Solution to Problem 80-11: Extreme values of an integral

Citation for published version (APA):

Lossers, O. P. (1981). Solution to Problem 80-11: Extreme values of an integral. *SIAM Review*, 23(3), 392-393.
<https://doi.org/10.1137/1023075>

DOI:

[10.1137/1023075](https://doi.org/10.1137/1023075)

Document status and date:

Published: 01/01/1981

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
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Show that

$$4K^2 \left\{ \sqrt{\frac{1 - \sqrt{1 - 2z}}{2}} \right\} = \pi^2 \sum_{k=0}^{\infty} \frac{[(2k-1)!!]^3 z^k}{[k!]^2 [(4k)!!]},$$

where $(-1)!! = 1$, $(4k)!! = 3 \cdot 8 \cdots (4k)$, etc.

Identity for Hypergeometric Functions

Problem 81-15, by O. G. RUEHR (Michigan Technological University).

Show that

$$\begin{aligned} (a+b+1)_3 F_2 \left[-c, -a, 1; b+2, \frac{1-a-c}{2}; \frac{1}{2} \right] \\ = (b+1)_3 F_2 \left[-\frac{a}{2}, \frac{1-a}{2}, 1; \frac{1-a-b}{2}, \frac{1-a-c}{2}; 1 \right], \end{aligned}$$

where $(-1)!! = 1$, $(4k)!! = 3 \cdot 8 \cdots (4k)$, etc.

The identity arose in solving Problem 71-13.

SOLUTIONS

Extreme Value of an Integral

Problem 80-11, by W. W. MEYER (General Motors Research Laboratories).

Find the infimum of

$$I(f) = \int_0^a \sqrt{(f(t)-t)^2 + (f'(t))^2} dt$$

over all real-valued and differentiable functions f on a given interval $(0, a)$.

Solution by O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, the Netherlands).

We may assume that $f(t)-t$ has a zero in the interval $[0, a]$. Define

$$x := (f(t)-t) \cos t + \sin t,$$

$$y := (f(t)-t) \sin t - \cos t.$$

We then have to find the infimum of

$$\int_0^a \sqrt{\dot{x}^2 + \dot{y}^2} dt.$$

This is the length of a curve and suggests a straight line for the infimum. However, the curve is subjected to the conditions that, first, $x \sin t - y \cos t = 1$, i.e., the point $(x(t), y(t))$ lies on the line that touches the circle $x^2 + y^2 = 1$ in the point $(\sin t, -\cos t)$; and second, $x^2 + y^2 = (f(t)-t)^2 + 1$, i.e., the curve lies outside the unit circle. In view of the opening remark, the curve touches the unit circle at least once.

Now if $a < \pi$, the shortest curve joining the tangents corresponding to $t=0$ and $t=a$ via a point of the circle consists of two straight pieces of equal length meeting on the circle. This gives the minimum length $2(1 - \cos \frac{1}{2}a)$. If $a > \pi$, then the first condition forces the minimal curve to wind around the circle up to the point from where the final tangent can be reached by a straight line. In this case it is easy to show that the length of the shortest curve is $a - \pi + 2$.

Also solved by H. G. MOYER (Grumman Aerospace, NY) and the proposer, who also finds the infimum of

$$\int_0^a \sqrt{|f(t) - g(t)|^2 + |f'(t)|^2} dt,$$

where g is a given continuous nonnegative, monotone increasing function on $[0, a)$.

Partially solved by J. E. WILKINS (EG & G Idaho, Inc.).

An Infinite Triple Summation

Problem 80-13, by M. L. GLASSER (Clarkson College).

Show that

$$S = \sum_{i,j,k} (\operatorname{sgn} i)(\operatorname{sgn} j)(\operatorname{sgn} k)(\operatorname{sgn} (i+j-k))/i^2 j^2 = \pi^2 \ln 2,$$

where the sums are over all positive and negative odd integers, and each is understood to be a sum from $-N$ to N with the limit $N \rightarrow \infty$ taken at the end.

This result is needed in calculating the free energy of superfluid helium [1], [2].

REFERENCES

- [1] D. RAINER AND J. W. SERENE, Phys. Rev., B13 (1976), pp. 4745-4748.
- [2] J. C. RAINWATER, Phys. Rev., B18 (1978), pp. 3728-3729.

Solution by the proposer.

In [2] it is shown that

$$(1) \quad S = -8 \int_0^1 \frac{\tanh^{-1} x \ln x}{x(1-x^2)} dx - \frac{7}{2} \zeta(3).$$

By noting that $2 \tanh^{-1} x = \ln(1+x) - \ln(1-x)$ and using partial fractions, the integral in (1) can be broken up into the six integrals over the interval $(0, 1)$:

$$\begin{aligned} I_1 &= \int [\ln x \ln(1+x)/(1-x)] dx, & I_4 &= \int [\ln x \ln(1-x)/(1-x)] dx, \\ I_2 &= \int [\ln x \ln(1+x)/x] dx, & I_5 &= \int [\ln x \ln(1-x)/x] dx, \\ I_3 &= \int [\ln x \ln(1+x)/(1+x)] dx, & I_6 &= \int [\ln x \ln(1-x)/(1+x)] dx. \end{aligned}$$