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Process Algebra with Conditionals
in the Presence of Epsilon

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Abstract. In a previous paper, we presented several extensions of ACP
with conditional expressions, including one with a retrospection opera-
tor on conditions to allow for looking back on conditions under which
preceding actions have been performed. In this paper, we add a constant
for a process that is only capable of terminating successfully to those
extensions of ACP, which can be very useful in applications. It happens
that in all cases the addition of this constant is unproblematic.

Keywords: empty process, retrospective conditions, condition evaluation,
state operators, signal emission, splitting bisimulation, process algebra.


1 Introduction

In [9], we presented several extensions of ACP [7, 6] with conditional expressions. The main extensions of ACP presented in [9] are ACP\textsuperscript{cr}, an extension of ACP with conditional expressions of the form \( \zeta \rightarrow p \) in which the conditions are taken from a free Boolean algebra over a set of generators, ACP\textsuperscript{s}, an extension ACP\textsuperscript{cr} with a signal emission operator on processes, and ACP\textsuperscript{r}, an extension of ACP\textsuperscript{cr} with a retrospection operator on conditions. Signal emission is usable for a special kind of condition evaluation. Retrospection allows for looking back on conditions under which preceding actions have been performed. We also extended ACP\textsuperscript{r} and ACP\textsuperscript{cr} with operators devised for condition evaluation and we outlined an application of ACP\textsuperscript{cr} in which it allows for using conditions which express that a certain number of steps ago a certain action must have been performed.

In this paper, a constant for a process that is only capable of terminating successfully is added to the different extensions of ACP presented in [9]. This constant is often referred to as the empty process constant. In the past, the addition of the empty process constant to ACP has been treated in several ways.
The treatment in [16] yields a non-associative parallel composition operator. The first treatment that yields an associative parallel composition operator [19] is from 1986, but was not published until 1997. The addition of the empty process constant to different extensions of ACP in this paper is based on [5].

It is clear from early work [16, 19] that the addition of the empty process constant to ACP was rather problematic. Its addition to the different extensions of ACP with conditional expressions presented in [9] turns out to present no additional complications. For that reason, we look upon this paper in its current form primarily as supplementary material to [9].

The structure of this paper is as follows. First of all, we introduce ACP ɛ , the extension of ACP ɛ with the empty process constant (Section 2). After that, we introduce conditional transition systems and splitting bisimilarity of conditional transition systems (Section 3) and the full splitting bisimulation models of ACP ɛ , the main models of ACP ɛ (Section 4). Following this, we have a closer look at splitting bisimilarity based on structural operational semantics (Section 5). Next, we extend ACP ɛ with guarded recursion (Section 6). Thereupon, we extend ACP ɛ with condition evaluation operators (Section 7), with state operators (Section 8) and with a signal emission operator (Section 9); and analyse how those operators are related. We also adapt the full splitting bisimulation models of ACP ɛ to the full signal-observing splitting bisimulation models of ACP ɛ , the extension of ACP ɛ with signal emission (Section 10). After that, we extend ACP ɛ with a retrospection operator (Section 11) and adapt the full splitting bisimulation models of ACP ɛ to the full retrospective splitting bisimulation models of ACP ɛ , the extension of ACP ɛ with retrospection (Section 12). Thereupon, we extend ACP ɛ with condition evaluation operators as well (Section 13). We also outline an interesting application of ACP ɛ (Section 14). Finally, we make some concluding remarks (Section 15).

Some familiarity with Boolean algebras is desirable. The definitions of all notions concerning Boolean algebras that are used can be found in [17].

We thank Jan van Eijck. He communicated an application of ACP ɛ to us which involves a register update mechanism that cannot be dealt with in full generality without the empty process constant. This forms the greater part of our motivation to work out the addition of the empty process constant to ACP ɛ .

2 ACP ɛ with Conditions

In this section, we present ACP ɛ , an extension of ACP ɛ [5, 6] with conditional expressions of the form ζ ::= p. ACP ɛ can be regarded as an extension of ACP ɛ [9] with the empty process constant too. In ACP ɛ , as in ACP ɛ , it is assumed that a fixed but arbitrary finite set of actions A, with δ, ε ∉ A, and a fixed but arbitrary commutative and associative communication function |: Aδ × Aδ → Aδ, such that δ | a = δ for all a ∈ Aδ, have been given. The function | is regarded to give the result of synchronously performing any two actions for which this is possible, and to be δ otherwise. Moreover, it is assumed that a fixed but arbitrary set of atomic conditions C at has been given.

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Let \( \kappa \) be an infinite cardinal. Then \( C_\kappa \) is the free \( \kappa \)-complete Boolean algebra over \( C_{\aleph_0} \). As usual, we identify Boolean algebras with their domain. Thus, we also write \( C_\kappa \) for the domain of \( C_\kappa \). If \( \kappa \) is regular,\(^2\) then \( C_\kappa \) is isomorphic to the Boolean algebra of equivalence classes with respect to logical equivalence of the set of all propositions with elements of \( C_{\aleph_0} \) as propositional variables and with conjunctions and disjunctions of less than \( \kappa \) propositions (see e.g. [17]). In ACP\(^c\), conditions are taken from \( C_{\aleph_0} \). If \( C_{\aleph_0} \) is a finite set, then \( C_\kappa = C_{\aleph_0} \) for all cardinals \( \kappa > \aleph_0 \). We are also interested in \( C_\kappa \) for cardinals \( \kappa > \aleph_0 \) because it permits us to consider infinitely branching processes in the case where \( C_{\aleph_0} \) is an infinite set. Henceforth, we write \( C \) for \( C_{\aleph_0} \).

The algebraic theory ACP\(^c\) has two sorts:

- the sort \( P \) of processes;
- the sort \( C \) of conditions.

The algebraic theory ACP\(^c\) has the following constants and operators to build terms of sort \( P \):

- the deadlock constant \( \delta : P \);
- the empty process constant \( \epsilon : P \);
- for each \( a \in A \), the action constant \( a : P \);
- the binary alternative composition operator \( + : P \times P \to P \);
- the binary sequential composition operator \( \cdot : P \times P \to P \);
- the binary guarded command operator \( : C \times P \to P \);
- the binary parallel composition operator \( \parallel : P \times P \to P \);
- the binary left merge operator \( \lfloor \lfloor : P \times P \to P \);
- the binary communication merge operator \( \rfloor \rfloor : P \times P \to P \);
- for each \( H \subseteq A \), the unary encapsulation operator \( \partial H : P \to P \).

The algebraic theory ACP\(^c\) has the following constants and operators to build terms of sort \( C \):

- the bottom constant \( \bot : C \);
- the top constant \( \top : C \);
- for each \( \eta \in C_{\aleph_0} \), the atomic condition constant \( \eta : C \);
- the unary complement operator \( - : C \to C \);
- the binary join operator \( \sqcup : C \times C \to C \);
- the binary meet operator \( \sqcap : C \times C \to C \).

We use infix notation for the binary operators. The following precedence conventions are used to reduce the need for parentheses. The operators to build terms of sort \( C \) bind stronger than the operators to build terms of sort \( P \). The operator \( \cdot \) binds stronger than all other binary operators to build terms of sort \( P \) and the operator \( + \) binds weaker than all other binary operators to build terms of sort \( P \).

\(^1\) For a definition of free \( \kappa \)-complete Boolean algebras, see e.g. [17].
\(^2\) For a definition of regular cardinals, see e.g. [18, 13]. They include \( \aleph_0, \aleph_1, \aleph_2, \ldots \).
The constants and operators of $\text{ACP}_\epsilon^c$ to build terms of sort $P$ are the constants and operators of $\text{ACP}_\epsilon$, and additionally the guarded command operator. Let $p$ and $q$ be closed terms of sort $P$ and $\zeta$ and $\xi$ be closed terms of sort $C$, $a \in A$, $H \subseteq A$, and $\eta \in C_\text{at}$. Then, intuitively, the constants and operators to build terms of sort $P$ can be explained as follows:

- $\delta$ can neither perform an action nor terminate successfully;
- $\epsilon$ terminates successfully, unconditionally;
- $a$ first performs action $a$ and then terminates successfully, both unconditionally;
- $p + q$ behaves either as $p$ or as $q$, but not both;
- $p \cdot q$ first behaves as $p$, but when $p$ terminates successfully it continues by behaving as $q$;
- $\zeta : \forall p$ behaves as $p$ under condition $\zeta$;
- $p \parallel q$ behaves as the process that proceeds with $p$ and $q$ in parallel;
- $p\parallel q$ behaves the same as $p\parallel q$, except that it starts with performing an action of $p$;
- $p \parallel q$ behaves the same as $p\parallel q$, except that it starts with performing an action of $p$ and an action of $q$ synchronously;
- $\partial_H(p)$ behaves the same as $p$, except that actions from $H$ are blocked.

Intuitively, the constants and operators to build terms of sort $C$ can be explained as follows:

- $\eta$ is an atomic condition;
- $\bot$ is a condition that never holds;
- $\top$ is a condition that always holds;
- $-\zeta$ is the opposite of $\zeta$;
- $\zeta \cup \xi$ is either $\zeta$ or $\xi$;
- $\zeta \cap \xi$ is both $\zeta$ and $\xi$.

Some earlier extensions of $\text{ACP}$ include conditional expressions of the form $p < \zeta > q$; see e.g. [2]. Just as in [9], we treat conditional expressions of the form $p < \zeta > q$, where $p$ and $q$ are terms of sort $P$ and $\zeta$ is a term of sort $C$, as abbreviations. That is, we write $p < \zeta > q$ for $\zeta : \forall p + -\zeta : \forall q$.

The axioms of $\text{ACP}_\epsilon^c$ are given in Table 1. CM3, CM7, C1–C3 and D1–D2 are actually axiom schemas in which $a$, $b$ and $c$ stand for arbitrary constants of $\text{ACP}_\epsilon$ that differ from $\epsilon$ (i.e. $a, b, c \in A_\delta$). In D0–D4, $H$ stands for an arbitrary subset of $A$. So, D0, D3 and D4 are axiom schemas as well. Axioms A1–A9, CM1T, TM2, CM3, CM4, TM5, TM6, CM7–CM9, C1–C3 and D0–D4 are the axioms of $\text{ACP}_\epsilon$. Axioms BA1–BA8 are the axioms of Boolean Algebras (BA). So $\text{ACP}_\epsilon^c$ imports the (equational) axioms of both $\text{ACP}_\epsilon$ and BA. The axioms of BA have been taken from [15]. Several alternatives for this axiomatization can be found in the literature. Axioms GC1–GC11 have been taken from [2], but the axiom $x \cdot z < \phi > y \cdot z = (x < \phi > y) \cdot z$ (CO5) is replaced by the simpler axiom $\phi : \forall x \cdot y = (\phi : \forall x) \cdot y$ (GC5) and similarly for axioms GC8–GC11.

The terms of sort $C$ are interpreted in $C$ as usual.
We proceed to the presentation of the structural operational semantics of \( \text{ACP}_\varepsilon \). The following relations on closed terms of sort \( \mathbf{P} \) from the language of \( \text{ACP}_\varepsilon \) are used:

- for each \( \alpha \in C \setminus \{ \perp \} \), a unary relation \( [\alpha]_\perp \);
- for each \( \ell \in (C \setminus \{ \perp \}) \times A \), a binary relation \( \ell \).

We write \( p [\alpha] \) instead of \( p \in [\alpha] \) and \( p [\alpha] [\alpha] a, q \) instead of \( (p, q) \in \ell \). The relations \([\alpha]_\perp\) and \( \ell \) can be explained as follows:

- \( p [\alpha] \): \( p \) is capable of terminating successfully under condition \( \alpha \);
- \( p [\alpha] [\alpha] a, q \): \( p \) is capable of performing action \( a \) under condition \( \alpha \) and then proceeding as \( q \).

<table>
<thead>
<tr>
<th>Table 1. Axioms of ( \text{ACP}<em>\varepsilon ) (( a, b, c \in A</em>\varepsilon ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + y = y + x )</td>
</tr>
<tr>
<td>( (x + y) + z = x + (y + z) )</td>
</tr>
<tr>
<td>( x + x = x )</td>
</tr>
<tr>
<td>( (x + y) \cdot z = x \cdot z + y \cdot z )</td>
</tr>
<tr>
<td>( x \cdot (y + z) = x \cdot y + x \cdot z )</td>
</tr>
<tr>
<td>( x + \delta = x )</td>
</tr>
<tr>
<td>( \delta \cdot x = \delta )</td>
</tr>
<tr>
<td>( x \cdot \epsilon = x )</td>
</tr>
<tr>
<td>( \epsilon \cdot x = x )</td>
</tr>
<tr>
<td>( x \parallel y = x \parallel y \parallel x + x \parallel y + \partial_H(x) \cdot \partial_H(y) )</td>
</tr>
<tr>
<td>( \epsilon \parallel x = \delta )</td>
</tr>
<tr>
<td>( a \cdot x \parallel y = a \cdot (x \parallel y) )</td>
</tr>
<tr>
<td>( (x + y) \parallel z = x \parallel z + y \parallel z )</td>
</tr>
<tr>
<td>( \epsilon \parallel x = \delta )</td>
</tr>
<tr>
<td>( x \parallel \epsilon = \delta )</td>
</tr>
<tr>
<td>( a \cdot x \parallel b \cdot y = (a \mid b) \cdot (x \parallel y) )</td>
</tr>
<tr>
<td>( (x + y) \parallel z = x \parallel z + y \parallel z )</td>
</tr>
<tr>
<td>( x \parallel (y + z) = x \parallel y + x \parallel z )</td>
</tr>
<tr>
<td>( \phi \cup \psi = \psi \cup \phi )</td>
</tr>
<tr>
<td>( a \mid b = b \mid a )</td>
</tr>
<tr>
<td>( (a \mid b) \mid c = a \mid (b \mid c) )</td>
</tr>
<tr>
<td>( \delta \mid a = \delta )</td>
</tr>
<tr>
<td>( \phi \cap \psi = \psi \cap \phi )</td>
</tr>
<tr>
<td>( \phi \cap (\psi \cup \chi) = (\phi \cap \psi) \cup (\phi \cap \chi) )</td>
</tr>
</tbody>
</table>

5
3 Transition Systems and Splitting Bisimilarity for $\text{ACP}_c$

In this section, we adapt the definitions of conditional transition systems and splitting bisimilarity of conditional transition systems from [9] to the presence of a process that is only capable of terminating successfully. In Section 4, we will make use of conditional transition systems and splitting bisimilarity of conditional transition systems as defined in this section to construct models of $\text{ACP}_c$.

The transitions of conditional transition systems have labels that consist of a condition different from $\perp$ and an action. Labels of this kind are sometimes called guarded actions. Henceforth, we write $C_\kappa$ for $C_\kappa \setminus \{\perp\}$.

Let $\kappa$ be an infinite cardinal. Then a $\kappa$-conditional transition system $T$ consists of the following:

- a set $S$ of states;
- a set $\ell \to S \times S$, for each $\ell \in C_\kappa \times A$;
- a set $[\alpha] \subseteq S$, for each $\alpha \in C_\kappa$;
- an initial state $s^0 \in S$.

The structural operational semantics of $\text{ACP}_c$ is described by the transition rules given in Table 2.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon \vdash [\perp]_\epsilon$</td>
<td>$a \vdash [\perp]_a$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$</td>
<td>$x + y \vdash [\perp]_{x + y}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$</td>
<td>$x + y \vdash [\perp]_{x + y}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$, $y \vdash [\perp]_y$</td>
<td>$x \cdot y \vdash [\perp]_{x \cdot y}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$, $y \vdash [\perp]_y$</td>
<td>$x \cdot y \vdash [\perp]_{x \cdot y}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$</td>
<td>$\psi :\Rightarrow x \vdash [\psi \land \phi \land \psi \neq \perp]_{\psi \land \phi \land \psi \neq \perp}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$, $y \vdash [\perp]_y$</td>
<td>$x \parallel y \vdash [\perp]_{x \parallel y}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$, $y \vdash [\perp]_y$</td>
<td>$x \parallel y \vdash [\perp]_{x \parallel y}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$, $y \vdash [\perp]_y$</td>
<td>$x \parallel y \vdash [\perp]_{x \parallel y}$</td>
</tr>
<tr>
<td>$x \vdash [\perp]_x$</td>
<td>$\partial H(x) \vdash [\perp]_{\partial H(x)}$, $a \notin H$</td>
</tr>
</tbody>
</table>

### Table 2. Transition rules for $\text{ACP}_c$

The transitions of conditional transition systems have labels that consist of a condition different from $\perp$ and an action. Labels of this kind are sometimes called guarded actions. Henceforth, we write $C_\kappa$ for $C_\kappa \setminus \{\perp\}$.
If $(s, s') \in \ell \to$ for some $\ell \in C^- \times A$, then we say that there is a transition from $s$ to $s'$. We usually write $s \xrightarrow{[\alpha]} a s'$ instead of $(s, s') \in \ell \to (\alpha, a)$ and $s \xrightarrow{[\alpha]}$ instead of $s \in [\alpha]$. Furthermore, we write $\to$ for the family of sets $\ell \to$ and $\downarrow$ for the family of sets $\alpha \downarrow$.

The relations $[\alpha] \downarrow$ and $\ell \to$ can be explained as follows:

- $s \xrightarrow{[\alpha]}$: in state $s$, it is possible to terminate successfully under condition $\alpha$;
- $s \xrightarrow{[\alpha]} a s'$: in state $s$, it is possible to perform action $a$ under condition $\alpha$, and by doing so to make a transition to state $s'$.

A conditional transition system may have states that are not reachable from its initial state by a sequence of transitions. Unreachable states, and the transitions between them, are not relevant to the behaviour represented by the transition system.

Let $T = (S, \to, \downarrow, s^0)$ be a $\kappa$-conditional transition system (for an infinite cardinal $\kappa$). Then the reachability relation of $T$ is the smallest relation $\to \subseteq S \times S$ such that:

- $s \to s$;
- if $s \xrightarrow{\ell} s'$ and $s' \to s''$, then $s \to s''$.

We write $\text{RS}(T)$ for $\{ s \in S \mid s^0 \to s \}$. $T$ is called a connected $\kappa$-conditional transition system if $S = \text{RS}(T)$.

Henceforth, we will only consider connected conditional transition systems. However, this often calls for extraction of the connected part of a conditional transition system resulting from composition of connected conditional transition systems.

Let $T = (S, \to, \downarrow, s^0)$ be a $\kappa$-conditional transition system (for an infinite cardinal $\kappa$) that is not necessarily connected. Then the connected part of $T$, written $\Gamma(T)$, is defined as follows:

$$\Gamma(T) = (S', \to', \downarrow', s^0),$$

where

$$S' = \text{RS}(T),$$

and for every $\ell \in C^- \times A$ and $\alpha \in C^-$:

$$\xrightarrow{\ell} \to' = \ell \to \cap (S' \times S'),
\xrightarrow{[\alpha]} \downarrow' = [\alpha] \downarrow \cap S'.$$

It is assumed that for each infinite cardinal $\kappa$ a fixed but arbitrary set $S_\kappa$ with the following properties has been given:

- the cardinality of $S_\kappa$ is greater than or equal to $\kappa$;
– if $S_1, S_2 \subseteq S_\kappa$, then $S_1 \uplus S_2 \subseteq S_\kappa$ and $S_1 \times S_2 \subseteq S_\kappa$. \footnote{We write $A \uplus B$ for the disjoint union of sets $A$ and $B$, i.e. $A \uplus B = (A \times \{\emptyset\}) \uplus (B \times \{\emptyset\})$. We write $\mu_1$ and $\mu_2$ for the associated injections $\mu_1 : A \rightarrow A \uplus B$ and $\mu_2 : B \rightarrow A \uplus B$, defined by $\mu_1(a) = (a, \emptyset)$ and $\mu_2(b) = (b, \emptyset)$.}

Let $\kappa$ be an infinite cardinal. Then $\text{CTS}_\kappa^\kappa$ is the set of all connected $\kappa$-conditional transition systems $T = (S, \rightarrow_1, s^0)$ such that $S \subseteq S_\kappa$ and the branching degree of $T$ is less than $\kappa$, i.e. for all $s \in S$, the cardinality of the set $\{(t, s') \in (C_\kappa \times A) \times S \mid (s, s') \in \beta \} \cup \{\alpha \in C_\kappa \mid s \in [\alpha]_\kappa\}$ is less than $\kappa$.

The condition $S \subseteq S_\kappa$ guarantees that $\text{CTS}_\kappa^\kappa$ is indeed a set.

A conditional transition system is said to be finitely branching if its branching degree is less than $\aleph_0$. Otherwise, it is said to be infinitely branching.

The identity of the states of a conditional transition system is not relevant to the behaviour represented by it. Conditional transition system that differ only with respect to the identity of the states are isomorphic.

Let $T_1 = (S_1, \rightarrow_1, s^1_0)$ and $T_2 = (S_2, \rightarrow_2, s^2_0)$ be $\kappa$-conditional transition systems (for an infinite cardinal $\kappa$). Then $T_1$ and $T_2$ are isomorphic, written $T_1 \cong T_2$, if there exists a bijective function $b : S_1 \rightarrow S_2$ such that:

- $b(s^0_1) = s^0_2$;
- $s_1 \rightarrow_1 s'_1$ if $b(s_1) \rightarrow_2 b(s'_1)$;
- $s \in [\alpha]_\kappa$ if b(s) $\in [\alpha]_\kappa$.

Henceforth, we will always consider two conditional transition systems essentially the same if they are isomorphic.

Remark 3.1. The set $\text{CTS}_\kappa^\kappa$ is independent of $S_\kappa$. By that we mean the following. Let $\text{CTS}_\kappa^\kappa$ and $\text{CTS}_\kappa^\kappa'$ result from different choices for $S_\kappa$. Then there exists a bijection $b : \text{CTS}_\kappa^\kappa \rightarrow \text{CTS}_\kappa^\kappa'$ such that for all $T \in \text{CTS}_\kappa^\kappa$, $T \cong b(T)$.

Bisimilarity has to be adapted to the setting with guarded actions. In the definition given below, we use two well-known notions from the field of Boolean algebras: a partial order relation $\subseteq$ on $C_\kappa$ and a unary operation $\bigcup$ on the set of all subsets of $C_\kappa$ of cardinality less than $\kappa$ (for each infinite cardinal $\kappa$). The relation $\subseteq$ and the operation $\bigcup$ are defined by

$\alpha \subseteq \beta$ if $\alpha \cup \beta = \beta$ and $\bigcup C$ is the supremum of $C$ in $(C_\kappa, \subseteq)$, respectively. The operation $\bigcup$ is defined for all subsets of $C_\kappa$ of cardinality less than $\kappa$ because $C_\kappa$ is $\kappa$-complete.

Let $T_1 = (S_1, \rightarrow_1, s^1_0)$ \in $\text{CTS}_\kappa^\kappa$ and $T_2 = (S_2, \rightarrow_2, s^2_0)$ \in $\text{CTS}_\kappa^\kappa$ (for an infinite cardinal $\kappa$). Then a splitting bisimulation $B$ between $T_1$ and $T_2$ is a binary relation $B \subseteq S_1 \times S_2$ such that $B(s^1_0, s^2_0)$ and for all $s_1, s_2$ such that $B(s_1, s_2)$:

- if $s_1 \xrightarrow{[\alpha]_\kappa} s'_1$, then there is a set $CS^2_2 \subseteq C^-_\kappa \times S_2$ of cardinality less than $\kappa$ such that $\alpha \subseteq \bigcup \text{dom}(CS^2_2)$ and for all $(\alpha', s'_2) \in CS^2_2$, $s_2 \xrightarrow{[\alpha']_\kappa} s'_2$ and $B(s'_1, s'_2)$.

\footnote{We write $A \uplus B$ for the disjoint union of sets $A$ and $B$, i.e. $A \uplus B = (A \times \{\emptyset\}) \uplus (B \times \{\emptyset\})$. We write $\mu_1$ and $\mu_2$ for the associated injections $\mu_1 : A \rightarrow A \uplus B$ and $\mu_2 : B \rightarrow A \uplus B$, defined by $\mu_1(a) = (a, \emptyset)$ and $\mu_2(b) = (b, \emptyset)$.}
The branching degree is less than $\kappa$ is isomorphically embedded in a full splitting bisimulation model. Expressing that there exist other splitting bisimulation models, but each of them transitional transition systems modulo splitting bisimilarity. The qualification "full" expresses that there exist other splitting bisimulation models, but each of them isomorphic conditional transition systems of which the set of states is a subset of $\mathcal{D}$. The latter is guaranteed by considering only conditional transition systems with unreachable states as well, each conditional transition system would be splitting bisimilar to its connected part. It is also easy to see that isomorphic conditional transition systems are splitting bisimilar.

In Section 4, we will use $\mathcal{CTS}_\kappa$ as the domain of a structure that is a model of $\text{ACP}_\kappa^\epsilon$. As the domain of a structure, $\mathcal{CTS}_\kappa^\epsilon / \equiv$ must be a set. That is the case because $\mathcal{CTS}_\kappa^\epsilon$ is a set. The latter is guaranteed by considering only conditional transition systems of which the set of states is a subset of $\mathcal{S}_\kappa$.

Remark 3.2. The question arises whether $\mathcal{S}_\kappa$ is large enough if its cardinality is greater than or equal to $\kappa$. This question can be answered in the affirmative. Let $T = (\mathcal{S}, \rightarrow, \rightarrow', s^0)$ be a connected $\kappa$-conditional transition system of which the branching degree is less than $\kappa$. Then there exists a connected $\kappa$-conditional transition system $T' = (\mathcal{S}', \rightarrow', \rightarrow', s'^0)$ of which the branching degree is less than $\kappa$ such that $T \equiv T'$ and the cardinality of $\mathcal{S}'$ is less than $\kappa$.

It is easy to see that, if we would consider conditional transition systems with unreachable states as well, each conditional transition system would be splitting bisimilar to its connected part. It is also easy to see that isomorphic conditional transition systems are splitting bisimilar.

4 Full Splitting Bisimulation Models of $\text{ACP}_\kappa^\epsilon$

In this section, we introduce the full splitting bisimulation models of $\text{ACP}_\kappa^\epsilon$. They are models of which the domain consists of equivalence classes of conditional transition systems modulo splitting bisimilarity. The qualification “full” expresses that there exist other splitting bisimulation models, but each of them is isomorphically embedded in a full splitting bisimulation model.

The models of $\text{ACP}_\kappa^\epsilon$ are structures that consist of the following:

- a non-empty set $\mathcal{D}$, called the domain of the model;
- for each constant of $\text{ACP}_\kappa^\epsilon$, an element of $\mathcal{D}$;
– for each $n$-ary operator of $\text{ACP}^c_\kappa$, an $n$-ary operation on $D$.

In the full splitting bisimulation models of $\text{ACP}^c_\kappa$ that are introduced in this section, the domain is $\text{CTS}^c_\kappa / \approx$ for some infinite cardinal $\kappa$. We obtain the models concerned by associating certain elements of $\text{CTS}^c_\kappa / \approx$ with the constants of $\text{ACP}^c_\kappa$ and certain operations on $\text{CTS}^c_\kappa / \approx$ with the operators of $\text{ACP}^c_\kappa$. We begin by associating elements of $\text{CTS}^c_\kappa$ and operations on $\text{CTS}^c_\kappa$ with the constants and operators. The result of this is subsequently lifted to $\text{CTS}^c_\kappa / \approx$.

It is assumed that for each infinite cardinal $\kappa$ a fixed but arbitrary function $\text{ch}_\kappa : (\mathcal{P}(\mathcal{S}_\kappa) \setminus \{\emptyset\}) \rightarrow \mathcal{S}_\kappa$ such that for all $S \in \mathcal{P}(\mathcal{S}_\kappa) \setminus \{\emptyset\}$, $\text{ch}_\kappa(S) \in S$ has been given.

We associate with each constant $c$ of $\text{ACP}^c_\kappa$ an element $\hat{c}$ of $\text{CTS}^c_\kappa$ and with each operator $f$ of $\text{ACP}^c_\kappa$ an operation $\hat{f}$ on $\text{CTS}^c_\kappa$ as follows.

– \(\hat{\delta} = (\{s^0\}, \emptyset, \emptyset, s^0)\),

where

\[ s^0 = \text{ch}_\kappa(\mathcal{S}_\kappa). \]

– \(\hat{\epsilon} = (\{s^0\}, \emptyset, \downarrow, s^0)\),

where

\[ s^0 = \text{ch}_\kappa(\mathcal{S}_\kappa), \]

\[ [\top] = \{s^0\}, \]

and for every $\alpha \in \mathcal{C}_\kappa^\kappa \setminus \{\top\}$:

\[ [\alpha] = \emptyset. \]

– \(\hat{a} = (\{s^0, s^\vee\}, \rightarrow, \downarrow, s^0)\),

where

\[ s^0 = \text{ch}_\kappa(\mathcal{S}_\kappa), \]

\[ s^\vee = \text{ch}_\kappa(\mathcal{S}_\kappa \setminus \{s^0\}), \]

\[ [\top] = \{(s^0, s^\vee)\}, \]

\[ [\top] = \{s^\vee\}, \]

and for every \((\alpha', a') \in (\mathcal{C}_\kappa^\kappa \times \mathcal{A}) \setminus \{(\top, a)\}\) and $\alpha'' \in \mathcal{C}_\kappa^\kappa \setminus \{\top\}$:

\[ [\alpha'] = \emptyset, \]

\[ [\alpha''] = \emptyset. \]

– Let $T_i = (S_i, \rightarrow_i, \downarrow_i, s^0_i) \in \text{CTS}_\kappa$ for $i = 1, 2$. Then

\[ T_1 \ast T_2 = \Gamma(S, \rightarrow, \downarrow, s^0), \]

where

\[ s^0 = \text{ch}_\kappa(\mathcal{S}_\kappa \setminus (S_1 \uplus S_2)), \]

\[ S = \{s^0\} \cup (S_1 \uplus S_2). \]
and for every \( (\alpha, a) \in C^- \times A \) and \( \alpha' \in C^- \):

\[
\frac{(\alpha, a)}{\alpha'} = \{ (s^0, \mu_1(s)) \mid s^0 \xrightarrow{\alpha} s \}
\cup \{ (s^0, \mu_2(s)) \mid s^2 \xrightarrow{\alpha} s \}
\cup \{ (\mu_1(s), \mu_1(s')) \mid s \xrightarrow{\alpha} s' \}
\cup \{ (\mu_2(s), \mu_2(s')) \mid s \xrightarrow{\alpha} s' \},
\]

\[
\alpha' = \{ (s^0 | s^1_{\alpha' \mid 1}) \}
\cup \{ s^0 | s^2_{\alpha' \mid 2} \}
\cup \{ \mu_1(s) \mid s \xrightarrow{\alpha'} \}
\cup \{ \mu_2(s) \mid s \xrightarrow{\alpha'} \}.
\]

- Let \( T_i = (S_i, \xrightarrow{\cdot}, 1_i, s^0_i) \in \text{CTS}^- \) for \( i = 1, 2 \). Then

\[
T_1 \parallel T_2 = \Gamma(S, \xrightarrow{\cdot}, 1, s^0_1),
\]

where

\[
S = S_1 \uplus S_2,
\]

and for every \( (\alpha, a) \in C^- \times A \) and \( \alpha' \in C^- \):

\[
\frac{(\alpha, a)}{\alpha'} = \{ (\mu_1(s), \mu_1(s')) \mid s \xrightarrow{\alpha} s' \land \exists \beta \bullet s' \xrightarrow{\beta \mid 1}
\cup \{ (\mu_1(s), \mu_2(s_2)) \mid \exists s', \beta \bullet s \xrightarrow{\alpha} s' \land s' \xrightarrow{\beta \mid 2}
\cup \{ (\mu_2(s), \mu_2(s')) \mid \exists s, \beta, \beta' \bullet s \xrightarrow{\beta \mid 1} \land s_2 \xrightarrow{\beta' \mid 2} s' \land \alpha = \beta \cap \beta'
\cup \{ (\mu_2(s), \mu_2(s')) \mid s \xrightarrow{\alpha} s' \land s \neq s_2 \},
\]

\[
\alpha' = \{ \mu_2(s_2') \mid \exists s, \beta, \beta' \bullet s \xrightarrow{\beta \mid 1} \land s_2 \xrightarrow{\beta' \mid 2} \land \alpha' = \beta \cap \beta'
\cup \{ \mu_2(s) \mid s \xrightarrow{\alpha'} \}.
\]

- Let \( T = (S, \xrightarrow{\cdot}, 1, s^0) \in \text{CTS}^- \). Then

\[
\alpha \vdash T = \Gamma(S, \xrightarrow{\cdot}, 1, s^0),
\]

where for every \( (\alpha', a) \in C^- \times A \) and \( \alpha'' \in C^- \):

\[
\frac{(\alpha', a)}{\alpha''} = \{ (s^0, s') \mid \exists \beta \bullet s^0 \xrightarrow{\beta} s' \land \alpha'' = \alpha \cap \beta
\cup \{ (s, s') \mid s \xrightarrow{\alpha''} s' \land s \neq s_0 \},
\]

\[
\alpha'' = \{ s^0 | \exists \beta \bullet s \xrightarrow{\beta \mid 1} \land \alpha'' = \alpha \cap \beta
\cup \{ s | s \xrightarrow{\alpha''} \land s \neq s_0 \}.
\]

- Let \( T_i = (S_i, \xrightarrow{\cdot}, 1_i, s^0_i) \in \text{CTS}^- \) for \( i = 1, 2 \). Then

\[
T_1 \parallel T_2 = (S, \xrightarrow{\cdot}, 1, s^0),
\]
where
\[ s^0 = (s^0_1, s^0_2), \]
\[ S = S_1 \times S_2, \]
and for every \((\alpha, a) \in C^-_\kappa \times A\) and \(\alpha'' \in C^-_\kappa:\)
\[ \overset{\alpha,a}{\rightarrow} = \{((s_1, s_2), (s'_1, s'_2)) \mid s_1 \overset{[\alpha]_a}{\rightarrow} s'_1 \land s_2 \in S_2 \} \]
\[ \cup \{((s_1, s_2), (s_1, s'_2)) \mid s_1 \in S_1 \land s_2 \overset{[\alpha]_a}{\rightarrow} s'_2 \} \]
\[ \cup \{((s_1, s_2), (s'_1, s_2)) \mid (s_1, s'_1) \in \bigvee_{\alpha', \beta' \in C^-_\kappa, \alpha', \beta' \in A} (s \overset{[\alpha']_a}{\rightarrow} s'_1 \land s_2 \overset{[\beta']_b}{\rightarrow} s' \land \alpha' \land \beta' = \alpha \land \alpha' = \alpha \land b' = a) \}, \]
\[ \{\alpha''\} \overset{\alpha,a}{\rightarrow} = \{((s_1, s_2)) \mid \bigvee_{\alpha', \beta' \in C^-_\kappa} (s \overset{[\alpha']_a}{\rightarrow} s'_1 \land s_2 \overset{[\beta']_b}{\rightarrow} s' \land \alpha' \land \beta' = \alpha'' \land \alpha' = \alpha \land b' = a) \}. \]

- Let \(T_i = (S_i, \rightarrow, 1, s^0_i) \in CTS^-_\kappa\) for \(i = 1, 2\). Suppose that \(T_1 \parallel T_2 = (S, \rightarrow, 1, s^0)\). Then
\[ T_1 \parallel T_2 = \Gamma(S', \rightarrow', 1, s^{0'}) , \]
where
\[ s^{0'} = ch_\kappa(S \setminus S), \]
\[ S' = \{s^{0'}\} \cup S, \]
and for every \((\alpha, a) \in C^-_\kappa \times A:\)
\[ \overset{\alpha,a}{\rightarrow}' = \{((s^{0'}, s_2^0)) \mid s^0_1 \overset{[\alpha]_a}{\rightarrow} s^0_1 \land (\alpha,a), (\alpha,a)' \}. \]

- Let \(T_i = (S_i, \rightarrow, 1, s^0_i) \in CTS^-_\kappa\) for \(i = 1, 2\). Suppose that \(T_1 \parallel T_2 = (S, \rightarrow, 1, s^0)\). Then
\[ T_1 \parallel T_2 = \Gamma(S', \rightarrow', 1, s^{0'}) , \]
where
\[ s^{0'} = ch_\kappa(S \setminus S), \]
\[ S' = \{s^{0'}\} \cup S, \]
and for every \((\alpha, a) \in C^-_\kappa \times A:\)
\[ \overset{\alpha,a}{\rightarrow}' = \{((s^{0'}, s_2^0)) \mid s^0_1 \overset{[\alpha']_a}{\rightarrow} s^0_1 \land s_2^0 \overset{[\beta']_b}{\rightarrow} s^0_2 \land \alpha' \land \beta' = \alpha \land \alpha' = \alpha \land b' = a \}. \]
Let $T = (S, \rightarrow, \bot, s^0) \in \text{CTS}_\kappa$. Then

$$\delta_H(T) = \Gamma(S, \rightarrow', \bot, s^0),$$

where for every $(\alpha, a) \in C^- \times (A \setminus H)$:

$$\hat{(\alpha, a)}_T = (\hat{\alpha}, \hat{a}),$$

and for every $(\alpha, a) \in C^- \times H$:

$$\hat{(\alpha, a)}_T = \emptyset.$$

In the definition of alternative composition on $\text{CTS}_\kappa$, a new initial state is introduced because, in $T_1$ and/or $T_2$, there may exist a transition back to the initial state. The connected part of the resulting conditional transition system is extracted because the initial states of $T_1$ and $T_2$ may be unreachable from the new initial state.

**Remark 4.1.** The elements of $\text{CTS}_\kappa$ and the operations on $\text{CTS}_\kappa$ defined above are independent of $\epsilon_H$. Different choices for $\epsilon_H$ lead for each constant of $\text{ACP}_\kappa$ to isomorphic elements of $\text{CTS}_\kappa$ and lead for each operator $\text{ACP}_\kappa$ to operations on $\text{CTS}_\kappa$ with isomorphic results.

We can show that splitting bisimilarity is a congruence with respect to the operations on $\text{CTS}_\kappa$ associated with the operators of $\text{ACP}_\kappa$.

**Proposition 4.1 (Congruence).** Let $\kappa$ be an infinite cardinal. Then for all $T_1, T_2, T'_1, T'_2 \in \text{CTS}_\kappa$ and $\alpha \in C_\kappa$, $T_1 \cong T'_1$ and $T_2 \cong T'_2$ imply $T_1 \hat{\cdot} T_2 \cong T'_1 \hat{\cdot} T'_2$, $T_1 \hat{\cdot} T_2 \cong T'_1 \hat{\cdot} T'_2$, $\alpha \hat{\cdot} T_1 \cong \alpha \hat{\cdot} T'_1$, $T_1 \hat{\cdot} T_2 \cong T'_1 \hat{\cdot} T'_2$, $T_1 \hat{\cdot} T_2 \cong T'_1 \hat{\cdot} T'_2$, and $\delta_H(T_1) \cong \delta_H(T'_1)$.

**Proof.** For all operations except $\hat{\cdot}$, witnessing splitting bisimulations are constructed in the same way as in the congruence proofs for the corresponding operations on $\text{CTS}_\kappa$ given in [9]. For $\hat{\cdot}$, the construction of a witnessing splitting bisimulation is easier than in [9].

1 Let $R_1$ and $R_2$ be splitting bisimulations witnessing $T_1 \cong T'_1$ and $T_2 \cong T'_2$, respectively. Then we construct relations $R_{\hat{\cdot}}$ as follows:

$$R_{\hat{\cdot}} = \{((s_1, s_2), (s'_1, s'_2)) \mid (s_1, s'_1) \in R_1, (s_2, s'_2) \in R_2\}.$$

Given the definition of parallel composition, it is easy to see that $R_{\hat{\cdot}}$ is a splitting bisimulation witnessing $T_1 \hat{\cdot} T_2 \cong T'_1 \hat{\cdot} T'_2$. \Box

The full splitting bisimulation models $\text{CT}^{\kappa \times \kappa}$, one for each infinite cardinal $\kappa$, consist of the following:

1 Because the relation constructed in [9] is by mistake the same as the one constructed in this paper, we should actually say “in the revision of [9] that can be found at www.win.tue.nl/~keesm/sbrc.pdf.”
a set \( \mathcal{P} \), called the domain of \( \Psi_\kappa^c \);

- for each constant \( c \) of \( \text{ACP}_c^\varepsilon \), an element \( \tilde{c} \) of \( \mathcal{P} \);

- for each \( n \)-ary operator \( f \) of \( \text{ACP}_c^\varepsilon \), an \( n \)-ary operation \( \tilde{f} \) on \( \mathcal{P} \);

where those ingredients are defined as follows:

\[
\mathcal{P} = \text{CTS}_\kappa^c / \equiv,
\]

\[
\tilde{\delta} = \left[ \Delta \right], \quad \alpha \vdash \left[ T_1 \right] = \left[ \alpha : \vdash T_1 \right].
\]

\[
\tilde{\varepsilon} = \left[ \varepsilon \right], \quad [T_1] \parallel [T_2] = [T_1 \parallel T_2].
\]

\[
\tilde{a} = \left[ \tilde{\alpha} \right], \quad [T_1] \parallel [T_2] = [T_1 \parallel T_2].
\]

\[
[T_1] \parallel [T_2] = [T_1 \parallel T_2].
\]

\[
[T_1] \parallel [T_2] = [T_1 \parallel T_2].
\]

\[
\tilde{\partial}_H([T_1]_{\equiv}) = [\tilde{\partial}_H(T_1)]_{\equiv}.
\]

The operations on \( \text{CTS}_\kappa^c / \equiv \) are well-defined because \( \equiv \) is a congruence with respect to the corresponding operations on \( \text{CTS}_\kappa^c \).

The structures \( \Psi_\kappa^c \) are models of \( \text{ACP}_c^\varepsilon \).

**Theorem 4.1 (Soundness of ACP^\varepsilon).** For each infinite cardinal \( \kappa \), we have \( \Psi_\kappa^c \models \text{ACP}_c^\varepsilon \).

**Proof.** Because \( \equiv \) is a congruence, it is sufficient to show that all additional axioms are sound. The soundness of all additional axioms follows easily from the definition of \( \Psi_\kappa^c \). \( \square \)

For all axioms that are in common with \( \text{ACP}_c^\varepsilon \), the proof of soundness with respect to \( \Psi_\kappa^c \) follows the same line as the proof of soundness with respect to \( \Psi_\kappa^c \).

The full splitting bisimulation models are related by isomorphic embeddings.

**Theorem 4.2 (Isomorphic Embedding).** Let \( \kappa \) and \( \kappa' \) be infinite cardinals such that \( \kappa < \kappa' \). Then \( \Psi_\kappa^c \) is isomorphically embedded in \( \Psi_{\kappa'}^c \).

**Proof.** The proof is analogous to the proof of the corresponding property for the full splitting bisimulation models of \( \text{ACP}^\varepsilon \) given in [9]. \( \square \)

### 5 SOS-Based Splitting Bisimilarity for ACP^\varepsilon

It is customary to associate transition systems with closed terms of the language of an ACP-like theory about processes by means of structural operational semantics and to identify closed terms if their associated transition systems are splitting bisimilar.

The structural operational semantics of \( \text{ACP}_c^\varepsilon \) presented in Section 2 determines a conditional transition system for each process that can be denoted by
a closed term of sort $P$. These transition systems are special in the sense that their states are closed terms of sort $P$.

Let $p$ be a closed term of sort $P$. Then the transition system of $p$ induced by the structural operational semantics of $\text{ACP}_c$, written $\text{CTS}(p)$, is the connected conditional transition system $\Gamma(S, \rightarrow, \downarrow, s^0)$, where:

- $S$ is the set of all closed terms of sort $P$;
- the sets $\frac{\alpha \cdot a}{\alpha} \subseteq S \times S$ and $\frac{\alpha}{\bot} \subseteq S$ for each $\alpha \in C \setminus \{\bot\}$ and $a \in A$ are the smallest subsets of $S \times S$ and $S$, respectively, for which the transition rules from Table 2 hold;
- $s^0 \in S$ is the closed term $p$.

Let $p$ and $q$ be closed terms of sort $P$. Then we say that $p$ and $q$ are splitting bisimilar, written $p \equiv q$, if $\text{CTS}(p) \equiv \text{CTS}(q)$.

Clearly, the structural operational semantics does not give rise to infinitely branching conditional transition systems. For each closed term $p$ of sort $P$, there exists a $T \in \text{CTS}_\aleph_0$ such that $\text{CTS}(p) \sim T$. In Section 4, it has been shown that it is possible to consider infinitely branching conditional transition systems as well.

6 Guarded Recursion

In order to allow for the description of (potentially) non-terminating processes, we add guarded recursion to $\text{ACP}_c$.

A recursive specification over $\text{ACP}_c$ is a set of equations $E = \{X = t_X \mid X \in V\}$ where $V$ is a set of variables and each $t_X$ is a term of sort $P$ that only contains variables from $V$. We write $V(E)$ for the set of all variables that occur on the left-hand side of an equation in $E$. A solution of a recursive specification $E$ is a set of processes (in some model of $\text{ACP}_c$) $\{P_X \mid X \in V(E)\}$ such that the equations of $E$ hold if, for all $X \in V(E)$, $X$ stands for $P_X$.

Let $t$ be a term of sort $P$ containing a variable $X$. We call an occurrence of $X$ in $t$ guarded if $t$ has a subterm of the form $a \cdot t'$ containing this occurrence of $X$. A recursive specification over $\text{ACP}_c$ is called a guarded recursive specification if all occurrences of variables in the right-hand sides of its equations are guarded or it can be rewritten to such a recursive specification using the axioms of $\text{ACP}_c$ and the equations of the recursive specification. We are only interested in models of $\text{ACP}_c$ in which guarded recursive specifications have unique solutions.

For each guarded recursive specification $E$ and each variable $X \in V(E)$, we introduce a constant of sort $P$ standing for the unique solution of $E$ for $X$. This constant is denoted by $\langle X \rangle E$. We often write $X$ for $\langle X \rangle E$ if $E$ is clear from the context. In such cases, it should also be clear from the context that we use $X$ as a constant.

We will also use the following notation. Let $t$ be a term of sort $P$ and $E$ be a guarded recursive specification over $\text{ACP}_c$. Then we write $\langle t \rangle E$ for $t$ with, for all $X \in V(E)$, all occurrences of $X$ in $t$ replaced by $\langle X \rangle E$.

The additional axioms for recursion are the equations given in Table 3. Both
Table 3. Axioms for recursion

\[
\langle X|E \rangle = \langle tX|E \rangle \quad \text{if} \ X = tX \in E \quad \text{RDP}
\]

\[
E \Rightarrow X = \langle X|E \rangle \quad \text{if} \ X \in V(E) \quad \text{RSP}
\]

Table 4. Transition rules for recursion

\[
\begin{array}{c}
\langle tX|E \rangle \xrightarrow{\phi} \langle X|E \rangle \\
\langle X|E \rangle \xrightarrow{\phi} x' \quad \text{if} \ X = tX \in E
\end{array}
\]

RDP and RSP are axiom schemas. A side condition is added to restrict the variables, terms and guarded recursive specifications for which \(X, tX\) and \(E\) stand. The additional axioms for recursion are known as the recursive definition principle (RDP) and the recursive specification principle (RSP). The equations \(\langle X|E \rangle = \langle tX|E \rangle\) for a fixed \(E\) express that the constants \(\langle X|E \rangle\) make up a solution of \(E\). The conditional equations \(E \Rightarrow X = \langle X|E \rangle\) express that this solution is the only one.

The structural operational semantics for the constants \(\langle X|E \rangle\) is described by the transition rules given in Table 4.

In the full splitting bisimulation models of \(ACP_c^\epsilon\), guarded recursive specifications have unique solutions.

**Theorem 6.1 (Unique solutions in \(P_k^\epsilon\)).** For each infinite cardinal \(\kappa\), guarded recursive specifications over \(ACP_c^\epsilon\) have unique solutions in \(P_k^\epsilon\).

**Proof.** The proof is analogous to the proof of the corresponding property for the full splitting bisimulation models of \(ACP^\epsilon\) given in [9].

Thus, the full splitting bisimulation models \(P_k^{\epsilon'}\) of \(ACP_c^\epsilon\) with guarded recursion are simply the expansions of the full splitting bisimulation models \(P_k^\epsilon\) of \(ACP^\epsilon\) obtained by associating with each constant \(\langle X|E \rangle\) the unique solution of \(E\) for \(X\) in the full splitting bisimulation model concerned.

7 Evaluation of Conditions

Guarded commands cannot always be eliminated from closed terms of sort \(P\) because conditions different from both \(\bot\) and \(\top\) may be involved. The condition evaluation operators introduced below, can be brought into action in such cases. These operators require to fix an infinite cardinal \(\lambda\). By doing so, full splitting bisimulation models with domain \(CTS_\kappa/\leq\) for \(\kappa > \lambda\) are excluded.

There are unary \(\lambda\)-complete condition evaluation operators \(CE_h: P \to P\) and \(CE_h: C \to C\) for each \(\lambda\)-complete endomorphisms \(h\) of \(C_\lambda\).

These operators can be explained as follows: \(CE_h(p)\) behaves as \(p\) with each condition \(\zeta\) occurring in \(p\) replaced according to \(h\). If the image of \(C_\lambda\) under \(h\)

\[\text{For a definition of \(\kappa\)-complete endomorphisms, see e.g. [17].}\]
Table 5. Axioms for condition evaluation ($\alpha \in \mathcal{A}_\lambda$, $\eta \in \mathcal{C}_\alpha$, $\eta' \in \mathcal{C}_\alpha \cup \{\bot, \top\}$)

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{CE}_h(\epsilon) = \epsilon$</td>
<td>CE1T</td>
</tr>
<tr>
<td>$\text{CE}_h(a \cdot x) = a \cdot \text{CE}_h(x)$</td>
<td>CE2</td>
</tr>
<tr>
<td>$\text{CE}_h(x + y) = \text{CE}_h(x) + \text{CE}_h(y)$</td>
<td>CE3</td>
</tr>
<tr>
<td>$\text{CE}_h(\phi ::= x) = \text{CE}_h(\phi) ::= \text{CE}_h(x)$</td>
<td>CE4</td>
</tr>
<tr>
<td>$\text{CE}<em>h(\text{CE}<em>h(x)) = \text{CE}</em>{h</em>{\alpha_{\eta'}}}(x)$</td>
<td>CE5</td>
</tr>
<tr>
<td>$\text{CE}_h(\bot) = \bot$</td>
<td>CE6</td>
</tr>
<tr>
<td>$\text{CE}_h(\top) = \top$</td>
<td>CE7</td>
</tr>
<tr>
<td>$\text{CE}_h(\eta) = \eta'$ if $h(\eta) = \eta'$</td>
<td>CE8</td>
</tr>
<tr>
<td>$\text{CE}_h(-\phi) = -\text{CE}_h(\phi)$</td>
<td>CE9</td>
</tr>
<tr>
<td>$\text{CE}_h(\phi \sqcup \psi) = \text{CE}_h(\phi) \sqcup \text{CE}_h(\psi)$</td>
<td>CE10</td>
</tr>
<tr>
<td>$\text{CE}_h(\phi \sqcap \psi) = \text{CE}_h(\phi) \sqcap \text{CE}_h(\psi)$</td>
<td>CE11</td>
</tr>
</tbody>
</table>

Table 6. Transition rules for condition evaluation

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \vdash [\phi]_1$</td>
<td>$h(\phi) \neq \bot$</td>
</tr>
<tr>
<td>$x \vdash [\phi]_{\alpha}$</td>
<td>$h(\phi) \neq \bot$</td>
</tr>
</tbody>
</table>

is $\mathbb{B}$, i.e. the Boolean algebra with domain $\{\bot, \top\}$, then guarded commands can be eliminated from $\text{CE}_h(p)$. In the case where the image of $C_\lambda$ under $h$ is not $\mathbb{B}$, $\text{CE}_h$ can be regarded to evaluate the conditions only partially.

Henceforth, we write $\mathcal{H}_\lambda$ for the set of all $\lambda$-complete endomorphisms of $C_\lambda$.

The additional axioms for $\text{CE}_h$, where $h \in \mathcal{H}_\lambda$, are the axioms given in Table 5.

The structural operational semantics of $\text{ACP}_\lambda$ extended with condition evaluation is described by the transition rules for $\text{ACP}_\lambda$ and the transition rules given in Table 6.

If $\lambda$ is a regular infinite cardinal, the elements of $C_\lambda$ can be used to represent equivalence classes with respect to logical equivalence of the set of all propositions with elements of $\mathcal{C}_\alpha$ as propositional variables and with conjunctions and disjunctions of less than $\lambda$ propositions. We write $\mathcal{P}_\lambda$ for this set of propositions. If $\lambda$ is a regular infinite cardinal, it is likely that there is a theory $\Phi$ about the atomic conditions in the shape of a set of propositions. Let $\Phi \subset \mathcal{P}_\lambda$, and let $h_\Phi \in \mathcal{H}_\lambda$ be such that for all $\alpha, \beta \in C_\lambda$:

$$\Phi \vdash \langle \langle h_\Phi(\alpha) \rangle \rangle \iff \langle \langle \alpha \rangle \rangle \quad \text{and} \quad h_\Phi(\alpha) = h_\Phi(\beta) \iff \Phi \vdash \langle \langle \alpha \rangle \rangle \iff \langle \langle \beta \rangle \rangle$$

(1)

where $\langle \langle \alpha \rangle \rangle$ is a representative of the equivalence class of propositions isomorphic to $\alpha$. Then we have $h_\Phi(\alpha) = \top$ iff $\langle \langle \alpha \rangle \rangle$ is derivable from $\Phi$ and $\neg \langle \langle \alpha \rangle \rangle$ is derivable from $\Phi$. The image of $C_\lambda$ under $h_\Phi$ is $\mathbb{B}$ iff $\Phi$ is a complete theory. If $\Phi$ is not a complete theory, then $h_\Phi$ is not uniquely determined by (1).

However, the images of $C_\lambda$ under the different endomorphisms satisfying (1) are isomorphic subalgebras of $C_\lambda$. Moreover, if both $h$ and $h'$ satisfy (1), then $\Phi \vdash \langle \langle h(\alpha) \rangle \rangle \iff \langle \langle h'(\alpha) \rangle \rangle$ for all $\alpha \in C_\lambda$.

Below, we show that condition evaluation on the basis of a complete theory can be viewed as substitution on the basis of the theory. That leads us to the use of the following convention: for $\alpha \in \mathcal{C}$, $\alpha$ stands for an arbitrary closed term of sort $\mathcal{C}$ of which the value in $\mathcal{C}$ is $\alpha$. 

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Proposition 7.1 (Condition evaluation on the basis of a theory). Assume that \( \lambda \) is a regular infinite cardinal. Let \( \Phi \subset \mathcal{P}_\lambda \) be a complete theory and let \( p \) be a closed term of sort \( \mathcal{P} \). Then \( \text{CE}_{h_\Phi}(p) = p' \) where \( p' \) is \( p \) with, for all \( \alpha \in C \), in all subterms of the form \( \overline{\alpha} : \rightarrow q \), \( \overline{\alpha} \) replaced by \( \top \) if \( \Phi \vdash \langle \langle \alpha \rangle \rangle \) and \( \overline{\alpha} \) replaced by \( \bot \) if \( \Phi \vdash \neg \langle \langle \alpha \rangle \rangle \).

Proof. This result follows immediately from the definition of \( h_\Phi \) and the distributivity of \( \text{CE}_{h_\Phi} \) over all operators of \( \text{ACP}^c \). \( \square \)

In \( \mu \text{CRL} \) [14], an extension of \( \text{ACP} \) which includes conditional expressions, we find a formalization of the substitution-based alternative for \( \text{CE}_{h_\Phi} \).

The substitution-based alternative works properly because condition evaluation by means of a \( \lambda \)-complete condition evaluation operator is not dependent on process behaviour. Hence, the result of condition evaluation is globally valid. Below, we will generalize the condition evaluation operators introduced above in such a way that condition evaluation may be dependent on process behaviour. In that case, the result of condition evaluation is in general not globally valid.

Remark 7.1. Assume that \( \lambda \) is a regular infinite cardinal. Let \( h \in \mathcal{H}_\lambda \). Then \( h \) induces a theory \( \Phi \subset \mathcal{P}_\lambda \) such that \( h = h_\Phi \), viz. the theory \( \Phi \) defined by

\[
\Phi = \{ \langle \langle h(\alpha) \rangle \rangle \iff \langle \langle \alpha \rangle \rangle \mid \alpha \in C_\lambda \} \cup \{ \langle \langle \beta \rangle \rangle \mid h(\alpha) = h(\beta) \}.
\]

Consequently, if \( \lambda \) is a regular infinite cardinal, condition evaluation by means of the \( \lambda \)-complete condition evaluation operators introduced above is always condition evaluation of which the result can be determined from a set of propositions. We will return to this observation in Section 9.

We proceed with generalizing the condition evaluation operators introduced above. It is assumed that a fixed but arbitrary function \( \text{eff} : \mathcal{A} \times \mathcal{H}_\lambda \to \mathcal{H}_\lambda \) has been given.

There is a unary generalized \( \lambda \)-complete condition evaluation operator \( \text{GCE}_h : \mathcal{P} \to \mathcal{P} \) for each \( h \in \mathcal{H}_\lambda \); and there is again the unary operator \( \text{CE}_h : \mathcal{C} \to \mathcal{C} \) for each \( h \in \mathcal{H}_\lambda \).

The \( \lambda \)-complete generalized condition evaluation operator \( \text{GCE}_h \) allows, given the function \( \text{eff} \), to evaluate conditions dependent of process behaviour. The function \( \text{eff} \) gives, for each action \( a \) and \( \lambda \)-complete endomorphism \( h \), the \( \lambda \)-complete endomorphism \( h' \) that represents the changed results of condition evaluation due to performing \( a \). The function \( \text{eff} \) is extended to \( \mathcal{A}_\delta \) such that \( \text{eff}(\delta, h) = h \) for all \( h \in \mathcal{H}_\lambda \).

The additional axioms for \( \text{GCE}_h \), where \( h \in \mathcal{H}_\lambda \), are the axioms given in Table 7 and axioms CE6–CE11 from Table 5.

The structural operational semantics of \( \text{ACP}^c \) extended with generalized condition evaluation is described by the transition rules for \( \text{ACP}^c \) and the transition rules given in Table 8.

We can add both the \( \lambda \)-complete condition evaluation operators and the generalized \( \lambda \)-complete condition evaluation operators to \( \text{ACP}^c \). However, Proposition 7.2 stated below makes it clear that the latter operators supersede the former operators.
Table 7. Axioms for generalized condition evaluation \((a \in A)\)

<table>
<thead>
<tr>
<th>Expression</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{GCE}_h(\epsilon) = \epsilon)  (\text{GCE1T})</td>
<td></td>
</tr>
<tr>
<td>(\text{GCE}<em>h(a \cdot x) = a \cdot \text{GCE}</em>{\text{eff}(a, h)}(x))</td>
<td>(\text{GCE2})</td>
</tr>
<tr>
<td>(\text{GCE}_h(x + y) = \text{GCE}_h(x) + \text{GCE}_h(y))  (\text{GCE3})</td>
<td></td>
</tr>
<tr>
<td>(\text{GCE}_h(\phi \rightarrow x) = \text{CE}_h(\phi) \rightarrow \text{GCE}_h(x))  (\text{GCE4})</td>
<td></td>
</tr>
</tbody>
</table>

Table 8. Transition rules for generalized condition evaluation

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x[{\phi}]\quad h(\phi) \neq \bot)</td>
<td>(1)</td>
</tr>
<tr>
<td>(GCE_h(x)[{h(\phi)}]\quad h(\phi) \neq \bot)</td>
<td>(2)</td>
</tr>
<tr>
<td>(GCE_h(x)[{h(\phi)}] \rightarrow_{a} \text{GCE}_{\text{eff}(a, h)}(x'))</td>
<td>(3)</td>
</tr>
</tbody>
</table>

The full splitting bisimulation models of \(\text{ACP}_c^\epsilon\) with condition evaluation and/or generalized condition evaluation are simply the expansions of the full splitting bisimulation models \(\Psi_{\kappa}^{\text{set}}\) of \(\text{ACP}_c^\epsilon\), for infinite cardinals \(\kappa \leq \lambda\), obtained by associating with each operator \(\text{CE}_h\) and/or \(\text{GCE}_h\) the corresponding re-labeling operation on conditional transition systems. As mentioned before, full splitting bisimulation models with domain \(\text{CTS}_{\kappa}/\equiv\) for \(\kappa > \lambda\) are excluded.

The equation \(\text{CE}_h(\text{CE}_{h'}(x)) = \text{CE}_{h \circ h'}(x)\) is an axiom, but the equation \(\text{GCE}_h(\text{GCE}_{h'}(x)) = \text{GCE}_{h \circ h'}(x)\) is not an axiom. The reason is that the latter equation is only valid if \(\text{eff}\) satisfies \(\text{eff}(a, h \circ h') = \text{eff}(a, h) \circ \text{eff}(a, h')\) for all \(a \in A\) and \(h, h' \in H^\lambda\).

As their name suggests, the generalized \(\lambda\)-complete condition evaluation operators are generalizations of the \(\lambda\)-complete condition evaluation operators.

**Proposition 7.2 (Generalization).** We can fix the function \(\text{eff}\) such that \(\text{GCE}_h(x) = \text{CE}_h(x)\) for all \(h \in H^\lambda\).

*Proof.* Clearly, if \(\text{eff}(a, h') = h'\) for all \(a \in A\) and \(h' \in H^\lambda\), then \(\text{GCE}_h(x) = \text{CE}_h(x)\) for all \(h \in H^\lambda\). \(\square\)

The \(\lambda\)-complete state operators that are added to \(\text{ACP}_c^\epsilon\) in Section 8 are in their turn generalizations of the generalized \(\lambda\)-complete condition evaluation operators.

We come back to the \(\lambda\)-complete condition evaluation operators \(\text{CE}_h\) for \(h \in H^\lambda\). The image of \(C^\lambda\) under the \(\lambda\)-complete endomorphism \(h\) is a subalgebra of \(C^\lambda\) that is \(\lambda\)-complete too. For that reason, we could have used \(\lambda\)-complete homomorphisms to subalgebras that are \(\lambda\)-complete instead of \(\lambda\)-complete endomorphisms. It would go beyond the models of the theory developed so far to generalize this in such a way that \(\lambda\)-complete homomorphisms to \(\lambda\)-complete Boolean algebras different from subalgebras of \(C^\lambda\) are also included.

However, in the case where we consider \(\lambda\)-complete homomorphisms between free \(\lambda\)-complete Boolean algebras over different sets of generators, we can relate the models for different choices for \(C_{\lambda t}\).
Let $C$ and $C'$ be different choices for $C_{at}$, and let $\mathfrak{P}_\kappa^c(C)$ and $\mathfrak{P}_\kappa^c(C')$, for $\kappa \leq \lambda$, be the full splitting bisimulation models $\mathfrak{P}_\kappa^c$ of ACP$_c$ for the different choices for $C_{at}$. Moreover, let $h$ be a $\lambda$-complete homomorphism from the free $\lambda$-complete Boolean algebra over $C$ to the free $\lambda$-complete Boolean algebra over $C'$. Then $h$ can be extended to a homomorphism $h^*$ from $\mathfrak{P}_\kappa^c(C)$ to $\mathfrak{P}_\kappa^c(C')$. This homomorphism is defined by

$$h^*([((S, \rightarrow, \downarrow, s^0)]_{\mathfrak{P}_\kappa^c}) = [\Gamma(S, \rightarrow', \downarrow', s^0)]_{\mathfrak{P}_\kappa^c} ,$$

where for every $(\alpha, a) \in C^-_{\kappa} \times A$ and $\alpha' \in C^-_{\kappa'}$:

$$\langle (\alpha, a) \rangle' = \{(s, s') | \exists \beta \cdot s \stackrel{[\beta]_a}{\rightarrow} s' \land \alpha = h(\beta) \} ,$$

$$[\alpha']' = \{s | \exists \beta \cdot s \stackrel{[\beta]}{\rightarrow} \land \alpha' = h(\beta) \} .$$

It is easy to see that $h^*$ is well-defined and a homomorphism indeed.

Thus, a $\lambda$-complete homomorphism between $\lambda$-complete Boolean algebras over different sets of generators can be used to translate conditions throughout a full splitting bisimulation model for one choice of $C_{at}$ in such a way that a full splitting bisimulation model for a different choice of $C_{at}$ is obtained.

8 State Operators

The state operators make it easy to represent the execution of a process in a state. The basic idea is that the execution of an action in a state has effect on the state, i.e. it causes a change of state. Besides, there is an action left when an action is executed in a state. The operators introduced here generalize the state operators added to ACP in [1]. The main difference with those operators is that guarded commands are taken into account. As in the case of the condition evaluation operators and the generalized condition evaluation operators, these state operators require to fix an infinite cardinal $\lambda$. By doing so, full splitting bisimulation models with domain $\mathcal{CTS}_{\kappa}/\mathcal{E}$ for $\kappa > \lambda$ are excluded.

It is assumed that a fixed but arbitrary set $S$ of states has been given, together with functions $\text{act} : A \times S \rightarrow A_S$, $\text{eff} : A \times S \rightarrow S$ and $\text{eval} : C_{\lambda} \times S \rightarrow C_{\lambda}$, where, for each $s \in S$, the function $h_s : C_{\lambda} \rightarrow C_{\lambda}$ defined by $h_s(\alpha) = \text{eval}(\alpha, s)$ is a $\lambda$-complete endomorphism of $C_{\lambda}$.

There are unary $\lambda$-complete state operators $\lambda_s : P \rightarrow P$ and $\lambda_a : C \rightarrow C$ for each $s \in S$.

The $\lambda$-complete state operator $\lambda_s$ allows, given the above-mentioned functions, processes to interact with a state. Let $p$ be a process. Then $\lambda_s(p)$ is the process $p$ executed in state $s$. The function $\text{act}$ gives, for each action $a$ and state $s$, the action that results from executing $a$ in state $s$. The function $\text{eff}$ gives, for

---

6 The interesting cases are those where the cardinalities of $C$ and $C'$ are different. Otherwise, the homomorphisms are isomorphisms.

7 Holding on to the usual conventions leads to the double use of the symbol $\lambda$: without subscript it stands for an infinite cardinal, and with subscript it stands for a state operator.
The functions $\text{eval}$ gives, for each condition $\alpha$ and state $s$, the condition that results from evaluating $\alpha$ in state $s$. The functions $\text{act}$ and $\text{eff}$ are extended to $A_\lambda$ such that $\text{act}(\delta, s) = \delta$ and $\text{eff}(\delta, s) = s$ for all $s \in S$.

The additional axioms for $\lambda_s$, where $s \in S$, are the axioms given in Table 9.

The structural operational semantics of $\text{ACP}_\lambda$ extended with state operators is described by the transition rules for $\text{ACP}_\lambda^c$ and the transition rules given in Table 10.

The full splitting bisimulation models of $\text{ACP}_\lambda^c$ with state operators are simply the expansions of the full splitting bisimulation models $\Psi_{\lambda}^c$ of $\text{ACP}_\lambda^c$ obtained by associating with each operator $\lambda_a$ the corresponding re-labeling operation on conditional transition systems.

We can add, in addition to the $\lambda$-complete state operators, the $\lambda$-complete condition evaluation operators and/or the generalized $\lambda$-complete condition evaluation operators from Section 7 to $\text{ACP}_\lambda^c$.

We write $\Psi_{\lambda}^{c,\text{ext}}$ for the expansion of $\Psi_{\lambda}^c$ for the $\lambda$-complete condition evaluation operators, the generalized $\lambda$-complete condition evaluation operators and the $\lambda$-complete state operators.

The $\lambda$-complete state operators are generalizations of the generalized $\lambda$-complete condition evaluation operators from Section 7.

**Proposition 8.1 (Generalization).** We can fix $S$, $\text{act}$, $\text{eff}$ and $\text{eval}$ such that, for some $f: \mathcal{H}_\lambda \rightarrow S$, $\lambda_{f(h)}(x) = \text{GCE}_h(x)$ holds for all $h \in \mathcal{H}_\lambda$ in all full splitting bisimulation models $\Psi_{\lambda}^{c,\text{ext}}$ with $\kappa \leq \lambda$.

**Proof.** Clearly, if $S = \mathcal{H}_\lambda$, $f$ is the identity function on $\mathcal{H}_\lambda$, and $\text{act}(a, s) = a$, $\text{eff}(a, s) = \text{eff}(a, f^{-1}(s))$ and $\text{eval}(\alpha, s) = f^{-1}(s)(\alpha)$ for all $a \in A$, $s \in S$ and

| $\lambda_a(\epsilon) = \epsilon$ | SOIT | $\lambda_a(\bot) = \bot$ | SO5 |
| $\lambda_a(a \cdot x) = \text{act}(a, s) \cdot \lambda_{\text{eff}(a, s)}(x)$ | SO2 | $\lambda_a(\top) = \top$ | SO6 |
| $\lambda_a(x + y) = \lambda_a(x) + \lambda_a(y)$ | SO3 | $\lambda_a(\eta) = \eta'$ if $\text{eval}(\eta, s) = \eta'$ | SO7 |
| $\lambda_a(\phi :\rightarrow x) = \lambda_a(\phi) :\rightarrow \lambda_a(x)$ | SO4 | $\lambda_a(\neg \phi) = \neg \lambda_a(\phi)$ | SO8 |
| $\lambda_a(\phi \sqcup \psi) = \lambda_a(\phi) \sqcup \lambda_a(\psi)$ | SO9 | $\lambda_a(\phi \sqcap \psi) = \lambda_a(\phi) \sqcap \lambda_a(\psi)$ | SO10 |

**Table 9.** Axioms for state operators ($a \in A_\lambda$, $\eta \in C_\lambda$, $\eta' \in C_\lambda \cup \{\bot, \top\}$)

**Table 10.** Transition rules for state operators

- $x^{[\eta]} \quad \text{eval}(\phi, s) \neq \bot$
- $x^{[\phi]}_{\lrcorner} \quad \text{act}(a, s) \neq \delta$, $\text{eval}(\phi, s) \neq \bot$

$\lambda_a(x) = \text{eval}(\phi, s)$

$\lambda_a(x) = \text{act}(a, s) \cdot \lambda_{\text{eff}(a, s)}(x)$

Each action $a$ and state $s$, the state that results from executing $a$ in state $s$. The function $\text{eval}$ gives, for each condition $\alpha$ and state $s$, the condition that results from evaluating $\alpha$ in state $s$. The functions $\text{act}$ and $\text{eff}$ are extended to $A_\lambda$ such that $\text{act}(\delta, s) = \delta$ and $\text{eff}(\delta, s) = s$ for all $s \in S$.
\( \alpha \in C_\lambda \), then \( \lambda f(h)(x) = GCE_h(x) \) holds for all \( h \in H_\lambda \) in all full splitting bisimulation models \( \mathcal{P}_k^{\text{ext}} \) with \( \kappa \leq \lambda \). \( \square \)

9 Signal Emission

In Section 7, we made the observation that, if \( \lambda \) is a regular infinite cardinal, condition evaluation by means of the \( \lambda \)-complete condition evaluation operators \( CE_h \) from that section is always condition evaluation of which the result can be determined from a set of propositions (see Remark 7.1). A similar observation can be made about condition evaluation by means of the generalized \( \lambda \)-complete condition evaluation operators \( GCE_h \) from that section. In the case of condition evaluation by means of \( CE_h \), the set of propositions determining the result of condition evaluation does not change as a process proceeds. In the case of condition evaluation by means of \( GCE_h \), it may happen that the set of propositions determining the result of condition evaluation changes as a process proceeds. That is, the sets of propositions relevant to a process and its subprocesses may differ. This suggest that condition evaluation can also be dealt with by explicitly associating sets of propositions with processes. The intuition is, then, that all propositions from the set of propositions associated with a process holds at the start of the process.

Clearly, if we restrict ourselves to sets of propositions of cardinality less than a regular infinite cardinal \( \lambda \), we can associate elements of \( C_\lambda \) with processes instead. In line with [2], the element of \( C_\lambda \) associated with a process is called the signal emitted by the process. Because \( \perp \) represents the proposition \( F \), the proposition that cannot hold at the start of any process, we regard a process with which \( \perp \) is associated as an inconsistency. However, in an algebraic setting, we cannot exclude this inconsistency. Therefore, we consider it to be a special process, which is called the inaccessible process.\(^8\)

The idea to associate elements of \( C_\lambda \) with processes naturally suggests itself in the case where \( \lambda \) is a regular infinite cardinal. However, there are no trammels to drop the restriction that \( \lambda \) is regular.

All this leads us to an extension of \( ACP^e_\epsilon \), called \( ACP^{cs}_\epsilon \), with the following additional constants and operators:

- the inaccessible process constant \( \perp : P \);
- the binary signal emission operator \( \bowtie : C \times P \rightarrow P \).

The axioms of \( ACP^{cs}_\epsilon \) are the axioms of \( ACP^e_\epsilon \) with axioms CM2, CM3 and GC8–GC10 replaced by axioms CM2ST, CM3S and GC8S–GC10S from Table 11, and the additional axioms given in Table 12. Axioms NE1–NE3 and SE1–SE11 have been taken from [3] and axioms GC9S and GC10S have been taken from [3] with subterms of the form \( s(x) \bowtie \delta \) replaced by \( \partial_h(x) \). Axioms

\(^8\) In [12, 8], this process is rather contradictory called the non-existent process. Its new name was prompted by the fact that after performing an action no process will ever proceed as this process.
The proof is the same to the proof of the corresponding proposition in Proof.

In the case of the communication merge, always the meet of the signals emitted to the choice of having as the signal emitted by the left merge of two processes, as CM2ST, CM3S and GC8S differ really from the corresponding axioms in [3] due to the choice of having as the signal emitted by the left merge of two processes, as in the case of the communication merge, always the meet of the signals emitted by the two processes.

If \( \langle \langle \alpha \rangle \rangle \vdash \langle \langle \beta \rangle \rangle \Leftrightarrow \langle \langle \beta' \rangle \rangle \), then \( \alpha \triangleright (\beta \rightarrow x) = \alpha \triangleright (\beta' \rightarrow x) \).

The following gives a good picture of the nature of signals and conditions.

**Proposition 9.1 (Signals and conditions).** If \( \langle \langle \alpha \rangle \rangle \vdash \langle \langle \beta \rangle \rangle \), then \( \alpha \triangleright (\beta \rightarrow x) = \alpha \triangleright x \).

**Proof.** The proof is the same to the proof of the corresponding proposition in the setting of ACP^{cs} given in [9]. \(\square\)

We have the following corollaries from Proposition 9.1.

**Corollary 9.1.** If \( \langle \langle \alpha \rangle \rangle \vdash \langle \langle \beta \rangle \rangle \), then \( \alpha \triangleright (\beta \rightarrow x) = \alpha \triangleright x \). If \( \langle \langle \alpha \rangle \rangle \vdash \neg \langle \langle \beta \rangle \rangle \), then \( \alpha \triangleright (\beta \rightarrow x) = \alpha \triangleright \delta \).

Table 11. Axioms adapted to signal emission (\( \alpha \in A_3 \))

<table>
<thead>
<tr>
<th>( x \bot = \bot )</th>
<th>( x = \partial x )</th>
<th>( a \cdot x \bot = a \cdot (x \bot) + \partial x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NE1</td>
<td>CM2ST</td>
<td></td>
</tr>
<tr>
<td>( \bot \cdot x = \bot )</td>
<td>( \phi \triangleright (\psi \triangleright x) = (\phi \lor \psi \triangleright) \triangleright x )</td>
<td>SE5</td>
</tr>
<tr>
<td>SE1</td>
<td>( \phi \triangleright x = x )</td>
<td>SE8</td>
</tr>
<tr>
<td>( \partial \triangleright x = \bot )</td>
<td>( \phi \triangleright x \bot )</td>
<td>SE9</td>
</tr>
<tr>
<td>SE2</td>
<td>( \phi \triangleright x \bot )</td>
<td>SE10</td>
</tr>
<tr>
<td>( \phi \triangleright x + y = \phi \triangleright (x + y) )</td>
<td>( x \bot (\phi \triangleright y) )</td>
<td>SE11</td>
</tr>
<tr>
<td>(( \phi \triangleright x ) \cdot y = \phi \triangleright x \cdot y )</td>
<td>( \partial H (\phi \triangleright x) = \phi \triangleright \partial H (x) )</td>
<td>SE11</td>
</tr>
</tbody>
</table>

CM2ST, CM3S and GC8S differ really from the corresponding axioms in [3] due to the choice of having as the signal emitted by the left merge of two processes, as in the case of the communication merge, always the meet of the signals emitted by the two processes.

In the structural operational semantics of ACP^{cs}, unary relations \( s^\alpha \), one for each \( \alpha \in C \setminus \{ \bot \} \), are used in addition to the relations \( [\alpha] \) and \( \triangleright \). We write \( s(p) = \alpha \) instead of \( p \in s^\alpha \). The relation \( s^\alpha \) can be explained as follows:

\(- s(p) = \alpha \): \( p \) emits the signal \( \alpha \).

The structural operational semantics of ACP^{cs} is described by the transition rules given in Table 13. These transition rules include all transition rules from Table 2 with additional premises to exclude transitions from or to processes that emit the signal \( \bot \). There are additional transition rules describing the signals emitted by the processes. The transition rules for signal emission are new as well.

The following gives a good picture of the nature of signals and conditions.

**Table 12. Additional axioms for signal emission (\( \alpha \in A_3 \))**

<table>
<thead>
<tr>
<th>( x \bot = \bot )</th>
<th>( \phi \triangleright (\psi \triangleright x) = (\phi \lor \psi \triangleright) \triangleright x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NE1</td>
<td>SE5</td>
</tr>
<tr>
<td>( \bot \cdot x = \bot )</td>
<td>( \phi \triangleright x \bot )</td>
</tr>
<tr>
<td>SE1</td>
<td>( \phi \triangleright x \bot )</td>
</tr>
<tr>
<td>( \partial \triangleright x = \bot )</td>
<td>( \phi \triangleright x \bot )</td>
</tr>
<tr>
<td>SE2</td>
<td>( \phi \triangleright x \bot )</td>
</tr>
</tbody>
</table>

\( x \bot (\phi \triangleright y) \) | \( \partial H (\phi \triangleright x) = \phi \triangleright \partial H (x) \) | SE11 |
Table 13. Transition rules for $\text{ACP}_s$

\[
\begin{align*}
& x \xrightarrow{[1]} a \xrightarrow{[1]} \epsilon \\
& x \xrightarrow{[a]} s(x + y) \neq \perp & y \xrightarrow{[a]} s(x + y) \neq \perp \\
& x + y \xrightarrow{[a]} x' & x + y \xrightarrow{[a]} y' \\
& x \xrightarrow{[a]} x', s(x + y) \neq \perp & y \xrightarrow{[a]} y', s(x + y) \neq \perp \\
& x + y \xrightarrow{[a]} x' & x + y \xrightarrow{[a]} y' \\
& x \xrightarrow{[a]} x', y \xrightarrow{[a]} y' & x \xrightarrow{[a]} x', y \xrightarrow{[a]} y' \\
& x \cdot y \xrightarrow{[a]} x' & x \cdot y \xrightarrow{[a]} y' \\
& x \xrightarrow{[a]} x' & y \xrightarrow{[a]} y' \\
& \psi \xrightarrow{[a]} x \xrightarrow{[a]} x' & \phi \cap \psi \neq \perp \\
& \psi \xrightarrow{[a]} x \xrightarrow{[a]} x' & \phi \cap \psi \neq \perp \\
& x \xrightarrow{[a]} x', s(x \parallel y) \neq \perp & y \xrightarrow{[a]} y', s(x \parallel y) \neq \perp \\
& x \parallel y \xrightarrow{[a]} x' \parallel y & x \parallel y \xrightarrow{[a]} x' \parallel y \\
& x \xrightarrow{[a]} x', y \xrightarrow{[a]} y', s(x \parallel y) \neq \perp & s(x' \parallel y') \neq \perp \\
& x \parallel y \xrightarrow{[a]} x' \parallel y' \parallel y & a = b = c, \phi \cap \psi \neq \perp \\
& x \xrightarrow{[a]} x', s(x \parallel y) \neq \perp & y \xrightarrow{[a]} y' \\
& x \parallel y \xrightarrow{[a]} x' \parallel y' & a = b = c, \phi \cap \psi \neq \perp \\
& x \xrightarrow{[a]} x' & x \xrightarrow{[a]} x' \\
& \partial H(x) \xrightarrow{[a]} \partial H(x) & a \notin H \\
& s(\perp) = \perp & s(\top) = \top \\
& s(a) = \top & s(x + y) = \phi \cap \psi \\
& s(x \cdot y) = \phi & s(x) = \phi, s(y) = \psi \\
& s(x \parallel y) = \phi & s(\psi : \rightarrow y) = \phi \parallel \phi \\
& s(x) = \phi, s(y) = \psi & s(x) = \phi, s(y) = \psi \\
& s(x \parallel y) = \phi \parallel \psi & s(x \parallel y) = \phi \parallel \psi \\
& s(\phi \parallel \psi) = \phi & s(\partial H(x)) = \phi
\end{align*}
\]

Corollary 9.2. If $\text{eff}(h, a)$ is the identity endomorphism on $C$ for all endomorphisms $h$ on $C$ and $a \in A$, then we have $GCE_{\text{eff}(h, a)}(\beta : \rightarrow x) = \beta' : \rightarrow GCE_{\text{eff}(h, a)}(x)$ implies $\alpha \times (\beta : \rightarrow x) = \alpha \times (\beta' : \rightarrow x)$.
10 Full Signal-Observing Splitting Bisimulation Models of $\text{ACP}^\epsilon_c$

In this section, we introduce conditional transition systems with signals, signal-observing splitting bisimilarity of conditional transition systems with signals, and the full signal-observing splitting bisimulation models of $\text{ACP}^\epsilon_c$.

Conditional transition systems with signals generalize conditional transition systems.

Let $\kappa$ be an infinite cardinal. Then a $\kappa$-conditional transition system with signals $T$ is a tuple $(S, \rightarrow, \downarrow, s, s^0)$ where

- $(S, \rightarrow, \downarrow, s, s^0)$ is a $\kappa$-conditional transition system;
- $s$ is a function from $S$ to $C^-_\kappa$;

and for all $\ell \in C^-_\kappa \times A$ and $\alpha \in C^-_\kappa$:

- $\{(s, s') \in S | s(s) = \bot \lor s(s') = \bot\} = \emptyset$;
- $\{s \in [\alpha] | s(s) = \bot\} = \emptyset$.

We say that $s(s)$ is the signal emitted by the state $s$.

For conditional transition systems with signals, reachability and connectedness are defined exactly as for conditional transition systems.

Let $(S, \rightarrow, \downarrow, s, s^0)$ be a $\kappa$-conditional transition system with signals (for an infinite cardinal $\kappa$) that is not necessarily connected. Then the connected part of $T$, written $\Gamma(T)$, is simply defined as follows:

$$\Gamma(T) = (S', \rightarrow', \downarrow', s', s^0),$$

where

$(S', \rightarrow', \downarrow', s^0) = \Gamma(S, \rightarrow, \downarrow, s^0)$,

$s'$ is the restriction of $s$ to $S'$.

Let $\kappa$ be an infinite cardinal. Then $\text{CTS}^\epsilon_c(\kappa)$ is the set of all $\kappa$-conditional transition systems with signals $(S, \rightarrow, \downarrow, s, s^0)$ for which $(S, \rightarrow, \downarrow, s, s^0) \in \text{CTS}^\epsilon_c(\kappa)$.

Isomorphism between conditional transition systems with signals is defined as between conditional transition systems, but with the additional condition that $s_1(s) = s_2(b(s))$. Splitting bisimilarity has to be adapted to the setting with signals.

Let $T_1 = (S_1, \rightarrow_1, \downarrow_1, s_1, s_1^0) \in \text{CTS}^\epsilon_c(\kappa)$, $T_2 = (S_2, \rightarrow_2, \downarrow_2, s_2, s_2^0) \in \text{CTS}^\epsilon_c(\kappa)$ (for an infinite cardinal $\kappa$). Then a signal-observing splitting bisimulation $B$ between $T_1$ and $T_2$ is a binary relation $B \subseteq S_1 \times S_2$ such that $B(s_1^0, s_2^0)$ and for all $s_1, s_2$ such that $B(s_1, s_2)$:

- $s_1(s_1) = s_2(s_2)$;
- if $s_1 \xrightarrow{[\alpha]a_1} s_1'$, then there is a set $CS_2' \subseteq C^-_\kappa \times S_2$ of cardinality less than $\kappa$ such that $s_1(s_1) \cap \alpha \subseteq \bigcup \text{dom}(CS_2')$ and for all $(\alpha', s_2') \in CS_2'$, $s_2 \xrightarrow{[\alpha']a_2} s_2'$ and $B(s_1', s_2')$;
if \( s_2 \xrightarrow{[\alpha]a} s_2' \), then there is a set \( CS'_1 \subseteq C^-_\kappa \times S_1 \) of cardinality less than \( \kappa \) such that \( s_2(s_2) \cap \alpha \subseteq \bigcup \text{dom}(CS'_1) \) and for all \( (\alpha', s'_1) \in CS'_1, s_1 \xrightarrow{[\alpha']a} s'_1 \) and \( B(s'_1, s_2') \);

- if \( s_1 \xrightarrow{[\alpha]1} \), then there is a set \( C' \subseteq C^-_\kappa \) of cardinality less than \( \kappa \) such that \( s_1(s_1) \cap \alpha \subseteq \bigcup C' \) and for all \( \alpha' \in C', s_2 \xrightarrow{[\alpha']2} \);

- if \( s_2 \xrightarrow{[\alpha]2} \), then there is a set \( C' \subseteq C^-_\kappa \) of cardinality less than \( \kappa \) such that \( s_2(s_2) \cap \alpha \subseteq \bigcup C' \) and for all \( \alpha' \in C', s_1 \xrightarrow{[\alpha']1} \).

Two conditional transition systems with signals \( T_1, T_2 \in \text{CTS}_{\kappa}^a \) are \textit{signal-observing splitting bisimilar}, written \( T_1 \cong^a T_2 \), if there exists a signal-observing splitting bisimulation \( B \) between \( T_1 \) and \( T_2 \). Let \( B \) be a signal-observing splitting bisimulation between \( T_1 \) and \( T_2 \). Then we say that \( B \) is a splitting signal-observing bisimulation \textit{witnessing} \( T_1 \cong^a T_2 \).

It is straightforward to see that \( \cong^a \) is an equivalence on \( \text{CTS}_{\kappa}^a \). Let \( T \in \text{CTS}_{\kappa}^a \). Then we write \( [T]_{\cong^a} \) for \( \{ T' \in \text{CTS}_{\kappa}^a \mid T \cong^a T' \} \), i.e. the \( \cong^a \)-equivalence class of \( T \). We write \( \text{CTS}_{\kappa}^a / \cong^a \) for the set of equivalence classes \( \{ [T]_{\cong^a} \mid T \in \text{CTS}_{\kappa}^a \} \).

The elements of \( \text{CTS}_{\kappa}^a \) and operations on \( \text{CTS}_{\kappa}^a \) to be associated with the constants and operators of \( \text{ACP}_\kappa^a \) as the elements of \( \text{CTS}_{\kappa}^a \) and operations on \( \text{CTS}_{\kappa}^a \) associated with them, but with all relations \( \xrightarrow{[\alpha]1} \) and \( \xrightarrow{a} \) restricted to states that emit a signal different from \( \bot \) and with the additional function \( s \) as suggested by the structural operational semantics of \( \text{ACP}_\kappa^a \).

We associate with the additional constant \( \bot \) an element \( \hat{\bot} \) of \( \text{CTS}_{\kappa}^a \) and with the additional operator \( \hat{\star} \) an operation \( \hat{\alpha} \) on \( \text{CTS}_{\kappa}^a \) as follows.

- \( \hat{\bot} = (\{ s^0 \}, \emptyset, \emptyset, s^0) \),
  where \( s(s^0) = \bot \).

- Let \( T = (S, \rightarrow, \bot, s, s^0) \in \text{CTS}_{\kappa}^a \). Then
  \( \alpha \hat{\star} T = \Gamma(S, \rightarrow', \hat{\bot}, s', s^0) \),
  where
  \( s'(s) = s(s) \quad \text{for } s \in S \setminus \{ s^0 \}, \)
  \( s'(s^0) = \alpha \cap s(s^0), \)
  and for every \( (\alpha, a) \in C^-_\kappa \times A \) and \( \alpha' \in C^-_\kappa : \)
  \( \xrightarrow{(\alpha, a)}' = \{ (s, s') \mid s \xrightarrow{[\alpha]a} s' \land s'(s) \neq \bot \land s'(s') \neq \bot \} \),
  \( \xrightarrow{[\alpha']1} = \{ s \mid s \xrightarrow{[\alpha']1} s' \land s'(s) \neq \bot \} . \)

We can easily show that signal-observing splitting bisimilarity is a congruence with respect to the operations on \( \text{CTS}_{\kappa}^a \) associated with the operators of \( \text{ACP}_\kappa^a \).
Proposition 10.1 (Congruence). Let \( \kappa \) be an infinite cardinal. Then for all \( T_1, T_2, T_1', T_2' \in \text{CTS}_\kappa \) and \( \alpha \in C_\kappa \), \( T_1 \equiv^s T_1' \) and \( T_2 \equiv^s T_2' \) imply \( T_1 \hat{=} T_2 \equiv^s T_1' \hat{=} T_2' \), \( T_1 \cdot \hat{=} T_2 \equiv^s T_1' \cdot \hat{=} T_2' \), \( \hat{\alpha} \equiv^s \hat{\alpha}' \), \( \hat{\alpha} \equiv_{\kappa} \hat{\alpha}' \), \( \hat{\alpha} \equiv_{\kappa} \hat{\alpha}' \), \( \hat{\alpha} \equiv_{\kappa} \hat{\alpha}' \), and \( \hat{\alpha} \equiv_{\kappa} \hat{\alpha}' \).

Proof. For \( \hat{\alpha} \equiv_{\kappa} \hat{\alpha}' \), witnessing signal-observing splitting bisimulations are constructed in the same way as witnessing bisimulations are constructed in the proof of Proposition 4.1. What remains is to construct a witnessing signal-observing splitting bisimulation for \( \hat{\alpha} \). Let \( R \) be a signal-observing splitting bisimulation witnessing \( T_1 \equiv^s T_1' \). Then we construct a relation \( R_{\hat{\alpha}} \) as follows:

\[- R_{\hat{\alpha}} = R \cap (S \times S'), \]

where \( S \) and \( S' \) are the sets of states of \( \hat{\alpha} \equiv_{\kappa} T_1 \) and \( \hat{\alpha} \equiv_{\kappa} T_1' \), respectively.

Given the definition of signal emission, it is easy to see that \( R_{\hat{\alpha}} \) is a signal-observing splitting bisimulation witnessing \( \hat{\alpha} \equiv_{\kappa} T_1 \equiv^s \hat{\alpha} \equiv_{\kappa} T_1' \).

The ingredients of the full signal-observing splitting bisimulation models \( \Psi_{\kappa}^{\text{cs}} \) of \( \text{ACP}^\kappa \), one for each infinite cardinal \( \kappa \), are defined as follows:

\[
\begin{align*}
P &= \text{CTS}_\kappa / \equiv^s, \\
\hat{\alpha} &= \{ \hat{\alpha} \}^{\equiv^s}, \\
\hat{\alpha} &= \{ \hat{\alpha} \}^{\equiv^s}, \\
\hat{\alpha} &= \{ \hat{\alpha} \}^{\equiv^s}, \\
\hat{\alpha} &= \{ \hat{\alpha} \}^{\equiv^s}, \\
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\hat{\alpha} &= \{ \hat{\alpha} \}^{\equiv^s}, \\
\hat{\alpha} &= \{ \hat{\alpha} \}^{\equiv^s}, \\
\hat{\alpha} &= \{ \hat{\alpha} \}^{\equiv^s}.
\end{align*}
\]

The operations on \( \text{CTS}_\kappa / \equiv^s \) are well-defined because \( \equiv^s \) is a congruence with respect to the corresponding operations on \( \text{CTS}_\kappa \).

The structures \( \Psi_{\kappa}^{\text{cs}} \) are models of \( \text{ACP}^\kappa \).

Theorem 10.1 (Soundness of \( \text{ACP}^\kappa \)). For each infinite cardinal \( \kappa \), we have \( \Psi_{\kappa}^{\text{cs}} \models \text{ACP}^\kappa \).

Proof. Because \( \equiv^s \) is a congruence, it is sufficient to show that all axioms are sound. The soundness of all axioms follows straightforwardly from the definition of \( \Psi_{\kappa}^{\text{cs}} \).

For all axioms that are in common with \( \text{ACP}^\kappa \), the proof of soundness with respect to \( \Psi_{\kappa}^{\text{cs}} \) follows the same line as the proof of soundness with respect to \( \Psi_{\kappa} \).
11 ACP£ with Retrospective Conditions

In this section, we present an extension of ACP£ with a retrospection operator on conditions. The retrospection operator allows for looking back on conditions under which preceding actions have been performed. The extension of ACP£ with the retrospection operator is called ACP£r.

ACP£r has the constants and operators of ACP£ and in addition:

- the unary retrospection operator ∼ : C → C;
- the unary retrospection shift operator Π+ : P → P;
- for each n ∈ N, the unary restricted retrospection shift operator Π+ n : P → P;
- for each n ∈ N, the unary restricted retrospection shift operator Π+ n : C → C.

In the parallel composition of two processes, when an action of one of the processes is performed, the retrospections of the other process that are not internal should go one step further. This is accomplished by the retrospection shift operator. The restricted retrospection shift operators, on processes and conditions, are needed for the axiomatization of the retrospection shift operator. The retrospection shift operator Π+ is similar to the history pointer shift operator hps from [4].

The axioms of ACP£r are the axioms of ACP£ with axiom CM3 replaced by axiom CM3R from Table 14, and the additional axioms for retrospection given in Table 15. The crucial axiom is R6, which shows that a conditional expression of the form ∼ζ :→ p gives a retrospection at the condition under which the immediately preceding action has been performed. Axiom CM3R shows that retrospections are adapted if two processes proceed in parallel. Axioms RS0,
Table 16. Transition rules adapted to retrospection

<table>
<thead>
<tr>
<th>Rule</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \rightarrow [0] \rightarrow x' )</td>
<td>( x \parallel y \rightarrow [0] \rightarrow x' \parallel y' )</td>
</tr>
<tr>
<td>( y \rightarrow [0] \rightarrow y' )</td>
<td>( x \parallel y \rightarrow [0] \rightarrow x' \parallel y' )</td>
</tr>
<tr>
<td>( x \rightarrow [0] \rightarrow x' )</td>
<td>( x \parallel y \rightarrow [0] \rightarrow x' \parallel y' )</td>
</tr>
</tbody>
</table>

RS1T and RS2–RS12 state that this happens as explained above. By means of axioms RS5–RS12, the retrospection shift operators on conditions can be eliminated from all terms of sort \( C \).

Recall that we write \( p \triangleleft \zeta \triangleright q \) for \( \zeta : p + \neg \zeta : q \). An interesting equation is \( a \cdot (x \triangleleft \neg \phi \triangleright y) = a \cdot x \triangleleft \phi \triangleright a \cdot y \). This equation is a generalization of axiom R6: axiom R6 is derivable from the other axioms of ACP \( cr \) and this equation by substituting \( \delta \) for \( y \) and applying axioms GC3 and A6. It is not immediately clear that this equation is derivable from the axioms of ACP \( cr \).

Proposition 11.1 (Derivability Generalization Axiom R6). The equation
\[
a \cdot (x \triangleleft \neg \phi \triangleright y) = a \cdot x \triangleleft \phi \triangleright a \cdot y
\] (R6"

Proof. The proof is the same to the proof of the corresponding proposition in the setting of ACP \( cr \) given in [9].

Because of the addition of the retrospection operator, we cannot use the Boolean algebras \( C_\kappa \) here. The algebras \( C'_\kappa \) that we use here can be characterized as the free \( \kappa \)-complete algebras over \( C_{at} \) from the class of algebras with interpretations for the constants and operators of Boolean algebras and the retrospection operator that satisfy the axioms of Boolean algebras (Table 1) and axioms R1–R5 from Table 15. We do not make this fully precise, but give an explicit construction of the algebras \( C'_\kappa \) instead. Important to bear in mind is that not only the atomic conditions, but also the results of applying the operation associated with the retrospection operator a finite number of times to atomic conditions, should not satisfy any equations except those derivable from the axioms.

Let \( C'_{at} = \bigcup \{ C_{at} \times \{ i \} \mid i \in \omega \} \) and define \( \text{prev} : C'_{at} \rightarrow C'_\kappa \) by \( \text{prev}(\eta, i) = (\eta, i + 1) \). For any infinite cardinal \( \kappa \), let \( C'_\kappa \) be the free \( \kappa \)-complete Boolean algebra over \( C'_{at} \). Then the function \( \text{prev} \) extends to a unique \( \kappa \)-complete endomorphism \( \text{prev}^* \) of \( C'_\kappa \). This endomorphism is a unary operation on \( C'_\kappa \) that satisfies axioms R1–R5 from Table 15 and preserves \( \bigcup C' \) for every \( C' \subseteq C'_\kappa \) of cardinality less then \( \kappa \). The algebra \( C'_\kappa \) is the expansion of \( C'_\kappa \) obtained by associating the operation \( \text{prev}^* \) with the operator \( \neg \). We write \( C' \) for \( C'_{\aleph_0} \). The structural operational semantics of ACP \( cr \) is described by the transition rules for ACP \( cr \) with the second and third transition rule for parallel composition and the one transition rule for left merge replaced by the transition rules given in Table 16, and the additional transition rules for retrospection given in Table 17. Of course, the conditions involved are now taken from \( C' \) instead of \( C \).
12 Full Retrospective Splitting Bisimulation Models of ACP^cr

The construction of the full splitting bisimulation models of ACP^cr differs from the construction of the full splitting bisimulation models of ACP^r in the conditions involved and in the notion of splitting bisimulation used. The conditions are now taken from C^r_κ instead of C_κ. Henceforth, we write C^r_κ for C^r_κ \ {⊥}.

Let κ be an infinite cardinal. Then a κ-conditional transition system with retrospection T consists of the following:

- a set S of states;
- a set ℓ ⊆ S × S, for each ℓ ∈ C^r_κ × A;
- a set [α] ⊆ S, for each α ∈ C^r_κ;
- an initial state s^0 ∈ S.

For conditional transition systems with retrospection, reachability, connectedness and connected part are defined exactly as for conditional transition systems.

Let κ be an infinite cardinal. Then CTS^r_κ is the set of all connected κ-conditional transition systems with retrospection T = (S, →, ↓, s^0) such that S ⊆ S_κ and the branching degree of T is less than κ.

Isomorphism between conditional transition systems with retrospection is defined exactly as for conditional transition systems. Splitting bisimilarity has to be adapted to the setting with retrospection.

Let T_1 = (S_1, →_1, ↓, s^0_1) ∈ CTS^r_κ and T_2 = (S_2, →_2, ↓, s^0_2) ∈ CTS^r_κ (for an infinite cardinal κ). Then a retrospective splitting bisimulation B between T_1 and T_2 is a ternary relation B ⊆ S_1 × C^r_κ × S_2 such that B(s_1^0, ⊤, s_2^0) and for all s_1, β, s_2 such that B(s_1, β, s_2):

- if s_1 [α] →_1 s_1', then there is a set CS'_2 ⊆ C^r_κ × S_2 of cardinality less than κ such that α ∩ β ⊆ \bigcup \text{dom}(CS'_2) and for all (α', s_2') ∈ CS'_2, s_2 [α'] →_2 s_2' and B(s_1', α', s_2');
- if s_2 [α] →_2 s_2', then there is a set CS'_1 ⊆ C^r_κ × S_1 of cardinality less than κ such that α ∩ β ⊆ \bigcup \text{dom}(CS'_1) and for all (α', s_1') ∈ CS'_1, s_1 [α'] →_1 s_1' and B(s_1', α', s_2');

Table 17. Additional transition rules for retrospection

<table>
<thead>
<tr>
<th>x [α]</th>
<th>x [α]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Pi^+(x)[\Pi^+(\alpha)])</td>
<td>(\Pi^+ (x)[\Pi^+(\alpha)])</td>
</tr>
<tr>
<td>x [α]</td>
<td>x [α]</td>
</tr>
<tr>
<td>(\Pi^+(x)[\Pi^+(\alpha)])</td>
<td>(\Pi^+ (x)[\Pi^+(\alpha)])</td>
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<tr>
<td>x [α]</td>
<td>x [α]</td>
</tr>
<tr>
<td>(\Pi^+(x)[\Pi^+(\alpha)])</td>
<td>(\Pi^+ (x)[\Pi^+(\alpha)])</td>
</tr>
</tbody>
</table>
bisimulation witnessing transition system, for each different path that ends in state s.

In order to unfold a transition system, we need for each state the smallest subset of \( P(T) \) and for all \( \alpha' \in C' \), \( s_1[\alpha]_1 \).

Two conditional transition systems with retrospection \( T_1, T_2 \in CTS^r_{\kappa} \) are retrospective splitting bisimilar, written \( T_1 \equiv^r T_2 \), if there exists a retrospective splitting bisimulation \( B \) between \( T_1 \) and \( T_2 \). Let \( B \) be a retrospective splitting bisimulation between \( T_1 \) and \( T_2 \). Then we say that \( B \) is a retrospective splitting bisimulation witnessing \( T_1 \equiv^r T_2 \).

It is straightforward to see that \( \equiv^r \) is an equivalence on \( CTS^r_{\kappa} \). Let \( T \in CTS^r_{\kappa} \). Then we write \([T]\equiv^r \) for \( \{ T' \in CTS^r_{\kappa} \mid T \equiv^r T' \} \), i.e. the \( \equiv^r \)-equivalence class of \( T \). We write \( CTS^r_{\kappa} \equiv^r \) for the set of equivalence classes \( \{ [T]\equiv^r \mid T \in CTS^r_{\kappa} \} \).

The elements of \( CTS^r_{\kappa} \) and operations on \( CTS^r_{\kappa} \) to be associated with the constants and operators of \( ACP^r_{\kappa} \) are defined exactly as the elements of \( CTS^r_{\kappa} \) and operations on \( CTS^r_{\kappa} \) associated with them, except for \( \|, \| \) and \( | \). The operations on \( CTS^r_{\kappa} \) that we associate with \( \|, \|, \| \) and \( \Pi^+ \) and \( 2n \) call for unfolding of transition systems from \( CTS^r_{\kappa} \).

For the sake of unfolding, it is assumed that, for each infinite cardinal \( \kappa \), \( S_\kappa \) has the following closure property:\footnote{We write \( (\cdot) \) for the empty sequence, \( (e) \) for the sequence having \( e \) as sole element and \( \sigma \circ \sigma' \) for the concatenation of sequences \( \sigma \) and \( \sigma' \); and we use \( (e_1, \ldots, e_n) \) as a shorthand for \( (e_1) \circ \ldots \circ (e_n) \).}

for all \( S \subseteq S_\kappa \), \( \{ \pi \circ (s) \mid \pi \in (S \times (C_\kappa \times A))^\ast, s \in S \} \subseteq S_\kappa \).

We write \( P'(S) \) for the set \( \{ \pi \circ (s) \mid \pi \in (S \times (C_\kappa \times A))^\ast, s \in S \} \). The function \( \# : P'(S) \to N \) is defined by

\[
\#(\langle s \rangle) = 0, \\
\#(\pi \circ (s, \ell, s')) = \#(\pi \circ (s)) + 1.
\]

The elements of \( P'(S) \), for an \( S \subseteq S_\kappa \), can be looked upon as potential paths of a \( \kappa \)-conditional transition system with \( S \) as set of states. A path of a transition system \( (S, \rightarrow, \downarrow, s^0) \in CTS^r_{\kappa} \) is a finite alternating sequence \( (s_0, \ell_1, s_1, \ldots, \ell_n, s_n) \) of states from \( S \) and labels from \( C_\kappa \times A \) such that \( s_0 = s^0 \) and \( s_i \overset{\ell_{i+1}}{\rightarrow} s_{i+1} \) for all \( i < n \). The state \( s_n \) is called the state in which the path ends.

Let \( T = (S, \rightarrow, \downarrow, s^0) \in CTS^r_{\kappa} \). Then the set of paths of \( T \), written \( P(T) \), is the smallest subset of \( P'(S) \) such that:

- \( (s^0) \in P(T) \).
- if \( \pi \circ (s) \in P(T) \) and \( s \rightarrow s' \), then \( \pi \circ (s, \ell, s') \in P(T) \).

In order to unfold a transition system, we need for each state \( s \) of the original transition system, for each different path that ends in state \( s \), a different state in
the unfolded transition system. The obvious choice is to take the paths concerned as states. 

Let $T = (S, \rightarrow, ↓, s^0) \in \text{CTS}_κ^\text{ct}$. Then the unfolding of $T$, written $\Upsilon(T)$, is defined as follows:

$$\Upsilon(T) = (S', \rightarrow', ↓', s'^{0'}) ,$$

where $S' = P(T)$ ,

and for every $\ell \in Cκ^− \times A$ and $\alpha \in Cκ^−$:

$$\overset{\ell}{\overset{\rightarrow}{\rightarrow}}' = \{ (\pi \cap \langle s \rangle, \pi \cap \langle s, \ell, s' \rangle) \mid \pi \cap \langle s \rangle \in P(T), s \overset{\ell}{\rightarrow} s' \} ,$$

$$\overset{\alpha}{\overset{\rightarrow}{\rightarrow}}' = \{ (\pi \cap \langle s \rangle) \mid \pi \cap \langle s \rangle \in P(T), s \overset{\alpha}{\rightarrow} s' \} ,$$

$$s'^{0'} = \langle s^0 \rangle .$$

The functions $\text{upd}_1$ and $\text{upd}_2$ defined next will be used in the definition of parallel composition on $\text{CTS}_κ^\text{ct}$ to adapt the retrospection in steps originating from the first operand and the second operand, respectively.

Let $S_1, S_2 \subseteq Sκ$. Then the functions $\text{upd}_i : Cκ^− \times P(S_1 \times S_2) \to Cκ^−$, for $i = 1, 2$, are defined by

$$\text{upd}_i(\alpha, (\langle s_1, s_2 \rangle)) = \alpha ,$$

$$\text{upd}_i(\alpha, (\langle s_1, s_2 \rangle, \ell, (s'_1, s'_2) \cap \pi')) = \text{upd}_i(\alpha, (\langle s'_1, s'_2 \rangle) \cap \pi')$$

if $s_i \neq s'_i$,

$$\text{upd}_i(\alpha, (\langle s_1, s_2 \rangle, \ell, (s'_1, s'_2) \cap \pi')) = \text{upd}_i(\Pi^+_\kappa((s'_1, s'_2) \cap \pi'))$$

if $s_i = s'_i$.

Henceforth, we write $\text{upd}(\alpha_1, \alpha_2, \pi)$ for $\text{upd}_1(\alpha_1, \pi) \cap \text{upd}_2(\alpha_2, \pi)$.

We proceed with associating operations on $\text{CTS}_κ^\text{ct}$ with the operators $\parallel$, $\parallel'$, $\Pi^+$ and $\Pi^+_\kappa$.

We associate with the additional operator $\parallel$ an operation $\parallel'$ on $\text{CTS}_κ^\text{ct}$ as follows.

- Let $T_1, T_2 \in \text{CTS}_κ^\text{ct}$. Suppose that $\Upsilon(T_i) = (S_i, \rightarrow_i, ↓_i, s^0_i)$ for $i = 1, 2$, and $\Upsilon(\Upsilon(T_1) \parallel \Upsilon(T_2)) = (S, \rightarrow, ↓, s^0)$. Then

$$T_1 \parallel' T_2 = (S, \rightarrow', ↓', s^0) ,$$

where for every $(\alpha, a) \in Cκ^− \times A$ and $\alpha'' \in Cκ^−$: 32
The operations on $\mathsf{CTS}$ cannot be determined uniquely. To have a well-defined adaptation, the conditions under which the action of $\mathsf{T}$ can be performed are synchronously, the condition under which the action of $\mathsf{T}$ may be different for the different paths that end in the state from which the step starts.

Somewhat surprisingly, in addition, $T_1$ and $T_2$ must be unfolded before the actual composition takes place. In a step where an action of $T_1$ and an action of $T_2$ are performed synchronously, the condition under which the action of $T_1$ can be performed and the condition under which the action of $T_2$ can be performed are needed to adapt the retrospection in that step correctly. If $T_1$ and $T_2$ are not unfolded before the actual composition takes place, in general, those conditions cannot be determined uniquely.

The operations on $\mathsf{CTS}_k^\mathsf{tr}$ to be associated with the additional operators $\parallel$ and $\|$ are defined analogously. The operations on $\mathsf{CTS}_k^\mathsf{tr}$ to be associated with the additional operators $\partial_H$ are defined exactly as the operations on $\mathsf{CTS}_k^\mathsf{tr}$ associated with them. We associate with the additional operators $\Pi_1^\mathsf{tr}$ operations $\Pi_2^\mathsf{tr}$ on $\mathsf{CTS}_k^\mathsf{tr}$ as follows.

Let $T \in \mathsf{CTS}_k^\mathsf{tr}$. Suppose that $\Upsilon(T) = (S, \rightarrow, |, s^0)$. Then

$$\Pi_2^\mathsf{tr}(T) = (S, \rightarrow', |', s^0).$$
where for every \((\alpha, a) \in C_k^- \times A\) and \(\alpha'' \in C_k^+\):

\[
\frac{(\alpha, a)_P}{\big(\pi \mapsto (s), \pi' \mapsto (s')\big)} = \big\{ (\pi \mapsto (s)) \mid \alpha' = \alpha'' \big\},
\]

\[
\big[\alpha''\big]_P = \big\{ (\pi \mapsto (s)) \mid (\pi \mapsto |\alpha''|) \land \Pi_P^{n+1}(\alpha) = \alpha'\big\}.
\]

The operation on \(\mathit{CTS}^k_\mathit{cr}\) to be associated with the additional operator \(\Pi^+\) is the same as the operation on \(\mathit{CTS}^k\) associated with \(\Pi^+\).

We can show that retrospective splitting bisimilarity is a congruence with respect to the operations on \(\mathit{CTS}^k_\mathit{cr}\) associated with the operators of \(\mathit{ACP}_\mathit{cr}\).

**Proposition 12.1 (Congruence).** Let \(\kappa\) be an infinite cardinal. Then for all \(T_1, T_2, T_1', T_2' \in \mathit{CTS}_k^\mathit{cr}\) and \(a \in C_\mathit{cr}\), \(T_1 \bowtie T_2\) and \(T_1' \bowtie T_2'\) imply \(T_1 \bowtie T_2 \bowtie T_1' \bowtie T_2'\). \(\alpha \bowtie \alpha'\) \(T_1 \bowtie T_2 \bowtie T_1' \bowtie T_2'\) and \(\Pi^+\) of \(\Pi^\alpha\) and \(\Pi^\alpha\) are well-defined because \(\bowtie\) is a congruence, it is sufficient to show that all axioms are sound.

The ingredients of the full retrospective splitting bisimulation models \(\mathcal{P}_k^\mathit{cr}\) of \(\mathit{ACP}_\mathit{cr}\), one for each infinite cardinal \(\kappa\), are defined as follows:

\[
\begin{align*}
\mathcal{P} &= \mathit{CTS}_k^\mathit{cr} / \bowtie, \\
\hat{\delta}' &= \hat{\delta}'_{\bowtie}, \\
\hat{\varepsilon}' &= \hat{\varepsilon}'_{\bowtie}, \\
\hat{\alpha}' &= \hat{\alpha}'_{\bowtie}, \\
\Pi^\alpha(T_1) &= \Pi^\alpha(T_1)_{\bowtie}.
\end{align*}
\]

The axioms of \(\mathcal{P}_k^\mathit{cr}\) are well-defined because \(\bowtie\) is a congruence with respect to the corresponding operations on \(\mathit{CTS}_k^\mathit{cr}\).

The structures \(\mathcal{P}_k^\mathit{cr}\) are models of \(\mathit{ACP}_\mathit{cr}\).

**Theorem 12.1 (Soundness of \(\mathit{ACP}_\mathit{cr}\)).** For each infinite cardinal \(\kappa\), we have \(\mathcal{P}_k^\mathit{cr} \models \mathit{ACP}_\mathit{cr}\).

**Proof.** Because \(\bowtie\) is a congruence, it is sufficient to show that all axioms are sound. The soundness of all axioms follows straightforwardly from the definition of \(\mathcal{P}_k^\mathit{cr}\).
For all axioms that are in common with $\text{ACP}^c$, the proof of soundness with respect to $\mathcal{P}_\kappa^c$ follows the same line as the proof of soundness with respect to $\mathcal{P}_\kappa^c$.

In the full retrospective splitting bisimulation models of $\text{ACP}^c$, guarded recursive specifications over $\text{ACP}^c$ have unique solutions.

**Theorem 12.2 (Unique solutions in $\mathcal{P}_\kappa^c$).** For each infinite cardinal $\kappa$, guarded recursive specifications over $\text{ACP}^c$ have unique solutions in $\mathcal{P}_\kappa^c$.

**Proof.** The proof is analogous to the proof of the corresponding property for the full retrospective splitting bisimulation models given in [9]. $\square$

Thus, the full retrospective splitting bisimulation models $\mathcal{P}_\kappa^c$ of $\text{ACP}^c$ with guarded recursion are simply the expansions of the full retrospective splitting bisimulation models $\mathcal{P}_\kappa^c$ of $\text{ACP}^c$ obtained by associating with each constant $\langle X \mid E \rangle$ the unique solution of $E$ for $X$ in the full retrospective splitting bisimulation model concerned.

### 13 Evaluation of Retrospective Conditions

In this section, we add condition evaluation operators and generalized condition evaluation operators to $\text{ACP}^c$. As in the case of $\text{ACP}^c$, these operators require to fix an infinite cardinal $\lambda$. By doing so, full retrospective splitting bisimulation models with domain $\text{CTS}_\kappa^c/\Sigma^c$ for $\kappa > \lambda$ are excluded.

Henceforth, we write $\mathcal{H}_\lambda^c$ for the set of all $\lambda$-complete endomorphisms of $C^c_\lambda$.

In the case of $\text{ACP}^c$, there are $\lambda$-complete condition evaluation operators $\text{CE}_A : P \rightarrow P$ and $\text{CE}_C : C \rightarrow C$, and generalized $\lambda$-complete condition evaluation operators $\text{GCE}_h : P \rightarrow P$ and $\text{GCE}_h : C \rightarrow C$, for each $h \in \mathcal{H}_\lambda^c$. We also need the following auxiliary operators:

- for each $h \in \mathcal{H}_\lambda^c$, $n \in \mathbb{N}$, the unary retrospective update operator $\Pi^n_h : P \rightarrow P$;
- for each $h \in \mathcal{H}_\lambda^c$, $n \in \mathbb{N}$, the unary retrospection update operator $\Pi^n_h : C \rightarrow C$.

In the case of $\text{ACP}^c$, it is assumed that a fixed but arbitrary function $\text{eff} : A \times \mathcal{H}_\lambda^c \rightarrow \mathcal{H}_\lambda^c$ has been given. The function $\text{eff}$ is extended to $A_\delta$ such that $\text{eff}(\delta, h) = h$ for all $h \in \mathcal{H}_\lambda^c$.

The condition evaluation operators and generalized condition evaluation operators cannot be added to $\text{ACP}^c$ in the same way as they are added to $\text{ACP}^c$. First of all, retrospective conditions may refer back too far to be evaluated. The effect is that, in condition evaluation or generalized condition evaluation of a process according to some endomorphism, the retrospective conditions that refer back further than the beginning of the process have to be left unevaluated. This is accomplished by the retrospection update operators mentioned above. In the case of generalized condition evaluation, there is another complication. Recall that generalized condition evaluation allows the results of condition evaluation to change by performing an action. In the presence of retrospection, different parts of a condition may have to be evaluated differently because of such changes.
Table 18. New axioms for (generalized) condition evaluation \((a \in A_h)\)

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(CE_h(\epsilon) = \epsilon)</td>
<td>CE1T</td>
</tr>
<tr>
<td>(CE_h(a \cdot x) = a \cdot CE_h(\Pi_h^1(x)))</td>
<td>CE2R</td>
</tr>
<tr>
<td>(CE_h(x + y) = CE_h(x) + CE_h(y))</td>
<td>CE3</td>
</tr>
<tr>
<td>(CE_h(\phi :\rightarrow x) = \Pi_h^0(\phi) :\rightarrow CE_h(x))</td>
<td>CE4R</td>
</tr>
<tr>
<td>(GCE_h(\epsilon) = \epsilon)</td>
<td>GCE1T</td>
</tr>
<tr>
<td>(GCE_h(a \cdot x) = a \cdot GCE_{\text{eff}(a,h)}(\Pi_h^1(x)))</td>
<td>GCE2R</td>
</tr>
<tr>
<td>(GCE_h(x + y) = GCE_h(x) + GCE_h(y))</td>
<td>GCE3</td>
</tr>
<tr>
<td>(GCE_h(\phi :\rightarrow x) = \Pi_h^0(\phi) :\rightarrow GCE_h(x))</td>
<td>GCE4R</td>
</tr>
</tbody>
</table>

Table 19. Axioms for retrospection update \((a \in A_h, \eta \in C_{at}, \eta' \in C_{at} \cup \{\bot, \top\}\))

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Pi_h^0(\eta) = \eta)</td>
<td>RU1T</td>
</tr>
<tr>
<td>(\Pi_h^0(a \cdot x) = a \cdot \Pi_h^{i+1}(x))</td>
<td>RU2</td>
</tr>
<tr>
<td>(\Pi_h^0(x + y) = \Pi_h^0(x) + \Pi_h^0(y))</td>
<td>RU3</td>
</tr>
<tr>
<td>(\Pi_h^0(\phi :\rightarrow x) = \Pi_h^0(\phi) :\rightarrow \Pi_h^0(x))</td>
<td>RU4</td>
</tr>
<tr>
<td>(\Pi_h^0(\Pi_h^0(\eta)) = \eta')</td>
<td>RU7</td>
</tr>
<tr>
<td>(\Pi_h^0(\Pi_h^0(\eta)) = \eta)</td>
<td>RU8</td>
</tr>
<tr>
<td>(\Pi_h^0(\Pi_h^0(\eta)) = \eta')</td>
<td>RU9</td>
</tr>
<tr>
<td>(\Pi_h^0(\Pi_h^0(\eta)) = \eta)</td>
<td>RU10</td>
</tr>
<tr>
<td>(\Pi_h^0(\Pi_h^0(\eta)) = \eta')</td>
<td>RU11</td>
</tr>
<tr>
<td>(\Pi_h^0(\Pi_h^0(\eta)) = \eta)</td>
<td>RU12</td>
</tr>
<tr>
<td>(\Pi_h^0(\Pi_h^0(\eta)) = \eta')</td>
<td>RU13</td>
</tr>
</tbody>
</table>

The effect is that, in generalized condition evaluation of a process according to some endomorphism, after an action of the process is performed, the subsequent retrospective conditions that refer back to the beginning of the process have to be evaluated according to that endomorphism as well. This is also accomplished by the retrospection update operators mentioned above.

In the case of \(ACP_{\text{cr}}\), the additional axioms for \(CE_h\) and \(GCE_h\), where \(h \in H_{r,\lambda}\), are the axioms given in Tables 18 and 19. These additional axioms differ from the additional axioms in the absence of retrospection (Tables 5 and 7) in that axioms CE2, CE4, GCE2 and GCE4 have been replaced by axioms CE2R, CE4R, GCE2R and GCE4R, and axioms CE6–CE11 by axioms RU1T and RU2–RU13. Axioms CE2R and CE4R, together with axioms RU1T and RU2–RU13, show that, in condition evaluation of a process, retrospective conditions that refer back further than the beginning of the process are not at all evaluated. Similarly, axioms GCE2R and GCE4R, together with axioms RU1T and RU2–RU13, show that, in generalized condition evaluation of a process, retrospective conditions that refer back further than the beginning of the process are not at all evaluated. Moreover, axiom GCE2R, together with axioms RU1T and RU2–RU13, shows that, in generalized condition evaluation of a process according to some endomorphism, after an action of the process is performed, the subsequent retrospective conditions that refer back to the beginning of the process are evaluated according to that endomorphism as well.
The structural operational semantics of ACP\textsubscript{cr}\textsubscript{\textepsilon} extended with condition evaluation and generalized condition evaluation is described by the transition rules for ACP\textsubscript{cr}\textsubscript{\textepsilon} and the transition rules given in Table 20.

The full retrospective splitting bisimulation models of ACP\textsubscript{cr}\textsubscript{\textepsilon} with condition evaluation and/or generalized condition evaluation are not simply the expansions of the full retrospective splitting bisimulation models \( \Psi_{\kappa}^{\text{cr}} \) of ACP\textsubscript{cr}\textsubscript{\textepsilon}, for infinite cardinals \( \kappa \leq \lambda \), obtained by associating with each operator \( \text{CE}_h \) and/or \( \text{GCE}_h \) the corresponding re-labeling operation on conditional transition systems with retrospection. As suggested by the structural operational semantics of ACP\textsubscript{cr}\textsubscript{\textepsilon} extended with condition evaluation and generalized condition evaluation, these re-labeling operations have to be adapted in a way similar to the way in which parallel composition had to be adapted to the case with retrospection in Section 12. As mentioned before, full retrospective splitting bisimulation models with domain \( \text{CTS}_{r}^{\text{cr}} / \text{CTS} \) for \( \kappa > \lambda \) are excluded.

Proposition 7.2, stating that the generalized \( \lambda \)-complete condition evaluation operators supersede the \( \lambda \)-complete condition evaluation operators in the setting of ACP\textsubscript{\textepsilon}, goes through in the setting of ACP\textsubscript{cr}\textsubscript{\textepsilon}.

Adding state operators to ACP\textsubscript{cr}\textsubscript{\textepsilon} can be done on the same lines as adding generalized evaluation operators to ACP\textsubscript{\textepsilon}, but is more complicated. Roughly speaking, signal emission can be added to ACP\textsubscript{cr}\textsubscript{\textepsilon} in the same way as it is added to ACP\textsubscript{\textepsilon} provided that signals are taken from \( \mathcal{C} \). No adaptations like for generalized condition evaluation are needed because signal emission corresponds to condition evaluation that does not persist over performing an action. This property also points at one of the differences between the signal-emission approach to condition evaluation and the other approaches treated in this paper: retrospection has to be resolved in the signal-emission approach before condition evaluation can take place. The case where signals are taken from \( \mathcal{C} \) is expected to be too complicated to handle.

### 14 An Application of ACP\textsubscript{cr}\textsubscript{\textepsilon}

The ultimate applications of a process algebra that includes conditional expressions of some form are the ones that remain entirely within the domain of process
conditions

interesting application of this kind in the setting of ACP algebra. Such applications are by their nature extensions as well. We outline one problematic in alternative composition.

corresponding atomic conditions because their use without restrictions would be ∼J conditions. We mean by this that

by axioms RS7Ja and RS7Jb because of the retrospective nature of last action are LAU1T and LAU2–LAU14 from Table 22. Axiom RS7 must be replaced

a equation

and Π a cate that

a just now by the processes originally following a after performing RS7Jb from Table 22. Axiom CM7 must be replaced by axiom CM7J because,

(1) and RS7 (Table 15) must be replaced by axioms CM7J and RS7Ja–

that a certain number of steps ago a certain action must have been performed.

now. The retrospection operator now allows for using conditions which express

ated analogously. This is accomplished by the auxiliary operators Π n a , where η indicates that action a is performed next. We have not introduced corresponding atomic conditions because their use without restrictions would be problematic in alternative composition.

From the axioms of BPA aJ and the additional axiom J, we can derive the equation a·x+b·y = (a+b)·(J a:=→x+J b:=→y). It can be used to reduce the number

Table 21. Additional axioms for last action conditions (a ∈ A)

| a · x | b · y = (a | b) · (Π n a (x) || Π n b (y)) | CM7J | Π n a (⊥) = ⊥ | LAU5 |
| π n (T) = T | LAU6 |
| Π n a (J c) = ∼J c | RS7Ja | Π n b (J c) = ⊥ | if a ≠ c | LAU7 |
| Π n a (J c) = J c | RS7Jb | Π n b (J c) = T | if a = c | LAU8 |
| Π n+1 (J c) = J c | LAU9 |
| Π n a (−φ) = −Π n b (φ) | LAU10 |
| Π n a (ε) = ε | LAU1T |
| Π n a (φ ∨ ψ) = Π n a (φ) ∨ Π n a (ψ) | LAU11 |
| Π n a (b · x) = b · Π n a+1 (x) | LAU2 |
| Π n a (φ ∧ ψ) = Π n a (φ) ∧ Π n a (ψ) | LAU12 |
| Π n a (x + y) = Π n a (x) + Π n a (y) | LAU3 |
| Π n a (φ :=→ x) = Π n a (φ) :=→ Π n a (x) | LAU4 |
| Π n+1 a (−φ) = −Π n a (φ) | LAU13 |

Table 22. Axioms adapted to last action conditions (a, b ∈ A a, c ∈ A)

algebra. Such applications are by their nature extensions as well. We outline one interesting application of this kind in the setting of ACP a.

We take the set \{J a | a ∈ A\} of last action conditions as the set of atomic conditions C a. The intuition is that J a indicates that action a is performed just now. The retrospection operator now allows for using conditions which express that a certain number of steps ago a certain action must have been performed.

Because we remain entirely within the domain of process algebra some additional axioms are needed. They are given in Table 21. Moreover, axioms CM7 (Table 1) and RS7 (Table 15) must be replaced by axioms CM7J and RS7Ja–RS7Jb from Table 22. Axiom CM7 must be replaced by axiom CM7J because, after performing a | b, it makes no sense to refer back to the actions performed just now by the processes originally following a and b in the process following a | b. Retrospective conditions in the process originally following a that indicate that a is performed just now should be evaluated to T and the ones that indicate that another action is performed just now should be evaluated to ⊥. Retrospective conditions in the process originally following b should be evaluated analogously. This is accomplished by the auxiliary operators Π n a : P → P and Π n b : C → C (for each a ∈ A a and n ∈ N) of which the defining axioms are LAU1T and LAU2–LAU14 from Table 22. Axiom RS7 must be replaced by axioms RS7Ja and RS7Jb because of the retrospective nature of last action conditions. We mean by this that J a can be viewed as a condition of the form ∼η, where η indicates that action a is performed next. We have not introduced corresponding atomic conditions because their use without restrictions would be problematic in alternative composition.

From the axioms of BPA aJ and the additional axiom J, we can derive the equation a·x+b·y = (a+b)·(J a:=→x+J b:=→y). It can be used to reduce the number
of subprocesses of a process. For example, \(a \cdot (a_1 \cdot a_1' + a_2 \cdot a_2') + b \cdot (b_1 \cdot b_1' + b_2 \cdot b_2') = (a+b) \cdot (\mathcal{J}_a : \rightarrow (a_1 + a_2) \cdot (\mathcal{J}_a : \rightarrow a_1' + \mathcal{J}_a : \rightarrow a_2') + \mathcal{J}_b : \rightarrow (b_1 + b_2) \cdot (\mathcal{J}_b : \rightarrow b_1' + \mathcal{J}_b : \rightarrow b_2'))\) shows a reduction from 7 subprocesses to 4 subprocesses.

In order to obtain the full retrospective splitting bisimulation models of the extension of ACP\(^n\) with last action conditions, retrospective splitting bisimilarity has to be adapted: in the definition of retrospective splitting bisimulation (see Section 12), the two occurrences of \(B(s'_1, -a' \mathbin{\sqcap} \mathcal{J}_a, s'_2)\) must be replaced by \(B(s'_1, -a' \sqcup \mathcal{J}_a, s'_2)\).

The operators \(\Pi^a_n\) are reminiscent of the operators \(\Pi^a_0\). In fact, if we would exclude full retrospective splitting bisimulation models with domain \(\mathcal{CTS}^n/\mathcal{ES}\) for \(\kappa\) greater than some infinite cardinal \(\lambda\), \(\Pi^a_0\) could have been replaced by \(\Pi^a_n\), where \(h_a \in \mathcal{H}^a_\lambda\) for \(a \in A\) is defined by \(h_a(\mathcal{J}_a) = \top\) and \(h_b(\mathcal{J}_b) = \bot\) if \(a \neq b\) and \(h_b\) is defined by \(h_b(\mathcal{J}_a) = \bot\).

We conclude with an example of the use of the retrospection operator together with last action conditions.

**Example 14.1.** The example concerns a service that resembles the services considered in [10, 11]. For any command \(m\) from some set \(M\), the service can be requested to process command \(m\) and it can be requested to report back what the reply would be to the request to process command \(m\). We suppose that the service can be described by a function \(F: M^+ \rightarrow \{\top, F, B\}\) with the property that \(F(\alpha) = B \Rightarrow F(\alpha \& \langle m \rangle) = B\). This function is called the reply function of the service. Given a reply function \(F\) and a command \(m\), the derived reply function of \(F\) after processing \(m\), written \(\overline{F}_m F\), is defined by \(\overline{F}_m F(\alpha) = F(\langle m \rangle \& \alpha)\). The connection between a reply function \(F\) and the service described by it can be understood as follows:

- if \(F(\langle m \rangle) \neq B\), the request to process command \(m\) is accepted by the service, the reply is \(F(\langle m \rangle)\) and the service proceeds as described by \(\overline{F}_m F\);
- if \(F(\langle m \rangle) = B\), the request to process command \(m\) is not accepted by the service, the reply is \(F(\langle m \rangle)\) and the service proceeds as described by \(F\);
- the request to report back what the reply would be to the request to process command \(m\) is always accepted by the service, the reply is \(F(\langle m \rangle)\) and the service proceeds as described by \(F\).

Hence, the service can be viewed as the process defined by the guarded recursive specification that consists of an equation

\[
P_G = \sum_{m \in M} (r(m) + r(?m)) \cdot s(G(\langle m \rangle)) \cdot (P_{\overline{F}_m G} \triangleleft \mathcal{J}_{r(\langle m \rangle)} \sqcup \mathcal{J}_{s(B)}) \triangleright P_G
\]

for each reply function \(G\). Here, we write \(r(m)\) for the action of receiving a request to process command \(m\), \(r(?m)\) for the action of receiving a request to report back what the reply would be to the request to process command \(m\), and \(s(v)\) for the action of sending reply \(v\).
Concluding Remarks

We have added the empty process constant to the different extensions of ACP with conditional expressions presented in [9]. In the past, the addition of the empty process constant to ACP was rather problematic. Its current addition to the different extensions of ACP with conditional expressions presented in [9] turns out to present no additional complications.

The addition of the empty process constant to different extensions of ACP in this paper is based on the treatment of the empty process constant in the setting of ACP that is chosen in [5]. If it was based on the treatment of the empty process constant chosen in [19] instead, the addition of the empty process constant to different extensions of ACP in this paper would have been slightly different. For example, with the treatment from [5], no special additional axioms concerning conditional expressions are needed when adding the empty process constant, whereas with the treatment from [19], the special additional axiom $\epsilon \parallel (\phi \Rightarrow \epsilon) = \phi \Rightarrow \epsilon$ is needed.

In [11], we showed that threads, as found in programming languages such as Java and C#, and services used by them can be viewed as processes that are definable over $\text{ACP}^c$, and that thread-service composition on those processes can be expressed in terms of operators of $\text{ACP}^c$ extended with action renaming. In fact, the termination behaviour of the composition of a thread with the services used by it can be dealt with more directly, and without action renaming, in $\text{ACP}^c_\epsilon$.

References

9. J. A. Bergstra and C. A. Middelburg. Splitting bisimulations and retrospective
conditions. Computer Science Report 05-03, Department of Mathematics and
interleaving. In B. Löwe S. B. Cooper and L. Torenvliet, editors, CiE 2005, volume
11. J. A. Bergstra and C. A. Middelburg. Thread algebra with multi-level strate-
gies. Computer Science Report 05-08, Department of Mathematics and Computer
and the semantics of SDL. In J. A. Bergstra, A. Ponse, and S. A. Smolka, editors,
13. C. C. Chang and H. J. Keisler. Model Theory, volume 73 of Studies in Logic and
data. In D. J. Andrews, J. F. Groote, and C. A. Middelburg, editors, Semantics of
Specification Languages, Workshops in Computing Series, pages 232–251. Springer-
Verlag, 1994.
15. P. R. Halmos. Lectures on Boolean Algebras. Mathematical Studies. Van Nostrand,
16. C. P. J. Koymans and J. L. M. Vrancken. Extending process algebra with the
empty process $\epsilon$. Logic Group Preprint Series 1, Department of Philosophy, Utrecht
University, Utrecht, 1985.
18. J. R. Shoenfield. Mathematical Logic. Addison-Wesley Series in Logic. Addison-

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