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RECIPROCAL BASES FOR THE INTEGERS

P. J. VAN ALBADA AND J. H. VAN LINT, Technical University,
Eindhoven, Netherlands

1. Introduction.

(1.1) DEFINITION. A sequence $S: \{n_1, n_2, \dots\}$ of distinct integers is called an R -basis if every positive integer is the sum of reciprocals of finitely many integers of S .

It is well known that the set of all positive integers is an R -basis. In [1] H. S. Wilf suggested the following problems:

- (1.2) Are the odd numbers an R -basis?
- (1.3) Is every arithmetic progression an R -basis?
- (1.4) Does an R -basis necessarily have positive density?
- (1.5) If S contains all positive integers and $f(n)$ is the least number required to represent n what is the asymptotic behaviour of $f(n)$?

Problem (1.2) was proposed by E. P. Starke in this MONTHLY 59 (1952). A complete solution was given by R. Breusch [2], and by B. M. Stewart [3]. In solutions given by E. P. Churchill and B. M. Stewart the proof depended on the theorem: *For every odd A there are infinitely many odd multiples M of A each of which has the property: every integer n such that $2 < n < \sigma(M) - 2$, is a sum of distinct positive divisors of M .* In this paper we give an affirmative answer to (1.3) by generalizing the theorem cited above.

It is easily seen that the answer to (1.4) is negative. Many examples of bases of zero density can be found. Mr. Wilf informed us that this question was also answered by S. Scheinberg and by S. Stein. For the function f of (1.5) we prove

$$\log f(n) = n - \gamma + O\left(\frac{1}{n}\right).$$

Here γ is Euler's constant. Mr. Wilf informed us that $f(n) \sim e^{n-\gamma}$ was also found by P. Erdős.

In the following "integer" always means "positive integer."

2. R -bases of zero density. Obviously, for every integer a , the sequence $S_a = \{a, 2a, 3a, \dots\}$ is an R -basis. Using this we now construct an R -basis of zero-density. Let $S(1)$ be the set $\{1\}$ and suppose $S(n)$ is a set of integers such that every integer $m \leq n$ can be expressed as the sum of reciprocals of distinct elements of $S(n)$. Let α_n be the largest element of $S(n)$. We now take an integer $a(n)$ satisfying:

- (1) $a(n) > \alpha_n \geq n$,
- (2) the set $S(n)$ has less than $a(n)/n$ elements.

There is a representation of $n+1$ by reciprocals of elements of $S_{a(n)}$. We add these elements to $S(n)$ to form the set $S(n+1)$. By induction we have thus

defined an R -basis $S = \bigcup_{n=1}^{\infty} S(n)$ and clearly the condition (2) implies that S has zero density.

3. Every arithmetic progression is an R -basis. For every integer a , the arithmetic progression S_a is an R -basis. This suggests that every arithmetic progression is an R -basis. To prove this we first prove two lemmas.

Consider the arithmetic progression $a_0, a_1, a_2, \dots$, where

$$a_i = 1 + iv \quad (i = 0, 1, 2, \dots; v \geq 2).$$

We define $P_n = a_1 a_2 \cdots a_n$. We shall call an integer d^* a *stardivisor* of m if d^* is a divisor of m and d^* is a product of different integers a_i . Let N be an integer satisfying the conditions

$$(3.1) \quad \begin{aligned} & (a) \quad N > v^2 \\ & (b) \quad 1 + Nv > (1 + 2v)(1 + 3v) \\ & (c) \quad \sum_{i=1}^N a_i^{-1} > 1 \\ & (d) \quad (1 + v)(1 + 2v) \cdots \{1 + (N - 1)v\} > 1 + Nv. \end{aligned}$$

Condition (d) is trivial. A consequence of conditions (c) and (d) is that the sum of the stardivisors of P_N , which are greater than a_N , is greater than P_N . We take $N' = v^2$. We prove first the

LEMMA (3.2). *Every integer m with $\frac{1}{2}v^3 \leq m \leq v^4$ can be written as the sum of $2v$ or less distinct terms a_i ($i \leq N'$).*

Obviously the sum of k or $v+k$ terms a_i is congruent to $k \pmod{v}$. So an integer $m \equiv k \pmod{v}$ can be written as the sum of k terms a_i ($i \leq N'$) if and only if

$$\sum_{i=0}^{k-1} (1 + iv) \leq m \leq \sum_{i=N'-k+1}^{N'} (1 + iv),$$

and m can be written as the sum of $v+k$ terms a_i ($i \leq N'$) if and only if

$$\sum_{i=0}^{v+k-1} (1 + iv) \leq m \leq \sum_{i=N'-v-k+1}^{N'} (1 + iv).$$

Since

$$\begin{aligned} \sum_{i=0}^{v+k-1} (1 + iv) &= \frac{1}{2}(v+k)(v^2 + vk - v + 2) \leq \frac{1}{2}k(2v^3 - kv + v + 2) \\ &= \sum_{i=N'-k+1}^{N'} (1 + iv), \end{aligned}$$

the two intervals have a nonempty intersection. Now, taking $k = 1, 2, \dots, v$

we find a set of integers which can be represented as the sum of $2v$ or less distinct terms a_i ($i \leq N'$). This set contains the interval $\frac{1}{2}v^3 \leq m \leq v^4$ which proves the lemma.

LEMMA (3.3). *If $v^4 < \sigma \leq P_N$ then σ can be written as the sum of distinct stardivisors of P_N .*

Proof. We wish to show that by successively subtracting stardivisors from σ we can find a number which is the interval of Lemma (3.2) and then (3.3) would follow. In this process no stardivisor may be used twice: Let $d_1^* < d_2^* < \dots < d_k^*$ be the stardivisors of P_N which are greater than a_N . If $\alpha = \sum_{i=1}^k \epsilon_i 2^{i-1}$, ($\epsilon_i = 0$ or 1), we define $t(\alpha) = \sum_{i=1}^k \epsilon_i d_i^*$. So now we must show that there is a number α for which

$$(1) \quad \sigma - v^4 \leq t(\alpha) \leq \sigma - \frac{1}{2}v^3.$$

Now $t(1) = d_1^* = v^3 + v + 1 \leq \sigma - \frac{1}{2}v^3$ and $t(2^k - 1) = d_1^* + d_2^* + \dots + d_k^* > P_N > \sigma - \frac{1}{2}v^3$ (by 3.1). Let α be the largest integer for which $t(\alpha) \leq \sigma - \frac{1}{2}v^3$. If $t(\alpha)$ does not satisfy (1) this would mean $t(\alpha) < \sigma - v^4$ and $t(\alpha + 1) > \sigma - \frac{1}{2}v^3$. Hence there is a stardivisor d_j^* for which

$$(2) \quad d_j^* - \sum_{i=1}^{j-1} d_i^* > v^4 - \frac{1}{2}v^3 + 1.$$

We now show that (2) is not true and therefore there is a $t(\alpha)$ satisfying (1) which completes the proof of (3.3). As d_1^* and d_2^* are both less than $v^4 - \frac{1}{2}v^3 + 1$ the stardivisor d_j^* cannot be a stardivisor less than a_N because every a_i is less than the sum of a_{i-1} and a_{i-2} . This means that (2) implies $j > 2$ and that $d_j^* > a_N$. So d_j^* is the product of at least 2 different numbers a_i ($i > 0$). Write $d_j^* = a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_l}$ with $\lambda_1 < \lambda_2 < \dots < \lambda_l$ ($l \geq 2$). We use:

- (a) $a_{\lambda_1-1} + a_1 > a_{\lambda_1}$ if $\lambda_1 > 2$,
- (b) $a_{\lambda_1} a_{\lambda_2-1} + a_1 a_2 > a_{\lambda_1} a_{\lambda_2}$ if $\lambda_1 \leq 2$, $\lambda_2 > 4$,
- (c) if $\lambda_1 \leq 2$, $\lambda_2 \leq 3$ then $a_{\lambda_1} a_{\lambda_2} \leq (1+2v)(1+3v) < a_N$ by (3.1.b) and therefore $a_{\lambda_1} a_{\lambda_2} < (a_{\lambda_1} a_{\lambda_2} - v) + a_1 = a_v + a_1$. This shows that we can find two stardivisors, both less than d_j^* , whose sum is greater than d_j^* and this contradicts (2). We now prove the main

THEOREM (3.4). *Every arithmetic progression of positive integers is an R-basis.*

Proof. If $an = \sum_i (1 + \lambda_i v)^{-1}$ then $n = \sum_i (a + a\lambda_i v)^{-1}$ and therefore it is sufficient to prove the theorem for arithmetic progressions with first term $a_0 = 1$. If $a_0, a_1, \dots$, where $a_i = a + iv$ is the arithmetic progression then $\sum_{i=0}^{\infty} a_i^{-1}$ is divergent. For every integer m we can find N_0 such that

$$(3) \quad \sum_{i=0}^{N_0} a_i^{-1} \leq m < \sum_{i=0}^{N_0+1} a_i^{-1}$$

If equality occurs the proof is complete. If not, then $m = \sum_{i=0}^{N_0} a^{-1} + \nu/P_{N_0}$. We write

$$\frac{\nu}{P_{N_0}} = \frac{\sigma}{P_N},$$

where $\nu^4 < \sigma < P_N$ and N satisfies the conditions (3.1). By Lemma (3.3) the numerator σ is the sum of distinct stardivisors of P_N and hence σ/P_N is the sum of reciprocals of distinct terms of the arithmetic progression. Condition (3) guarantees that these terms have index greater than N_0 . In this way m has been represented as the sum of reciprocals of distinct terms of the arithmetic progression.

4. Asymptotic behaviour of $f(n)$. We consider the R -basis S consisting of all positive integers. Let $f(n)$ be the least number of elements of S required to represent n as a sum of reciprocals. We prove the

THEOREM (4.1). $\log f(n) = n - \gamma + O(1/n)$.

Proof. Let k be the integer satisfying

$$(4) \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} < n < 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1}.$$

Then

$$n = 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{a}{b}$$

in which $1 \leq a < b$ and b is the least common multiple of $1, 2, \dots, k$. In [4] P. Erdős proved that a/b can be represented as the sum of $N(a, b)$ reciprocals of distinct integers in which

$$(5) \quad N(a, b) \leq \frac{c \log b}{\log \log b}.$$

As

$$\log b = \sum_{p \leq k} \sum \log p = \psi(k) = k + O\left(\frac{k}{\log k}\right)$$

we find

$$(6) \quad N(a, b) = O\left(\frac{k}{\log k}\right).$$

We use the well-known theorem

$$(7) \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \log k + \gamma + O\left(\frac{1}{k}\right).$$

By (4) we have $f(n) > k \geq n$. By (4) and (6) on the other hand: $f(n) < k + O(k/\log k)$. (4) and (7) lead to $\log k = n - \gamma + O(1/k)$. Combining these we have

$$\log f(n) = n - \gamma + O\left(\frac{1}{n}\right) + O\left(\frac{1}{\log k}\right) = n - \gamma + O\left(\frac{1}{n}\right).$$

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