Self-decomposable distributions and branching processes
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Published: 01/01/1989

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Memorandum COSOR 89-06

Self-decomposable distributions
and branching processes

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Eindhoven, April 1989
The Netherlands
SELF-DECOMPOSABLE DISTRIBUTIONS

AND

BRANCHING PROCESSES

by

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1. INTRODUCTION

A random variable \( X \) (or its distribution) is said to be self-decomposable if for every \( t \in \mathbb{R}_+ := [0, \infty) \) there exists a random variable \( X_t \) such that

\[
X = e^{-t} X' + X_t,
\]

where \( X, X' \) and \( X_t \) are independent and \( X \) and \( X' \) are identically distributed. Let \( X \) be non-negative and denote its Laplace-Stieltjes transform by \( f \). In terms of Laplace-Stieltjes transforms equation (1.1) is equivalent to

\[
f(t) = f(e^{-t} \tau) f_i(t), \quad t \geq 0,
\]

where \( f_i \) is the Laplace-Stieltjes transform of \( X_i \). If \( f_i(t) = f((1-e^{-\alpha \tau})^{1/\alpha} \tau) \) (or equivalently \( X_i = (1-e^{-\alpha \tau})^{1/\alpha} X \) in (1.1)) for some \( \alpha > 0 \), then \( f \) (or its distribution) is called stable with exponent \( \alpha \). If \( f \) is stable, then (1.2) is equivalent to (see for example van Harn et al. (1982))

\[
f(t) = f e^{-\alpha \tau}(e^{-t} \tau), \quad t \geq 0.
\]

Stable and self-decomposable distributions are widely studied as they are obtained as limit distributions of normalized partial sums of independent (and in the case of stability, also identically distributed) random variables. For more details we refer to Loève (1977).

From (1.1) it follows that there are no discrete random variables which are self-decomposable (in fact all self-decomposable random variables are absolutely continuous (cf. Fisz and Varadarajan (1963)). Steutel and van Harn (1979) introduced a discrete analogue of self-decomposability and stability as follows; a Laplace-Stieltjes transform \( f \) (or its distribution) of an \( \mathbb{N}_0 = \{0, 1, \ldots\} \)-valued random variable is said to be discrete self-decomposable if for every \( t \geq 0 \) there exists a Laplace-
Stieltjes transform $f_t$ of an $\mathbb{N}_0$-valued random variable such that

$$f(t) = f(-\ln(1-e^{-\alpha(-\ln(1-e^{-\tau}))})) f_t(t), \quad t \geq 0,$$

(1.4)

If $f_t(t) = f(-\alpha^{-1} \ln(1-e^{-\alpha(-\ln(1-e^{-\tau}))}))$, $\alpha > 0$, then $f$ is called discrete stable with exponent $\alpha$. The random variable with Laplace-Stieltjes transform $f(-\ln(1-e^{-\alpha(-\ln(1-e^{-\tau}))}))$ is interpreted as the discrete analogue of $e^{-t} X$ (cf. (1.1), (1.2) and (1.4)), i.e., of multiplication of a random variable $X$ by a scalar $e^{-t}$. Thus, denoting the Laplace-Stieltjes transform of $X$ by $f_X$, they defined (in distribution) the random variable $e^{-t} \otimes X$ by

$$f_{e^{-t} \otimes X}(t) = f_X(-\ln(1-e^{-\alpha(-\ln(1-e^{-\tau}))})).$$

(1.5)

In van Harn et al. (1982) this notion of scalar multiplication is generalized, by defining operators $(T_t)_{t \geq 0}$ acting on the set of Laplace-Stieltjes transforms of $\mathbb{N}_0$-valued random variables (denoted by $LST(\mathbb{N}_0)$) as follows;

$$f_{e^{-t} \otimes X}(t) = T_t f_X(t)$$

(1.6)

where the operators $(T_t)_{t \geq 0}$ satisfy the following conditions:

1. $T_t$ maps $LST(\mathbb{N}_0)$ into $LST(\mathbb{N}_0)$;
2. $T_t T_s = T_{t+s}, \quad t \geq 0, s \geq 0$;
3. $T_t(fg) = T_t f \cdot T_t g, \quad f, g \in LST(\mathbb{N}_0)$;
4. $T_t(pf + (1-p)g) = p T_t f + (1-p) T_t g, \quad p \in [0,1], f, g \in LST(\mathbb{N}_0)$;
5. $T_t$ is continuous in the topology of pointwise convergence.

(1.7)

It is then shown that if $(T_t)_{t \geq 0}$ satisfies (1.7) then

$$T_t f(t) = f(F_t(t)),$$

(1.8)

where $F = (F_t)_{t \geq 0}$ is a composition semigroup of cumulant generating functions, with $F_t(t) = -\ln g_t(t), \quad g_t \in LST(\mathbb{N}_0)$. Under the additional assumption that $T_t$ tends to the identity mapping as $t \downarrow 0$ (i.e., $F_t(t) \rightarrow \tau$ as $t \downarrow 0$) they showed that $g_t$ is the Laplace-Stieltjes transform of a continuous time, discrete $\mathbb{N}_0$-valued state space branching process. Analogous with (1.4) and (1.5) van Harn et al. (1982) call a Laplace-Stieltjes transform $f$ (or its distribution) (discrete) $F$-self-decomposable if it satisfies

$$f(t) = f(F_t(t)) f_t(t), \quad t \geq 0,$$

(1.9)

where $f_t \in LST(\mathbb{N}_0)$.

Another generalization of self-decomposability is given in O'Connor (1979) and (1981) for $\alpha \in [-2,1]$, Jurek (1989) for $\alpha \in [-2,\infty)$ and independently of Jurek (1989), Hansen (1988) (also for $\alpha \in [-2,\infty)$): A characteristic function $\phi$ of an $\mathbb{R} = (-\infty, \infty)$-valued random variable is said to be $\alpha$-self-decomposable if for every $t \geq 0$ there exists a characteristic function $\phi_t$ such that

$$\phi(t) = \phi^{\alpha_\alpha} (e^{-\alpha t}) \phi_t(t).$$

(1.10)
Again, there are no \( \mathbb{N}_0 \)-valued random variables who’s characteristic function satisfies (1.10). In Hansen (1988) the analogue of (1.10) for \( \mathbb{N}_0 \)-valued random variables in the sense of (1.4) is also studied.

In this paper we introduce operators \((T_t)_{t \geq 0}\) acting on the set of Laplace-Stieltjes transforms of either \( \mathbb{N}_0 \)-valued or \( \mathbb{R}_+ \)-valued random variables satisfying conditions similar to (1.7). As in van Harn et al. (1982) it is shown that there is a one-to-one correspondence between operators \((T_t)_{t \geq 0}\) satisfying these conditions and composition semigroups \((F_t)_{t \geq 0}\) of cumulant generating functions, where \( F_t \) is the cumulant generating function of a continuous time, continuous or discrete state space branching process. We call a Laplace-Stieltjes transform \( f \) (which can stem from either an \( \mathbb{N}_0 \)-valued or \( \mathbb{R}_+ \)-valued random variable) \( F, \alpha \)-self-decomposable if it satisfies

\[
f(t) = f^{\ast \alpha}(F_t(t)) f_t(t), \quad t \geq 0,
\]

where \( f_t \) is a Laplace-Stieltjes transform for every \( t \geq 0 \). If \( f_t = 1 \) for every \( t \geq 0 \) then \( f \) is said to be \( F \)-stable (cf. (1.3)). In Section 3 we give a review of some basic properties of branching process. These results are used in Section 4 to define and characterize so-called \( \alpha \)-discounted branching process with immigration. In Sections 5 and 6 we derive a canonical representation for respectively the \( F \)-stable Laplace-Stieltjes transforms and the \( F, \alpha \)-self-decomposable Laplace-Stieltjes transforms. It is also shown that the \( F, \alpha \)-self-decomposable distributions (\( F \)-stable distributions) are the invariant distributions of \( \alpha \)-discounted branching processes with (without) immigration. Analogous to Urbanik (1973) we define, in Section 7, the sets of \( n \)-times-\( F, \alpha \)-self-decomposable Laplace-Stieltjes transforms. Also here a canonical representation is derived and it is shown that the sets of \( n \)-times-\( F, \alpha \)-self-decomposable Laplace-Stieltjes transforms provide us with a classification of the set of infinitely divisible Laplace-Stieltjes transforms. We conclude in Section 8 with some central limit theorems for \( F, \alpha \)-self-decomposable distributions. In the following section we give some preliminaries.

2. PRELIMINARIES

Let \( I \in \{ \mathbb{N}_0, \mathbb{R}_+ \} \). A random variable \( X \) is said to be \( I \)-valued if \( \mathbb{P}(X \in I) = 1 \). Denote the set of Laplace-Stieltjes transforms \( f \) of \( I \)-valued random variables with \( f \neq 1 \) by \( \text{LST}(I) \). Also let \( \text{LST}^c(I) \) denote the set of functions of the form \( f^c \), with \( f \in \text{LST}(I) \). Analogous to van Harn et al. (1982) we generalize the notion of scalar multiplication of an \( I \)-valued random variable by a scalar \( e^{-t} \) by defining operators \((T_t)_{t \geq 0}\) acting on \( \text{LST}(I) \) as follows;

\[
f_{e^{-\alpha}X}(t) = T_t f_X(t)
\]

where the operators \((T_t)_{t \geq 0}\) satisfy
$T_t$ maps $\text{LST}^c(I) \cup \{1\}$ into $\text{LST}^c(I) \cup \{1\}$, $c \in I$; \hspace{1cm} (2.2a)

$T_t T_s = T_{t+s}$, $t \geq 0$, $s \geq 0$; \hspace{1cm} (2.2b)

$T_t(f^c) = (T_tf)^c$, $c \in I$, $f \in \text{LST}(I) \cup \{1\}$; \hspace{1cm} (2.2c)

$T_t(pf + (1-p)g) = p T_t f + (1-p) T_t g$, $p \in [0,1]$, $f, g \in \text{LST}(I) \cup \{1\}$; \hspace{1cm} (2.2d)

$T_t$ is continuous in the topology of pointwise convergence. \hspace{1cm} (2.2e)

We now have the following generalization of Theorem 2.1 in van Harn et al. (1982).

**THEOREM 2.1.** Let $I \in \{\mathbb{N}_0, \mathbb{R}_+\}$. Collections of operators $(T_t)_{t \geq 0}$ satisfying (2.2) correspond one-to-one to collections $(F_t)_{t \geq 0}$ of composition semigroups, where $F_t = -\ln g_t$, $g_t \in \text{LST}(I)$, and necessarily, $g_t^c$ is a Laplace-Stieltjes transform for every $c \in I$. The correspondence is given by

$$T_t f = f \circ F_t, \hspace{0.5cm} t \geq 0. \hspace{1cm} (2.3)$$

**PROOF.** Let $t$ be fixed, $\delta_c(t) := \exp(-c \tau)$, $c \in I$, $g_t := T_t \delta_1$ and let $F_t := -\ln g_t$. By (2.2a), $g_t \in \text{LST}(I)$. From (2.2c) we have that for any $c \in I$

$$T_t \delta_c = T_t \delta_1^c = (T_t \delta_1)^c = g_t^c = e^{-cF_t} = \delta_c \circ F_t. \hspace{1cm} (2.4)$$

Observe that $g_t^c$ is a Laplace-Stieltjes transform for every $c \in I$. Let $f := \sum_{k=1}^n p_k \delta_{c_k}$, $\sum_{k=1}^n p_k = 1$. By (2.2d) and (2.4)

$$T_t f = \sum_{k=1}^n p_k T_t \delta_{c_k} = \sum_{k=1}^n p_k \delta_{c_k} \circ F_t = f \circ F_t.$$ 

Since any Laplace-Stieltjes transform $f$ of an $I$-valued random variable can be approximated by a sequence of simple Laplace-Stieltjes transforms with $c_k \in I$, we have, by (2.2e), that (2.3) holds. By (2.2b) it follows that for $s, t \geq 0$ (cf. (2.4))

$$F_t \circ F_s = -\ln T_t \delta_1 \circ F_s = -\ln T_t \delta_1 (-\ln T_s \delta_1) = -\ln T_t(T_s \delta_1) = -\ln T_{t+s} \delta_1 = F_{t+s}.$$ 

The converse is easily proved.

We will henceforth also assume that $T_t$ tends to the identity mapping as $t \downarrow 0$. \hspace{1cm} (2.2f)

From (2.2) and Theorem 2.1 it now follows that $(F_t)_{t \geq 0}$ is a continuous composition semigroup of cumulant generating functions satisfying

$$F_t(F_s(\tau)) = F_{t+s}(\tau), \hspace{0.5cm} s, t \geq 0; \hspace{1cm} (2.5a)$$

$$F_0(\tau) = \tau, \hspace{0.5cm} \tau \geq 0; \hspace{1cm} (2.5b)$$
\[ F_t(\tau) \to \tau \text{ as } t \downarrow 0; \quad (2.5c) \]
\[ F_t(0) = 0, \quad t \geq 0, \quad (2.5d) \]
where \( F_t = -\ln g_t \), with \( g_t \in \text{LST}(I) \) and necessarily \( g_t^c \) is a Laplace-Stieltjes transform for every \( c \in I \). In the next section we show that \( (g_t)_{t \geq 0} \) is the Laplace-Stieltjes transforms of a branching process. We conclude this section with a representation theorem for infinitely divisible Laplace-Stieltjes transforms. For a proof we refer to van Harn (1978).

**Theorem 2.2.** A function \( f \) is the Laplace-Stieltjes transform of a non-negative infinitely divisible random variable if and only if it can be written in the form
\[
\ln f(\tau) = -\gamma \tau + \int_0^\infty (e^{-\tau x} - 1) \, dM(x), \quad (2.6)
\]
where \( \gamma \geq 0 \) and the function \( M \) (called the Lévy spectral function) satisfies
\[
M(x) \text{ is non-decreasing and } M(\infty) = 0; \quad (2.7)
\]
\[
\int_0^1 x \, dM(x) < \infty. \quad (2.8)
\]
The representation is unique.

The couple \( (\gamma, M) \) is called the Lévy couple of \( f \) and it determines \( f \) uniquely. Let \( ID(\mathbb{N}_0) \) denote the set of infinitely divisible Laplace-Stieltjes transforms whose random variables are \( \mathbb{N}_0 \)-valued. Similarly, let \( ID(\mathbb{R}_+) \) denote the set of infinitely divisible Laplace-Stieltjes transforms whose random variables are \( \mathbb{R}_+ \)-valued.

**3. CONTINUOUS TIME BRANCHING PROCESSES**

In this section we give a review of some properties of continuous time, continuous and discrete state space, Markov branching processes which will be used in the following sections. For a more rigorous introduction to discrete state space branching processes we refer to Asmussen and Hering (1983), which also includes the results on discrete state space branching processes stated in this section. There is no one work giving a comprehensive introduction to continuous state space branching processes. As a starting point we refer to Pakes and Trajstman (1985) and their reference list. For non-elementary results on continuous state space branching processes stated in this section, which are not proved in Pakes and Trajstman (1985), we will give a reference which does.
Let \( I \in \{ \mathbb{N}_0, \mathbb{R}_+ \} \). An \( I \)-valued stochastic process \((Y_t(c))_{t \geq 0}\) with Laplace-Stieltjes transforms \((g_t^c)_{t \geq 0}\) is said to be a branching process with state space \( I \), if
\[
\begin{align*}
Y_t(Y_s(c)) &= Y_{t+s}(c), \ s, t \geq 0, \ c \in I; \\
Y_0(c) &= c \text{ a.s.} .
\end{align*}
\]  
Property (3.1a) is the basic branching property of the process \((Y_t(c))_{t \geq 0}\). One often makes one of the following three assumptions on the stochastic process \((Y_t(c))_{t \geq 0}\):
\[
\begin{align*}
Y_t(1) &\to 1 \text{ as } t \downarrow 0; \\
\mathbb{P}(Y_t(1) < \infty) &= 1, \ t \geq 0; \\
\mathbb{E}(Y_1(1)) &= e^m < \infty.
\end{align*}
\]
If \((Y_t(c))_{t \geq 0}\) satisfies (3.1a) through (3.1d), then we write \((g_t^c)_{t \geq 0} \in BP(I)\). The branching process is called supercritical, critical or subcritical according to whether \(m > 0, m = 0\) or \(m < 0\), respectively. Let \(g_t\) be the Laplace-Stieltjes transform of the random variable \(Y_t(1)\). The function \(g_t(\tau)\) has no zeros for \(\tau \geq 0\) so the cumulant generating function
\[
F_t(\tau) := -\ln g_t(\tau) = -\ln \mathbb{E}(e^{-\tau Y_t(1)})
\]
is a well defined function. In terms of \(F_t\) conditions (3.1a) through (3.1d) are equivalent to (2.5) and condition (3.1e) is equivalent to
\[
F_1'(0) = e^m
\]
being finite. The continuity condition (3.1c) (or (2.5c)) implies that \(F_t\) is a differentiable function of \(t\) (cf. for example Kawazu and Watanabe (1971)) with
\[
\frac{\partial}{\partial t} F_t(\tau) = U(F_t(\tau)) = U(\tau)F'_t(\tau), \quad (3.2)
\]
where \(F'_t(\tau) := \frac{\partial}{\partial \tau} F_t(\tau)\) and
\[
U(\tau) := \frac{\partial}{\partial t} F_t(\tau) \bigg|_{t=0} = \lim_{t \downarrow 0} t^{-1}(F_t(\tau) - \tau). \quad (3.3)
\]
The function \(U\) in (3.3) has been characterized for both \(BP(\mathbb{N}_0)\) and \(BP(\mathbb{R}_+)\). For \(BP(\mathbb{N}_0)\), \(U\) takes the form
\[
U(\tau) = -\lambda (e^\tau h(\tau) - 1), \quad (3.4)
\]
where \(\lambda > 0\) and \(h \in LST(\mathbb{N}_0)\). If (3.1d) is satisfied then \(U\) satisfies the non-explosion condition
\[
\int_0^\epsilon U(x)^{-1} dx \bigg|_{0+} = \infty \quad \text{for all sufficiently small } \epsilon > 0. \quad (3.5)
\]
For \(BP(\mathbb{R}_+)\), Silverstein (1968) showed that if (3.1d) holds (such processes are called conservative) then,
\[
U(\tau) = a\tau - \frac{1}{2}\sigma^2 \tau^2 - \int_0^\infty (e^{-\tau x} - 1 + \tau x/(1+x^2)) dN(x), \quad (3.6)
\]
where $U$ must satisfy the non-explosion condition (3.5) and $N$ is a Lévy spectral measure

$$
\int_{-\infty}^{\infty} x^2 \, dN(x) < \infty,
$$

for every $\varepsilon > 0$. Conversely, any function of the form (3.4) or (3.6) generates a continuous composition semigroup $\left(F_t\right)_{t \geq 0}$ through (3.2). From (3.4) and (3.6) it follows that $BP(\mathbb{N}_0) \cap BP(\mathbb{R}_+)$ is non-empty and that $BP(\mathbb{N}_0) \not\subseteq BP(\mathbb{R}_+)$. We note that the cumulant generating function $U$ in (3.6) can be written as a limit of cumulant generating functions of the form (3.4), with $h \in LST(\mathbb{N}_0)$ (cf. Takacs (1967)). The function $U$ in (3.4) and (3.6) is minus the logarithm of an infinitely divisible Laplace-Stieltjes transform. Hence

$$
U(t) \text{ is concave;}
$$

$$
U(t)/t \text{ is decreasing.}
$$

Equation (3.2) is equivalent to

$$
\int_{\tau} U(x)^{-1} \, dx = t, \quad t \geq 0.
$$

If (2.5e) holds then on differentiating (2.5a) in $t$ it follows that

$$
F_t'(0) = e^{mt}, \quad t \geq 0,
$$

and hence that (cf. (3.3))

$$
U'(0) = m.
$$

We will use the probability that the process $(Y_t(c))_{t \geq 0}$ goes 'instinct' in 'finite time' and in 'infinite time', that is the quantities

$$
p := \lim_{t \to \infty} \mathbb{P}(Y_t(1) = 0);
$$

$$
r := \lim_{t \to \infty} \mathbb{P}(\lim_{t \to \infty} Y_t(1) = 0) = \lim_{t \to \infty} g_t(\tau), \quad \tau > 0.
$$

If there is a $\tau_0 > 0$ such that $U(\tau_0) < 0$ then $-\ln r$ is the largest zero of $U(\tau)$. If $U(\tau) > 0$ for all $\tau > 0$ then $r = 0$. Furthermore, we have for $BP(\mathbb{N}_0)$ that

$$
\text{if } m \leq 0 \text{ then } r = p = 1, \quad (BP(\mathbb{N}_0)),
$$

and for $BP(\mathbb{R}_+)$ with $F_1(\tau) \neq \tau$ (cf. Grey (1974))

$$
\text{if } m \leq 0 \text{ then } r = 1 \text{ and } p = 0 \text{ or } p = 1, \quad (BP(\mathbb{R}_+)).
$$

It is evident from (3.12) and (3.13) that $p \leq r$.

Let $\tau, \tau_0 \in (0, \infty)$ be such that $-\ln r \in [\tau_0, \tau]$ if $r \in (0, 1)$. Then

$$
A_\alpha(\tau) := \exp \left\{ \alpha \int_{\tau_0}^{\tau} U(x)^{-1} \, dx \right\},
$$
is a well defined function. Observe that (cf. (3.9))

\[ A_\alpha(F_\tau(t)) = e^{\alpha t} A_\alpha(\tau), \quad t \geq 0. \tag{3.16} \]

We are now ready to prove two lemmas which we will need in Sections 6, 7 and 8. The proof of the second lemma is similar to that of Theorem 4.2 in Pakes and Trajstman (1985).

**Lemma 3.1.** Let \( A_\alpha \) be given by (3.15) with \( \alpha \in \mathbb{R} \) and \( \tau, \tau_0 \in (0, \infty) \) being such that \(-\ln r \notin [\tau_0, \tau].\)

(i) If \( m \neq 0 \), then there exists a \( c > 0 \) such that

\[ A_\alpha(\tau) - c \tau^{\alpha/m} \text{ as } \tau \downarrow 0. \]

(ii) Let \( \tau_0 > -\ln r \) if \( r > 0 \). If \( U(\tau)/\tau \to a \neq 0 \text{ as } \tau \to \infty \), then there exists a \( c > 0 \) such that

\[ A_\alpha(\tau) - c \tau^{\alpha/a} \text{ as } \tau \to \infty. \]

If we define \( a/\infty := 0 \), then (i) and (ii) are also valid for \( m = \infty \) and \( a = \infty \), respectively.

**Proof.** Observe that

\[
(\tau/\tau_0)^{-\alpha/b} \exp \left\{ \alpha \int_{\tau_0}^{\tau} \frac{1}{U(x)} \, dx \right\} = \exp \left\{ \int_{\tau_0}^{\tau} \left[ \alpha \frac{x}{U(x)} - \frac{\alpha}{b} \right] \frac{1}{x} \, dx \right\}.
\]

The lemma now follows from Theorem 1.3.1, p. 12 of Bingham et al. (1987).

**Lemma 3.2.** Let \( A_\alpha \) be given by (3.15) with \( \alpha \in \mathbb{R} \) and \( \tau, \tau_0 \in (0, \infty) \) being such that if \( r > 0 \), then \( \tau_0 > -\ln r \) and \( \tau > -\ln r \) or \( \tau_0 < -\ln r \) and \( \tau < -\ln r \).

(i) If \( m < 0 \), then

\[
\frac{F_\tau(t)}{F_\tau(\tau_0)} \to A_m(\tau) \text{ as } t \to \infty;
\]

(ii) If \( U(\tau)/\tau \to a > 0 \text{ as } \tau \to \infty \), then

\[
\frac{F_\tau(t)}{F_\tau(\tau_0)} \to A_a(\tau) \text{ as } t \to \infty;
\]

**Proof.** (i). Let \( \alpha < 0 \). From (3.15) it follows that \( A_\alpha \) has an inverse, \( A_\alpha^{-1} \), say, on \((-\ln r, \infty)\) or on \((0, -\ln r)\) if \( r > 0 \) and on \((0, \infty)\) if \( r = 0 \) or \( r = 1 \). By (3.16)
By Lemma 3.1 (i), \( A_\alpha \) is regularly varying with exponent \( \alpha/m \) at the origin, so \( A_\alpha^{-1} \) is regularly varying with exponent \( m/\alpha \) at the origin. Hence

\[
\frac{F_t(\tau)}{F_t(\tau_0)} = \frac{A_\alpha^{-1}(e^{\alpha t} A_\alpha(\tau))}{A_\alpha^{-1}(e^{\alpha t} A_\alpha(\tau_0))} - \frac{e^{mt} A_m(\tau)}{e^{mt} A_m(\tau_0)} = A_m(\tau) \text{ as } t \to \infty.
\]

(ii). This part of the proof is almost identical with the proof of part (i), except that \( \alpha > 0 \) is chosen and Lemma 3.1 (ii) is used.

For our purposes not only the state space of a given branching process is of interest, but also its divisibility properties. This will become clear in the following sections. We therefore introduce the following notation.

**NOTATION 3.3.** Let \( I_1, I_2 \in \{ N_0, \mathbb{R}_+ \} \). Let

\[
BP(I_1, I_2) := \{ (g_t)_{t \geq 0} \mid (g_t)_{t \geq 0} \in BP(I_1) \text{ and } g_c^e \in \text{LST}(I_1) \text{ for all } c \in I_2 \},
\]

with \( I_2 \) the largest of the two sets which has this property. For ease of notation we will also say that the family \( (F_t)_{t \geq 0} \) of cumulant generating functions is a member of \( BP(\cdot ; I) \) if \( F_t = -\ln g_t \) and \( (g_t)_{t \geq 0} \in BP(\cdot ; I) \).

We thus have that

\[
BP(\mathbb{R}_+ ; \mathbb{R}_+) = BP(\mathbb{R}_+);
BP(\mathbb{N}_0 ; \mathbb{R}_+) = BP(\mathbb{N}_0) \cap BP(\mathbb{R}_+);
BP(\mathbb{N}_0 ; \mathbb{N}_0) = BP(\mathbb{N}_0) \cup BP(\mathbb{N}_0 ; \mathbb{R}_+).
\]

We conclude this section with two examples of \( F = (F_t)_{t \geq 0} \). The first example gives the semigroup which generates the classical self-decomposable Laplace-Stieltjes transforms if \( m < 0 \) and the reverse self-decomposable Laplace-Stieltjes transforms as introduced in Hansen (1989b) when \( m > 0 \). The second example corresponds to discrete self-decomposability as defined in Steutel and van Harn (1979). For other examples of \( BP(\mathbb{N}_0) \) we refer to van Harn et al. (1982) and of \( BP(\mathbb{R}_+) \) we refer to Lamperti (1967).

**EXAMPLE 3.4.** Take \( a = m \neq 0, \sigma = 0 \) and \( N \equiv 0 \) in (3.6). Then \( U(\tau) = m \tau \) and \( F_t(\tau) = e^{mt} \tau \).

**EXAMPLE 3.5.** Take \( m < 0 \) and \( H \equiv 1 \) in (3.4). Then \( U(\tau) = m(e^{\tau} - 1) \) and \( F_t(\tau) = 1 - e^{mt}(1 - e^{-\tau}) \).
4. DISCOUNTED BRANCHING PROCESSES WITH IMMIGRATION

In this section we define a stochastic process \((Z_t(1))_{t \geq 0}\) called an \(\alpha\)-discounted branching process with immigration. This definition generalizes the notion of continuous state space branching processes with immigration as introduced by Kawazu and Watanabe (1971) and it also generalizes the definition of discrete state space branching processes with immigration as given in Steutel et al. (1983). These generalized branching processes are related with \(F\)-stable distributions (cf. Section 5) and \(F\), \(\alpha\)-self-decomposable distributions (cf. Section 6) and will be studied in greater detail in the near future.

Let \(I \in \{N_0, \mathbb{R}_+\}\), let \(F_t = -\ln g_t\) with \((g_t)_{t \geq 0} \in BP(\cdot ; I)\) and let \(X\) be a random variable with Laplace-Stieltjes transform \(f \in ID(I)\). A stochastic process \((Z_t(1))_{t \geq 0}\), with Laplace-Stieltjes transforms \((h_t)_{t \geq 0}\) is said to be an \(\alpha\)-discounted branching process with immigration having an initial distribution \(X\) and \((h_t)_{t \geq 0}\) are said to belong to \(BPI_{\alpha}(\cdot ; I)\), if

\[
\begin{align*}
    h_t(\tau) &= f e^{-\alpha \tau} (F_t(\tau)) f_t(\tau), \quad t \geq 0; \\
    f_{t+s}(\tau) &= f_t(\tau) f_s(\tau), \quad t, s \geq 0; \\
    f_0(\tau) &= 1,
\end{align*}
\]

with \(f_t \in ID(I)\). In terms of random variables (4.1) and (4.2) are equivalent to

\[
Z_t(1) = Y_t(X(e^{-\alpha t})) + X_t(1), \quad t \geq 0,
\]

where \(X(e^{-\alpha t})\) has Laplace-Stieltjes transform \(f e^{-\alpha t}\), \(Y_t(\cdot), X(e^{-\alpha t})\) and \(X_t(1)\) are mutually independent, \((Y_t(1))_{t \geq 0}\) satisfies (3.1a) through (3.1d) and \(X_t(1)\), the total amount of discounted living immigration at time \(t\), generates a stochastic process \(X_t(u)\) with independent stationary increments satisfying

\[
\begin{align*}
    X_{t+s}(1) &= Y_s(X_t(e^{-\alpha s})) + X_s(1); \\
    X_0(1) &= 0 \text{ a.s. .}
\end{align*}
\]

If \(\alpha = 0\), then it is no longer necessary to assume that \(X(1)\) (or its Laplace-Stieltjes transform \(f\)) and \(X_t(1)\) (or its Laplace-Stieltjes transform \(f_t\)) are infinitely divisible. Before proving a representation theorem for the Laplace-Stieltjes transform \(f_t\), we prove a preparatory lemma.

**Lemma 4.1.** Let \(f_t\) be a Laplace-Stieltjes transform satisfying (4.2). Then \(\partial/\partial t f_t(\tau)\) exists for all \(t \geq 0\) and all \(\tau \geq 0\).

**Proof.** The proof of Lemma 1.2 in Kawazu and Watanabe (1971) can easily be adapted to show that \(\partial/\partial t \ln f_t(\tau) \mid_{t=0}\) exists for all \(\tau \geq 0\). From (4.2a) it follows that
\[ t^{-1}(\ln f_{t+s}(\tau) - \ln f_s(\tau)) = t^{-1} e^{-as} (\ln f_t(F_s(\tau))) . \tag{4.5} \]

The limit of the right hand side of (4.5) exists for all \( s \geq 0 \) as \( t \downarrow 0 \), hence so does the limit of the left hand side.

**Theorem 4.2.** Let \( I \in \{ \mathbb{N}_0, \mathbb{R}_+ \} \). If \((h_t)_{t \geq 0} \in \text{BPI}_\alpha(\cdot; I)\) then the Laplace-Stieltjes transform \( f_t \) is of the form

\[ \ln f_t(\tau) = \int_0^t \ln q(F_s(\tau)) e^{-as} \, dv , \tag{4.6} \]

for some \( q \in \text{ID}(I) \). Conversely, if \( f_t \) is of the form (4.6) for some \( q \in \text{ID}(I) \) then \((h_t)_{t \geq 0} \) satisfies (4.2) and it generates a family of Laplace-Stieltjes transforms \((h_t)_{t \geq 0} \in \text{BPI}_\alpha(\cdot; I)\) through (4.1) for any \( f \in \text{ID}(I) \).

**Proof.** Let \((t_n)_{n=0}^\infty\) be a sequence of non-negative numbers such that \( t_n \downarrow 0 \) and \( x t_n^{-1} \in \mathbb{N}_+ := \{ 1, 2, \ldots \} \) for \( x > 0 \). Then (cf. (4.5))

\[ x t_n^{-1} (\ln f_{t_n+s}(\tau) - \ln f_s(\tau)) = x t_n^{-1} e^{-as} (\ln f_t(F_s(\tau))) , \tag{4.7} \]

where \( x t_n^{-1} \ln f_t(F_s(\tau)) \) is the logarithm of a Laplace-Stieltjes transform in \( \text{LST}(\cdot) \). By Lemma 4.1 we can let \( t_n \downarrow 0 \) in (4.7), yielding

\[ x \frac{\partial}{\partial s} \ln f_s(\tau) = \ln q_x(F_s(\tau)) , \tag{4.8} \]

for every \( x \in \mathbb{R}_+ \) and some function \( q_x \). Integrating (4.8) in \( s \) over \((0, t)\) yields

\[ x \ln f_t(\tau) = \int_0^t \ln q_x(F_s(\tau)) \, ds . \]

Since \( f_t(0+) = 1 \) and \( \ln q_x(\tau) \leq 0 \), we have that \( q_x(0+) = 1 \) and hence by the continuity theorem for Laplace-Stieltjes transforms, \( q_x \) is a Laplace-Stieltjes transform for every \( x > 0 \), and so by (4.8) \( q := q_1 \) is infinitely divisible (and hence so is \( f_t \)). Since \( q \) is the limit of the right hand side of (4.7) with \( s = 0 \) as \( n \to \infty \) and \( f_t \in \text{ID}(I) \), \( q \in \text{ID}(I) \).

The converse is easily verified. \( \square \)

**Remark 4.3.** Let the operator \( T_t^q \), acting on infinitely divisible Laplace-Stieltjes transforms, be defined by (cf. (4.1))

\[ T_t^q f = f e^{-as} \circ F_t . \tag{4.9} \]

From (2.5a) it follows that \((T_t^q)_{t \geq 0}\) forms a composition semigroup, i.e.,

\[ T_t^q T_s^q = T_{t+s}^q , \quad s, t \geq 0 ; \tag{4.10a} \]

\[ T_0^q = \text{identity mapping} . \tag{4.10b} \]

Furthermore, \( e^{-as} \) is the only depreciation function which makes \((T_t^q)_{t \geq 0} a\)
composition semigroup.

REMARK 4.4. In the proofs of Theorem 4.2 and Lemma 4.1 we do not use the assumption that \( f_t \) is infinitely divisible. The infinite divisibility of \( f_t \) is in fact a consequence of (4.8).

5. THE STABLE DISTRIBUTIONS

Theorem 2.1 gives us a one-to-one correspondence between collections of operators \((T_t)_{t \geq 0}\) satisfying (1.11) and continuous semigroups of cumulant generating functions \((F_t)_{t \geq 0}\) satisfying (2.5a) through (2.5d). In Section 3 we saw that \((F_t)_{t \geq 0}\) satisfies (2.5) if and only if it is a family of cumulant generating functions corresponding to a Markov branching process. We shall therefore henceforth assume that \((F_t)_{t \geq 0}\) satisfies (2.5a) through (2.5d). If \( m \) is allowed to take the value \( \infty \), then (2.5e) is always satisfied.

A Laplace-Stieltjes transform \( f \) is stable with exponent \( \alpha \) if and only if it satisfies (1.3) for some \( \alpha > 0 \). This result leads us to the following definition of a stable Laplace-Stieltjes transform with exponent \( \alpha \), corresponding to the branching process \((F_t)_{t \geq 0}\).

DEFINITION 5.1. Let \( I \in \{ \mathbb{N}_0, \mathbb{R}_+ \} \) and let \( F = (F_t)_{t \geq 0} \in BP(\cdot ; I) \). A function \( f \) is said to be \( F \)-stable with exponent \( \alpha \) and belong to the set \( U_{\alpha}^* (F) \) if \( f \in \text{LST}(I) \) and for every \( t \geq 0 \) and some \( \alpha \in \mathbb{R} \)

\[ f(t) = f^{e^{\alpha t}}(F_t(t)), \quad \tau \geq 0. \tag{5.1} \]

Since (5.1) holds for every \( t \geq 0 \), \( f \) is infinitely divisible if \( \alpha \neq 0 \). From (4.1) it now follows that a Laplace-Stieltjes transform is \( F \)-stable with exponent \( \alpha \neq 0 \) if and only if it is an invariant distribution of an \( \alpha \)-discounted branching process without immigration, where \( \alpha \neq 0 \). If \( F_t = -\ln g_t \), with \( g_t \in \text{LST}(\mathbb{N}_0) \), then, since \( f \) is infinitely divisible for \( \alpha \neq 0 \), \( f \in \text{LST}(\mathbb{N}_0) \). If \( g_t \notin \text{LST}(\mathbb{N}_0) \) then neither is \( f \). We will consider (5.1) for a fixed semigroup \( F \). We therefore often write \( U_{\alpha}^* F \) instead of \( U_{\alpha}^* (F) \). If \( F_t(t) = t \) for all \( t \geq 0 \), then \( f \equiv 1 \) is the only solution of (5.1) for \( \alpha \neq 0 \) and all \( f \) satisfy (5.1) if \( \alpha = 0 \). We will therefore henceforth exclude this case. We also exclude the trivial case \( F_t(t) \equiv 0 \) for all \( t \geq 0 \).

It can be shown (cf. van Harn et al. (1982) or Remark 5.8) that if \( \alpha < 0 \), then (5.1) is equivalent to
for some sequence \((t_n)_{n=0}^\infty\) with \(t_n \in \mathbb{R}_+\). If \(a > 0\), then we can rewrite (5.1) as
\[
f(t) = f^{1/n}(F_{t_n}(x)), \quad t \geq 0,
\]
for some non-negative sequence \((t_n)_{n=0}^\infty\). Furthermore (5.2) corresponds to the classical definition of stability (cf. Feller (1971)). If \(F_t(x)\) is as in Example 3.4 with \(m > 0\), then rewriting (5.3) we can obtain (5.2) with \(F_t(x) = e^{-mt}\). Thus in this case (5.2) and (5.3) give rise to the same \(F\)-stable distributions. We now prove a representation theorem for \(F\)-stable Laplace-Stieltjes transforms.

**Theorem 5.2.** Let \(\alpha \in \mathbb{R}\). \(f \in U_\alpha^*\) if and only if \(m \neq 0\), \(U\) and \(\alpha\) satisfy

1. if \(m < 0\), then \(m \leq \alpha < 0\);
2. if \(0 < m < \infty\) then \(0 < \alpha \leq m\) and \(U(x)/\alpha \rightarrow a \geq \alpha\) as \(\tau \rightarrow \infty\);
3. if \(m = \infty\), then \(0 < \alpha < \infty\) and \(U(x)/\alpha \rightarrow a \geq \alpha\) as \(\tau \rightarrow \infty\),

and \(f\) is of the form
\[
\ln f(\tau) = -\lambda(\tau_0) \exp \left\{ \alpha \int_{\tau_0}^\tau U(x)^{-1} \, dx \right\}, \quad \tau \geq 0,
\]
with \(\lambda(\tau_0) > 0\). Furthermore, either \(p = 1\) (and we may choose \(\tau_0 = \infty\)) or \(p = 0\) (and we may choose \(\tau_0 = 1\)). With this choice of \(\tau_0\), the representation (5.4) is unique.

**Proof.** Let \(f \in U_\alpha^*\) for some fixed \(\alpha \in \mathbb{R}\). Logarithmic differentiation of (5.1) with respect to \(t\) and evaluation in \(t = 0\), yields
\[
U(\tau) \frac{f'(\tau)}{f(\tau)} - \alpha \ln f(\tau) = 0. \tag{5.5}
\]
If \(\alpha = 0\) then we must have \(U(\tau) = 0\), which is not possible. If \(U(\tau_0) = 0\) for some \(\tau_0 \in (0, \infty)\), then \(f(\tau_0) = 1\) and so \(f \equiv 1\), which is not possible. Hence \(U(\tau)/\alpha \neq 0\) and so (5.4) is the solution of (5.5). It now follows that \(f\) is infinitely divisible. This implies that \(\ln f(\tau)\) is negative and convex and so
\[
[\ln f(\tau)/\tau] \left[ \frac{d}{d\tau} \ln f(\tau) \right]^{-1} \geq 1. \tag{5.6}
\]
It follows from (5.5) that
\[
\frac{U(\tau)}{\alpha} = \left[ \ln f(\tau)/\tau \right] \left[ \frac{d}{d\tau} \ln f(\tau) \right]^{-1}. \tag{5.7}
\]
Letting \(\tau \rightarrow 0\) and \(\tau \rightarrow \infty\) in (5.7) now yields (i), (ii) and (iii) and the condition that \(m \neq 0\) (cf. (3.8) and (3.11)). If \(m \leq 0\), then by (3.14) either \(p = 0\) or \(p = 1\). If \(m > 0\), then, since \(U(\tau)\) has no zeros on \((0, \infty)\), we have \(p < r = 0\). If \(p = 1\) then \(\lim_{\tau \rightarrow \infty} f(\tau) > 0\) and so we may choose \(\tau_0 = \infty\). If \(p = 0\) then this limit is zero, so we
may choose $\tau_0 < \infty$.

Conversely, if $f$ has the form (5.4), then $f$ satisfies (5.1) (cf. (3.15) and (3.16)). Furthermore

$$\frac{\ln f(\tau)}{\tau} = -\lambda(\tau_0) \exp \left\{ \int_{\tau_0}^\tau \frac{\alpha x - U(x)}{xU(x)} \, dx - \ln \tau_0 \right\}.$$  \hspace{1cm} (5.8)

Conditions (i), (ii) and (iii) imply that the integrand in (5.8) is negative and so (5.8) is increasing. By Seneta and Vere-Jones (1968) for $m < 0$ and $BP(\mathbb{R}_+)$, Pakes (1979) for $m > 0$ and $BP(\mathbb{R}_+)$ and Seneta (1969) for $m \neq 0$ and $BP(\mathbb{N}_0)$, we have that $f$ is a Laplace-Stieltjes transform. Hence $f \in U_\alpha^*$. \hfill \Box

**COROLLARY 1.** If $f \in U_\alpha^* \text{ and } -f'(0) < \infty$, then $\alpha = m < \infty$.

**PROOF.** If $-f'(0) < \infty$, then the right hand side of (5.7) tends to one and the left hand side tends $m/\alpha$ (cf. (3.11)) as $\tau \downarrow 0$. \hfill \Box

Before continuing we introduce some notation.

**NOTATION 5.3.** Denote by $f_\alpha^*$ the Laplace-Stieltjes transform of an $F$-stable Laplace-Stieltjes transform with exponent $\alpha$ if $m \neq 0$ and $\alpha$, $m$ and $U$ satisfy conditions (i) and (ii) of Theorem 5.2 (and we call $f_\alpha^*$ a proper $F$-stable Laplace-Stieltjes transform) and if $m = 0$ or $\alpha$, $m$ or $U$ do not satisfy (i), (ii) or (iii) of Theorem 5.2, let $f_\alpha^* \equiv 1$.

Suppose that $m \neq 0$, $m < \infty$, that $\alpha$, $U$ and $m$ satisfy (i), (ii) and (iii) of Theorem 5.2 and let $A_\alpha$ be defined by (3.15) (cf. Theorem 5.2), with $\tau_0$ chosen as in Theorem 5.2. Then $A_\alpha \in U_\alpha^*$ and hence $A_\alpha$ is infinitely divisible. Let $F = (F_t)_{t \geq 0} \in BP(I_1, I_2)$, $I_1, I_2 \in \{\mathbb{N}_0, \mathbb{R}_+\}$. For $\theta > 0$ define the map $\pi_\theta^F : \text{LST}(I_2) \rightarrow \text{LST}(I_1)$ (as in van Harn et al. (1982)) as follows:

$$\pi_\theta^F g(\tau) := g(\theta A_\alpha(\tau)) = \int_0^\infty (f_\alpha^*(\tau))^x \, dG(x),$$  \hspace{1cm} (5.9)

with $f_\alpha^* := \theta A_\alpha$ and $G$ the distribution function corresponding to $g$. Hence, $f$ is a powermixture of $F$-stable Laplace-Stieltjes transforms with exponent $\alpha$. Let $f \in U_\alpha^*$. From (5.4) and (3.15) it follows that

$$\ln f(\tau) = -\lambda A_m(\tau)^{\alpha/m}, \quad \alpha/m \in (0, 1],$$  \hspace{1cm} (5.10)

for some $\lambda > 0$, i.e., $f(\tau) = g(A_m(\tau))$ with $g$ a stable Laplace-Stieltjes transform with exponent $\alpha/m$. Conversely, if $g$ is stable with exponent $\alpha/m \in (0, 1)$, then $f(\tau) := g(A_m(\tau)) \in U_\alpha^*$. We have thus proved
THEOREM 5.4. Let $m \neq 0$, $m < \infty$ and let $\alpha$, $U$ and $m$ satisfy conditions (i) and (ii) of Theorem 5.2.

(i) $f \in U^{*}_{\alpha}$ if and only if $f = \pi^{F}_{0} \cdot m g$, with $g \in U^{*}_{\alpha/m} \cdot ((e^{-t})_{t \geq 0})$, i.e., $g$ is stable with exponent $\alpha/m$, for some, and hence all, $\theta > 0$.

(ii) $g \in U^{*}_{\alpha/m} \cdot ((e^{-t})_{t \geq 0})$ if and only if $\pi^{F}_{0} \cdot m g \in U^{*}_{\alpha} \cdot F$ for some, and hence all, $\theta > 0$.

The analogy between the classical stable and the $F$-stable distributions is further emphasized in the following theorem.

THEOREM 5.5. Let $f$ be a Laplace-Stieltjes transform with $f \neq 1$.

(i) Let $m < 0$. $f$ is $F$-stable if and only if there exists a Laplace-Stieltjes transform $g$ and a sequence $(t_{k})_{0}^{\infty}$ with $t_{k} \in \mathbb{R}_{+}$ such that

$$
\ln f(t) = \lim_{k \to \infty} k \ln g(F_{t_{k}}(t)) , \quad t \geq 0 .
$$

(ii) Let $m > 0$. $f$ is $F$-stable if and only if there exists an infinitely divisible Laplace-Stieltjes transform $g$ and a sequence $(t_{k})_{0}^{\infty}$ with $t_{k} \in \mathbb{R}_{+}$ such that

$$
\ln f(t) = \lim_{k \to \infty} \frac{1}{k} \ln g(F_{t_{k}}(t)) , \quad t \geq 0 .
$$

PROOF. The necessity follows from Definition 5.1 with $(t_{k})_{0}^{\infty}$ chosen such that

$$
\exp\{-\alpha t_{k}\} = k \text{ if } \alpha < 0 \text{ and } \exp\{\alpha t_{k}\} = k \text{ if } \alpha > 0 .
$$

Conversely, let $m < 0$ and let $f$ be of the form (5.11). For the limit to exist $F_{t_{k}}(t)$ must tend to zero as $k \to \infty$, i.e., $t_{k} \to \infty$ as $k \to \infty$ (cf. (3.13)). Also

$$
\lim_{k \to \infty} (k+1) \ln g(F_{t_{k}}(F_{t_{k+1}}(t))) = \ln f(t) .
$$

Hence $t_{k+1} - t_{k} \to 0$ as $k \to \infty$ (cf. (2.5c)). Thus for any $t > 0$ there exists a subsequence $(t_{l(k)})_{0}^{\infty}$ with $l(k) < k$ such that $t_{k} - t_{l(k)} \to t$ as $k \to \infty$. Observe that

$$
\ln f(t) = \lim_{k \to \infty} \frac{k}{l(k)} \ln g(F_{t_{l(k)}}(F_{t_{l(k)}}(t))) .
$$

Necessarily $k/l(k)$ converges as $k \to \infty$. Call this limit $S(t)$. By the semigroup property (2.5a) of $(F_{t})_{t \geq 0}$ we have that $S(t_{1}) = S(t_{2})$ for any $t_{1}, t_{2} > 0$. Hence $S(t) = e^{-\alpha t}$ for some $\alpha < 0$ and so $f$ satisfies (5.1). Since $f \neq 1$, $\alpha/m \in (0, 1]$ by Theorem 5.2. Similarly for (ii). \qed
REMARK 5.6. Let $q$ be an infinitely divisible Laplace-Stieltjes transform. Suppose $\alpha > m > 0$ and $r = 0$. Then $U(\tau) > 0$ for $\tau > 0$. Hence (cf. (3.2)) $F_t(\tau)$ is an increasing function in $t$ and $F_t(\tau) \to \infty$ as $t \to \infty$ for any $\tau > 0$. From (3.2) we also see that

$$F_{t''}(\tau) = \frac{\partial}{\partial \tau} \frac{U(\tau)}{U(F_t(\tau))} = \frac{U(F_t(\tau))}{U(\tau)^2} \left( U'(F_t(\tau)) - U'(\tau) \right),$$

which is negative by (3.7). Hence $F_t'(\tau)$ is decreasing in $\tau$ and so $F_t'(\tau) \leq F_t'(0) = e^{mt}$. Also, since $\ln q(\tau)$ is convex and negative, $d/d\tau \ln q(\tau)$ is increasing and negative. Hence, by L'Hôpital's rule, (3.10) and (3.11),

$$\lim_{t \to \infty} e^{-\alpha t} \ln q(F_t(\tau)) = \alpha \lim_{t \to \infty} e^{-\alpha t} \frac{q'(F_t(\tau))}{q(F_t(\tau))} U(\tau) F_t'(\tau) \geq C U(\tau) \lim_{t \to \infty} e^{-\alpha t} e^{mt} = 0,$$

for some $C < 0$. If $r > 0$ (this is for example the case when $m \leq 0$ and $F_t(\tau) \neq \tau$) then $F_t(\tau) \to -\ln r$ as $t \to \infty$ and hence, for $\alpha > 0$

$$\lim_{t \to \infty} e^{-\alpha t} \ln q(F_t(\tau)) = 0 \quad \Box$$

REMARK 5.7. Theorem 5.5 (i) can be expressed in terms of random variables as follows: A random variable $X$ is $F$-stable if and only if there exists a sequence of independent and identically distributed random variables $(X_j)_{j=1}^\infty$ and a non-negative sequence $(r_k)_{k=0}^\infty$ such that

$$\sum_{j=1}^k Y_{i_1}^{(j)}(X_j) \to X \text{ as } k \to \infty,$$

where $Y_{i_1}^{(j)}(\cdot), j = 1, 2, \ldots$ are mutually independent, independent of $X_j, j = 1, 2, \ldots$ and all distributed as $Y_{i_1}(\cdot)$. The triangular array $(Y_{i_1}^{(j)}(X_j))_{j=1}^k, k \in \mathbb{N}$ is necessarily uan. Thus $X$ is a limit of a sequence of partial sums of independent identically distributed normed random variables, where the norming is done via the branching process $(Y_{i_1}(\cdot))_{i=0}^\infty$. \Box

REMARK 5.8. The method used to prove Theorem 5.5 can be used to prove the equivalence between (5.1) and equation (5.2) (if $m < 0$) or (5.3) (if $m > 0$). \Box

We conclude this section with two remarks and a generalization of the definition of the domain of normal attraction of a (stable) Laplace-Stieltjes transform (cf. Feller (1971)).

DEFINITION 5.9. A Laplace-Stieltjes transform $g$ is said to be in the domain of normal attraction with respect to $F$ of the Laplace-Stieltjes transform $f$ if there exists a non-
negative sequence \((t_k)_{k=0}^\infty\) such that \((5.11)\) is satisfied if \(m < 0\) and \((5.12)\) is satisfied if \(m > 0\). In the latter case \(g\) must be an infinitely divisible Laplace-Stieltjes transform. □

REMARK 5.10. By Theorem 5.5 and Remark 5.6 the only Laplace-Stieltjes transforms which are not identically one, having a domain of normal attraction with respect to \(F\) are the \(F\)-stable Laplace-Stieltjes transforms. □

REMARK 5.11. From the proof of Theorem 5.2, it follows that \((5.4)\) is the (up to multiplicative constants) unique solution of \((5.1)\), even if \(f\) is not a priori assumed to be a Laplace-Stieltjes transform. □

6. THE SELF-DECOMPOSABLE DISTRIBUTIONS

We begin this section with a definition which generalizes \((1.9)\) and \((1.10)\) and which combines the, until now, separate approaches to self-decomposability of \(\mathbb{N}_0\)-valued random variables and \(\mathbb{R}_+\)-valued random variables.

DEFINITION 6.1. Let \(I \in \{\mathbb{N}_0, \mathbb{R}_+\}\) and let \(F = (F_t)_{t \geq 0} \in BP(\cdot; I)\). A function \(f\) is said to be \(F, \alpha\)-self-decomposable and belong to the set \(U_\alpha(F)\) if \(f \in LST(I)\) and for every \(t \geq 0\) and some \(\alpha \in \mathbb{R}\), there exists an \(f_t \in LST(I)\) such that

\[
f(t) = f e^{-\alpha t} (F_t(t)) f_t(t), \quad t \geq 0. \tag{6.1}
\]

As in Section 5, we consider \((6.1)\) for a fixed semigroup \(F\). We therefore often write \(U_\alpha\) instead of \(U_\alpha(F)\). Let \(\alpha(1) > \alpha(2)\). By \((6.1)\)

\[
f(t) = f e^{-\alpha(2) t} (F_t(t)) \cdot f e^{-\alpha(2) t} - e^{-\alpha(2) t} (F_t(t)) f_t(t)
\]

If \(f\) is infinitely divisible then it follows that \(U_{\alpha(1)} \supset U_{\alpha(2)}\) for all \(\alpha(1) \geq \alpha(2)\). Hence \((U_\alpha)_{\alpha \in \mathbb{R}}\) is an increasing sequence of sets.

REMARK 6.2. It is evident from Definition 6.1 that \(U_\alpha(F) = U_1(G), \quad \alpha > 0\), where \(G = (G_t)_{t \geq 0}\) and \(G_t = F_t/\alpha\). Similarly if \(\alpha < 0\). It would therefore be sufficient to consider the cases \(\alpha = -1, \alpha = 0\) and \(\alpha = 1\). Since the analysis is not more complicated when considering \(\alpha \in \mathbb{R}\) and since \((6.1)\) provides us with some interesting interpretations when \(\alpha\) is allowed to take values in \(\mathbb{R}\) (cf. Section 7), we consider \((6.1)\) for \(\alpha \in \mathbb{R}\). □
We are now ready to prove the analog of Theorem 5.3 in van Ham et al. (1982).

**Theorem 6.3.** Let \( m \neq 0, m < \infty \) and let \( \alpha, U \) and \( m \) satisfy conditions (i) and (ii) of Theorem 5.2.

(i) If \( m < 0 \), then \( g \in U \alpha((e^{-t})_{t \geq 0}) \), i.e., \( g \) is \( \alpha \)-self-decomposable if and only if \( \pi_0^{E^m} g \in U_{-\alpha/m}(F) \) for all \( \theta > 0 \).

(ii) If \( m > 0 \), then \( g \in U \alpha((e^t)_{t \geq 0}) \) if and only if \( \pi_0^{E^m} g \in U_{\alpha/m}(F) \) for all \( \theta > 0 \).

**Proof.** (i). Let \( A_m(t) \) be given by (3.15) with \( m < 0 \). Hence \( r = 1 \) (cf. (3.14a)), \( F_t(t) \rightarrow 0 \) as \( t \rightarrow \infty \) and \( A_m(t) \) is a proper \( F \)-stable Laplace-Stieltjes transform (cf. Notation 5.3). Let \( g \) be self-decomposable, i.e., let \( g \) be a Laplace-Stieltjes transform satisfying (1.10) and let \( f_\theta := \pi_0^{E^m} g, \theta > 0 \). Then (cf. (5.1) and (3.16))

\[
\int_0^\infty g(\theta (A_m(t))) = g(e^{-\theta A_m(t)}) = \int_0^\infty (F(t)) f_\theta(t)
\]

with \( f_\theta(t) = g(-i/m A_m(t)) \). Hence \( f_\theta \in U_{-\alpha/m}(F) \) for all \( \theta > 0 \).

Conversely, if \( g \) is a Laplace-Stieltjes transform, \( A_m(t) \) is a proper \( F \)-stable Laplace-Stieltjes transform and \( g(\theta A_m(t)) \in U_{-\alpha/m} \) for all \( \theta > 0 \), then

\[
g(\theta A_m(F_1/\ln(\theta_0) x)) = g(e^{-\theta A_m(F_1/\ln(\theta_0) x)}) g(\theta A_m(F_1/\ln(\theta_0) x))
\]

By Lemma 2.1 (i), \( A_m(x) \sim \tau \) as \( x \downarrow 0 \) and hence \( A_m(x) - A_m(x) \tau \) as \( x \downarrow 0 \). Letting \( x = F_1/\ln(\theta_0) \) we have (cf. (3.16))

\[
\theta A_m(F_1/\ln(\theta_0) x) \sim \tau \text{ as } \theta \rightarrow \infty.
\]

Letting \( \theta \rightarrow \infty \) shows that \( g \) is an \( \alpha \)-self-decomposable Laplace-Stieltjes transform.

(ii). Let \( A_\alpha(t) \) be given by (3.15) with \( \alpha > 0 \). Under the conditions of the theorem \( U(\tau) \) has no zeros on \((0, \infty)\), so \( r = 0 \). Hence \( F_t(t) \rightarrow \infty \) as \( t \rightarrow \infty \) (cf. (3.13)). Let \( g \) be reverse \( \alpha \)-self-decomposable (cf. Hansen (1989b)), i.e., let \( g \) be a Laplace-Stieltjes transform satisfying

\[
g(\tau) = g(e^{i\alpha} g(t), t \geq 0
\]

with \( g \in \text{LST}(\mathbb{R}_+) \). The rest of the proof is almost identical with the proof of part (i), except that we use Lemma 2.1 (ii), and therefore omitted. \( \square \)

Theorem 6.3 shows that there exists a one-to-one correspondence between \( \alpha \)-self-decomposable Laplace-Stieltjes transforms and the Laplace-Stieltjes transforms in \( U_\alpha \) of the form \( g(A_\alpha(t)) \) with \( g \) a Laplace-Stieltjes transform and where the underlying branching process \((Y_\tau(c))_{t \geq 0}\) is such that proper \( F \)-stable Laplace-Stieltjes transforms exist. A similar relation holds between \( F, \alpha \)-self-decomposable Laplace-Stieltjes transforms with \( 0 < m < \infty \) and reverse self-decomposable Laplace-Stieltjes transforms with \( 0 < m < \infty \) and reverse self-decomposable Laplace-Stieltjes transforms.
transforms as introduced in Hansen (1989b). We note that not all Laplace-Stieltjes transforms in $U_\alpha$ are of the form $g(\theta A_m(t))$ with $g$ a Laplace-Stieltjes transform (cf. Example 6.6, van Harn et al. (1982)). Furthermore, not all branching processes give rise to proper $F$-stable Laplace-Stieltjes transforms (cf. Theorem 5.2). Theorem 6.3 provides us with a way of constructing Laplace-Stieltjes transforms in $U_\alpha$ for branching processes whose generators $U$ (cf. (3.4) and (3.6)) satisfy the conditions of Theorem 5.2, but it does not give us a characterization of $U_\alpha$, even if the conditions of Theorem 5.2 are met.

Before proving a representation theorem for Laplace-Stieltjes transforms in $U_\alpha$ we state an assumption. The assumption is made for analytic purposes only.

**ASSUMPTION 6.4.** If $m > 0$ and $U(\tau)/\tau \rightarrow a \in [0, \alpha)$, then we assume that $f$ in Definition 5.1 is infinitely divisible. □

**THEOREM 6.5.** Let $I \in \{N_0, \mathbb{R}_+\}$ and let $F = (F_t)_{t \geq 0} \in BP(\cdot; I)$. $f \in U_\alpha(F)$ if and only if

(i) if $m \leq 0$, then $m \leq \alpha$;

(ii) if $m > 0$, then $\alpha > 0$,

and there exists a $q \in ID(I)$ such that

$$\ln f(t) = \int_0^\infty \ln q(F_v(t)) e^{-\alpha v} dv + \ln f^*(\tau),$$

(6.2)

with the integral converging (cf. Theorem 6.6) and $f^*$ defined in Notation 4.3. The representation in (6.2) is unique. Furthermore, if $f \in U_\alpha$, then $f_i$ in (6.1) is infinitely divisible.

**PROOF.** Suppose $f \in U_\alpha$, i.e., suppose $f$ satisfies (6.1). From (6.1) it follows that

$$e^{-\alpha(s+t)} \ln f(F_{s+t}(\tau)) + \ln f_{s+t}(\tau) = f(\tau)$$

$$= e^{-\alpha s} e^{-\alpha t} \ln f(F_t(F_s(\tau))) + e^{-\alpha s} \ln f_t(F_s(\tau)) + \ln f_s(\tau).$$

Hence $f_t$ satisfies (4.2). By Theorem 4.2 and Remark 4.4, $f_t$ is of the form (4.6) and so $f_t$ is infinitely divisible and (cf. (6.1))

$$\ln f(\tau) = e^{-\alpha t} \ln f(F_t(\tau)) + \int_0^t \ln q(F_v(\tau)) e^{-\alpha v} dv.$$  

(6.3)

Since both terms on the right hand side of (6.3) are negative, the limit of each term separately exists as $t \to \infty$. Let

$$\lim_{t \to \infty} e^{-\alpha t} \ln f(F_t(\tau)) := \ln x(\tau).$$  

(6.4)
Necessarily, \( x(\tau) \) satisfies (5.1). Since \( f(0^+)=q(0^+)=1 \) we have from (6.3) with \( t=\infty \) that \( x(0^+)=1 \). If \( \alpha \leq 0 \), then we can define a sequence \((\eta_n)_{n=1}^\infty\) such that \( \exp\{-\alpha \eta_n\} \in \mathbb{N}_+ \). Hence \( e^{-\alpha \eta_n} \ln f(F_{\eta_n}(\tau)) \) is the logarithm of a Laplace-Stieltjes transform and so by the continuity theorem for Laplace-Stieltjes transforms \( x \) is a Laplace-Stieltjes transform. If \( \alpha > 0 \) and \( r > 0 \) then by (3.13), \( x(\tau) \equiv 1 \). If \( \alpha > 0 \) and \( U(\tau) \tau \to a \geq \alpha \), then by Remark 5.11 and Theorem 5.2 \( x \in U^*_\alpha \). If \( \alpha > 0 \), \( r = 0 \) and \( U(\tau) \tau \to a \in [0, \alpha) \), then by Assumption 6.4, \( e^{-\alpha t} \ln f(F_\tau(\tau)) \) is a Laplace-Stieltjes transform, and hence, by the continuity theorem for Laplace-Stieltjes transforms, so is \( x \). By Theorem 5.2, \( x \equiv 1 \). In any case \( f \) is in the domain of normal attraction with respect to \( F \) of \( f^*_\alpha \). Hence either \( x \in U^*_\alpha \) or \( x \equiv 1 \), and so \( f \) is of the form (6.3). It follows that \( f \) is infinitely divisible, so \( \ln f(\tau) \) is negative and convex and hence \( f \) satisfies (5.6). Differentiating (6.3) in \( t \), evaluating in \( t=0 \) and rearranging as in the proof of Theorem 5.6 yields

\[
U(\tau)/\tau \leq \alpha [\ln f(\tau)/\tau] [\frac{d}{d\tau} \ln f(\tau)]^{-1}.
\]

If \( m > 0 \), then necessarily \( \alpha > 0 \). Let \( m \leq 0 \). Letting \( \tau \downarrow 0 \) yields (cf. (3.11) and (5.6)) \( m \leq \alpha \).

Conversely, if \( f \) is of the form (6.2) then \( f \) is an infinitely divisible Laplace-Stieltjes transform and by (5.1) and (2.5a), \( f \) satisfies (6.1).

**COROLLARY 1.** If \( f \in U_\alpha \) and \(-f'(0) < \infty\), then \( \alpha = m < \infty \) if \( f \in U^*_\alpha \) and \( \alpha > m \) if \( f \notin U^*_\alpha \).

**PROOF.** If \(-f'(0) < \infty\), then the right hand side of (6.5) tends to \( \alpha \) as \( \tau \downarrow 0 \). Since we have equality in (6.5) if and only if \( f \in U^*_\alpha \) (cf. (5.5)), the corollary follows.

**COROLLARY 2.** If \( f \in U_\alpha \), then \( f \) is in the domain of normal attraction with respect to \( F \) (cf. Definition 5.9) of \( f^*_\alpha \) (cf. Notation 5.3).

**COROLLARY 3.** Let \( I \in \{\mathbb{N}_0, \mathbb{R}_+\} \). \( U_\alpha(F), F = (F_i)_{i \geq 0} \in BP(I; \cdot) \) is a multiplication semigroup, closed under limits, with \( U_\alpha(F) \subset ID(I) \).

From Theorem 6.5 it follows that if \((F_i)_{i \geq 0} \in BP(\mathbb{N}_0, \cdot) \), then \( f, f_i \in LST(\mathbb{N}_0) \). Also if \( g_i \notin LST(\mathbb{N}_0) \), then neither is \( f \) nor \( f_i \). It is curious to note that if \((F_i)_{i \geq 0} \in BP(\mathbb{N}_0; \mathbb{R}_+) \), then \( q \) is not necessarily the Laplace-Stieltjes transform of an \( \mathbb{N}_0 \)-valued random variable. Thus in this case the set \( U_0(F) \) is larger than the set of \( F \)-self-decomposable distributions obtained by van Harn et al. (1982). The following theorem gives us necessary and sufficient conditions for the convergence of the integral in (6.2).
THEOREM 6.6. Let \((\gamma, M)\) be the Lévy couple of the infinitely divisible Laplace-Stieltjes transform \(q\) in (6.2) (cf. Theorem 2.2).

(i) If \(m \leq \alpha < 0\), then the integral in (6.2) converges if and only if
\[
\int x^{\alpha/m} \, dM(x) < \infty. \tag{6.6}
\]

(ii) If \(m < \alpha = 0\), then the integral in (6.2) converges if and only if
\[
\int \ln x \, dM(x) < \infty. \tag{6.7}
\]

(iii) If \(m = \alpha = 0\), then for the integral in (6.2) to converge it is necessary that (6.7) holds and \(U''(0) = \infty\) and sufficient that for some \(\delta \in [0, 1)\) and some \(c \geq 0\)
\[
\frac{\ln q(x)}{U(x)} - c x^{-\delta} \text{ as } x \downarrow 0.
\]

(iv) If \(m > 0\) and \(U(\tau)/\tau \to a > \alpha\) as \(\tau \to \infty\), then the integral in (6.2) converges if and only if \(\gamma = 0\) and
\[
\int_0^1 x^{\alpha/a} \, dM(x) < \infty. \tag{6.8}
\]

(v) In all other cases the integral in (6.2) converges for any infinitely divisible Laplace-Stieltjes transform \(q\).

PROOF. (i). Let \(m \leq \alpha < 0\). By Theorem 5.2, \(f_x^*\) is proper. Suppose that the integral in (6.2) converges. Then (cf. (5.1) and (3.2))

\[
\ln f(\tau) = \int_0^\infty \ln q(F_\nu(\tau)) e^{-\nu} \, d\nu
\]

\[
= \int_0^\infty \ln q(F_\nu(\tau)) \left[ \ln f_x^* (\tau) / \ln f_x^* (F_\nu(\tau)) \right] d\nu
\]

\[
= \ln f_x^* (\tau) \int_0^\infty \int \tau \left[ \ln q(F_\nu(x)) / \ln f_x^* (F_\nu(x)) \right] dx \, d\nu
\]

\[
+ \ln f_x^* (\tau) \ln f(\tau_0) / \ln f_x^* (\tau_0)
\]

\[
= \ln f_x^* (\tau) \int_0^\tau \int \tau_0^\infty \left[ \ln q(F_\nu(x)) / \ln f_x^* (F_\nu(x)) \right] U(x)^{-1} \, d\nu \, dx
\]

Since both terms on the right hand side of (6.9) are negative, the limit of each term separately exists as $\tau_0 \downarrow 0$. By Lemma 2.1 (i), $\ln f_\alpha^*(x) \to -\lambda x^{\alpha/m}$ and by (3.11), $U(t) \sim mx$ as $x \downarrow 0$. Hence the integral in (6.9) converges if and only if
\[
- \int_0^\tau \ln q(x) x^{-\alpha/m} \, dx < \infty.
\]
(6.10)
This is the case (cf. Hansen (1989a)) if and only if (6.6) holds. Conversely, if (6.6) holds, then (6.10) holds and so the integral on the right hand side of (6.9) converges with $\tau_0 = 0$. As in (6.9) we have
\[
- \ln f_\alpha^*(t) \int_0^\tau [\ln q(x)/U(x) \ln f_\alpha^*(x)] \, dx = \int_0^\tau \ln q(F_t(x)) e^{-\alpha v} \, dv
\]
\[
- \ln f_\alpha^*(t) \int_0^\tau [\ln q(F_t(x))/U(x) \ln f_\alpha^*(F_t(x))] \, dx.
\]
(6.11)
Again, both terms on the right hand side of (6.11) converge as $t \to \infty$, and so the integral in (6.2) converges.

(ii). Let $m < 0$ and $\alpha = 0$. Suppose the integral in (6.2) converges. Then, as in the proof of part (i), we have
\[
\int_0^\infty \ln q(F_t(x)) \, dv = - \int_0^\tau [\ln q(x)/U(x)] \, dx.
\]
(6.12)
Since $U(x) \sim mx$ as $x \downarrow 0$, the above integral converges if and only if
\[
- \int_0^\tau \ln q(x) x^{-1} \, dx < \infty.
\]
(6.13)
This is the case (cf. Hansen (1989a)) if and only if (6.7) holds. The converse is proved as in part (i), by using the equivalence between (6.13) and the convergence of the right hand side of (6.12).

(iii). The proof of this part is almost identical with the proof of Theorem 6.2 (i), van Harn et al. (1982), and therefore omitted.

(iv). Let $m > 0$ and $U(t)/t \to a > \alpha$. By Theorem 5.2, $f_\alpha^*$ is proper. From the proof of part (i) we know that if (6.2) converges, then (6.9) holds, i.e.,
\[
\ln f_t(\tau)/\ln f^*_\alpha(\tau) + \int_\tau^\infty \frac{\ln q(x)/U(x) \ln f^*_\alpha(x)}{\ln f^*_\alpha(x)} \, dx = \ln f(\tau_0)/\ln f^*_\alpha(\tau_0). 
\]

(6.14)

Since both terms on the left hand side are positive, the limit of each term separately exists as \( \tau \to \infty \). By Lemma 2.1 (ii), \( \ln f^*_\alpha(x) = -\lambda x^{\alpha/a} \) and by (3.11), \( U(\tau) = mx \) as \( x \to \infty \). Hence if (6.14) holds then

\[
- \int_\tau^\infty \ln q(x) x^{-\alpha/a-1} \, dx < \infty. 
\]

(6.15)

This is the case if and only if \( \gamma = 0 \) and (6.8) holds (cf. Hansen (1989b)). Conversely, if (6.8) holds, then (6.15) holds and so the integral in (6.14) converges with \( \tau = \infty \). As in (6.9) we have

\[
-\ln f^*_\alpha(\tau) \int_\tau^\infty \frac{\ln q(x)/U(x) \ln f^*_\alpha(x)}{\ln f^*_\alpha(x)} \, dx = -\int_0^\tau \ln q(F_v(\tau_0)) e^{-\alpha \nu} \, d\nu 
\]

\[
-\ln f^*_\alpha(\tau) \int_\tau^\infty \frac{\ln q(F_v(\tau_0))/U(x) \ln f^*_\alpha(F_v(\tau_0))}{\ln f^*_\alpha(\tau_0)} \, dx. 
\]

(6.16)

Again, both terms on the right hand side of (6.16) converge as \( t \to \infty \), and so the integral in (6.2) converges.

(v). Let \( q \) be an infinitely divisible Laplace-Stieltjes transform. If \( r > 0 \) (which is the case when \( m \leq 0 \)) then, by Remark 5.6,

\[
\lim_{t \to \infty} e^{-\delta t} \ln q(F_t(\tau)) = 0, \quad \tau > 0, 
\]

for any \( \delta > 0 \). Letting \( \delta = \alpha/2, \alpha > 0 \), it follows that the integral in (6.2) is bounded from above. Let \( r = 0 \). Then \( U(\tau) \) has no zeros on \( (0, \infty) \), so the function \( f \) in (5.4) is well defined (but not a Laplace-Stieltjes transform). If \( U(\tau)/\tau \to a \in (0, \alpha] \), as \( \tau \to \infty \), then from Theorem 2.2 it is clear that (6.8) holds for any Lévy spectral function \( M \). As in the proof of (iv), it follows that (6.2) converges. If \( U(\tau)/\tau \to 0 \), as \( \tau \to \infty \), then on defining \( U_\alpha \to U \) as \( a \downarrow 0 \) with \( U_\alpha(\tau)/\tau \to a > 0 \) as \( \tau \to \infty \), it follows that (6.2) converges. \( \square \)

**Corollary 1.** If \( f \in U_m, m < 0 \), then \( f \) must be F-stable with exponent \( m \), or, equivalently, then \( f \) is of the form (6.2) with \( q \equiv 1 \).

**Proof.** Let \( m < 0 \) and suppose \( f \in U_m \). By the proof of Theorem 6.6 (i), \( q \) must satisfy (6.10) with \( \alpha = m \). Hence

\[
\lim_{x \downarrow 0} \ln q(x) x^{-1} = 0. 
\]

This implies that \( q'(0) = 0 \) and so \( q \equiv 1 \). \( \square \)
REMARK 6.7. We can drop the assumption that $f$ is infinitely divisible in Assumption 6.4 if it can be shown that $f$ is infinitely divisible if and only if $f^c$ is a Laplace-Stieltjes transform for every $c \geq 1$. 

REMARK 6.8. From Theorems 4.2 and 6.4 we see that a Laplace-Stieltjes transform is $F, \alpha$-self-decomposable if and only if its distribution is an invariant distribution of an $\alpha$-discounted branching process with immigration.

We conclude this section with a remark and a limit theorem for $\alpha$-discounted branching processes with immigration, which is analogous to Theorem 3.2 in Steutel et al. (1983).

THEOREM 6.9. Let $(Z_t(1))_{t \geq 0}$ be an $\alpha$-discounted branching process with immigration and let $X$ be an $F, \alpha$-self-decomposable random variable. Then

(i) If there is a random variable $Z_{\infty}(1)$ such that $Z_t(1) \xrightarrow{w} Z_{\infty}(1)$ as $t \to \infty$, then $Z_{\infty}(1)$ is an $F, \alpha$-self-decomposable random variable.

(ii) $X$ is the weak limit of an $\alpha$-discounted branching process with immigration.

PROOF. (i). Let $h_t$ be the Laplace-Stieltjes transform of $Z_t(1)$. Hence $h_t$ is of the form (4.1) with $f$ some (infinitely divisible if $\alpha \neq 0$) Laplace-Stieltjes transform and $f_t$ given by (4.6). Obviously

$$\ln h_t(\tau) := \lim_{t \to \infty} \ln h_t(\tau) = \ln x(\tau) + \int_0^\infty \ln q(F^c(\nu)) e^{-\nu v} \, dv,$$

with $x$ a Laplace-Stieltjes transform satisfying (5.1). By Theorem 5.2, $x(\tau) \equiv 1$ or $x \in U^*_\alpha$, and so by Theorem 6.5, $h \in U_\alpha$.

(ii). Let $f$ be the Laplace-Stieltjes transform of $X$. Since $f \in U_\alpha$, $f$ satisfies (6.1). Define $h_t$ by (4.1). Since $f_t$ is of the form (4.6), $f_t$ satisfies (4.2) (cf. Theorem 4.2). Hence $(h_t)_{t \geq 0}$ is the Laplace-Stieltjes transforms of an $\alpha$-discounted branching process with immigration. Obviously $h_t \to f$ as $t \to \infty$.

REMARK 6.10. Necessary and sufficient conditions for the existence of $Z_{\infty}(1)$ in Theorem 6.9 can be derived from Theorem 5.5 and Theorem 6.6.
7. MULTIPLY SELF-DECOMPOSABLE DISTRIBUTIONS

Urbanik (1973) considered distributions (on $\mathbb{R}$) whose characteristic function $\phi$ satisfies (1.10) with $\alpha=0$. If the characteristic functions $\phi_t$ also satisfy (1.10) with $\alpha=0$ for every $t \geq 0$, then Urbanik (1973) called $\phi$ 1-times-self-decomposable. Proceeding inductively he defines the set of $n$-times-self-decomposable characteristic functions. In a similar way Berg and Forst (1983) consider $n$-times-(discrete)-self-decomposable Laplace-Stieltjes transforms, with the functional equation (1.4) as their starting point. We generalize both definitions (for distributions on $\mathbb{N}_0$ and $\mathbb{R}_+$) in the following definition.

DEFINITION 7.1. Let $F = (F_t)_{t \geq 0}$ be a continuous composition semigroup of cumulant generating functions satisfying (2.5). Define the sets $U_\alpha^n(F)$, $n \in \mathbb{N}_0$, inductively by $U_\alpha^0(F) := U_\alpha(F)$ and

$$U_\alpha^n(F) := \{ f \in U_\alpha^{n-1}(F) \mid \text{for all } t \in (0, \infty), f_t \in U_\alpha^{n-1}(F) \} , n \in \mathbb{N}_+ .$$

We say that $f$ is $n$-times-$F$, $\alpha$-self-decomposable if $f \in U_\alpha^n(F)$.

Multiply self-decomposable distributions have also been studied by Hong and Thu (1985), where the discrete parameter $n$ is replaced by a continuous parameter $c \geq 0$. It is evident from Definition 7.1 that $U_\alpha^n$, $n \in \mathbb{N}_0$, form a decreasing sequence of sets for each fixed $\alpha$, i.e.,

$$U_\alpha := U_\alpha^0 \supseteq U_\alpha^1 \supseteq \ldots \supseteq U_\alpha^n \supseteq \ldots \supseteq U_\alpha := \bigcap_{n \in \mathbb{N}_0} U_\alpha^n . \quad (7.1)$$

From Corollary 3 to Theorem 6.5 and Definition 7.1 it follows that $U_\alpha^1$ is a multiplication semigroup, closed under limits. Proceeding by induction the following lemma is easily proved.

LEMMA 7.2. $U_\alpha^n$, $n \in \mathbb{N}_0$ is a multiplication semigroup, closed under limits.

We are now ready to prove

THEOREM 7.3. Let $f$ and $q$ be related by (6.2) with the integral converging and let $n \in \mathbb{N}_+$. $f \in U_\alpha^n$ if and only if $q \in U_\alpha^{n-1}$ with $q$ having no $F$-stable component with exponent $\alpha$.

PROOF. Let $f \in U_\alpha^n$, i.e., let $f$ satisfy (6.1) with $f_t \in U_\alpha^{n-1}$ for every $t > 0$. From the proof of Theorem 5.2 we have that
\[
\ln q(\tau) = \lim_{t \to 0} t^{-1} \ln f_t(\tau), \quad \tau \geq 0. \tag{7.2}
\]

Since \( f_t \in U^{n-1}_\alpha \subset ID(\mathbb{R}_+), \) we have that \( f^{1/t}_t \in U^{n-1}_\alpha. \) By Lemma 7.2, \( U^{n-1}_\alpha \) is closed under limits and hence \( q \in U^{n-1}_\alpha. \)

Conversely, let \( q \in U^{n-1}_\alpha \) (with no \( F \)-stable component with exponent \( \alpha \)). Observe that
\[
\ln f_t(\tau) = \ln f(\tau) - e^{-\alpha t} \ln f(F_t(\tau))
\]
\[
= \int_0^\infty \left[ \ln q(F_v(\tau)) - e^{-\alpha t} \ln q(F_t(F_v(\tau))) \right] e^{-av} dv
\]
\[
= \int_0^\infty \ln q_t(F_v(\tau)) e^{-av} dv.
\]

Hence \( f_t \in U_\alpha \) for every \( t \geq 0. \) Obviously, if \( q \) can be decomposed \( n \) times in this fashion, then so can \( f. \)

Theorem 7.3 provides us with a simple way of characterizing \( U^n_\alpha, \ n \in \mathbb{N}_0. \) This is done in the following theorem.

**Theorem 7.4.** Let \( I \in \{ \mathbb{N}_0, \mathbb{R}_+ \}, \) let \( F = (F_t)_{t \geq 0} \in BP(\cdot ; I) \) and let \( n \in \mathbb{N}_0. \) \( f \in U^n_\alpha(F) \) if and only if

(i) If \( m \leq 0, \) then \( m \leq \alpha; \)

(ii) If \( m > 0, \) then \( \alpha > 0, \)

and there exists a \( q \in ID(I) \) such that
\[
\ln f(\tau) = \int_0^\infty \ln q(F_v(\tau)) v^n e^{-av} dv + \ln f^*_\alpha(\tau), \tag{7.3}
\]

with the integral converging (cf. Theorem 7.5) and \( f^*_\alpha \) defined in Notation 4.3. The representation in (7.3) is unique.

**Proof.** We prove the theorem by induction. The theorem is true for \( n = 0 \) by Theorem 6.5. Assume that the theorem holds for \( n = 0, 1, \ldots, N-1. \) By Theorem 7.3 \( f \in U^N_\alpha \) if and only if \( q \in U^{N-1}_\alpha. \) Substituting (7.3) with \( n = N-1 \) into (6.2) for \( q, \) changing the variable of integration and evaluating yields (7.3) with \( n = N. \)

We now give a theorem which gives necessary and sufficient conditions for the convergence of the integral in (7.3).
THEOREM 7.5. Let \((\gamma, M)\) be the Lévy couple of the infinitely divisible Laplace-Stieltjes transform \(q\) in (7.3) (cf. Theorem 2.2).

(i) If \(m \leq \alpha < 0\), then the integral in (7.3) converges if and only if
\[
\int_{1}^{\infty} (\ln x)^n x^{\alpha/m} dM(x) < \infty. \tag{7.4}
\]

(ii) If \(m < \alpha = 0\), then the integral in (7.3) converges if and only if
\[
\int_{1}^{\infty} (\ln x)^{n+1} dM(x) < \infty. \tag{7.5}
\]

(iii) If \(m = \alpha = 0\), then for the integral in (7.3) to converge it is necessary that (7.5) holds and \(U''(0) = 0\) and, on letting
\[
\ln q_k(\tau) := \int_{0}^{\infty} \ln q(F_v(\tau)) v^{k-1} dv,
\]
if \(k > 0\) and \(q_0 := q\), sufficient that for every \(k=0,1,...,n\) there exists a \(\delta_k \in [0, 1)\) and some \(c \geq 0\) such that
\[
\frac{\ln q_k(x)}{U(x)} \sim c x^{-\delta_k} \text{ as } x \downarrow 0.
\]

(iv) If \(m > 0\) and \(U(\tau)/\tau \to a > \alpha\) as \(\tau \to \infty\), then the integral in (7.3) converges if and only if \(\gamma = 0\) and
\[
\int_{0}^{1} (\ln x^{-1})^n x^{\alpha/a} dM(x) < \infty.
\]

(v) In all other cases the integral in (7.3) converges for any infinitely divisible Laplace-Stieltjes transform \(q\).

PROOF. (i). Let \(m \leq \alpha < 0\). By Theorem 5.2, \(f_\alpha^*\) is proper. As in the proof of Theorem 6.6 (i) we see that for \(n > 0\),
\[
\ln f(\tau) = \int_{0}^{\infty} \ln q(F_v(\tau)) v^n e^{-av} dv
\]
\[
= \int_{0}^{\infty} \left[ \int_{u}^{\infty} n \ln q(F_v(\tau)) e^{-av} dv \right] u^{n-1} du
\]
\[
= \ln f_\alpha^*(\tau) \int_{0}^{\infty} \int_{0}^{\infty} \frac{\partial}{\partial v} \left[ n \ln q(F_v(\tau)) / f_\alpha^*(F_v(\tau)) \right] u^{n-1} / U(x) dx dv \] du
The integral converges (cf. proof of Theorem 6.6 (i)) if and only if
\[
\int_0^\tau \left[ \int_0^\infty n \ln q(F_u(x)) u^{n-1} e^{-\omega u} \, du \right] \left[ U(x) f_{\alpha}^*(x) \right]^{-1} \, dx < \infty.
\] (7.6)

By iteration we have that (7.6) holds if and only if (cf. proof of Theorem 6.6 (i) and (6.10))
\[
\int_0^\tau \cdots \int_0^{x_1} \cdots \int_0^{x_{n-1}} n! \ln q(x_n) (x_1 \cdots x_{n-1})^{-1} x_n^{-\alpha/m-1} \, dx_n \cdots \, dx_1 < \infty,
\] (7.7)
which is equivalent to
\[
\int_0^1 \ln q(x_n) (\ln x_n)^n x_n^{-\alpha/m-1} \, dx_n < \infty.
\]
This is true if and only if (7.4) holds (cf. Hansen (1989a)).

(ii). Let \( \ln f_0^*(\tau) := 1 \). As in the proof of Theorem 6.6, we see that the integral in (7.3) converges if and only if (7.6) is true or equivalently (7.7) holds, both with \( \alpha = 0 \). Again, (7.7) with \( \alpha = 0 \) is equivalent to (7.5).

(iii). As in the proof of part (i) we see that
\[
\int_0^\infty \ln q(F_v(\tau)) v^n \, dv = \int_0^\tau \left[ \int_0^\infty n \ln q(F_u(x)) u^{n-1} \, du \right] \left[ U(x) \right]^{-1} \, dx.
\]
This part now follows as in the proof of Theorem 6.3 (ii) in van Harn et al. (1982).

(iv). This part can be proved by applying the same method of proof used to prove Theorem 6.6 (iv) to part (i).

(v). This part of the proof is almost identical with the proof of Theorem 6.6 (v) and therefore omitted.

Let \( X \) have Laplace-Stieltjes transform \( f \in U^a_\alpha \), with \( \alpha > 0 \) and \( f \) having no \( F \)-stable component with exponent \( \alpha \). Since the set of compound Poison distributions is dense, in the topology of weak convergence, in the set of infinitely divisible distributions, we have that
\[
\ln f(\tau) = \lim_{k \to \infty} -\lambda_k \left( 1 - \int_0^\infty g_k(F_v(\tau)) v^n \alpha^{n+1} e^{-\alpha v} (n!)^{-1} \, dv \right),
\]
for some sequence of Laplace-Stieltjes transforms \( (g_k)_{0}^\infty \). Hence the set of random variables of the form \( Y_{\Gamma}(Z(N)) \), with \( N \) Poison, \( \Gamma \) a gamma \((\alpha, n+1)\) distributed random variable and \( Y, \Gamma, Z \) and \( N \) independent, is dense, in the topology of weak
convergence, in the set of \( n \)-times-\( F \), \( \alpha \)-self-decomposable random variables with no \( F \)-stable component with exponent \( \alpha \). In fact, if \( X \) is compound Poison (which is the case if \( X \) is \( \mathbb{N}_0 \)-valued and infinitely divisible), then \( X = Y_\Gamma(Z(N)) \). Let \( (F_t)_{t \geq 0} \) be as in Example 3.4 with \( m = -1 \). Then

\[
Y_\Gamma(Z) = U_1^{\frac{1}{\alpha}} \cdots U_n^{\frac{1}{\alpha}} Z, \tag{7.8}
\]

where \( U_i, i = 1, 2, \ldots, n+1 \) are uniformly distributed on \([0, 1]\), mutually independent and independent of \( Z \). Similarly if \( (F_t)_{t \geq 0} \) is as in Example 3.5 with \( m = -1 \) and \( \emptyset \) is defined (in distribution) by (1.5), then

\[
Y_\Gamma(Z) = U_1^{\frac{1}{\alpha}} \cdots \emptyset U_{n+1}^{\frac{1}{\alpha}} \emptyset Z, \tag{7.9}
\]

with the same conditions on \( U_i \) and \( Z \) as above. When \( n = 0 \), then \( Y_\Gamma(Z) \) in (7.8) is \( \alpha \)-unimodal (with mode at zero) in the sense of Olshen and Savage (1970) and \( Y_\Gamma(Z) \) in (7.9) is discrete \( \alpha \)-unimodal (at zero) in the sense of Abouammoh (1987) and Steutel (1988). We will call a random variable \( n \)-times-\( \alpha \)-unimodal (at zero) if it is of the form (7.8) (if it is absolutely continuous) or (7.9) (if it is \( \mathbb{N}_0 \)-valued). Let \( f \) be compound Poisson. We then have that \( f \in U^\alpha_\alpha \) with \( f \) having no \( F \)-stable component with exponent \( \alpha \) if and only if the Lévy spectral function of \( f \) is equal to a constant times the distribution function of an \( n \)-times-\( \alpha \)-unimodal (at zero) random variable. In Hansen (1989a) and Hansen (1989b) a more general result is obtained characterizing all Laplace-Stieltjes transforms in \( U^\alpha_\alpha \), with \( (F_t)_{t \geq 0} \) as in Example 3.4 with, respectively, \( m = -1 \) and \( m = 1 \), in terms of an \( \alpha \)-unimodality condition of the Lévy spectral function.

The sets \( U^\alpha_\alpha \) are increasing in \( \alpha \) and decreasing in \( n \). It is therefore natural to consider the limits of the sequence of sets \( U^\alpha_\alpha \) as \( \alpha \to \infty \) and \( n \to \infty \). In the following theorem we find the former.

**THEOREM 7.6.** Let \( I \in \{ \mathbb{N}_0, \mathbb{R}_+ \} \), \( n \in \mathbb{N}_0 \) and let \( F = (F_t)_{t \geq 0} \in \mathcal{B}P(\cdot ; I) \). The sequence of sets \( (U^\alpha_\alpha(F))_{\alpha \in \mathbb{R}} \) is an increasing sequence of sets with

\[
\bigcup_{\alpha \in \mathbb{R}} U^\alpha_\alpha(F) = ID(I). \tag{7.10}
\]

**PROOF.** Let \( q \in ID(I) \) and denote by \( \Gamma_\alpha(v) \) the distribution function of a gamma distributed \((\alpha, n+1)\) random variable. For \( \alpha > \max(0, m) \), let

\[
\ln q_\alpha(\tau) = \int_0^\infty \ln q(F_\alpha(v)) \, d\Gamma_\alpha(v),
\]

with the integral converging by Theorem 7.5. From Theorem 7.4, \( q_\alpha \in U^\alpha_\alpha(F) \). Obviously \( \Gamma_\alpha(v) \) tends to a distribution function with a jump of size one at the origin as \( \alpha \to \infty \). By Helly's second theorem \( q_\alpha \to q \) as \( \alpha \to \infty \) and the theorem is proved. □
We next characterize $U^R_{\alpha} := \lim_{n \to \infty} U^n_{\alpha}$. The proof is similar to the proof of Theorem 3.3 in Hong and Thu (1985).

**Theorem 7.7.** Let $\alpha \in \mathbb{R}$. \( f \in U^R_{\alpha} := \bigcap_{n \in \mathbb{N}_0} U^n_{\alpha} \) if and only if there exists a distribution function $G$ such that

\[
\ln f(\tau) = \int_{(\min(0,m), \alpha]} \ln f^*_x(\tau) \, dG(x),
\]

with $f^*_x$ defined in Notation 5.3.

**Proof.** Let $\ln U^R_{\alpha} := \{ \ln f \mid f \in U^R_{\alpha} \}$ and let $K = \{ \ln f \in \ln U^R_{\alpha} \mid f(1) = e^{-1} \}$. It is easily checked that $K$ is compact (in the topology of pointwise convergence), convex and spans $\ln U^R_{\alpha}$. By Choquet's theorem (cf. Choquet (1960)) every point of $K$ is the barycenter of a probability measure concentrated on the set of extreme points of $K$.

Let $f \in U^R_{\alpha}$ and suppose that $f$ is not $F$-stable with exponent $\alpha$. For each $t \in \mathbb{R}_+$, let

\[
c(t) = -e^{-\alpha t} \ln f(F_t(1))
\]

and let

\[
\begin{align*}
\ln g_{1,t}(\tau) &= c(t)^{-1} e^{-\alpha t} \ln f(F_t(\tau)); \\
\ln g_{2,t}(\tau) &= (1 - c(t))^{-1} [ \ln f(\tau) - e^{-\alpha t} \ln f(F_t(\tau))] ,
\end{align*}
\]

for some function $c(t) \in (0, 1)$. Obviously $g_{1,t}, g_{2,t} \in K$. Hence

\[
\ln f(\tau) = c(t) \ln g_{1,t}(\tau) + (1 - c(t)) \ln g_{2,t}(\tau).
\]

If $\ln f_e$ is an extreme point of $K$ and $f_e \neq f^*_x$, then necessarily $\ln f_e = \ln g_{1,t} = \ln g_{2,t}$ for all $t \in \mathbb{R}_+$. By (7.11)

\[
\ln f_e(\tau) = c(t)^{-1} e^{-\alpha t} \ln f_e(F_t(\tau)),
\]

for all $t \in \mathbb{R}_+$. Necessarily $c(t_1 + t_2) = c(t_1)c(t_2)$ and hence $c(t) = e^{-at}$, for some $a > 0$. From Theorem 5.2 it follows that $f_e = f^*_{\alpha-a}$, (cf. Notation 5.3). Applying Choquet's theorem, we see that $f$ is of the form (7.10).

The converse is easily verified.

**Remark 7.8.** The sets $U^n_{\alpha}(F)$, with $m < 0$ are similar to the set of $n$-times-$\alpha$-self-decomposable Laplace-Stieltjes transforms and the sets $U^n_{\alpha}(F)$, with $m > 0$ are similar to the set of reverse $n$-times-$\alpha$-self-decomposable Laplace-Stieltjes transforms as introduced in Hansen (1989a) and Hansen (1989b), respectively.
8. SOME CENTRAL LIMIT PROBLEMS

It is well known (cf. Loève (1977)) that a random variable $X$ is infinitely divisible if and only if there exists a triangular array, $(X_{jk})_{j=1}^k$, $k \in \mathbb{N}_+$, of independent random variables satisfying

$$\lim \sup_{k \to \infty} \mathbb{P}( |X_{jk}| \geq \varepsilon) = 0,$$

for all $\varepsilon > 0$, such that

$$\sum_{j=1}^k X_{jk} \xrightarrow{w} X \text{ as } k \to \infty.$$  

Let $X_{jk}$ be non-negative and have Laplace-Stieltjes transform $\psi_{jk}$. As mentioned in van Harn et al. (1982), Section 23.A of Loève (1977) can be adapted to show that (8.1) is equivalent to

$$\lim \inf_{k \to \infty} \psi_{jk}(t) = 1.$$  

If $(X_{jk})_{j=1}^k$, $k \in \mathbb{N}_+$ satisfies (8.1) or $(f_{jk})_{j=1}^k$, $k \in \mathbb{N}_+$ satisfies (8.3), then the triangular array is said to be uniformly asymptotically negligible (uan for short). If $X_{jk} = a_k X_j$, $a_k > 0$, then $X$ is self-decomposable and if in addition the $X_j$ are identically distributed, then $X$ is stable (cf. Loève (1977)). In this section we give necessary and sufficient conditions in terms of $X_{jk}$ ensuring that $X$ in (8.2) has a Laplace-Stieltjes transform in $U_\alpha$. We also characterize the random variables $X$ in (8.2) when $X_{jk} = \mathcal{Y}_k(X/(1/k))$, i.e., when $X$ is the weak limit of a sequence of partial sums of independent normed random variables, where the norming is done via the ($\alpha$-discounted) branching process $\mathcal{Y}_k(\cdot)$. Writing the integral in Theorem 7.4 as a limit of Riemann sums we have (cf. Notation 5.3)

**Theorem 8.1.** Let $I \in \{\mathbb{N}_0, \mathbb{R}_+\}$, let $F = (F_t)_{t \geq 0} \in \text{BP}(\cdot; I)$, let $n \in \mathbb{N}_0$ and let $f$ be a Laplace-Stieltjes transform with $f$ not identically one. $f \in U_\alpha(F)$ if and only if there exists a $q \in \text{ID}(I)$ such that

$$\ln f(\tau) = \ln f^*_\alpha(\tau) + \lim_{k \to \infty} (1/k)^{\alpha} \sum_{j=1}^k \ln q(F_{\ln k - \ln j}(\tau)) j^{\alpha - 1} (\ln k/j)^n.$$  

By Remark 6.2 and Theorem 8.1 it follows that $f \in U_\alpha(F)$ if and only if

$$\ln f(\tau) = \ln f^*_\alpha(\tau) + \lim_{k \to \infty} (1/k)^{\alpha} \sum_{j=1}^k \ln q^{\text{sign}(\alpha)} (F_{\alpha(\ln k - \ln j)}(\tau)) (\ln k/j)^n.$$  

Here we use the function $\text{sign}$ defined by $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = 0$ if $x = 0$ and $\text{sign}(x) = -1$ if $x < 0$. In Theorem 8.1 we see that the logarithm of any Laplace-
Stieltjes transform in $U_{\alpha}$ can be written as a limit of a weighted sum of logarithms of identical Laplace-Stieltjes transforms. In the next theorem we obtain a less restrictive limiting form of $f$ in $U_{\alpha}$ than in Theorem 8.1.

**Theorem 8.2.** Let $I \in \{\mathbb{N}_0, \mathbb{R}_+\}$, let $F = (F_t)_{t \geq 0} \in \text{BP}(\cdot ; I)$ and let $f$ be a Laplace-Stieltjes transform with $f \neq 1$. \( f \in U_{\alpha}(F) \) for some $\alpha \in \mathbb{R}$ if and only if there exists a sequence $(q_j)$, with $q_j \in \text{ID}(I)$ and a non-decreasing sequence $(t_k)_1^\infty$, with $t_k \to \infty$ and $t_{k+1} - t_k \to 0$ as $k \to \infty$, such that

$$\ln f(\tau) = \lim_{k \to \infty} (1/k) \sum_{j=1}^{k} \ln q_j(F_{t_k-t_j}(\tau)). \quad (8.6)$$

If $\alpha \leq 0$, then it is not necessary to assume that $q_j$ is infinitely divisible.

**Proof.** Let $t = \ln k$ in (5.1). Then $\ln f^*_\alpha = (1/k)^\alpha \ln f^*_\alpha \circ f_{\ln k}$. Letting $t_k = \ln k$, $q_j^*_{\text{ID}(\cdot)} = q_j$, $j \geq 2$ and $q_1 = q \cdot f^*_\alpha$ in (8.5) proves the only if part. Conversely, let $l(k)$ be a sequence with $l(k) \leq k$ and $t_k - t_{l(k)} \to t$ for some $t \in \mathbb{R}_+$. Observe that

$$(1/k)^\alpha \sum_{j=1}^{k} \ln q_j(F_{t_k-t_j}(\tau))$$

$$= (l(k)/k)^\alpha (l(k))^\alpha \sum_{j=1}^{l(k)} \ln q_j(F_{t_{l(k)}-t_j}(F_{t_{l(k)}-t_j}(\tau)))$$

$$+ (1/k)^\alpha \sum_{j=1}^{k} \ln q_j(F_{t_k-t_j}(\tau)). \quad (8.7)$$

Letting $k \to \infty$ in (8.7), it follows that

$$\ln f(\tau) = S(\tau) \ln f_i(\tau) + \ln f_i(\tau), \quad (8.8)$$

for some function $f_i(\tau)$. Since $f(0+) = 1$, $f_i(0+) = 1$. By the continuity theorem for Laplace-Stieltjes transforms, $f_i$ is a Laplace-Stieltjes transform. Let $i(k) \leq k$ be such that $t_k - t_{i(k)} \to s + t$ as $k \to \infty$ for some $s \in (0, \infty)$. Necessarily $t_{i(k)} - t_{i(k)-1} \to s$ as $k \to \infty$ and hence

$$\lim_{k \to \infty} (l(k)-1)^{-\alpha} \sum_{j=i(k)}^{l(k)-1} \ln q_j(F_{t_{i(k)}-t_j}(\tau)) = \ln f(\tau).$$

Observe that

$$(1/k)^\alpha \sum_{j=i(k)}^{k} \ln q_j(F_{t_k-t_j}(\tau))$$

$$= ((l(k)-1)/k)^{\alpha} (l(k)-1)^{-\alpha} \sum_{j=i(k)}^{l(k)-1} \ln q_j(F_{t_{i(k)}-t_j}(F_{t_{i(k)}-t_{i(k)-1}}(\tau)))$$

$$+ (1/k)^\alpha \sum_{j=i(k)}^{k} \ln q_j(F_{t_k-t_j}(\tau)). \quad (8.9)$$
Letting $k \to \infty$ in (8.9) we see that

$$\ln f_{t+s}(\tau) = S(t) \ln f_s(F_t(\tau)) + \ln f_t(\tau).$$

From (8.8) we have

$$S(t+s) \ln f(F_{t+s}(\tau)) + \ln f_{t+s}(\tau) = S(t) \ln f(F_t(\tau)) + \ln f_t(\tau) + S(s) \ln f(F_s(\tau)) + \ln f_s(\tau).$$

It now follows that $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$ and so $S(t) = e^{-at}$ with $\text{sign}(a) = \text{sign}(\alpha)$. Hence $f \in U_\alpha(F)$. \hfill \qed

**COROLLARY 1.** Let $f$ be a Laplace-Stieltjes transform with $f \not\equiv 1$. $f \in U_\alpha((e^{\nu})_{\nu \geq 0})$ for some $\alpha \in \mathbb{R}$ and for some $m \neq 0$ if and only if there exists a sequence $(q_j)$, with $q_j \in \mathbb{ID}(I)$ and a non-decreasing sequence $(t_k)_m$, with $t_k \to \infty$ and $t_{k+1} - t_k \to 0$ as $k \to \infty$, such that

$$\ln f(\tau) = \lim_{k \to \infty} (1/k)^{\text{sign}(\alpha)} \sum_{j=1}^{k} \ln q_j(e^{\nu} \tau).$$

If $\alpha \leq 0$, then it is not necessary to assume that $q_j$ is infinitely divisible.

**REMARK 8.3.** Let

$$\ln f_{j,k}(\tau) := (1/k)^{\text{sign}(\alpha)} \ln q(F_{\alpha(lnk-lnj)}(\tau)) j^{\text{sign}(\alpha)-1} (lnk/j)^{\alpha}.$$ 

Since the limit in (8.4) is a limit of a sequence of Riemann sums, necessarily (8.3) holds and so $(f_{j,k})_j^k$, $k \in \mathbb{N}_+$ is a uan triangular array. Now suppose $f$ is of the form (8.6) with $(f_{j,k})_j^k$, $k \in \mathbb{N}_+$ a uan triangular array defined by

$$\ln f_{j,k}(\tau) := (1/k)^{\text{sign}(\alpha)} \ln q_j(F_{t_{i,j}-j}(\tau)).$$

(8.10)

Observe that

$$(1/k+1)^{\text{sign}(\alpha)} \sum_{j=1}^{k+1} \ln q_j(F_{t_{i,j}-j}(\tau))$$

$$= (1/k+1)^{\text{sign}(\alpha)} \ln q_{k+1}(\tau) + (k/k+1)^{\text{sign}(\alpha)} (1/k)^{\text{sign}(\alpha)} \sum_{j=1}^{k} \ln q_j(F_{t_{i,j}-j}(F_{t_{i,j}-j}(\tau)))$$

By the uan property $(1/k+1)^{\text{sign}(\alpha)} \ln q_{k+1}(\tau) \to 0$ as $k \to \infty$. Hence, on letting $k \to \infty$, it follows that $F_{t_{i,j}-j}(\tau) \to \tau$, which implies that $t_{k+1} - t_k \to 0$ as $k \to \infty$. Let $\alpha \leq 0$ and suppose $t_k \to t < \infty$ as $k \to \infty$. Since $f \not\equiv 1$, there exists a $q_j(0) \not\equiv 1$. Hence

$$0 = \lim_{k \to \infty} \inf_{1 \leq j \leq k} (1/k+1)^{\text{sign}(\alpha)} \ln q_j(F_{t_{i,j}-j}(\tau))$$

$$\leq \lim_{k \to \infty} \ln q_j(0)(F_{t_{i,j}-j}(\tau)) = \ln q_j(0)(F_{t_{i,j}}(\tau)).$$

Since $q_j(0) \not\equiv 1$, $F_{t_{i}}(\tau) \equiv 0$, which has been excluded. So $t_k \to \infty$ as $k \to \infty$. The condition that $t_k \to \infty$ and $t_{k+1} - t_k \to 0$ as $k \to \infty$ in Theorem 8.2 can therefore be
replaced by the condition that \((f_{jk})_{j=1}^{k}, k \in \mathbb{N}_+, \) with \(f_{jk}\) defined by (8.10), is a uan triangular array. If \(\alpha > 0\) and \(t_k \to t < \infty\) as \(k \to \infty\), then for any \(q \in ID(I)\), it can be shown that

\[
(1/k) \sum_{j=1}^{k} \ln q(F_{k-t_j}(\tau)) \to \ln q(\tau) \quad \text{as} \quad k \to \infty.
\]

So if \(\alpha > 0\), then the condition that \((f_{jk})_{j=1}^{k}, k \in \mathbb{N}_+, \) with \(f_{jk}\) defined by (8.10), is a uan triangular array can only replace the condition that \(t_k^+ - t_k \to 0\) as \(k \to \infty\) in Theorem 8.2, but not the condition that \(t_k \to \infty\) as \(k \to \infty\).

We next consider the special case where \(q_j = q_j^* \circ F_{t_j}\) in Theorem 8.2. This approach is similar to that in Section 8 of van Harn et al. (1982). We first give a definition.

**Definition 8.3.** Let \(I \in \{\mathbb{N}_0, \mathbb{R}_+\} \), let \(F = (F_t)_{t \in \mathbb{R}} \in BP(\cdot; I)\) and let \(f\) be a Laplace-Stieltjes transform with \(f \neq 1\). Denote by \(L_\alpha(F)\) the set of Laplace-Stieltjes transforms \(f\) having the form

\[
\ln f(\tau) = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \ln q_j(F_{t_j}(\tau)),
\]

for some sequence of Laplace-Stieltjes transforms \((q_j)\), with \(q_j \in ID(I)\) if \(\alpha > 0\) and some non-decreasing sequence \((t_k)_{k \in I}^\infty\) with \(t_k \to \infty\) and \(t_{k+1} - t_k \to 0\) as \(k \to \infty\). Also let

\[
\pi_{\varnothing}^{F,b}(U_{\alpha}(G)) := \{ \pi_{\varnothing}^{F,b} g \mid g \in U_{\alpha}(G) \}.
\]

If \(r \in (0, 1)\), then by (3.13) the set \(L_\alpha(F)\) is empty. We will therefore hence forth only consider branching processes with \(r = 0\) or \(r = 1\), that is processes where \(U(\tau)\) has no zeros on \((0, \infty)\). Hence either \(m \leq 0\) or \(m > 0\) and \(U(\tau)/\tau \to a \geq 0\) as \(\tau \to \infty\). Before characterizing the sets \(L_\alpha(F)\) we prove a lemma.

**Lemma 8.4.** Let \(\tau_0 \in (0, \infty)\) and \(A_\alpha\) be defined by (3.15). If \(m = 0\) or \(U(\tau)/\tau \to 0\) as \(\tau \to \infty\), then for all \(\varepsilon \in (0, \tau_0)\)

\[
\frac{F_t(\tau)}{F_t(\tau_0)} \to 1 \quad \text{as} \quad t \to \infty \quad \text{uniformly on} \quad (\varepsilon, \tau_0);
\]

**Proof.** By (3.8), \(U(\tau)/\tau\) is decreasing and since \(F_t(\tau) < F_t(\tau_0)\) for all \(\tau \in (0, \tau_0)\) we have

\[
\frac{\partial}{\partial t} \frac{F_t(\tau)}{F_t(\tau_0)} = \frac{1}{F_t(\tau_0)^2} \left[ U(F_t(\tau))F_t(\tau_0) - U(F_t(\tau_0))F_t(\tau) \right] \geq 0.
\]
Hence the limit in the lemma, say $F(t)$, must exist on $(0, \tau_0)$. As in the proof of Theorem 5.5, it can be verified that $\exp-F(t)$ satisfies (5.1) for some $\alpha \in \mathbb{R}$. Let $m=0$. If $\alpha \neq 0$, then $F(0^+)=0$. By (3.14a) and (3.13), $F_t(t) \to 0$ as $t \to \infty$. So there exists a sequence $(t_k)^\infty_{k=1}$ such that $F_{t_k}(\tau_0)=k^{-1}$. Hence by the continuity theorem for Laplace-Stieltjes transforms $\exp-F$ is a Laplace-Stieltjes transform. By Theorem 5.2, we must have $F \equiv 0$ which is impossible ($F(\tau_0)=1$). Hence $\alpha=0$ and so $F \equiv 1$. Let $U(t) \to 0$ as $t \to \infty$. If $(F_t)_{t \geq 0} \in BP(\mathbb{N}_0)$, then by (3.4), $h(t)=\sum_{k=1}^{\infty} h_k e^{-t_k}$ and so $U$ is also of the form (3.6). We thus have that $(F_t)_{t \geq 0} \in BP(\mathbb{R}_+)$, which implies that $\exp-F_t$ is infinitely divisible. Hence $F_t(t)/F_t(\tau_0)$ is a cumulant generating function. As in the case when $m=0$, it can be shown that $F \equiv 1$. 

\begin{theorem}
Let $L\alpha(F)$ and $\pi^{E,b}_a(U\alpha(G))$ be as in Definition 8.3.

(i) If $m < 0$, then $L\alpha(F)=\pi^{E,m}_b(U\alpha((e^{-t})_{t \geq 0}))$ for all $\theta > 0$;

(ii) If $m = 0$, then $L\alpha(F)$ is empty;

(iii) If $m > 0$ and $U(t)/t \to a \geq 0$ as $t \to \infty$, then $L\alpha(F)=\pi^{E,a}_b(U\alpha((e^{t})_{t \geq 0}))$ for all $\theta > 0$;

(iv) If $m > 0$ and $U(t)/t \to a \in [0, \alpha)$ as $t \to \infty$, then $L\alpha(F)$ is empty.

\end{theorem}

\begin{proof}
(i) and (iii). Suppose $f \in L\alpha(F)$. Let $b=m$ if $m < 0$ and $b=a > 0$ if $m > 0$. By Lemma 3.2, $F_t(t)/F_t(\tau_0) \to A_b(t)$ as $t \to \infty$. Observe that

\begin{equation}
(1/k)^{\text{sign}(\alpha)} \sum_{j=1}^{k} \ln q_j(F_{t_k}(\tau)) = (1/k)^{\text{sign}(\alpha)} \sum_{j=1}^{k} \ln q_j(F_{t_k}(\tau) e^{-m \tau} F_{t_k}(\tau_0) e^{m \tau_k}).
\end{equation}

Let $c_k$ be a non-decreasing sequence such that $e^{-m \tau_k} F_{t_k}(\tau_0) \to \theta$ for some $\theta > 0$ (by (3.13) it is always possible to find such a sequence). Necessarily $c_{k+1} - c_k \to 0$ as $k \to \infty$. Since the limit of the left hand side of (8.11) exists (by assumption), so does the limit of the right hand side as $k \to \infty$, and hence the function $g$ defined by

\begin{equation}
\ln g(t) := \lim_{k \to \infty} (1/k)^{\text{sign}(\alpha)} \sum_{j=1}^{k} \ln q_j(e^{m \tau_j} e^{-c_k})
\end{equation}

is a well defined Laplace-Stieltjes transform. Since $f \not\equiv 1$, $g \not\equiv 1$ and so by Corollary 1 to Theorem 8.2, $g \in U\alpha((e^{mt})_{t \geq 0})$ for some $\alpha \in \mathbb{R}$. It is now clear that (cf. (8.11))

\begin{equation}
\ln f(t) = \ln g(\theta A_b(t)) = \ln \pi^{E,b}_a g(t).
\end{equation}

The converse is proved similarly.

\end{proof
(iv). If \( m > 0 \) and \( U(t)/t \rightarrow a \in (0, \alpha) \), then as above \( f \) must also be of the form (8.12), but since \( A_\alpha \) is not a Laplace-Stieltjes transform we must have \( g \equiv 1 \). If \( a = 0 \), then, by Lemma 8.4 we have that \( F_\alpha(t) \sim F_\alpha(t_0) \) uniformly on \((\varepsilon, \tau_0)\). Hence for all \( \tau \in (\varepsilon, \tau_0) \)

\[
\ln f(\tau) = \lim_{k \to \infty} (1/k)^{\text{sign}(\alpha)} \sum_{j=1}^{k} \ln q_j(F_\alpha(\tau))
\]

\[
= \lim_{k \to \infty} (1/k)^{\text{sign}(\alpha)} \sum_{j=1}^{k} \ln q_j(F_\alpha(t_0)) = \ln f(t_0).
\]

Hence \( f \equiv 1 \) and so, in this case, \( L_\alpha(F) \) is empty.

(ii). The proof of this part is almost identical with the proof of (iv) when \( a = 0 \), and therefore omitted.

\[
\square
\]

References


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