
Y_3 \\
0
\end{bmatrix}
\subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_1 & 0 \\
T_2 & 0 \\
T_3 & 0
\end{bmatrix}
\]

It is clear that there exist matrices \(D_{13}, D_{23}, Y_3, E_{31}, E_{32}\) and \(U_3\) such that

\[
\begin{bmatrix}
X_3 \\
Y_3 \\
0
\end{bmatrix} = \begin{bmatrix}
D_{13} \\
D_{23} \\
0
\end{bmatrix} \quad \text{with} \quad \text{im} \begin{bmatrix}
D_{13} \\
D_{23} \\
0
\end{bmatrix} \subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_1 & 0 \\
T_2 & 0 \\
T_3 & 0
\end{bmatrix}
\]

and, dually,

\[
[T_3, 0] = [E_{31}, U_3] + [E_{32}, -U_3] \quad \text{with} \quad \ker [E_{31}, U_3] \subseteq \ker [E_{31}, U_3],
\]

\[
\ker [E_{32}, -U_3] \subseteq \ker [E_{32}, -U_3].
\]

Hence, the matrices \(D_{13}, D_{23}, Y_3, E_{31}, E_{32}\) and \(U_3\) satisfy

\[
\begin{bmatrix}
X_0 \\
X_1 \\
Y_3
\end{bmatrix}
\subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_1 & 0 \\
U_3 & 0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
X_0 \\
X_1 \\
-Y_3
\end{bmatrix}
\subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_2 & 0 \\
E_{32} & -U_3
\end{bmatrix}
\]

with \(D_{13} + D_{23} = X_3\) and \(E_{31} + E_{32} = T_3\).

Now, let \(D_1, D_2, E_1, E_2 \in \mathbb{R}^{n \times n}\) and \(R, P \in \mathbb{R}^{n \times n}\) be matrices such that

\[
D_1 [X_0, X_1, X_2, X_3] = [X_0, X_1, 0, D_{13}] , \quad D_2 [X_0, X_1, X_2, X_3] = [X_0, 0, X_2, D_{23}] , \quad R [X_0, X_1, X_2, X_3] = [0, 0, 0, Y_3] ,
\]

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix} E_1 = \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix} E_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix} P = \begin{bmatrix}
0 \\
0 \\
0 \\
U_3
\end{bmatrix}
\]
It is immediate that 
\[(D_1 - I)[X_0, X_1] = 0, \quad (D_2 - I)[X_0, X_2] = 0 \]
and
\[(D_1 + D_2 - I)[X_0, X_1, X_2, X_3] = [X_0, 0, 0, 0] \]
which implies that 
\[(D_1 - I)S^* = 0, \quad (D_2 - I)S^* = 0 \]
and 
\[(D_1 + D_2 - I)S^* = S^* \cap S^* \]. Hence, 
\[(D_1, D_2) \in \phi(S^*, S^*, S^*)\]. 

Dually, it can be shown that 
\[(E_1^T, E_2^T) \in \phi(V_1^\perp, V_2^\perp, V^\perp)\].

Furthermore, note that 
\[D_1 X + D_2 X = D_1 \tilde{X} + D_2 \tilde{X} = X, \quad TE_1 + TE_2 = TE_1 + TE_2 = T, \]
\[RX = RX \]
and 
\[TP = TP\].

From the previous it is now clear that

\[\text{im } \begin{bmatrix} D_1 X \\ RX \end{bmatrix} \subseteq \omega_1, e \subseteq \ker (TE_1, TP) \]

and

\[\text{im } \begin{bmatrix} D_2 X \\ -RX \end{bmatrix} \subseteq \omega_2, e \subseteq \ker (TE_2, TP), \]

which implies that 
\[T(E_1 D_1 + PR)X = 0 \]
and 
\[T(E_2 D_2 + PR)X = 0. \]

By the latter two expressions it also follows that

\[0 = \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} (E_1 D_1 + PR)X = \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} D_1 X, \]

which implies that 
\[D_1 S^* \subseteq V_1^\ast\]. Analogously, it can be shown that 
\[D_2 S^* \subseteq V_2^\ast, \quad E_1 S^* \subseteq V_1^\ast, \quad E_2 S^* \subseteq V_2^\ast. \]

Also, it is clear that

\[Q_T(E_1 D_1 + PR)XQ_T = \tilde{T}(E_2 D_2 + PR)X = T(E_1 D_1 + PR)X + T(I - D_1 - E_1)X = T(I - D_1 - E_1)X = 0, \]

which implies that 
\[(D_1 + E_1 - I)S^* \subseteq V^\ast.\]

Finally, recall that the subspaces \(\omega_1, e\) and \(\omega_2, e\) are \(A_e\)-invariant, from which it is immediate that

\[A_e \text{ im } \begin{bmatrix} D_1 X \\ RX \end{bmatrix} \subseteq \ker (TE_1, TP) \]
and
\[A_e \text{ im } \begin{bmatrix} D_2 X \\ -RX \end{bmatrix} \subseteq \ker (TE_2, TP). \]

In turn, these two subspace inclusions are equivalent to
\[
[\begin{bmatrix} D_1 X \\ RX \end{bmatrix}]_c = \begin{bmatrix} TE_1 A D_1 X + [TE_1 B, TP] \end{bmatrix}_{\begin{bmatrix} N \\ L \end{bmatrix}} = 0
\]
and
\[
[\begin{bmatrix} D_2 X \\ RX \end{bmatrix}]_c = \begin{bmatrix} TE_2 A D_2 X + [TE_2 B, TP] \end{bmatrix}_{\begin{bmatrix} N \\ L \end{bmatrix}} = 0 .
\]

Observe that the matrix \( \begin{bmatrix} N \\ L \end{bmatrix} \) is a common solution to the two linear matrix equations. Therefore, by Theorem 4.6 it follows that \( \tilde{A}_1 \ker \tilde{C}_1 \subseteq \operatorname{im} \tilde{B}_1 \). Here, we have denoted
\[
\tilde{A}_1 := \begin{bmatrix} TE_1 A D_1 X & 0 \\ 0 & -TE_2 A D_2 X \end{bmatrix}, \quad \tilde{B}_1 := \begin{bmatrix} TE_1 B & TP \\ TE_2 B & -TP \end{bmatrix}
\]
and
\[
\tilde{C}_1 := \begin{bmatrix} CD_1 X & CD_2 X \\ RX & -RX \end{bmatrix}.
\]

Let
\[
\tilde{U}_1 := \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{V}_1 := \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}.
\]

Then \( \tilde{A}_1 \ker \tilde{C}_1 \subseteq \operatorname{im} \tilde{B}_1 \) if and only if \( \tilde{A}_2 \ker \tilde{C}_2 \subseteq \operatorname{im} \tilde{B}_2 \) where \( \tilde{A}_2 := \tilde{V}_1 \tilde{A}_1 \tilde{U}_1 \), \( \tilde{A}_2 := \tilde{V}_1 \tilde{A}_1 \tilde{U}_1 \) and \( \tilde{C}_2 := \tilde{C}_1 \tilde{U}_1 \). Straightforward calculation shows that
\[
\tilde{A}_2 = \begin{bmatrix} \tau(AD_1 + E_1 A - A) X & -TE_2 A D_2 X \\ -TE_2 A D_2 X & -TE_2 A D_2 X \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} TB & 0 \\ TE_2 B & -TP \end{bmatrix}
\]
and
\[
\tilde{C}_2 := \begin{bmatrix} CX & CD_2 X \\ 0 & -RX \end{bmatrix}.
\]

Now, note that
\[
\begin{bmatrix} \tau(AD_1 + E_1 A - A) X \\ -TE_2 A D_2 X \end{bmatrix} \quad \ker CX \subseteq \tilde{A}_2 \ker \tilde{C}_2 \subseteq \operatorname{im} \tilde{B}_2 .
\]
Thus, it immediately follows that \( T(AD_1 + E_1^A - A)X \ker CX \subseteq \text{im } TB \).

The latter implies that \((AD_1 + E_1^A - A)(S^* \cap \ker C) \subseteq (V^* + \text{im } B)\), which completes the proof of the (only if)-part of Theorem 4.16.

\((\text{if})\) Because \((D_1, D_2) \in \Phi(S^*_1, S^*_2, S^*)\), there exists a representation of \( S^*_1, S^*_2 \) and \( S^* \), as indicated in Remark 4.17, such that

\[
D_1[X_0, X_1, X_2, X_3] = [X_0, X_1, 0, D_{13}]
\]

and

\[
D_2[X_0, X_1, X_2, X_3] = [X_0, 0, X_2, D_{23}]
\]

with \( D_{13} + D_{23} = X_3 \).

Dually, because \((E_1^T, E_2^T) \in \Phi(V^*_{11}, V^*_{22}, V^*)\), there exists a representation of \( V^*_{11}, V^*_{22} \) and \( V^* \), as indicated in Remark 4.17, such that

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
= \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
E_{31}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
= \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
E_{32}
\end{bmatrix}
\]

with \( E_{31} + E_{32} = T_3 \).

With \( X \) and \( T \) as introduced in Remark 4.17, the subspace inclusions of Theorem 4.16 read as follows:

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2
\end{bmatrix} D_1X = 0, \quad \begin{bmatrix}
T_0 \\
T_1 \\
T_2
\end{bmatrix} D_2X = 0, \quad \text{TE}_1[X_0, X_1] = 0, \quad \text{TE}_2[X_0, X_2] = 0,
\]

\( T(D_1 + E_1^A - I)X = 0 \) \quad \text{and} \quad T(AD_1 + E_1^A - A)X \ker CX \subseteq \text{im } TB \).

Decompose \(-E_{32}D_{23} = U_3Y_3\) with \( U_3 \) a surjective matrix and \( Y_3 \) an injective matrix and let \( v \) denote the number of rows of \( Y_3 \).
Now, consider the linear system obtained from (4.1) by adding a bank of $v$ integrators $\dot{v}_d: \dot{x}_d(t) = u_d(t), \ y_d(t) = x_d(t)$ with $x_d(t), u_d(t), y_d(t) \in \mathbb{R}^v$. The system composed of (4.1) and $\dot{v}_d$ is described by:

\begin{align*}
(4.3a) & \quad \dot{x}(t) = A \xi(t) + B \eta(t) + C_1 y_1(t) + C_2 y_2(t), \\
(4.3b) & \quad \dot{y}(t) = C \xi(t), \\
(4.3c) & \quad \dot{z}_1(t) = \hat{A}_1 \dot{x}(t), \quad \dot{z}_2(t) = \hat{A}_2 \dot{x}(t),
\end{align*}

where we have denoted

$$
\dot{x}(t) := \begin{bmatrix} x(t) \\ x_d(t) \end{bmatrix}, \quad \dot{y}(t) := \begin{bmatrix} u(t) \\ u_d(t) \end{bmatrix}, \quad \dot{y}(t) := \begin{bmatrix} y(t) \\ y_d(t) \end{bmatrix},
$$
$$
\hat{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{C}_i := \begin{bmatrix} G_i \\ 0 \end{bmatrix} \quad (i = 1, 2),
$$
$$
\hat{C} := \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{A}_i := \begin{bmatrix} \hat{A}_i \end{bmatrix} \quad (i = 1, 2).
$$

Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{V}_1$ and $\mathcal{V}_2$ be linear subspaces in $\mathbb{R}^{n+y}$ defined by

$$
\mathcal{S}_1 := \text{im} \begin{bmatrix} \hat{A}_1 \xi \\ \hat{B}_1 \xi \end{bmatrix}, \quad \mathcal{S}_2 := \text{im} \begin{bmatrix} \hat{A}_2 \xi \\ \hat{B}_2 \xi \end{bmatrix},
$$
$$
\mathcal{V}_1 := \ker \left[ T \mathcal{E}_1, T \mathcal{P} \right] \quad \text{and} \quad \mathcal{V}_2 := \ker \left[ T \mathcal{E}_2, -T \mathcal{P} \right].
$$

In order to complete the proof of the (ii)-part of the theorem, it suffices to show that the system (4.3) together with the linear subspaces $\mathcal{S}_1, \mathcal{S}_2, \mathcal{V}_1$ and $\mathcal{V}_2$ satisfy the conditions of Theorem 4.10.

Indeed, if this is the case, then by Theorem 4.10 there exist matrices $\hat{\xi}, \mathcal{E}, \mathcal{F}$ and $\mathcal{B}$ of suitable dimensions such that for $(i,j) = (1,2), (2,1)$:
Here, we have decomposed

\[ \hat{L} = [L_1, L_2], \quad \hat{\mathcal{A}} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad \text{and} \quad \hat{\mathcal{S}} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \]

From the latter we can conclude that the compensator given by

\[
\begin{align*}
\hat{\psi}_1(t) &= \begin{bmatrix} N_{22} & M_2 \end{bmatrix} \hat{\psi}_1(t) + \begin{bmatrix} N_{21} \\ L_1 \end{bmatrix} y(t), \\
\hat{\psi}_2(t) &= \begin{bmatrix} L_2 & \hat{R} \end{bmatrix} \hat{\psi}_2(t) + \begin{bmatrix} N_{12} \\ M_1 \end{bmatrix} y(t), \\
u(t) &= \begin{bmatrix} N_{12} & M_1 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1(t) \\ \hat{\psi}_2(t) \end{bmatrix} + N_{11} y(t),
\end{align*}
\]

achieves noninteraction and therefore solves (NICPM2).

Thus, it remains to be shown that the system (4.3) together with the linear subspaces \( \tilde{S}_1, \tilde{S}_2, \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \) satisfy the conditions of Theorem 4.10. To this end, note that

\[ \text{im} \tilde{\mathcal{C}}_1 = (\text{im} \, G_1 \tilde{\mathcal{S}}_1) \subseteq (\hat{\mathcal{S}}_1^* \tilde{\mathcal{S}}_1) = \text{im} \begin{bmatrix} D_1 \\ R \end{bmatrix} [X_0, X_1] \subseteq \hat{\mathcal{S}}_1. \]

Analogously, it can be shown that \( \text{im} \tilde{\mathcal{C}}_2 \subseteq \hat{\mathcal{S}}_2 \), \( \tilde{\nu}_1 \subseteq \ker \hat{\mathcal{A}}_2 \) and \( \tilde{\nu}_2 \subseteq \ker \hat{\mathcal{A}}_1 \). Next, we claim that \( \tilde{\mathcal{S}}_1 \subseteq \tilde{\mathcal{V}}_1 \) and \( \tilde{\mathcal{S}}_2 \subseteq \tilde{\mathcal{V}}_2 \). It is clear that the latter is equivalent to \( T(E_1D_1 + PR)X = 0 \) and \( T(E_2D_2 + PR)X = 0 \). Now, observe that

\[
T(E_2D_2 + PR)X = \begin{bmatrix}
T_0X_0 & 0 & T_0X_2 & T_0D_{23} \\
0 & 0 & \hat{T} & 0 \\
T_2X_0 & 0 & T_2X_2 & T_2D_{23} \\
E_{32}X_0 & 0 & E_{32}X_2 & E_{32}D_{23} + U_3Y_3
\end{bmatrix}.
\]
Since
\[
\begin{bmatrix}
\tau_0 \\
\tau_2
\end{bmatrix}
D_2 X = 0 , \quad TL_2 [X_0, X_2] = 0 \quad \text{and} \quad E_{12} D_{23} + U_3 Y_3 = 0
\]
it follows that \( T(E_2 D_2 + PR) X = 0 \). The latter also implies that
\[
Q_1 T(E_2 D_2 + PR) X = \tilde{T} (E_2 D_2 + PR) \tilde{X} = 0 ,
\]
where we have used \( \tilde{T} \), \( \tilde{X} \), \( \tilde{Q} \) and \( \tilde{Q} \) as introduced in Remark 4.17. Note that
\( RX = \tilde{R} X \), \( TP = \tilde{T} P \), \( D_1 X + D_2 X = D_1 \tilde{X} + D_2 \tilde{X} = X \) and \( TL_1 + TL_2 = \tilde{T} L_1 + TL_2 = T \).
Then, it follows that
\[
0 = \tilde{T} (E_2 D_2 + PR) \tilde{X} = \tilde{Y} (1 - D_1 - E_1) X + T(E_1 D_1 + PR) X .
\]
Since \( T(D_1 + E_1 - 1) X = 0 \), we obtain that \( T(E_1 D_1 + PR) X = 0 \). So, \( \tilde{S}_1 \subseteq \tilde{U}_1 \) and \( \tilde{S}_2 \subseteq \tilde{U}_2 \).
Next, we claim that the subspaces \( \tilde{S}_1 \), \( \tilde{S}_2 \) and \( \tilde{S}_1 + \tilde{S}_2 \) are \( (C, \tilde{A}) \)-invariant subspaces in \( W^{D+V} \). Indeed, because the matrix \( Y_3 \) is injective, we have
\[
\begin{align*}
\tilde{A}(\tilde{S}_1 \cap \ker \tilde{C}) &= \tilde{A} \left( \text{im} \begin{bmatrix} X_0 & X_1 & D_1  \\
0 & 0 & Y_3 \end{bmatrix} \cap \ker \tilde{C} \right) = \tilde{A} \left( \text{im} \begin{bmatrix} X_0 \end{bmatrix} \cap \ker \tilde{C} \right) = \\
&= \tilde{A}(\text{im} \begin{bmatrix} X_0 \end{bmatrix} \cap \ker \tilde{C}) \cap \tilde{0} 0) = \\
&= (\lambda(S_1^* \cap \ker \tilde{C}) \cap \tilde{0}) \subseteq (S_1^* \cap \tilde{0}) \subseteq \tilde{S}_1 ,
\end{align*}
\]
and
\[
\begin{align*}
\tilde{A}(\tilde{S}_1 + \tilde{S}_2) \cap \ker \tilde{C} &= \tilde{A} \left( \text{im} \begin{bmatrix} X_0 & X_1 & X_2 & X_3 & D_1  \\
0 & 0 & Y_3 & 0 & 0 \end{bmatrix} \cap \ker \tilde{C} \right) = \\
&= \tilde{A} \left( \text{im} \begin{bmatrix} X_0 & X_1 & X_2 & X_3 \end{bmatrix} \cap \ker \tilde{C} \right) = \\
&= \tilde{A}(\text{im} \begin{bmatrix} X_0, X_1, X_2, X_3 \end{bmatrix} \cap \ker \tilde{C}) \cap \tilde{0} = \\
&= \tilde{A}(S_1^* \cap \ker \tilde{C}) \cap \tilde{0}) = \\
&= (\lambda(S_1^* \cap \ker \tilde{C}) \cap \tilde{0}) \subseteq (S_1^* \cap \tilde{0}) \subseteq \tilde{S}_1 + \tilde{S}_2 .
\end{align*}
\]
Hence, $S_1$, $S_2$ and $S_1 + S_2$ are $(\mathcal{C}, \mathcal{A})$-invariant subspaces in $\mathbb{R}^{n+x}$. Dually, we can prove that $V_1$, $V_2$ and $V_1 \cap V_2$ are $(\mathcal{A}, \mathcal{B})$-invariant subspaces in $\mathbb{R}^{n+x}$.

Finally, we have to prove that

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} ((S_1 \oplus S_2) \cap \ker [\mathcal{C}, \mathcal{C}]) \subseteq (V_1 \oplus V_2) + \im \begin{bmatrix} \hat{B} \\ \hat{B} \end{bmatrix}.$$ 

To do this, denote

$$\tilde{\mathcal{A}}_0 := \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}, \quad \tilde{\mathcal{B}}_0 := \begin{bmatrix} \hat{B} \\ \hat{B} \end{bmatrix}, \quad \tilde{\mathcal{C}}_0 := [\mathcal{C}, \mathcal{C}],$$

$$\tilde{S} := \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

and

$$\mathcal{T} := \begin{bmatrix} \mathcal{T}E_1 & \mathcal{T}P & 0 & 0 \\ 0 & 0 & \mathcal{T}E_2 & -\mathcal{T}P \end{bmatrix}.$$ 

Then the above inclusion reads $\tilde{\mathcal{A}}_0 (\im \tilde{S} \cap \ker \tilde{\mathcal{C}}_0) \subseteq \ker \mathcal{T} + \im \tilde{\mathcal{B}}_0$.

It is easy to see that the latter subspace inclusion is equivalent to $\tilde{\mathcal{A}}_1 \ker \tilde{\mathcal{C}}_1 \subseteq \im \tilde{\mathcal{B}}_1$, where

$$\tilde{\mathcal{A}}_1 := \tilde{\mathcal{T}}_0 \tilde{\mathcal{A}}_0 \tilde{\mathcal{S}} = \begin{bmatrix} \mathcal{T}E_1 A_1 & 0 \\ 0 & -\mathcal{T}E_2 A_2 \end{bmatrix}, \quad \tilde{\mathcal{B}}_1 := \tilde{\mathcal{T}}_0 \tilde{\mathcal{B}}_0 = \begin{bmatrix} \mathcal{T}E_1 B & \mathcal{T}P \\ \mathcal{T}E_2 B & -\mathcal{T}P \end{bmatrix}.$$ 

and

$$\tilde{\mathcal{C}}_1 := \tilde{\mathcal{C}}_0 \tilde{\mathcal{S}} = \begin{bmatrix} CD_1 X & CD_2 X \\ 0 & 0 \end{bmatrix}. $$

Now, let

$$\tilde{U}_1 := \begin{bmatrix} I & 0 \\ Q_X & I \end{bmatrix}$$

and

$$\tilde{V}_1 := \begin{bmatrix} I & QT \\ 0 & I \end{bmatrix}.$$ 

Then $\tilde{\mathcal{A}}_1 \ker \tilde{\mathcal{C}}_1 \subseteq \im \tilde{\mathcal{B}}_1$ if and only if $\tilde{\mathcal{A}}_2 \ker \tilde{\mathcal{C}}_2 \subseteq \im \tilde{\mathcal{B}}_2$, where $\tilde{\mathcal{A}}_2 = \tilde{\mathcal{V}}_1 \tilde{\mathcal{A}}_1 \tilde{\mathcal{U}}_1$, $\tilde{\mathcal{B}}_2 = \tilde{\mathcal{V}}_1 \tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_1 \tilde{\mathcal{U}}_1$.

Straightforward calculation using $\mathcal{T}P = \mathcal{T}P$, $\mathcal{R}X = \mathcal{R}X$, $D_1 X + D_2 \mathcal{X} = X$ and $\mathcal{T}E_1 + \mathcal{T}E_2 = \mathcal{T}$, shows that (see also the (only if)-part)
\[
\tilde{\lambda}_2 = \begin{bmatrix}
T(\mathbf{A}_1 + \mathbf{E}_1 \mathbf{A} - \mathbf{A}) & -TE_2 \mathbf{A} \\
-TE_2 \mathbf{A} & -TE_2 \mathbf{A}
\end{bmatrix},
\]
\[
\tilde{\mathbf{v}}_2 = \begin{bmatrix}
\mathbf{T} \mathbf{B} \\
\mathbf{E}_2 \mathbf{B} - \mathbf{T} \mathbf{B}
\end{bmatrix}
\quad \text{and} \quad
\tilde{\mathbf{c}}_2 = \begin{bmatrix}
\mathbf{C} \\
\mathbf{G} \mathbf{D} \mathbf{x} \mathbf{X}
\end{bmatrix}.
\]

Recall that
\[
\begin{bmatrix}
\mathbf{D}_2 \mathbf{x} \\
-\mathbf{R} \mathbf{x}
\end{bmatrix} = \begin{bmatrix}
\mathbf{x}_0 & \mathbf{x}_2 & \mathbf{D}_2 \mathbf{x} \\
0 & 0 & 0 & -\mathbf{y}_3
\end{bmatrix}
\]

where the matrix \( \mathbf{y}_3 \) is injective.

Note that any vector in \( \text{ker} \tilde{\mathbf{v}}_2 \) consists of 8 'components'.

Because of the injectivity of the matrix \( \mathbf{y}_3 \) the 8-th component of any vector in \( \text{ker} \tilde{\mathbf{v}}_2 \) is zero. This means that the 8-th column of \( \tilde{\lambda}_2 \) does not play a role in the description of \( \tilde{\lambda}_2 \) \( \text{ker} \tilde{\mathbf{v}}_2 \).

Furthermore, it follows that the 6-th component of any vector in \( \text{ker} \tilde{\mathbf{v}}_2 \) can be chosen completely arbitrary because the 6-th component of \( \tilde{\lambda}_2 \) is a zero column. However, the contribution of this 6-th component to the description of \( \tilde{\lambda}_2 \) \( \text{ker} \tilde{\mathbf{v}}_2 \) is annihilated because the 6-th component of \( \tilde{\lambda}_2 \) is a zero column. Hence, we can delete the 6-th as well as the 8-th column in both \( \tilde{\lambda}_2 \) and \( \tilde{\mathbf{v}}_2 \) and still have a description of the subspace \( \tilde{\lambda}_2 \) \( \text{ker} \tilde{\mathbf{v}}_2 \).

By dual argumentation it can be shown that we can delete the 6-th as well as the 8-th row in both \( \tilde{\lambda}_2 \) and \( \tilde{\mathbf{v}}_2 \) and still be left with a subspace inclusion equivalent to \( \tilde{\lambda}_2 \) \( \text{ker} \tilde{\mathbf{v}}_2 \subseteq \text{im} \tilde{\mathbf{B}}_2 \).

In view of all this, denote
\[
\tilde{\lambda}_3 := \begin{bmatrix}
T(\mathbf{A}_1 + \mathbf{E}_1 \mathbf{A} - \mathbf{A}) & -TE_2 \mathbf{A} \\
-TE_2 \mathbf{A} & -TE_2 \mathbf{A}
\end{bmatrix},
\quad
\tilde{\mathbf{v}}_3 := \begin{bmatrix}
\mathbf{T} \mathbf{B} \\
\mathbf{T}' \mathbf{B}
\end{bmatrix}
\quad \text{and} \quad
\tilde{\mathbf{c}}_3 := \{\mathbf{C} \mathbf{X}, \mathbf{G} \mathbf{D} \mathbf{x} \mathbf{X}')
\]

where
\[
\mathbf{x}' := [\mathbf{x}_0, \mathbf{x}_2] \quad \text{and} \quad \mathbf{T}' := \begin{bmatrix}
\mathbf{T}_0 \\
\mathbf{T}_2
\end{bmatrix}.
\]

We then have \( \tilde{\lambda}_2 \) \( \text{ker} \tilde{\mathbf{v}}_2 \subseteq \text{im} \tilde{\mathbf{B}}_2 \) if and only if \( \tilde{\lambda}_3 \) \( \text{ker} \tilde{\mathbf{v}}_3 \subseteq \text{im} \tilde{\mathbf{B}}_3 \).
Let \( Q'_X \) and \( Q'_T \) be matrices of suitable dimensions such that \( XQ'_X = X' \), \( Q'_T T = T' \), and let
\[
\tilde{u}_3 := \begin{bmatrix} 1 & -Q'_X \\ 0 & 1 \end{bmatrix}, \quad \tilde{v}_3 := \begin{bmatrix} 1 \\ -Q'_T \end{bmatrix}.
\]

Then, \( \tilde{u}_3 \ker \tilde{C}_3 \subseteq \tilde{E}_3 \) if and only if \( \tilde{u}_4 \ker \tilde{C}_4 \subseteq \tilde{E}_4 \). Here, \( \tilde{u}_4 := \tilde{v}_3 \tilde{u}_3 \), \( \tilde{B}_4 := \tilde{v}_3 \tilde{B}_3 \) and \( \tilde{C}_4 := \tilde{C}_3 \tilde{u}_3 \).

Again, straightforward calculation shows that
\[
\tilde{A}_4 = \begin{bmatrix} T(AD_1 + E_1 A - A) X & TAX'' \\ T''AX & 0 \end{bmatrix}, \quad \tilde{B}_4 = \begin{bmatrix} TB \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{C}_4 = [CX, 0]
\]
where
\[
X'' := [X_0, 0] \quad \text{and} \quad T'' := \begin{bmatrix} T_0 \\ 0 \end{bmatrix}.
\]

Because of the structure of the matrices \( \tilde{A}_4, \tilde{B}_4 \) and \( \tilde{C}_4 \) it follows that \( \tilde{A}_4 \ker \tilde{C}_4 \subseteq \tilde{B}_4 \) is equivalent to \( T(AD_1 + E_1 A - A) X \ker CX \subseteq \ker TB \), \( \ker CX \subseteq \ker T''AX \) and \( \ker TAX'' \subseteq \ker TB \).

Translated into the original spaces, the latter 3 conditions mean that
\[
(AD_1 + E_1 A - A)(S^* \cap \ker C) \subseteq V^* + \ker B,
\]
\[
A(S^* \cap \ker C) \subseteq V_1^* + V_2^* \quad \text{and} \quad A(S_1^* \cap S_2^*) \subseteq V^* + \ker B.
\]

At this point, we have proved that the subspace inclusion
\[
\begin{bmatrix} \tilde{A} & 0 \\ 0 & -\tilde{A} \end{bmatrix} \left( (\tilde{S}_1 \cap \tilde{S}_2) \cap \ker [C, C] \right) \subseteq (\tilde{v}_1 + \tilde{v}_2) \otimes \begin{bmatrix} \tilde{B} \\ \tilde{B} \end{bmatrix}
\]
is equivalent to
\[
(AD_1 + E_1 A - A)(S^* \cap \ker C) \subseteq V^* + \ker B,
\]
\[
A(S^* \cap \ker C) \subseteq V_1^* + V_2^* \quad \text{and} \quad A(S_1^* \cap S_2^*) \subseteq V^* + \ker B.
\]
Now, of the latter three, the first subspace inclusion holds by assumption, while the second and third subspace inclusion hold because it can be shown that \( S^* \subseteq V_1^* + V_2^* \) and \( S_1^* \cap S_2^* \subseteq V^* \). Indeed,

\[
S^* = (D_1 + D_2 - (D_1 + D_2 - 1)S^* \subseteq V_1^* + D_2S^* + (D_1 + D_2 - 1)S^* \subseteq V_1^* + V_2^* + (D_1S_1^* \cap D_2S_2^*) \subseteq V_1^* + V_2^*.
\]

By dual arguments it can be shown that \( S_1^* \cap S_2^* \subseteq V^* \).

Hence, the conditions of Theorem 4.10 are fulfilled and we have completed the proof of the (if)-part of Theorem 4.16.

\[\square\]

4.5. Special cases

In the present section we study two special cases of (NICPM2). In the first special case we restrict ourselves to state feedback and in the second special case we assume that the two exogenous outputs of the linear system (4.1) satisfy a so-called output complete assumption.

1) State feedback, \( C = 1 \)

In this case, it is clear that \( \ker C = 0 \), \( S_1^* = \im G_1 \), \( S_2^* = \im G_2 \) and \( S^* = \im [G_1, G_2] \). Then, by Theorem 4.16 we have that (NICPM2) with \( C = 1 \) is solvable if and only if there exist pairs of matrices

\[
(D_1, D_2) \in \Phi(\im G_1, \im G_2, \im [G_1, G_2]) \text{ and } (E_1, E_2) \in \Phi(V_1^*, V_2^*, V^*)
\]

such that \( D_1 \im [G_1, G_2] \subseteq V_1^*, D_2 \im [G_1, G_2] \subseteq V_2^*, E_1 \im G_1 \subseteq V_1^*, E_2 \im G_2 \subseteq V_2^* \) and \( (D_1 + E_1 - 1) \im [G_1, G_2] \subseteq V^* \).

Because \( (D_1 - 1) \im G_1 = 0 \) and \( (D_2 - 1) \im G_2 = 0 \), the solvability of (NICPM2) implies that \( \im G_1 = D_1 \im G_1 \subseteq V_1^* \) and \( \im G_2 = D_2 \im G_2 \subseteq V_2^* \). Furthermore, it follows that \( E_1 \im G_1 \subseteq V_1^* \) and \( E_2 \im G_2 \subseteq V_2^* \). Hence, \( (E_1 + E_2) \im G_1 \subseteq V^* \) or, equivalently,

\[
\im G_1 \subseteq (E_1 + E_2)^{-1} V^* := \{ x \in \mathbb{R}^n \mid (E_1 + E_2) x \in V^* \}.
\]
Because \((E_1^T, E_2^T) \in \Phi(V_1^\perp, V_2^\perp, V^\perp)\), it can be shown that \((E_1 + E_2)^{-1} V^* = V^*\).

Thus, if (NICPM2) is solvable and \(C = I\), then \(G_1 \subseteq V_1^*\), \(G_2 \subseteq V_2^*\) and \(G_1 \cap G_2 \subseteq V^*\).

Conversely, let \(\text{im } G_1 \subseteq V_1^*, \text{ im } G_2 \subseteq V_2^*\) and \(\text{im } G_1 \cap \text{ im } G_2 \subseteq V^*\). Let \([\overline{G}_0, \overline{G}_1, \overline{G}_2]\) be an injective matrix such that \(\text{im } \overline{G}_0 = \text{ im } G_1 \cap \text{ im } G_2\), \(\text{im } [\overline{G}_0, \overline{G}_1] = \text{ im } G_1\) and \(\text{im } [\overline{G}_0, \overline{G}_2] = \text{ im } G_2\). Furthermore, let \(V_1^*, V_2^*\) and \(V^*\) be represented as indicated in Remark 4.17. Then

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
\begin{bmatrix}
[\overline{G}_0, \overline{G}_1] = 0 \\
[\overline{G}_0, \overline{G}_2] = 0 \\
[\overline{G}_0, \overline{G}_1] = 0 \\
[\overline{G}_0, \overline{G}_2] = 0
\end{bmatrix}
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
\begin{bmatrix}
\overline{C}_0 = 0
\end{bmatrix}
\]

Now, let \(E_{31}\) be a matrix of suitable dimensions such that \(E_{31}[\overline{C}_0, \overline{C}_1, \overline{C}_2] = [0, T_3 \overline{C}_2]\) and let \(E_{32} := T_3 - E_{31}\). Furthermore, let \(D_1, D_2, E_1, E_2 \in \mathbb{R}^{n \times n}\) be matrices such that

\[
D_1[\overline{C}_0, \overline{C}_1, \overline{C}_2] = [\overline{C}_0, \overline{C}_1, 0], \quad D_2[\overline{C}_0, \overline{C}_1, \overline{C}_2] = [\overline{C}_0, 0, \overline{C}_2],
\]

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
\begin{bmatrix}
E_1 = \begin{bmatrix}
T_0 \\
T_1 \\
0 \\
T_3
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
\begin{bmatrix}
E_2 = \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
E_{31}
\end{bmatrix}
\end{bmatrix}
\]

Then, \((D_1, D_2) \in \Phi(\text{im } G_1, \text{ im } G_2, \text{ im } [G_1, G_2]), (E_1^T, E_2^T) \in \Phi(V_1^\perp, V_2^\perp, V^\perp)\), \(D_1 \text{ im } [G_1, G_2] \subseteq V^*\), \(D_2 \text{ im } [G_1, G_2] \subseteq V^*\), \(E_1 \text{ im } G_1 \subseteq V^*\), \(E_2 \text{ im } G_2 \subseteq V^*\) and \((D_1 + E_1 - I) \text{ im } [G_1, G_2] \subseteq V^*\). Hence, (NICPM2) with \(C = I\) is solved.

Thus, we have proved the following (see also Corollary 3.15.1).

**Corollary 4.18:**

(NICPM2) with \(C = I\) is solvable if and only if

\[
\text{im } G_1 \subseteq V_1^*, \quad \text{im } G_2 \subseteq V_2^* \quad \text{and} \quad \text{im } G_1 \cap \text{ im } G_2 \subseteq V^*.
\]
2) Outputs complete, \( \ker H_1 \cap \ker H_2 = 0 \) (cf. Wonham (1979))

It is clear that in this case \( \mathcal{V}_1^* \cap \mathcal{V}_2^* = 0 \). Assume there exist pairs of matrices \((D_1,D_2) \in \Phi(\mathcal{S}_1^*,\mathcal{S}_2^*,\mathbb{R}^n)\) and \((E_1^T,E_2^T) \in \Phi(\mathcal{V}_1^*,\mathcal{V}_2^*,\mathbb{R}^n)\) such that 
\[
D_1 S_1^* \subseteq \mathcal{V}_1^*, \quad D_2 S_2^* \subseteq \mathcal{V}_2^*, \quad E_1 S_1 = 0, \quad E_2 S_2 = 0, \quad (D_1 + E_1 - I) S^* = 0 \quad \text{and} \quad (AD_1 + E_1 A - A) (S^* \cap \ker C) \subseteq \text{im} \ B.
\]

Let 
\[
[T_0] \\
[T_1] \\
[T_2]
\]

be a square invertible matrix such that 
\[
\ker T_0 = \mathcal{V}_1^* \cap \mathcal{V}_2^*, \quad \ker \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = \mathcal{V}_1^* \quad \text{and} \quad \ker \begin{bmatrix} T_0 \\ T_2 \end{bmatrix} = \mathcal{V}_2^*.
\]

Then, 
\[
\begin{bmatrix} T_0 \\ T_1 \end{bmatrix} \begin{bmatrix} D_1 \\ E_1 \end{bmatrix} = \begin{bmatrix} T_0 \\ T_1 \end{bmatrix}, \quad \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} \begin{bmatrix} D_2 \\ E_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Thus, \( E_1 S_1^* = 0 \) and \( E_2 S_2^* = 0 \) if and only if \( S_1^* \subseteq \mathcal{V}_1^* \) and \( S_2^* \subseteq \mathcal{V}_2^* \). Therefore, since \( \mathcal{V}_1^* \cap \mathcal{V}_2^* = 0 \), also \( S_1^* \cap S_2^* = 0 \). Let \([X_1,X_2,X_3]\) be an injective matrix such that \( \text{im} X_1 = S_1^* \), \( \text{im} X_2 = S_2^* \) and \( \text{im} [X_1,X_2,X_3] = S^* \). Then, 
\[
D_1 [X_1,X_2,X_3] = [X_1,0,D_{13}] \quad \text{and} \quad D_2 [X_1,X_2,X_3] = [0,X_2,D_{23}]
\]

with \( D_{13} + D_{23} = X_3 \). Because \( D_1 S_1^* \subseteq \mathcal{V}_1^* \) and \( D_2 S_2^* \subseteq \mathcal{V}_2^* \) it follows that 
\[
S^* = (D_1 + D_2) S^* \subseteq D_1 S^* + D_2 S^* \subseteq \mathcal{V}_1^* + \mathcal{V}_2^*.
\]

Also, it follows that \( \text{im} D_{13} \subseteq D_1 S^* \subseteq \mathcal{V}_1^* \) and \( \text{im} D_{23} \subseteq D_2 S^* \subseteq \mathcal{V}_2^* \). From the latter it follows that, given the representation of \( S^* \) as described above, the matrices \( D_{13} \) and \( D_{23} \) are uniquely determined because \( \mathcal{V}_1^* \cap \mathcal{V}_2^* = 0 \).

In fact, the following holds. If \( Q_1, Q_2 \in \mathbb{R}^{n \times n} \) are matrices representing the projection of \( \mathcal{V}_1^* \cap \mathcal{V}_2^* \) onto \( \mathcal{V}_1^* \) along \( \mathcal{V}_2^* \) and the projection of \( \mathcal{V}_1^* \cap \mathcal{V}_2^* \) onto \( \mathcal{V}_2^* \) along \( \mathcal{V}_1^* \), respectively, then \( Q_1 [X_1,X_2,X_3] = [X_1,0,D_{13}] \) and 
\[
Q_2 [X_1,X_2,X_3] = [0,X_2,D_{23}].
\]

So, we can replace the pair \((D_1,D_2)\) by the pair \((Q_1,Q_2)\).
Now, let $R \in \mathbb{R}^{n \times n}$ be a matrix determined by $R[X_1, X_2, X_3] = [0, 0, X_3]$ and let $S'_1, S'_2$ be linear subspaces in $\mathbb{R}^n$ defined by

$$S'_1 := \text{im} \begin{bmatrix} Q_1 X \\ R X \end{bmatrix} \quad \text{and} \quad S'_2 := \text{im} \begin{bmatrix} Q_2 X \\ -R X \end{bmatrix},$$

where $X := [X_1, X_2, X_3]$.

Then, completely analogously as in the proof of the (only if)-part of Theorem 4.16, we can show that $(AD + E, A - H)(S^* \cap \ker C) \subseteq \text{im } R$ if and only if

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & -A & 0 \end{bmatrix} (S' \oplus S'_2) \cap \ker \begin{bmatrix} C & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \subseteq (V_1^* \oplus V_2^*) + \text{im } \begin{bmatrix} B \\ B \end{bmatrix}.$$

Thus we have proved the (only-if)-part of the following Corollary 4.19:

**Corollary 4.19:**

Consider the linear system (4.1). Assume that $\ker H_1 \cap \ker H_2 = 0$. Then (NICPM2) is solvable if and only if

$$S'_1 \subseteq V_1^*, \ S'_2 \subseteq V_2^*, \ S^* \subseteq V_1^* + V_2^*$$

and

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & -A & 0 \end{bmatrix} (S'_1 \oplus S'_2) \cap \ker \begin{bmatrix} C & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \subseteq (V_1^* \oplus V_2^*) + \text{im } \begin{bmatrix} B \\ B \end{bmatrix},$$

where $S'_1$ and $S'_2$ are as defined above.

A proof of the (if)-part of the above corollary can be given along the lines of the proof of the (if)-part of Theorem 4.16.

### 4.6. (Almost) noninteracting control and (almost) diagonal transfer preservation by measurement feedback

We shall conclude the present chapter by considering two control problems that are modifications of the two control problems studied in this chapter. Again, we assume that the linear system (4.1) is controlled by the measurement feedback compensator (1.9), resulting in the closed loop system (4.2).
We recall that for the closed loop system the following relations hold:

\[ T_{ij} = K_{ij} + L_i X M_j, \quad (i, j = 1, 2), \]

where \( X = (I - FP)^{-1} F. \) Here, \( K_{ij} \) and \( T_{ij} \) denote the transfer matrix between the \( j \)-th exogenous input and the \( j \)-th exogenous output in the to-be controlled system (4.1) and the closed loop system (4.2), respectively.

**Definition 4.20:**

The non-interacting control problem by measurement feedback with diagonal transfer preservation (NICPMADTP2), consists of finding a measurement feedback compensator (1.9) such that in the closed loop system (4.2) \( T_{12} = 0, T_{21} = 0, T_{11} = K_{11} \) and \( T_{22} = K_{22}. \)

**Definition 4.21:**

The almost non-interacting control problem by measurement feedback with almost diagonal transfer preservation, (ANICPMADTP2), consists of finding, for all \( \epsilon > 0, \) a measurement feedback compensator (1.9) such that in the closed loop system (4.2) \( \|T_{12}\|_\omega < \epsilon, \|T_{21}\|_\omega < \epsilon, \|T_{11} - K_{11}\|_\omega < \epsilon \) and \( \|T_{22} - K_{22}\|_\omega < \epsilon. \)

By Theorem 3.4 and the previous results of this chapter, the next corollary is immediate.

**Corollary 4.22:**

1) (NICPMADTP2) is solvable if and only if there exists a proper rational matrix \( X \in \mathbb{R}^{mp}(s) \) such that \( K_{12} + L_1 X M_2 = 0, K_{21} + L_2 X M_1 = 0, L_1 X M_1 = 0 \) and \( L_2 X M_2 = 0. \)

2) (ANICPMADTP2) is solvable if and only if there exists a rational matrix \( X \in \mathbb{R}^{mp}(s) \) such that \( K_{12} + L_1 X M_2 = 0, K_{21} + L_2 X M_1 = 0, L_1 X M_1 = 0 \) and \( L_2 X M_2 = 0. \)

Define

\[ \bar{A} := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} B \\ B \end{bmatrix}, \quad \bar{C} := [C, C]. \]
\[ \mathcal{G} := \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \quad \text{and} \quad \mathcal{H} := \begin{bmatrix} H_2 & 0 \\ 0 & H_1 \end{bmatrix}, \]

and observe that

\[
\mathcal{H}(sI - \mathcal{A})^{-1} \mathcal{B} \mathcal{X}(s)(sI - \mathcal{A})^{-1} \mathcal{G} + \mathcal{H}(sI - \mathcal{A})^{-1} \mathcal{G} = \\
\begin{bmatrix} L_2(s) \\ L_1(s) \end{bmatrix} \mathcal{X}(s) \left[ \begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix} \right] + \begin{bmatrix} K_{21}(s) & 0 \\ 0 & K_{12}(s) \end{bmatrix}.
\]

Theorem 4.23:

1) (NICPMOTP2) is solvable if and only if

\[ \mathcal{S}^*(\text{im } \mathcal{G}; \mathcal{A}, \mathcal{C}) \subseteq \mathbb{V}^*(\ker \mathcal{H}; \mathcal{A}, \mathcal{B}) . \]

2) (ANICPMOTP2) is solvable if and only if

\[ \mathcal{S}_0^*(\text{im } \mathcal{G}; \mathcal{A}, \mathcal{C}) \subseteq \ker \mathcal{H} \quad \text{and} \quad \text{im } \mathcal{G} \subseteq \mathbb{V}_b^*(\ker \mathcal{H}; \mathcal{A}, \mathcal{B}) . \]

Proof:

By the previous remark, it is clear that (NICPMOTP2) is solvable if and only if there exists a proper rational matrix \( \mathcal{X} \in \mathbb{H}_0^{\text{mxp}}(s) \) such that

\[ \mathcal{H}(sI - \mathcal{A})^{-1} \mathcal{B} \mathcal{X}(s)(sI - \mathcal{A})^{-1} \mathcal{G} + \mathcal{H}(sI - \mathcal{A})^{-1} \mathcal{G} = 0 . \]

Then, application of Theorem 2 of Ohms, Bhattacharyya and Howze (1984), Proposition 1.25 and 1.32 completes the proof of part 1) of this theorem. Part 2) can be proved in an analogous way using Lemma 4.4 and the proof of the equivalence of the second and third statement of Theorem 4.7. See also Willems (1982), Theorem 4.
CHAPTER 5
(ALMOST) TRIANGULAR DECOUPLING

In this chapter we shall be dealing with a number of control problems that arise in the context of triangular decoupling. The linear systems that we consider in the present chapter have the same structure as the linear systems that we have considered in Chapter 3. This means that we assume that the systems, apart from a control input and a measurement output, have \( u \) exogenous inputs and \( v \) exogenous outputs, where \( u \) is an integer larger than one.

In Chapter 3 we saw that controlling such a system by means of the type of measurement feedback compensator introduced in Section 1.5, results in a closed loop system that has \( u \) exogenous inputs and \( v \) exogenous outputs. Hence, the transfer matrix of the closed loop system can be partitioned according to the dimensions of the exogenous inputs and outputs as a \( u \times v \) block matrix.

The main control objective of Chapter 3 was finding measurement feedback compensators such that the off-diagonal blocks in the transfer matrix of the closed loop system are zero or have an arbitrarily small norm. In contrast with Chapter 3, in the present chapter we will be content if we can find measurement feedback compensators such that the blocks of the transfer matrix of the closed loop system that are positioned above the block diagonal are zero or have arbitrarily small norm. If such measurement feedback compensators exist, we say that it is possible to achieve triangular decoupling or almost triangular decoupling, respectively.

Problems concerning triangular decoupling have been studied before in the literature (Morse and Wonham (1971), Wonham (1979)). However, the point of view on triangular decoupling as exposed in these references is completely different from our point of view on triangular decoupling. Indeed, in contrast to the systems that we consider, in the above references it is assumed that the systems only have a control input and \( u \) exogenous outputs.
Figure 6

Figure 7

Triangular decoupling $\mu = 5$
The problem of triangular decoupling in the above references then consists of finding a state feedback compensator such that in the closed loop system the feature of triangular decoupling is achieved. In addition, in the closed loop system, certain output controllability requirements should be satisfied. All this means that, in contrast to the type of compensator that we apply, the type of compensator used in the above references has, in addition to an output serving as a control input for the system, $u + 1$ inputs. Of these $u + 1$ inputs, one is the state of the system, while the remaining $u$ inputs serve as the exogenous inputs for the closed loop system.

The outline of the present chapter is as follows. In Section 5.1 the problem described above concerning triangular decoupling and almost triangular decoupling in our context are formulated mathematically. In Section 5.2 we start with some preliminary results. Then, using these results, we derive necessary and sufficient conditions for the existence of measurement feedback compensators that achieve the feature of triangular decoupling and, in addition, stabilize the closed loop system. In Section 5.3 we also start with a preliminary result. From this result, necessary and sufficient conditions for the existence of measurement feedback compensators that achieve the feature of almost triangular decoupling can be derived immediately. We note that all the necessary and sufficient conditions that we derive for the solvability of the various problems of this chapter can be verified computationally.

5.1. Problem formulations

As announced, in this chapter we consider the linear system described by:

\[(3.1a) \quad \dot{x}(t) = Ax(t) + Bu(t) + \sum_{i \in \mu} G_i y_i(t), \]

\[(3.1b) \quad y(t) = Cx(t), \]

\[(3.1c) \quad y_i(t) = H_i x(t), \quad (i \in \nu), \]
and we assume that this system is controlled by means of a measurement feedback compensator given by:

\[(1.9a) \quad \dot{w}(t) = Kw(t) + Ly(t),\]
\[(1.9b) \quad u(t) = Mw(t) + Ny(t).\]

The resulting closed loop system is then described by:

\[(3.2a) \quad \dot{x}_c(t) = A_c x_c(t) + \sum_{i \in \mathcal{H}} C_{i,c} y_i(t),\]
\[(3.2b) \quad z_i(t) = H_{i,c} x_c(t), \quad (i \in \mathcal{U}).\]

Again, we let \(T\) denote the transfer matrix of the closed loop system \((3.2)\) and we partition \(T = (T_{ij}), (i,j \in \mathcal{U})\), where \(T_{ij}(s) := H_{i,c}(sI - A_c)^{-1} G_{j,c} G_{i,c}\).

In Chapter 3 we have established the following relations to hold for the closed loop system \((3.2)\):

\[T_{ij} = K_{ij} + L_i X M_j, \quad (i,j \in \mathcal{U})\]

where \(K = (I - F)^{-1} F\).

Assume that \(\mathcal{C}_g \subset \mathcal{E}\) is a given stability region, then we shall consider the following control problems:

**Definition 5.1:**
The triangular decoupling problem by measurement feedback, abbreviated \((\text{TDPMu})\), consists of finding a measurement feedback compensator \((1.9)\) such that in the closed loop system \((3.2)\) \(T_{ij} = 0\) for all \(i,j \in \mathcal{U}\) with \(i < j\).

If, in addition to the feature of triangular decoupling, we require internal stabilization we obtain:

**Definition 5.2:**
The triangular decoupling problem by measurement feedback with internal stabilization, abbreviated \((\text{TDPMu})'\), consists of finding a measurement feedback compensator \((1.9)\) such that in the closed loop system \((3.2)\) \(\sigma(A_c) \subseteq \mathcal{C}_g\) and \(T_{ij} = 0\) for all \(i,j \in \mathcal{U}\) with \(i < j\).
If we do not require exact triangular decoupling but are content when we can achieve the feature of triangular decoupling to any desired degree of accuracy, we arrive at the following:

Definition 5.3:
The almost triangular decoupling problem by measurement feedback, abbreviated (ATDPM̃p), consists of finding, for all $\epsilon > 0$, a measurement feedback compensator $(1.9)$ such that in the closed loop system $(3.2)$ $\|T_{ij}\|_\infty \leq \epsilon$

for all $i, j \in \mathbb{P}$ with $i < j$.

Analogously as in Corollary 3.3 and Corollary 2.7 we have:

Corollary 5.4:

1) (TDPM̃p) is solvable if and only if there exists a rational matrix $X \in \mathbb{B}_{\mathbb{R}^{n \times p}}(s)$ such that $k_{ij} + 1_i X m_j = 0$ for all $i, j \in \mathbb{P}$ with $i < j$.

2) (ATDPM̃p) is solvable if and only if there exists a rational matrix $X \in \mathbb{R}_{\mathbb{R}^{n \times p}}(s)$ such that $k_{ij} + 1_i X m_j = 0$ for all $i, j \in \mathbb{P}$ with $i < j$.

5.2. Triangular decoupling by measurement feedback with internal stability

In this present section we shall derive necessary and sufficient conditions formulated in state space terms for the solvability of (TDPM̃p) and (TDPM̃p) as defined in the previous section. However, before stating these conditions we present a preliminary result that will be useful in their derivation. We remark that this preliminary result was already used in Chapters 1 and 2 for the cases $\nu = 1$ and $\nu = 2$, respectively. In order to formulate the result we return to the linear system $(1.1)$ and we assume that $C, G \in \mathbb{O}$ is a given stability region.

Proposition 5.5:
Let the pair $(A, B)$ be stabilizable and the pair $(C, A)$ be detectable.

Let $\{V_i | i \in \mathbb{P}\}$ be a family of stabilizability subspaces in $\mathbb{R}^n$ and $\{S_i | i \in \mathbb{P}\}$ be a family of detectability subspaces in $\mathbb{R}^n$ such that $V_i \subseteq V_{i+1}$ and $S_i \subseteq S_{i+1}$ for all $i \in \mathbb{P}$ and $S_i \subseteq V_i$ for all $i \in \mathbb{P}$. 

Then there exist matrices $F \in \mathbb{R}^{m \times n}$, $J \in \mathbb{R}^{n \times p}$ and $N \in \mathbb{R}^{m \times p}$ such that

$$(A + BF) V_i \subseteq V_i,$$  
$$(A + JC) S_i \subseteq S_i$$  
and  
$$(A + BNC) S_i \subseteq V_i$$

for all $i \in \mu$, and

$$(A + BF) \subseteq \xi,$$

$$(A + JC) \subseteq \xi.$$

**Proof:**

Because $\{V_i \mid i \in \mu\}$ is a family of stabilizability subspaces and the pair $(A, B)$ is stabilizable, there exist matrices $F_1, F_2, \ldots, F_\mu \in \mathbb{R}^{m \times n}$ such that for all $i \in \mu$

$$(A + BF_i) V_i \subseteq V_i$$

and $$(A + BF_i) \subseteq \xi$$

(see section 1.4). Since $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_\mu \subseteq \mathbb{R}^n$, there exists a square invertible matrix $X$ partitioned as $X = \begin{bmatrix} X_1, X_2, \ldots, X_\mu, X_{\mu+1} \end{bmatrix}$ such that for all $i \in \mu$, $V_i = \text{im} \left[ X_1, X_2, \ldots, X_i \right]$. Note that, if $V_j = V_{j+1}$ for some $j \in \mu - 1$, then the matrix $X_{j+1}$ does not appear.

Let $F \in \mathbb{R}^{m \times n}$ be a matrix defined by

$$F = \begin{bmatrix} F_1, F_2, \ldots, F_\mu, F_{\mu+1} \end{bmatrix}.$$

Then it follows that

$$(A + BF) \subseteq \xi$$

and

$$(A + BF) V_i \subseteq V_i$$

for all $i \in \mu$.

Completely dually, we can prove the existence of a matrix $J \in \mathbb{R}^{n \times p}$ such that

$$(A + JC) \subseteq \xi$$

and

$$(A + JC) S_i \subseteq S_i$$

for all $i \in \mu$. The proof of the existence of such a matrix can be accomplished by induction. Again, we note that it is possible that for some $i \in \mu$ the matrix $P_i$ or the matrix $Q_i$ does not appear. Furthermore, we note that ker $C = \text{im} \left[ P_1, P_2, \ldots, P_\mu, P_{\mu+1} \right]$ and the matrix $[CQ_1, CQ_2, \ldots, CQ_\mu]$ is injective.

For all $i \in \mu$ we have

$$\text{im} \left( A P_i \right) = A \text{im} \left( P_i \right) \subseteq A \left( S_i \cap \text{ker} \left( C \right) \right) \subseteq S_i \subseteq V_i$$

and

$$\text{im} \left( A Q_i \right) = A \text{im} \left( Q_i \right) \subseteq A S_i \subseteq A V_i \subseteq V_i + \text{im} \left( B \right).$$

The latter implies the existence of matrices $U_1, U_2, \ldots, U_\mu$ of suitable dimensions such that in $\text{im} \left( A Q_i + B U_i \right) \subseteq V_i$ for all $i \in \mu$. Now, let $N \in \mathbb{R}^{m \times p}$ be a matrix such that $N[CQ_1, CQ_2, \ldots, CQ_\mu] = [U_1, U_2, \ldots, U_\mu]$. It follows that

$$\text{im} \left( A Q_i + B U_i \right) = \text{im} \left( A + BNC \right) Q_i = \left( A + BNC \right) \text{im} \left( Q_i \right) \subseteq V_i$$

for all $i \in \mu$. 

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Furthermore, it follows that

\[(A + B N C) \text{ im } P_i = \text{ im } (A + B N C) P_i = \text{ im } A P_i \subseteq V_i \text{ for all } i \in \mathcal{U} . \]

Hence,

\[(A + B N C) \text{ im } [P_i, Q_i] \subseteq V_i \text{ for all } i \in \mathcal{U} . \]

In fact, \((A + B N C) \text{ im } [P_i, Q_i] \subseteq V_i \) for all \(i, j \in \mathcal{U} \) with \(j \neq i \).

From the latter it is clear that \((A + B N C) S_i \subseteq V_i \) for all \(i \in \mathcal{U} . \)

We are now able to derive computable necessary and sufficient conditions in state space terms for the solvability of the control problems (TDPMu) and (TDPMu'). To this end, let the linear system \((3.1)\) be given, let \(\xi \in \mathcal{U} \) be a given stability region and for all \(i \in \mathcal{U} - 1\) denote:

\[ \tilde{I}^i_k := \{ \frac{g}{g} \} \text{ in } G^i , \quad \tilde{K}^i_k := \cap_{j=1}^i \ker H_j . \]

Then the main result of the present section reads as follows:

**Theorem 5.6:**

1) (TDPMu) is solvable if and only if \(S^*(\tilde{I}^i_k) \subseteq V^*(\tilde{K}^i_k) \) for all \(i \in \mathcal{U} - 1\).

2) (TDPMu') is solvable if and only if \(S^*_g(\tilde{I}^i_k) \subseteq V^*_g(\tilde{K}^i_k) \) for all \(i \in \mathcal{U} - 1\), the pair \((A, B)\) is stabilizable and the pair \((C, A)\) is detectable.

**Proof:**

1) Follows from 2) by specifying \(\xi \rightarrow \xi\).

2) (Only if) Assume that (TDPMu') is solvable. That is, assume there exists a measurement feedback compensator \((1.9)\) such that in the closed loop system \((3.2)\) we have \(\sigma(A_c) \subseteq \xi_g \) and \(T_{ij} = 0 \) for all \(i, j \in \mathcal{U} \) with \(i < j\).

Because \(\sigma(A_c) \subseteq \xi^*_g \), it follows from Theorem 1.44 that the pair \((A, B)\) is stabilizable and the pair \((C, A)\) is detectable.

For all \(i \in \mathcal{U} - 1\), let \(W_{i,c}\) be a linear subspace in \(\mathbb{R}^{n+k}\), the state space of \((3.2)\), defined as

\[ W_{i,c} := \langle A_c \mid \tilde{I}^i_k \otimes 0 \rangle . \]
Then, for all $i \in \mu - 1$

$$A_e \omega_{i,e} \subseteq \omega_{i,e}$$

and

$$\left( \omega_{i,e} \otimes 0 \right) \subseteq \omega_{i,e} \subseteq \left( \omega_{i,e} \otimes \mathbb{R}^k \right).$$

For all $i \in \mu - 1$, let $S_i$ and $V_i$ be linear subspaces in $\mathbb{R}^n$ defined as

$$S_i := \left\{ s \in \mathbb{R}^n \mid \begin{bmatrix} s \\ 0 \end{bmatrix} \in \omega_{i,e} \right\}$$

and

$$V_i := \left\{ v \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^k \begin{bmatrix} v \\ w \end{bmatrix} \in \omega_{i,e} \right\}.$$

Then we have $\widetilde{S}_i \subseteq S_i \subseteq \widetilde{V}_i \subseteq \widetilde{K}_i$ for all $i \in \mu - 1$.

From Theorem 1.44 it follows that the linear subspaces $S_1, S_2, \ldots, S_{\mu - 1}$ are detectability subspaces and the linear subspaces $V_1, V_2, \ldots, V_{\mu - 1}$ are stabilizability subspaces. Hence, by Propositions 1.37 and 1.42 it follows that $S_i^*(\widetilde{S}_i) \subseteq V_i^*(\widetilde{K}_i)$ for all $i \in \mu - 1$.

(if) Note that

$$S_i^*(\widetilde{S}_{i+1}) \subseteq S_i^*(\widetilde{S}_i) \quad \text{and} \quad V_i^*(\widetilde{K}_{i+1}) \subseteq V_i^*(\widetilde{K}_i) \quad \text{for all } i \in \mu - 2.$$

By Proposition 5.5 there exist matrices $F \in \mathbb{R}^{m \times n}$, $J \in \mathbb{R}^{n \times p}$ and $N \in \mathbb{R}^{m \times p}$ such that $\sigma(A + BF) \subseteq \zeta_\delta$, $\sigma(A + JC) \subseteq \zeta_\delta$ and for all $i \in \mu - 1$

$$(A + BF)V_i^*(\widetilde{K}_i) \subseteq V_i^*(\widetilde{K}_i), \quad (A + JC)S_i^*(\widetilde{S}_i) \subseteq S_i^*(\widetilde{S}_i),$$

$$(A + BNC)S_i^*(\widetilde{S}_i) \subseteq V_i^*(\widetilde{K}_i).$$

Now, as in (1.13), let $\hat{A}_e \in \mathbb{R}^{2n \times 2n}$ be the matrix defined as

$$\hat{A}_e := \begin{bmatrix} A+BNC & BF-BNC \\ BNC-JC & A+BF+JC-BNC \end{bmatrix}$$

and let $\omega_{1,e}, \omega_{2,e}, \ldots, \omega_{\mu-1,e}$ be linear subspaces in $\mathbb{R}^{2n}$ defined as

$$\omega_{i,e} := \{ \begin{bmatrix} s \\ 0 \end{bmatrix} \mid \begin{bmatrix} s \\ v \end{bmatrix} \in S_i^*(\widetilde{S}_i), \begin{bmatrix} v \\ w \end{bmatrix} \in V_i^*(\widetilde{K}_i) \} \quad \text{where } i \in \mu - 1.$$
In construction 1.48 we showed that $\sigma(H_{c}) \subseteq \mathbb{C}$, and $H_{c} w_{i, e} \subseteq w_{i, e}$ and $(\mathbf{1}_{i} \otimes 0) \subseteq w_{i, e} \subseteq (\mathbf{1}_{i} \otimes \mathbb{R}^{n})$ for all $i \in \mu^{-1}$. The latter implies that $\sigma(H_{c}) \subseteq \mathbb{C}$ and $(\text{im } H_{c} \otimes 0) \subseteq w_{i, e} \subseteq (\text{ker } H_{c} \otimes \mathbb{R}^{n})$ for all $i, j \in \mu$ with $i < j$, from this we can conclude that $H_{i, e} (\sigma - H_{c})^{-1} G_{i, e} = 0$ for all $i, j \in \mu$ with $i < j$.

5.3. Almost triangular decoupling by measurement feedback

In this last section we shall derive necessary and sufficient conditions in state space terms for the solvability of (ATDFMv) as described in Section 5.1. The conditions that will be obtained are a direct consequence of the result formulated below. In order to be able to state this result, we let $\mathbb{F}$ denote an arbitrary field. We shall use the notational conventions as mentioned in the beginning of Section 4.2.

Let $U_{i} \in \mathbb{F}^{n_{i} \times h}$, $V_{j} \in \mathbb{F}^{k \times d_{j}}$ and $W_{i, j} \in \mathbb{F}^{[d_{i} \times d_{j}]}$ for all $i, j \in \mu^{-1}$ be given matrices. For all $i, j \in \mu^{-1}$, denote

$$A_{i} := \begin{bmatrix} U_{i} \\ V_{i} \\ \vdots \\ U_{i} \end{bmatrix}, \quad B_{i,j} := \begin{bmatrix} W_{i,j} \\ W_{i,j} \\ \vdots \\ W_{i,j} \end{bmatrix},$$

$$\Lambda_{i} := [V_{i}, V_{i+1}, \ldots, V_{j}], \quad \Psi_{i,j} := [W_{i,1}, W_{i,2}, \ldots, W_{i,j}]$$

and

$$\Gamma_{i,j} := \begin{bmatrix} \Psi_{i,j} \\ \Psi_{i,j} \end{bmatrix}^{T} = \begin{bmatrix} W_{i,1} & W_{i,2} & \cdots & W_{i,j} \\ W_{i,1} & W_{i,2} & \cdots & W_{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ W_{i,1} & W_{i,2} & \cdots & W_{i,j} \end{bmatrix}. $$

Then we have:

**Theorem 5.7.**

The following statements are equivalent:

1) There exists a matrix $X \in \mathbb{F}^{h \times k}$ such that $U_{i} X V_{j} = W_{i,j}$ for all $i, j \in \mu^{-1}$ with $i+j \leq n$.
2) For all $i \in \mu^{-1}$ there exists a matrix $X \in \mathbb{R}^{b \times c}$ such that 
$$
\Gamma_{i \mu^{-1}} = \Lambda_i X \Delta_{i \mu^{-1}}.
$$
3) For all $i \in \mu^{-1}$, $im \Delta_i \supseteq im \Gamma_{i \mu^{-1}}$ and $ker \Delta_i \subseteq ker \Gamma_{i \mu^{-1}}$.

Proof:

2) $\Rightarrow$ 3) See Lemma 4.4.

1) $\Rightarrow$ There exists a matrix $X \in \mathbb{R}^{b \times c}$ such that $\Gamma_{i \mu^{-1}} = \Lambda_i X \Delta_{i \mu^{-1}}$ for all $i \in \mu^{-1}$. Hence, 1) $\Rightarrow$ 2).

3) $\Rightarrow$ 1) Observe that for all $j \in \mu^{-1}$, $im \Delta_{j^{-1}} \subseteq im \Delta_j$. Here we have defined $im \Delta_0 := 0$. Therefore, there exist injective matrices $\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_{\mu^{-1}}$ such that $im \tilde{V}_j \subseteq im \tilde{V}_{j+1}$, $im \Delta_j = im \Delta_{j^{-1}}$ and $im \Delta_{\mu^{-1}} \cap im \tilde{V}_j = 0$ for all $j \in \mu^{-1}$.

It follows that $[\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_{\mu^{-1}}]$ is an injective matrix and $im \Delta_j = im [\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_j]$ for all $j \in \mu^{-1}$. Furthermore, there exist square invertible matrices $T_1, T_2, ..., T_{\mu^{-1}}$ such that $\tilde{V}_j T_j = [\tilde{V}_j, \tilde{V}_j]$ with $im \tilde{V}_j \subseteq im \Delta_{j^{-1}}$ for all $j \in \mu^{-1}$. From the latter it follows that for all $j \in \mu^{-1}$ there exist $j^{-1}$ matrices $\tilde{T}_1, \tilde{T}_2, ..., \tilde{T}_{j^{-1}}$ such that $\tilde{V}_j = \tilde{T}_{j^{-1}} T_j \tilde{T}_{j^{-1}}$.

Analogously, we can derive the existence of $\mu^{-1}$ square invertible matrices $S_1, S_2, ..., S_{\mu^{-1}}$ and for all $i \in \mu^{-1}$ the existence of $i^{-1}$ matrices $\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_{i^{-1}}$ such that for all $i \in \mu^{-1}$,

$$
\tilde{S}_i \Upsilon_i = \begin{bmatrix} \tilde{U}_i^1 \\ \tilde{U}_i^2 \\ \vdots \\ \tilde{U}_i^{\mu^{-1}} \end{bmatrix}, \quad \Upsilon_i = \sum_{\ell=1}^{i^{-1}} \tilde{S}_i^\ell \tilde{U}_i^\ell, \quad ker \Lambda_i = ker \begin{bmatrix} \tilde{U}_1^1 \\ \tilde{U}_2^1 \\ \vdots \\ \tilde{U}_i^{\mu^{-1}} \end{bmatrix}
$$

and

$$
\begin{bmatrix}
\tilde{U}_1^1 \\
\tilde{U}_1^2 \\
\vdots \\
\tilde{U}_1^{\mu^{-1}}
\end{bmatrix}
$$

is a surjective matrix.

Define

$$
S_i W_{ij} T_j := \begin{bmatrix} \tilde{W}_{ij} & \tilde{W}_{ij} \\
\tilde{W}_{ij} & \tilde{W}_{ij} \\
\end{bmatrix}
$$

where $i, j \in \mu^{-1}$.
Furthermore, for all $i,j \in \mu - 1$ define

$$
\begin{align*}
\mathcal{U}_j :&= \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \vdots \\ \bar{U}_l \end{bmatrix}, \\
\mathcal{W}_j :&= \begin{bmatrix} \bar{W}_{ij} \\ \bar{W}_{j1} \\ \vdots \\ \bar{W}_{jl} \end{bmatrix}, \\
\mathcal{V}_j :&= \begin{bmatrix} \bar{V}_1, \bar{V}_2, \ldots, \bar{V}_j \end{bmatrix}, \\
\mathcal{W}_{i,j} :&= \begin{bmatrix} \bar{W}_{ij} \\ \bar{W}_{i1} \\ \vdots \\ \bar{W}_{il} \end{bmatrix}.
\end{align*}
$$

and

$$
\begin{align*}
\Psi_{i,j} :&= \begin{bmatrix} \bar{W}_{i1} & \bar{W}_{i2} & \cdots & \bar{W}_{ij} \\ \bar{W}_{j1} & \bar{W}_{j2} & \cdots & \bar{W}_{jl} \end{bmatrix}, \\
\Gamma_{i,j} :&= \begin{bmatrix} \Psi_{1,j} \\ \Psi_{2,j} \\ \vdots \\ \Psi_{l,j} \end{bmatrix}.
\end{align*}
$$

Claim 1:

For all $i,j \in \mu - 1$ with $i+j < \mu$, we have

$$
Q_{i,j} = \begin{bmatrix} \sum_{k=1}^{l} \bar{W}_{ik} \bar{T}_{kj} \end{bmatrix}, \\
W_{i,j} = \begin{bmatrix} \sum_{k=1}^{l} \bar{S}_{ik} \bar{W}_{kj} \end{bmatrix}, \\
\bar{W}_{i,j} = \begin{bmatrix} \sum_{k=1}^{l} \bar{S}_{ik} \bar{T}_{kj} \end{bmatrix}.
$$

Proof:

Let $j_0 \in \mu - 1$ be fixed and recall that $\ker \Delta_{j_0} \subseteq \ker \Gamma_{\mu - j_0,j_0}$.

Because the matrices $S_1, S_2, \ldots, S_{\mu - 1}$ and $T_1, T_2, \ldots, T_{\mu - 1}$ are square and invertible, it follows that $\ker \Delta'_{j_0} \subseteq \ker \Gamma'_{\mu - j_0,j_0}$. Now observe that
\[ \Delta'_j = [\Delta'_j]_{-1} \] with \[ \hat{w}_j = \sum_{z=1}^{j-1} \tilde{w}_z \bar{z} \hat{j}_j = 0. \]

Define

\[
\begin{bmatrix}
\tilde{z}'_{j,j} \\
\vdots \\
\tilde{z}'_{1,j} \\
\tilde{z}'_{0,j}
\end{bmatrix}
= \begin{bmatrix}
\hat{z}_{j,j} \\
\vdots \\
\hat{z}_{1,j} \\
\hat{z}_{0,j}
\end{bmatrix}

\]

where

\[
\tilde{z}'_{j,j} := \hat{z}_{j,j} - \sum_{z=1}^{j-1} \bar{w}_z \bar{z} \hat{j}_j
\]

and

\[
z'_i := w'_i - \sum_{z=1}^{j-1} \bar{w}_z \bar{z} \hat{j}_j
\]

for all \( i \in \mu - j_0 \).

It follows that

\[
\ker \Delta'_j = \ker [\Delta'_j]_{-1} \subseteq \ker [\Gamma'_{\mu - j_0, j_0 - 1}, \Gamma'_{\mu - j_0, j_0}] = \ker \Gamma'_{\mu - j_0, j_0}
\]

if and only if

\[
\ker [\Delta'_j]_{-1} \subseteq \ker [\Gamma'_{\mu - j_0, j_0 - 1}, \Gamma'_{\mu - j_0, j_0}].
\]

From this it is clear that \( \tilde{z}'_{i,j} = 0 \) and \( z'_i = 0 \) for all \( i \in \mu - j_0 \). Hence, for all \( i \in \mu - j_0 \):

\[
\tilde{w}_i = \sum_{z=1}^{j-1} \bar{w}_z \bar{z} \hat{j}_j \quad \text{and} \quad \tilde{w}_i = \sum_{z=1}^{j-1} \bar{w}_z \bar{z} \hat{j}_j.
\]
By dual reasoning we can prove that for all \( i_0, j \in \mathbb{Z}^+ \) with \( j \neq \mu - i_0 \):

\[
\tilde{W}_{i_0} = \sum_{k=1}^{i_0-1} \tilde{W}_{i_0+k} = \sum_{k=1}^{i_0-1} \tilde{W}_{i_0+k}.
\]

Then, by combining the previous results, the proof of the claim can be completed.

Now define

\[
U := \begin{bmatrix}
\tilde{U}_1 \\
\tilde{U}_2 \\
\vdots \\
\tilde{U}_{\mu-1}
\end{bmatrix}, \quad V := [\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_{\mu-1}]
\]

and

\[
W := \begin{bmatrix}
\tilde{W}_1 & \tilde{W}_{12} & \cdots & \tilde{W}_{1,\mu-2} & \tilde{W}_{1,\mu-1} \\
\tilde{W}_{21} & \tilde{W}_2 & \cdots & \tilde{W}_{2,\mu-2} & \tilde{W}_{2,\mu-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{W}_{\mu-21} & \tilde{W}_{\mu-2} & \cdots & \tilde{W}_{\mu-2,\mu-2} & \tilde{W}_{\mu-2,\mu-1} \\
\tilde{W}_{\mu-11} & \tilde{W}_{\mu-1} & \cdots & \tilde{W}_{\mu-1,\mu-2} & \tilde{W}_{\mu-1,\mu-1}
\end{bmatrix}.
\]

Because \( U \) is a surjective matrix and \( V \) is an injective matrix, there exists a matrix \( X \) such that \( U X V = W \) (see Lemma 4.4). We claim that the matrix \( X \) satisfies the following:

**Claim 2:**

\[
[\tilde{U}_{i_0}]_{i+j} = [\tilde{W}_{i_0}]_{i+j} \quad \text{for all } i, j \in \mathbb{Z}^+ \text{ with } i+j \neq \mu.
\]

**Proof:**

Let \( i_0, j_0 \in \mathbb{Z}^+ \) be such that \( i_0 + j_0 \neq \mu \). Note that

\[
S_{i_0} U_{i_0} X V_{j_0} T_{j_0} = \begin{bmatrix}
\tilde{U}_{i_0} \\
\tilde{V}_{j_0}
\end{bmatrix} X \begin{bmatrix}
\tilde{V}_{j_0} \\
\tilde{U}_{i_0}
\end{bmatrix} = \begin{bmatrix}
\tilde{W}_{i_0} \\
\tilde{W}_{j_0}
\end{bmatrix}.
\]
Because $S_{i_0}$ and $T_{j_0}$ are invertible matrices, claim 2 is now immediate. In fact, we have proved the implication $3) \Rightarrow 1)$, and consequently we have completed the proof of Theorem 5.7. 

We now return to the control problem (ATDP). 

For all $i,j \in \mathbb{N}$ let

$$\tilde{K}_{i,j}(s) := K_{i\mu-j+1}(s) = H_i(sI - A)^{-1} C_{\mu-j+1}$$

and

$$\tilde{M}_{i,j}(s) := M_{\mu-j+1}(s) = C(sI - A)^{-1} G_{\mu-j+1}.$$ 

By Corollary 5.4.2 it is clear that (ATDP) is solvable if and only if
there exists a rational matrix $X \in \mathbb{R}^{m \times p}(s)$ such that $\tilde{K}_{i,j} + L_i X \tilde{M}_j = 0$ for all $i,j \in \mu_-$ with $i+j \leq \mu$.

Now, for all $i,j \in \mu_-$, denote

$$L_i := [G_{i}, G_{i-1}, \ldots, G_{j+1}] \quad \text{and} \quad \tilde{K}_i := \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_i \end{bmatrix}.$$ 

Then it is clear that $\tilde{L}_i = \text{im} \tilde{L}_i$ and $\tilde{K}_i = \ker \tilde{K}_i$ for all $i \in \mu_-$, where $\tilde{L}_i, \tilde{K}_i$ ($i \in \mu_-$) are as defined in Section 5.2.

Furthermore, for all $i,j \in \mu_-$ we have

$$\begin{bmatrix} \tilde{K}_{11}(s) & \tilde{K}_{12}(s) & \cdots & \tilde{K}_{1j}(s) \\ \tilde{K}_{21}(s) & \tilde{K}_{22}(s) & \cdots & \tilde{K}_{2j}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{K}_{i1}(s) & \tilde{K}_{i2}(s) & \cdots & \tilde{K}_{ij}(s) \end{bmatrix} = \tilde{K}_i (sI - A)^{-1} \tilde{L}_{\mu-j},$$

$$\begin{bmatrix} L_1(s) \\ L_2(s) \\ \vdots \\ L_i(s) \end{bmatrix} = \tilde{K}_i (sI - A)^{-1} B.$$

and

$$[\tilde{N}_1(s), \tilde{N}_2(s), \ldots, \tilde{N}_j(s)] = C(sI - A)^{-1} \tilde{L}_{\mu-j}.$$ 

From Theorem 5.7 it follows now that (ATDPH) is solvable if and only if for every $i \in \mu_-$ there exists a rational matrix $X \in \mathbb{R}^{m \times p}(s)$ such that

$$\tilde{K}_i (sI - A)^{-1} \tilde{L}_i = \tilde{K}_i (sI - A)^{-1} B X(s) C(sI - A)^{-1} \tilde{L}_i.$$ 

Now, there exists a rational matrix $X \in \mathbb{R}^{m \times p}(s)$ such that

$$\tilde{K}_i (sI - A)^{-1} \tilde{L}_i = \tilde{K}_i (sI - A)^{-1} B X(s) C(sI - A)^{-1} \tilde{L}_i$$

if and only if $\text{im} \tilde{L}_i \subseteq \mathcal{V}_b^*(\ker \tilde{K}_i)$ and $S_b^*(\text{im} \tilde{L}_i) \subseteq \ker \tilde{K}_i$ (see Willems...
(1982) or the proof of Theorem 4.23). Thus, we have proved:

**Theorem 3.8:**

(ATDPM) is solvable if and only if

\[ s_b^*(L_i) \leq \tilde{K}_i \quad \text{and} \quad \tilde{e}_i \leq \nu_b^*(K_i) \quad \text{for all } i \in \mathbb{N} \]

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<td>exogenous input matrices</td>
<td>30, 33</td>
</tr>
<tr>
<td>C_{e_i}, C_{i,e}</td>
<td>closed loop system exogenous input matrices</td>
<td>31, 33</td>
</tr>
<tr>
<td>H, H_i</td>
<td>exogenous output matrices</td>
<td>30, 33</td>
</tr>
<tr>
<td>H_{e_i}, H_{i,e}</td>
<td>closed loop system exogenous output matrices</td>
<td>31, 33</td>
</tr>
<tr>
<td>J</td>
<td>output injection matrix</td>
<td>20</td>
</tr>
<tr>
<td>k</td>
<td>compensator order</td>
<td>21</td>
</tr>
<tr>
<td>(&lt;\ker C</td>
<td>A&gt;)</td>
<td>largest A-invariant subspace in (\ker C)</td>
</tr>
<tr>
<td>K, L, M, N</td>
<td>compensator matrices</td>
<td>21</td>
</tr>
<tr>
<td>R_{i,j}, L_{i,j}, N_{i,j}, P</td>
<td>system transfer matrices</td>
<td>58</td>
</tr>
<tr>
<td>(\mathbb{C}^r)</td>
<td>radical</td>
<td>68</td>
</tr>
<tr>
<td>m</td>
<td>number of control input variables</td>
<td>2</td>
</tr>
<tr>
<td>n</td>
<td>number of state variables</td>
<td>2</td>
</tr>
<tr>
<td>N</td>
<td>static measurement feedback</td>
<td>21</td>
</tr>
<tr>
<td>P</td>
<td>number of measurement output variables</td>
<td>2</td>
</tr>
<tr>
<td>(\mathbb{R})</td>
<td>set of real numbers</td>
<td>2</td>
</tr>
<tr>
<td>(\mathbb{R}(s))</td>
<td>set of rational functions with real coefficients</td>
<td>5</td>
</tr>
<tr>
<td>(\mathbb{R}_0(s))</td>
<td>set of proper rational functions with real coefficients</td>
<td>5</td>
</tr>
<tr>
<td>(\mathbb{R}_+(s))</td>
<td>set of strictly proper rational functions with real coefficients</td>
<td>5</td>
</tr>
<tr>
<td>(\mathbb{R}[s])</td>
<td>set of polynomials with real coefficients</td>
<td>60</td>
</tr>
</tbody>
</table>
\( S^*_a (\text{im } G) \) smallest almost conditioned invariant subspace containing \( \text{im } G \) 

\( S^*_b (\text{im } G) \) smallest \( L_p \)-almost conditioned invariant subspace containing \( \text{im } G \) 

\( S^*_f (\text{im } G) \) two-input constrained detectability subspace 

\( S^*_g (\text{im } G) \) smallest detectability subspace containing \( \text{im } G \) 

\( S^*_h (\text{im } G, \text{im } C) \) two-input constrained conditioned invariant subspace 

\( S^*_k (\text{im } G, \text{im } C) \) smallest conditioned invariant subspace containing \( \text{im } G \) 

\( T, T_0, T_{ij} \) closed loop system transfer matrices 

\( u \) control input 

\( v, v_1 \) exogenous input(s) 

\( V_{\text{a}} (\ker H) \) largest almost controlled invariant subspace in \( \ker H \) 

\( V_{\text{b}} (\ker H) \) largest \( L_p \)-almost controlled invariant subspace in \( \ker H \) 

\( V_{\text{e}} (\ker H, \ker H) \) two-output constrained stabilizability subspace 

\( V_{\text{g}} (\ker H) \) largest stabilizability subspace in \( \ker H \) 

\( V_{\text{h}} (\ker H, \ker H) \) two-output constrained controlled invariant subspace 

\( V^* (\ker H) \) largest controlled invariant subspace in \( \ker H \) 

\( x \) state of the system 

\( X_{\text{b}} (A) \) unstable eigenspace of the matrix \( A \) 

\( X_{\text{d}} (A, C) \) undetectable subspace of the pair \( (C, A) \) 

\( x_0 \) state of the closed loop system 

\( X_{\text{g}} (A) \) stable eigenspace of the matrix \( A \) 

\( X_{\text{stab}} (A, \tilde{H}) \) stabilizable subspace of the pair \( (A, \tilde{H}) \) 

\( y \) measurement output 

\( z, z_i \) exogenous output(s) 

\( \sigma(h) \) spectrum of Bohl function \( h \) 

\( \sigma(M) \) spectrum of matrix \( M \) 

\( \hat{\cup} \) disjoint union of sets 

\( \hat{\otimes} \) external direct sum 

\( \hat{1} \) the set \( \{1, 2, \ldots, n\} \) 

\( \hat{1} \) orthogonal complement 

\( \hat{\mathbb{M}} \) degree of rational matrix \( \mathbb{M} \) 

\( \hat{\mathbb{W}} \) \( H^\infty \)-norm of \( \mathbb{C} \)-stable strictly proper rational matrix \( \mathbb{W} \) 

\( \mathcal{O} \) set of stability regions.
SAMENVATTING

De systemen die in dit proefschrift bestudeerd zijn kunnen worden opgevat als 'processen' die van buitenaf beïnvloed worden en waarvan de uitkomsten naar buiten toe relevant zijn. Deze interactie met de 'buitenwereld' is in de systemen weergegeven door middel van zogenaamde exogene ingangen en uitgangen. Voorts is verondersteld dat de systemen bestuurd kunnen worden en dat aan de systemen metingen kunnen worden verricht, weergegeven door respectievelijk een besturingsingang en een metingsuitgang.

Voor de regeling van de systemen is in dit proefschrift uitgegaan van mechanisms die de metingen terugkoppelen naar de besturingen. Met dit soort van regelingen zijn de mogelijkheden onderzocht om, uitgaande van een gegeven systeem, te komen tot een geregeld systeem dat aan van tevoren vastgestelde eisen voldoet. Een dergelijk streven is in dit proefschrift geformuleerd als regelprobleem.

Van zowel de systemen als de terugkoppelingen is verondersteld dat deze beschreven kunnen worden door middel van gewone lineaire differentiaalvergelijkingen met constante coëfficiënten.

Om de regelproblemen op een elegante manier te kunnen (her)formuleren en te kunnen oplossen is gebruik gemaakt van de zogenaamde 'meetkundige benadering' van lineaire systemen. Daartoe worden in Hoofdstuk 1 de benodigde systeemtheoretische concepten met hun eigenschappen ingevoerd.


De Hoofdstukken 3 en 4 bevatten de uitbreiding naar metingsterugkoppeling van problemen betreffende 'diagonaal ontkoppeling' en 'bijna diagonaal ontkoppeling' zoals deze beschreven staan in Willems (1980).

Ten slotte worden in Hoofdstuk 5 een aantal nieuwe regelproblemen bestudeerd betreffende 'driehoeksontkoppeling' en 'bijna driehoeksontkoppeling'. Deze regelproblemen zijn geïnspireerd op problemen behandeld in Wonham (1979).
CURRICULUM VITAE

FEEDBACK DECOUPLING AND STABILIZATION FOR LINEAR SYSTEMS WITH MULTIPLE EXOGENOUS VARIABLES

van

Jacob van der Woude
1. Indien de radicaal $V^V$ van een familie $(A,B)$-invariante deelruimten

$\{V_1, V_2, ..., V_k\}$ zelf een $(A,B)$-invariante deelruimte is, en voor $i = 1, 2, ..., V$ voldoet aan $(V^V + V_i) \cap B \subseteq V_i$, dan is de familie $\{V_1, V_2, ..., V_k\}$ compatibel met betrekking tot het paar $(A,B)$.


2. Zij $A$ een reële $n \times m$-matrix, $B$ een reële $n \times m$-matrix en $p$ een reëel $n$-de graads monisch polynoom. Dan geldt: Er bestaat een reële $m \times n$-matrix $F$ zodanig dat $p(A + BF) < 0$ precies dan als $\text{im} p(A) \subseteq \text{im} \left[ B, AB, ..., A^{n-1} B \right]$. In tegenstelling tot de poolplaatsingsstelling in Wonham (1967), die universeel van aard is, kan de bovenstaande bewering opgevat worden als een resultaat betreffende individuele poolplaatsing. Vergelijk ook met Hautus (1970).


4. Zij $A$ een reële $n \times n$-matrix, $B$ een reële $n \times 1$-matrix en $C$ een reële $p \times n$-matrix zodanig dat rang $[b, Ab, ..., A^{n-1} b] = n$, en zij $p$ een reëel $n$-de graads monisch polynoom. Dan geldt: Er bestaat een reële $1 \times p$-matrix $k$ zodanig dat $\text{det} \left( sI - (A + bkC) \right) = p(s)$ precies dan als $p(A) \ker C \subseteq \text{im} \left[ b, Ab, ..., A^{n-2} b \right]$. Voor het bewijs van deze stelling kan gebruik gemaakt worden van de resultaten in Ackermann (1972).

5. Laat voor $i = 1, 2$, $A_i$ een reële $p_i \times n$-matrix, $B_i$ een reële $m \times q_i$-matrix en $C_i$ een reële $p_i \times q_i$-matrix zijn.
Er bestaat een reële $n \times m$-matrix $X$ zodanig dat $A_1XB_1 = C_1$ en $A_2XB_2 = C_2$
precies dan als voor $i = 1, 2$ geldt

$$\text{rang } A_i = \text{rang } [A_i, C_i], \text{ rang } B_i = \text{rang } \begin{bmatrix} B_i \\ C_i \end{bmatrix},$$

en

$$\text{rang } \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & 0 & 0 \\ 0 & B_1 & B_2 \end{bmatrix} = \text{rang } \begin{bmatrix} A_1 & C_1 & 0 \\ A_2 & 0 & -C_2 \\ 0 & B_1 & B_2 \end{bmatrix}.$$ 

Zie stelling 4.6 in dit proefschrift.

6. Zij $A$ een reële $n \times n$-matrix, $B$ een reële $n \times m$-matrix en $C$ een reële $p \times n$-
matrix. Laat $\{S_k, k \geq 0\}$ en $\{T_k, k \geq 0\}$ twee families lineaire deelruimten
zijn, gedefinieerd door

$$S_0 := \mathbb{R}^n, \quad S_{k+1} := \{x \mid x \in A(S_k \cap \ker C)\}$$

en

$$T_0 := \{0\}, \quad T_{k+1} := \{x \mid Ax \in (T_k + \text{im } B)\}.$$ 

Dan geldt het volgende:
Er bestaan reële matrices $K, L, M$ en $N$ van geschikte afmetingen zodanig dat

$$\begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}^\mu = 0,$$

precies dan als voor $k = 0, 1, \ldots, w$ geldt $S_{w-k} \subseteq T_k$.

7. Laat $A$ een reële $n \times n$-matrix, $B$ een reële $n \times m$-matrix en $C$ een reële $p \times n$-
matrix zijn.
Als $\text{im } A^2 \subseteq \text{im } [B, AB]$, $A \ker C = A^{-1} \text{im } B = \{x \mid Ax \in \text{im } B\}$ en
$A^2 \ker \begin{bmatrix} C \\ CA \end{bmatrix} = 0$, dan bestaat er een reële $m \times p$-matrix $N$ zodanig dat
$$(A + BNC)^2 = 0.$$
8. Laat $A_1, A_2$ reële $n \times n$-matrices en $B_1, B_2$ reële $m \times m$-matrices zijn.
Dan geldt: Voor iedere reële $n \times m$-matrix $C$ bestaat er een reële $n \times m$-matrix $X$ zodanig dat $A_1XB_1 + A_2XB_2 = C$ precies dan als rang $(A_1 \lambda_1 + A_2 \lambda_2) = n$ voor alle $\lambda_1, \lambda_2 \in \mathbb{C}$ met $(\lambda_1, \lambda_2) \neq (0,0)$ waarvoor rang $(B_1 \lambda_2 - B_2 \lambda_1) < m$.

9. Het oplossen van 'noninteracting control' problemen zoals beschreven in de hoofdstukken 3 en 4 van dit proefschrift, die uitgebreid zijn met eisen ten aanzien van uitgangsbestuurbaarheid, vereist onder andere het oplossen van 'minimal cover'-achtige problemen.


Het ligt in de verwachting dat bij de afleiding van correcte resultaten voor dit soort problemen de radicaal van een familie lineaire doelruimten een belangrijke rol speelt.

