Asymptotics of a certain integral

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ABSTRACT
The asymptotic behaviour for $t \to \infty$ of $\int_0^\infty \exp(tx - c(x)) \, dx$ is studied. The function $c$ is positive and $c'(x) \to \infty$ ($x \to \infty$). Sufficient conditions on $c$ are given in order that the method of Laplace is applicable.

1. INTRODUCTION
In his master's thesis ([1] p. 67) A.M. Peeters is confronted with an asymptotic problem of the following kind:
What is the asymptotic behaviour for $t \to \infty$ of
\[
(1) \quad F(t) := F(t; c) := \int_0^\infty \exp(tx - c(x)) \, dx,
\]
where
\[
(2) \quad c \in C([0,\infty) : \mathbb{R}), \quad c \in C^2([a, \infty)) \text{ for some } a \geq 0,
\]
\[
c''(x) > 0 \quad (x \geq a) \quad \text{and} \quad c'(x) \to \infty \quad (x \to \infty).
\]
The asymptotic behaviour of $F$ depends strongly on the behaviour of $c''$. We impose an extra condition on $c''$ in order to apply the method of Laplace ([2], Ch. 4).

(3) There is a positive function $\alpha$ on $[b, \infty)$ for some $b \geq 0$ such that
\[
(a) \quad \alpha(x) (c''(x))^{1/2} \to \infty \quad (x \to \infty),
\]
In section 2 we formulate the main result. In section 3 we give a proof under some simplifying conditions of which we will get rid in section 4. Section 5 is devoted to an example in which (3) is violated and the main result does not hold. In section 6 we describe an unsuccessful attempt to find an example in which (3) is not satisfied but nevertheless the main result holds.

2. RESULT

If $c$ satisfies conditions (2) and (3) then

$$F(t) = \sqrt{2\pi} \left( c''(x_0) \right)^{-1/2} \exp(t x_0 - c(x_0)) \quad (t \to \infty),$$

where $x_0 = x_0(t)$ is defined for $t$ sufficiently large by

$$c'(x_0) = t.$$

3. PROOF

We shall prove (4) under a simplified version of condition (2):

$$c \in C^2([0, \infty); \mathbb{R}), \quad c''(x) > 0 \quad (x > 0), \quad c'(x) \to \infty \quad (x \to \infty).$$

Without loss of generality we may and do assume that

$$c(0) = c'(0) = 0.$$

Indeed, $F$ has the transformation property

$$F(t; c) = e^{-c(0)} F(t - c'(0); \tilde{c}),$$

where

$$\tilde{c}(x) := c(x) - c(0) - x c'(0) \quad (x \geq 0).$$

Clearly $\tilde{c}$ satisfies condition (7). Moreover, formulas (4) and (5) are invariant under transformations of the kind (8), (9).

Now we will prove (4) under conditions (3), (6) and (7). The integrand of (1) reaches its maximum

$$M(t) := \exp(t x_0 - c(x_0))$$

at $x = x_0$ defined by (5). Therefore we substitute
\[(11) \quad x = x_0 + s\]

in (1) and find

\[(12) \quad F(t) = M(t) \int_{-x_0}^{\infty} \exp(h(s,x_0)) \, ds,\]

where

\[(13) \quad h(s) := h(s,x_0) := c'(x_0)s + c(x_0) - c(x_0 + s) \quad (s \geq -x_0).\]

The function \( h \) reaches its maximum value 0 at \( s = 0 \). We will now prove that the contribution of the interval \([-\alpha(x_0), \alpha(x_0)]\) to the integral in (12) is asymptotically equivalent with the whole integral.

We infer from (3) that for \( x_0 \) sufficiently large

\[(14) \quad c''(x_0 + s) \geq \frac{1}{2} c''(x_0) \quad (|s| \leq \alpha(x_0)).\]

Since \( h''(s) < 0 \) \( (s \geq -x_0) \) the graph of \( h \) lies below all tangents to this graph, especially the tangent at \( s = \alpha(x_0) \), i.e.

\[(15) \quad h(s) \leq h(\alpha(x_0)) + (s - \alpha(x_0)) h'(\alpha(x_0)) \quad (s \geq -x_0).\]

From (13), the mean value theorem and (14), we infer that for \( x_0 \) sufficiently large

\[(16) \quad h'(\alpha(x_0)) = c'(x_0) - c'(x_0 + \alpha(x_0)) \leq -\frac{1}{2} \alpha(x_0) c''(x_0).\]

Using (15) and (16) we get

\[(17) \quad \int_{\alpha(x_0)}^{\infty} \exp(h(s)) \, ds \leq |h'(\alpha(x_0))|^{-1} \exp[h(\alpha(x_0))] \leq 2(\alpha(x_0) c''(x_0))^{-1}.\]

Hence, by (3) (a)

\[(18) \quad \int_{\alpha(x_0)}^{\infty} \exp(h(s)) \, ds = o\left((c''(x_0))^{-\frac{1}{2}}\right) \quad (x_0 \to \infty).\]

Similarly

\[(19) \quad \int_{x_0}^{-\alpha(x_0)} \exp(h(s)) \, ds = o\left((c''(x_0))^{-\frac{1}{2}}\right) \quad (x_0 \to \infty).\]

The contribution of the remaining interval \([-\alpha(x_0), \alpha(x_0)]\) will be shown to be

\[\sqrt{2\pi} \quad (c''(x_0))^{-\frac{1}{2}} \quad (1 + o(1)) \quad (x_0 \to \infty).\]

Let \( 0 < \varepsilon < \frac{1}{2} \). Then, by (3)(c), there is a number \( A > 0 \) such that

\[(20) \quad (1 - \varepsilon) c''(x_0) \leq c''(x_0 + s) \leq (1 + \varepsilon) c''(x_0) \quad (|s| \leq \alpha(x_0))\]

holds for \( x_0 \geq A \).
From Taylor's formula and (20) it follows that for $x_0 \geq A$

$$-\frac{1}{2} (1+\varepsilon) c''(x_0) s^2 \leq h(s; x_0) \leq -\frac{1}{2} (1-\varepsilon) c''(x_0) s^2 \quad (|s| \leq \alpha(x)).$$

Using

$$\int_{|s| \geq \alpha(x_0)} \exp \left[ -\frac{1}{2} (1 \pm \varepsilon) c''(x_0) s^2 \right] ds$$

$$\leq 2 \int_{\alpha(x_0)} \exp \left[ -\frac{1}{4} c''(x_0) \alpha(x_0) s \right] ds = 8(c''(x_0) \alpha(x_0))^{-1} \exp[-\frac{1}{4} c''(x_0) \alpha^2(x_0)]$$

$$= o(c''(x_0))^{-1/2} \quad (x_0 \to \infty)$$

and

$$\int_{-\infty}^{\alpha(x_0)} \exp[-\frac{1}{2} (1 \pm \varepsilon) c''(x_0) s^2] ds = \sqrt{2\pi} (1 \pm \varepsilon)^{-1/2} (c''(x_0))^{-1/2},$$

and (21) we get

$$\int_{-\alpha(x_0)}^{\alpha(x_0)} \exp(h(s)) ds \geq \sqrt{2\pi} (1 \pm \varepsilon)^{-1/2} (c''(x_0))^{-1/2} (1 + o(1)) \quad (x_0 \to \infty).$$

Combining (24), (12), (18), (19) we get (4).

CONTINUATION OF THE PROOF

We have proved (4) under the simplifying conditions (6). Now we shall complete the proof of (4) using only the more general restriction (2) instead of (6).

Let $a \geq 0$ be such that $c''(x)$ exists and $c''(x) > 0$ if $x \geq a$. According to condition (2) such a number $a$ exists. Then we split the integral (1) into two integrals as follows:

$$F(t; c) = F_1(t) + F_2(t),$$

where

$$F_1(t) := \int_0^a \exp(tx - c(x)) dx$$

and

$$F_2(t) := \int_a^\infty \exp(tx - c(x)) dx.$$
\begin{align*}
(27) \quad F_2(t) &:= \int_a^\infty \exp(tx-c(x)) \, dx.
\end{align*}

Now $F_2(t)$ can be treated as $F$ in section 3 since

\begin{align*}
(28) \quad F_2(t) &= e^{at} F(t; \hat{c}),
\end{align*}

where $\hat{c}$ is defined by

\begin{align*}
(29) \quad \hat{c}(x) &:= c(x+a) \quad (x \geq 0)
\end{align*}

satisfies (2) and (6). Doing so we get

\begin{align*}
(30) \quad F_2(t) &= -\sqrt{2\pi} \left( c''(x_0) \right)^{-\frac{1}{2}} \exp(tx_0-c(x_0)) \quad (t \to \infty)
\end{align*}

where $x_0$ is defined by $c'(x_0) = t$.

Obviously we only have to prove that

\begin{align*}
(31) \quad F_1(t) &= o(F_2(t)) \quad (t \to \infty).
\end{align*}

Clearly

\begin{align*}
(32) \quad F_1(t) &= O(t^{-1} e^{at}) \quad (t \to \infty).
\end{align*}

We shall even prove that

\begin{align*}
(33) \quad \forall p \geq 0 \quad e^{pt} = o(F_2(t)) \quad (t \to \infty).
\end{align*}

We need some formulas.

\begin{align*}
(34) \quad x \ c'(x) - c(x) \to \infty \quad (x \to \infty).
\end{align*}

\textbf{Proof.} Let $a_1 > a$. Then

\begin{align*}
x \ c'(x) - c(x) &= a_1 \ c'(a_1) - c(a_1) + \int_{a_1}^x \ c''(s) \, ds \\
&\geq a_1 \ c'(a_1) - c(a_1) + \int_{a_1}^x \ c''(s) \, ds = a_1 \ c'(x) - c(a_1) \to \infty \quad (x \to \infty).
\end{align*}

\begin{align*}
(35) \quad c'(x) &= o(x \ c'(x) - c(x)) \quad (x \to \infty).
\end{align*}

\textbf{Proof.} Let $d > a$ be such that $d \ c'(d) - c(d) > 0$. Such a number $d$ exists according to (34). Let $0 < \varepsilon < 2d^{-1}$. Let $x_1 > 2 \varepsilon^{-1}$ be such that $c'(x) > 2 \ c'(2 \varepsilon^{-1}) \quad (x \geq x_1)$. Then
\[ x \, c'(x) - c(x) = d \, c'(d) - c(d) + \int_{d}^{x} s \, c''(s) \, ds > \int_{2e^{-1}}^{x} s \, c''(s) \, ds > 2e^{-1} \int_{2e^{-1}}^{x} c''(s) \, ds = 2e^{-1}(c'(x) - c'(2e^{-1})) > \epsilon^{-1} \, c'(x) \quad (x \geq x_1). \]

(36) \[ (c''(x))^{\frac{1}{2}} = o (c'(x)) \quad (x \to \infty). \]

**Proof.** Without loss of generality we may and do assume that \( \alpha(x) \leq \frac{1}{2}x \quad (x \geq b) \). Indeed, if \( \alpha \) meets the requirements of condition (3) then also \( \frac{1}{2}\alpha \). Then by (3)(c), there is a number \( x_2 > 0 \) such that \( c'(\frac{1}{2}x) > 0 \) and \( c''(s) > \frac{1}{2} \, c''(x) \quad (x - \alpha(x) \leq s \leq x) \) for \( x \geq x_2 \). Then

\[ c'(x) > c'(x - \alpha(x)) + \frac{1}{2} \alpha(x) \, c''(x) > \frac{1}{2} \alpha(x) \, c''(x) \quad (x \geq x_2). \]

Using now (3)(a) we infer (36).

(37) \[ \forall \beta > 0 \, \exp(\beta \, c'(x)) = o ((c''(x))^{1/4} \exp(x \, c'(x) - c(x))) \quad (x \to \infty). \]

**Proof.** For \( x \) sufficiently large, say \( x \geq x_3 \), the following inequalities hold: \( 0 < \log c'(x) \leq c'(x) \) and \( (c''(x))^{1/4} < c'(x) \). Then

\[ (c''(x))^{1/4} \exp[x \, c'(x) - c(x)] > (c'(x))^{-1} \exp[x \, c'(x) - c(x)] = \exp[x \, c'(x) - c(x) - \log c'(x)] \geq \exp[(x - 1) \, c'(x) - c(x)] \quad (x \geq x_3). \]

Now (37) follows from (35).

The proof of (33) is now easy. Writing \( t = c'(x_0) \) in (30), (32) and (33) and \( \beta = a \), \( x = x_0 \) in (37) we get

(38) \[ F_1(t) = o (t^{-1} \, F_2(t)) \quad (t \to \infty). \]

Hence the proof of (4) is completed.

(39) **Remark.** The conditions in (2) may still be slightly relaxed. For instance, \( c \) does not have to be continuous on \([0, a]\); the only requirement is that \( \int_{0}^{a} \exp[-c(x)] \, dx \) exists. Instead of (31) it is possible that \( F_1(t) = O(e^{at}) \quad (t \to \infty) \). Instead of (38) only (31) holds.
5. EXTRA CONDITION IS NOT SUPERFLUOUS

In order to appreciate the extra condition (3) we present an example in which the function \( c \) satisfies condition (2) but not (3) and such that the asymptotic behaviour of \( F \) differs from (4).

Let \( c \) be given by

\[
c(x) := x^2 + x - \sin x \quad (x \geq 0).
\]

This function \( c \) satisfies (2) but not (3). Indeed, since

\[
c''(x) = 2 + \sin x \quad (x \geq 0)
\]

is bounded, (3)(a) requires that \( \alpha(x) \to \infty \) \( (x \to \infty) \), but then (3)(c) is violated. According to (12), omitting the factor \( M(t) \), we have to study

\[
I(t) := \int_{-x_0}^{x_0} \exp(h(s)) \, ds,
\]

where

\[
h(s) := -s^2 - s \cos x_0 + \sin(x_0 + s) - \sin x_0,
\]

of which the domain of definition is taken to be \( \mathbb{R} \). From \( h(-x_0) \leq -x_0^2 + x_0 + 1 \) and \( h'(s) \geq -2s - 2 \) \( (s \in \mathbb{R}) \) it follows that

\[
h(s) \leq -x_0 + 1 - s^2 - 2s \quad (s \leq -x_0).
\]

Hence

\[
\int_{-x_0}^{x_0} \exp(h(s)) \, ds \leq e^{-x_0+1} \int_{-\infty}^{\infty} e^{-s^2-2s} \, ds = \sqrt{\pi} \, e^{-x_0+2}.
\]

It follows that

\[
I(t) = J(x_0) + O(e^{-x_0}) \quad (x_0 \to \infty),
\]

where

\[
J(x_0) := \int_{-x_0}^{x_0} \exp(h(s)) \, ds \quad (x_0 > 0).
\]

From the definition (42) of \( h \) it is clear that \( J \) is a periodic function with period \( 2 \pi \).

Now we suppose that (4) holds. Then

\[
I(t) \sim \sqrt{2\pi} \, (2 + \sin x_0)^{-\frac{1}{2}} \quad (t \to \infty).
\]

We take \( x_0 = 2\pi n + \phi \) and we let \( n \to \infty \). Then it follows from (43), (44) and (45) that
\( \int_{-\infty}^{\infty} \exp[-s^2 - s \cos \phi + \sin(\phi + s) - \sin \phi] \, ds = \sqrt{2} \pi \left( 2 + \sin \phi \right)^{1/2} \quad (\phi \in \mathbb{R}). \)

Taking \( \phi = 0 \) in (46) we get

\( \int_{-\infty}^{\infty} \exp[-s^2 - s + \sin s] \, ds = \sqrt{\pi} , \)

but, by taking the even part of the integrand, the left hand side of (47) can be written as

\( \int_{-\infty}^{\infty} e^{-s^2} \cosh(s - \sin s) \, ds , \)

which is obviously larger than \( \sqrt{\pi} \). Hence we have derived a contradiction. We conclude that (4) does not hold.

6. THE CONDITION (3) IS NOT NECESSARY?

Inspired by the foregoing section we try to find a function \( c \) of the form

\( c(x) = x^2 + x - p(x) , \)

where \( p \) is a nonconstant periodic function sufficiently smooth and satisfying

\( p''(x) < 2 \quad (x \in \mathbb{R}). \)

Clearly \( c \) satisfies (2) and not (3). We hope to find such a function \( p \) that (4) holds. In a similar fashion as in section 5 we arrive at the following equation for \( p \)

\( \int_{-\infty}^{\infty} \exp[-s^2 - p'(x)s - p(x) + p(x+s)] \, ds = \sqrt{2} \pi \left( 2 - p''(x) \right)^{1/2} \quad (x \in \mathbb{R}). \)

By the substitution \( s + \frac{1}{2}p'(x) = \sigma \) equation (51) transforms into

\( \int_{-\infty}^{\infty} \exp[-\sigma^2 + p(y(x) + \sigma)] \, d\sigma = \sqrt{\pi} \left( 1 - \frac{1}{2} p''(x) \right)^{1/2} \exp[p(x)-\frac{1}{4} (p'(x))^2] \quad (x \in \mathbb{R}) , \)

where

\( y(x) := x - \frac{1}{2} p'(x) . \)

The author has not succeeded in proving the existence of a nonconstant periodic function \( p \) satisfying (51) or (53).
REFERENCES
