Memorandum COSOR 88-12

Exact paired comparison designs
for quadratic models

by

E.E.M. van Berkum

Eindhoven, the Netherlands
May 1988
EXACT PAIRED COMPARISON DESIGNS FOR QUADRATIC MODELS

E.E.M. van Berkmr
Department of Mathematics and Computing Science
University of Technology
Eindhoven, The Netherlands

ABSTRACT

The construction of designs in paired comparison experiments is considered. The procedures that are used in paired comparisons to estimate the parameters yield a covariance matrix that depends on the unknown parameters.

The assumption of no treatment differences is made to construct designs. By use of $D$-optimal discrete designs, exact designs are constructed with a high efficiency and a relatively small number of pairs. This is done for a quadratic model for 2, 3, 4 and 5 dimensions and hypercube as experimental region.

AMS Subject Classification: Primary 62J15; Secondary 62K05.

Key words: Paired comparison; Bradley-Terry model; Optimal designs; Factorials.
1. Introduction

The paired comparison experiment has \( t \) treatments or items, \( T_1, \cdots, T_t \), with \( n_{ij} \) judgments or comparisons of \( T_i \) and \( T_j \), \( n_{ij} \geq 0 \), \( n_{ii} = 0 \), \( n_{ji} = n_{ij} \), \( i, j = 1, \cdots, t \).

Let \( n_{ij} \) be the number of times \( T_i \) has been preferred to \( T_j \) when \( T_i \) and \( T_j \) are compared. Bradley and Terry (1952) provided a paired comparison model, which postulates the existence of parameters, \( \pi_i \) for \( T_i, \pi_i > 0 \), such that the probability \( \pi_{ij} \) of selecting \( T_i \) when compared with \( T_j \) is

\[
\pi_{ij} = \frac{\pi_i}{(\pi_i + \pi_j)} \quad (i \neq j) .
\]  

(1.1)

The parameters \( \pi_i \) have to be estimated. Bradley (1976) gives a survey of estimation methods and extensions of the basic model. However, few results have been obtained in constructing optimal designs. Some of the results are listed in Berkum (1987). In that paper a method is given to construct discrete \( D \)-optimal designs for response surfaces.

Results are given in the case of a quadratic response surface. The designs given are not very useful for practical applications for two reasons. The number of pairs is large, and the pairs of a design have different weights. In this paper we give exact designs with a relatively small number of pairs.

2. \( D \)-optimal designs in the case of a quadratic response surface

We give some results of Berkum (1987).

The model considered is

\[
\ln \pi_x = \beta_1 x_1 + \cdots + \beta_n x_n + \beta_{11} x_1^2 + \cdots + \beta_{nn} x_n^2 + \beta_{12} x_1 x_2 + \cdots + \beta_{n-1,n} x_{n-1} x_n ,
\]  

(2.1)

where

\[ x = (x_1, \cdots, x_n)' , \quad x \in X , \]

and

\[ X = \{ x \in \mathbb{R}^n \mid -1 \leq x_i \leq 1 \text{ for all } i \} , \quad \text{the experimental region} . \]

The number of parameters is \( 2n + \frac{1}{2} n(n-1) \), so a discrete \( D \)-optimal design can be found with \( m \) pairs where

\[
m \leq \frac{1}{8} n(n+1)(n+2)(n+3) .
\]

(2.2)

The covariance matrix of the estimators of the unknown parameters \( \beta \) depends on the unknown \( \pi_i \). To construct designs the assumption of no treatment differences is made

\[
\pi_i = 1 , \quad i = 1, \cdots, t .
\]

(2.3)

It can be shown that in this case the covariance matrix \( \mathcal{M}^{-1} \) has the following structure:
It is useful to define the following sets.

**Definition 2.1** Let \((v, w)\) be a pair with \(v, w \in \mathbb{R}^n\).

The set \(S((v, w))\) contains the pair \((v, w)\) and all \(2^n - 1\) other pairs that can be found by multiplying one or more pairs of coordinates \((v_i, w_i)\) of \((v, w)\) by \(-1\). The set \(SP((v, w))\) is the union of all sets \(S((p(v), p(w)))\), where \(p(v)\) is a permutation of \(v\). The information matrix of \(SP((v, w))\) is denoted by \(MP((v, w))\).

There is a relation between the design matrix of the set \(S((v, w))\) and the design matrix \(X_1(n)\) of a \(2^n\)-factorial.

Define

\[
X_1(n) = (X_{11}(n) \mid J \mid X_{13}(n))
\]

where

\[
X_{11}(n) = \begin{bmatrix} X_{11}(n - 1) - u \\ X_{11}(n - 1) \\ u \end{bmatrix}, \quad X_{11}(1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

\[
u = (1, \cdots, 1)^T,
\]

\(J\) is a matrix with \(J_{ij} = 1\) for all \(i\) and \(j\),

\[
X_{13}(n) = \begin{bmatrix} X_{13}(n - 1) - X_{11}(n - 1) \\ X_{13}(n - 1) \\ X_{11}(n - 1) \end{bmatrix}, \quad X_{13}(1) = \emptyset
\]

\(X_{11}(n)\) is the notation of the main effects,

\(J\) is related to the quadratic effects,

\(X_{13}(n)\) is related to first order interactions.

It is known that
Now the design matrix $D$ of $S((v, w))$ can be expressed by

$$D = X_1(n) (V - W),$$

where

$$V = \text{diag} \left( v_1, \ldots, v_n \mid v_1^2, \ldots, v_n^2 \mid v_1v_2, \ldots, v_{n-1}v_n \right),$$

$$W = \text{diag} \left( w_1, \ldots, w_n \mid w_1^2, \ldots, w_n^2 \mid w_1w_2, \ldots, w_{n-1}w_n \right).$$

So the information matrix $M((v, w))$ of $S((v, w))$ is

$$M((v, w)) = 2^n (V - W) \left( V - W \right)' .$$

**Definition 2.2** The set $SP(k_1, k_2, k_3; v)$ is the set $SP((x, y))$ where the pair $(x, y)$ is defined as follows

\[
x = (1, \cdots, 1)' \\
y = (1, \cdots, 1, -1, \cdots, -1, v, \cdots, v)' ,
\]

and $k_1$ is the number of 1's in $y$, $k_2$ is the number of -1's in $y$, $k_3$ is the number of $v$'s in $y$, $-1 < v < 1$, $k_1 + k_2 + k_3 = n$.

The set $SP_l(0, 0, n; v)$ is the set containing the pairs of $SP(0, 0, n; v)$ and all pairs that can be found by replacing $l$ pairs of coordinates $(x_i, y_i)$ by $(y_i, x_i)$, i.e. exchanging $l$ coordinates of $x$ and $y$. The information matrices of these sets are denoted by replacing the letter $S$ by the letter $M$. If $k_3 = 0$, then we write $SP(k_1, k_2)$.

The optimal designs are listed as follows ($N_i$ is the number of pairs):

(i) $n$ odd, $n \geq 3$.

The design consists of

(a) the pairs of $SP(\frac{1}{2}(n - 1), \frac{1}{2}(n + 1))$ with weights $v_1$ and $N_1 = \left[ \frac{n}{(n-1)/2} \right] \cdot 2^{n-1},$
(b) the pairs of \(SP(0, 0, n; w_1)\) with weights \(\mu, -1 < w_1 < 1\) and \(N_2 = 2^n\),
(c) the pairs of \(SP_{(n-1)x2}(0, 0, n; w_1)\) with weights \(\lambda\) and \(N_3 = \left(\frac{n}{(n-1)/2}\right) 2^n\).

(ii) \(n = 2, 4\).

The design consists of
(a) the pairs of \(SP\left(\frac{1}{2}, \frac{1}{2}, n\right)\) with weights \(v_2\) and \(N_1 = \left(\frac{n}{n/2}\right) 2^{n-1}\),
(b) the pairs of \(SP(0, 0, n; w_1)\) with weights \(\mu\),
(c) the pairs of \(SP_{n/2}(0, 0, n; w_1)\) with weights \(\lambda\) and \(N_2 = \left(\frac{n}{n/2}\right) 2^{n-1}\),
(d) the pairs of \(SP\left(\frac{1}{2}, -1, \frac{1}{2}, n, 1; w_2\right)\) with weights \(\rho\) and \(N_3 = 8\) if \(n = 2\), and \(N_3 = 192\) if \(n = 4\).

The case \(n\) even, \(n \geq 6\) is not listed here.

The values of \(w_1, w_2\) and the weights of the designs above depend on \(n\) and are listed in Berkum (1987). The information matrices are also given there. The number of pairs \(N\) of the designs given above is - in most cases - large compared to the number \(m\) given in (2.2) as can be seen in table 1.

Table 1

<table>
<thead>
<tr>
<th>(n)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>20</td>
<td>44</td>
<td>304</td>
<td>512</td>
</tr>
<tr>
<td>(m)</td>
<td>15</td>
<td>45</td>
<td>105</td>
<td>210</td>
</tr>
</tbody>
</table>

In the next section we will construct \(D\)-optimal discrete designs with a reduced number of pairs.

3. Reduction of the number of pairs of discrete \(D\)-optimal designs

When \(n = 3\), the number of pairs of the design is smaller than \(m\). Therefore we will exclude this case. In Berkum (1985) a discrete \(D\)-optimal design with 15 pairs is given for the case \(n = 2\). However, that design is not very useful for the construction of exact designs: the weights have 9 different values. First we consider the reduction of the number of pairs of the set \(S((v, w))\) in general and of \(SP(k_1, k_2)\) in particular.
3.1. Half-replicates and quarter-replicated of \( S((v, w)) \)

As we have seen in (2.10), there is a relation between the design matrix of the set \( S((v, w)) \) and the design matrix of a \( 2^n \)-factorial, where all interactions between three or more factors are assumed negligible. The method to construct fractional factorial experiments can be used to reduce the number of pairs of \( S((v, w)) \) as follows: Let a half-replicate of a \( 2^n \)-factorial experiment exist, for which all main effects and all two-factor interactions are clear of one another. Now, by using the expressions (2.10) and (2.12) it is easy to see that a design can be constructed consisting of \( 2^{n-1} \) pairs of \( S((v, w)) \) and having an information matrix equal to \( \frac{1}{2} M((v, w)) \). The design matrix \( \bar{D} \) of this set of \( 2^{n-1} \) pairs is

\[
\bar{D} = \bar{X}_1(n)(V - W),
\]

where \( \bar{X}_1(n) \) consists of the rows of \( X_1(n) \) which are related to the pairs chosen in the half-replicate of the \( 2^n \)-factorial. The method to construct fractional factorial experiments is well-known and can be found in Davies (1963). A half-replicate of a \( 2^n \)-factorial experiment of which all main effects and \( \frac{n}{2} \) two-factor interactions are clear of one another can be found for \( n \geq 5 \).

We consider the case \( n = 5 \). If we choose as defining contrasts \( I \) and \( -ABCDE \), the principal block consists of \( (1, ab, ac, bc, ad, bd, cd, ae, de, abcd, abce, abde, acde, bcde) \) in the well-known notation. This gives half-replicates of \( SP(0, 0, 5; w_1) \) and \( SP_2(0, 0, 5; w_1) \), since these sets consist of subsets of the type \( S((v, w)) \). However, it does not give a half-replicate of \( SP(k_1, k_2) \). The set \( SP(k_1, k_2) \) consists of subsets of the type \( S((v, w)) \), where \( |v_i| = |w_i| = 1 \) for all \( 1 \leq i \leq n \). So all pairs occur twice and the number of different pairs of a subset is \( 2^{n-1} \). So the half-replicate of a \( 2^5 \) factorial does not necessarily entail a reduction of the number of pairs. As a matter of fact it entails a reduction if \( k_2 \) is even. However, we are interested in the set \( SP(2, 3) \). It is possible to reduce the number of pairs of this set also. This will be shown in section 3.2, where also the case \( n = 4 \) is dealt with.

More results for \( n \geq 6 \), can be found in Berkum (1985).

3.2. Reduction of the number of pairs of discrete \( D \)-optimal designs for \( n = 4 \) and \( n = 5 \)

First we discuss the reduction of the number of pairs when \( n = 5 \). The design given in section 2 consists of \( SP(2, 3) \) with 160 pairs, \( SP(0, 0, 5; w_1) \) with 32 pairs and \( SP_1(0, 0, 5; w_1) \) with 320 pairs.

By the method given in section 3.1 half-replicates can be found of \( SP(0, 0, 5; w_1) \) and \( SP_2(0, 0, 5; w_1) \). This yields a \( D \)-optimal design with 336 pairs. But a further reduction of the number of pairs is possible. First we consider the set \( SP_2(0, 0, 5; w_1) \). This set consists of the following subsets:
Each set $S_i$ consists of 32 pairs. A quarter-replicate of each set can be found by using the defining contrasts $I$, $CDE$, $ABD$, $ABCE$. The following quarter-replicates of a $2^5$-factorial experiment are obtained by using these defining contrasts:

(1): $(1), ab, acd, bcd, ce, abce, ade, bde$.
   Defining contrasts $I, -CDE, -ABD, ABCE$.

(II): $a, b, cd, abcd, ace, bce, de, abde$.
   Defining contrasts $I, -CDE, ABD, -ABCE$.

(III): $c, abc, ad, bd, e, abe, acde, acde, bcde$.
   Defining contrasts $I, CDE, -ABD, -ABCE$.

(IV): $ac, bc, d, abd, ae, be, cde, abcde$.
   Defining contrasts $I, CDE, ABD, ABCE$.

In these blocks some main effects and two-factor interactions are confounded:


All other main effects and two-factor interactions are clear of one another and of the main effects and interactions given above. We compute the information matrices $M_{(I)}$, $M_{(II)}$, $M_{(III)}$ and $M_{(IV)}$.

If all main effects and two-factor interactions would have been clear of one another, then the result would have been

$$M_{(i)} = \frac{1}{4} 2^n \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & J & \cdot \\ \cdot & \cdot & I \end{bmatrix},$$

where $i = I, II, III, IV$. Now, due to the fact that some main effects and two-factor interactions are confounded, we have

$$|M_{(i)}|_{k,l} = 8 \quad \text{for } i = I, II, III, IV.$$

where

$$(k,l) \in \{ (3,20), (4,19), (4,11), (11,19), (5,16), (1,15), (2,14), (12,18), (13,17) \}.$$

The signs of $(M_{(i)})_{k,l}$ are given in the following table.
Table 2

<table>
<thead>
<tr>
<th>(k,l)</th>
<th>i:</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>confounded effects</th>
</tr>
</thead>
<tbody>
<tr>
<td>(13,17)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>BC, AE</td>
</tr>
<tr>
<td>(12,18)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>AC, BE</td>
</tr>
<tr>
<td>(11,19)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>AB, CE</td>
</tr>
<tr>
<td>(4,11)</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>D, AB</td>
</tr>
<tr>
<td>(2,14)</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>B, AD</td>
</tr>
<tr>
<td>(1,15)</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>A, BD</td>
</tr>
<tr>
<td>(5,16)</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>E, CD</td>
</tr>
<tr>
<td>(4,19)</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>D, CE</td>
</tr>
<tr>
<td>(3,20)</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>C, DE</td>
</tr>
</tbody>
</table>

These signs can be found as follows. \((M_{(i)})_{13,17}\) is related to BC and AE, which are confounded. The defining contrasts of \((I)\) are \(I, -CDE, -ABD, ABCE\). Therefore, \(BC = AE\) and \((M_{(I)})_{13,17} = +8\). Similarly we find \((M_{(II)})_{13,17} = -8\). Now we can compute the information matrices \(M_i\) of quarter-replicates of \(S_i\). We define

\[
\delta_{i,j} = \begin{cases} 
+1, & \text{if the quarter-replicate } (i) \text{ or } (j) \text{ is chosen,} \\
-1, & \text{if not } (i) \text{ or } (j) \text{ is chosen.}
\end{cases}
\]

The expression (2.10) can be used to compute \(M_i\). We find for example for \(S_1\).

\[
V = (w_1,w_1,w_1,1,1 \mid w_1^2,w_1^2,w_1^2,1,1 \mid w_1^2,w_1^2,w_1^2,w_1,w_1,w_1,w_1,w_1,w_1,w_1). \\
W = (1,1,1,w_1,1 \mid 1,1,1,w_1^2,w_1^2 \mid 1,1,1,w_1,w_1,w_1,w_1,w_1,w_1,w_1). 
\]

So

\[
V-W = (w_1-1, w_1-1, w_1-1, 1-w_1,1-w_1 \mid w_1^2-1, w_1^2-1, w_1^2-1, 0,0,0,0,0,0,1-w_1^2), \\
1-w_1^2, 1-w_1^2 \mid w_1^2-1, w_1^2-1, w_1^2-1, 0,0,0,0,0,0,1-w_1^2), 
\]

and

\[
(M_1)_{13,17} = 0, \\
(M_1)_{4,11} = 8(1-w_1)(w_1^2-1)\delta_{II,III}, \\
(M_1)_{3,20} = 8(w_1-1)(1-w_1^2)\delta_{III,IV}. 
\]

In table 3 the signs are given of the elements of \(M_i\) which are of interest.
Table 3
Signs of the important elements of $M_i$

<table>
<thead>
<tr>
<th>i</th>
<th>(13,17)</th>
<th>(12,18)</th>
<th>(11,19)</th>
<th>(4,11)</th>
<th>(2,14)</th>
<th>(1,15)</th>
<th>(5,16)</th>
<th>(4,19)</th>
<th>(3,20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$-\delta_{I,IV}$</td>
<td></td>
<td>$-\delta_{III,IV}$</td>
<td></td>
<td>$-\delta_{III,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td>$-\delta_{II,IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now we choose the following quarter-replicate of $S_i$.

Table 4
Choice of quarter-replicate of $S_i$

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>quarter-replicate (j)</td>
<td>I</td>
<td>I</td>
<td>I</td>
<td>IV</td>
<td>I</td>
<td>III</td>
<td>II</td>
<td>III</td>
<td>I</td>
<td>H</td>
</tr>
</tbody>
</table>

As can be seen by inspecting table 3 we have constructed a quarter-replicate of $SP_2(0,0.5; w_1)$ for which the information matrix is equal to $\frac{1}{4} MP_2(0,0.5; w_1)$. A similar method can be used to reduce the number of pairs of $SP(2,3)$ which consists of 160 pairs. We consider the sets

$T_1 = S((-1,-1,-1,1,1),(1,1,1,1,1)),$

$T_2 = S((-1,-1,1,1,1),(1,1,1,1,1)),$

$T_3 = S((-1,1,-1,1,1),(1,1,1,1,1)),$

$T_4 = S((-1,1,-1,1,1),(1,1,1,1,1)),$

$T_5 = S((-1,1,-1,1,1),(1,1,1,1,1)),$

$T_6 = S((-1,1,-1,1,1),(1,1,1,1,1)),$

$T_7 = S((-1,1,-1,1,1),(1,1,1,1,1)),$

$T_8 = S((-1,1,-1,1,1),(1,1,1,1,1)),$

$T_9 = S((-1,1,-1,1,1),(1,1,1,1,1)),$

$T_{10} = S((-1,1,-1,1,1),(1,1,1,1,1)).$

In each of these sets every pair occurs twice. Now we choose the following quarter-replicate of $T_i$. 
Table 5
Choice of quarter-replicate of $T_i$

<table>
<thead>
<tr>
<th>quarter-replicate (j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>III</td>
<td>II</td>
<td>II</td>
<td>I</td>
<td>III</td>
<td>IV</td>
<td>II</td>
</tr>
</tbody>
</table>

This yields a half-replicate of $SP(2,3)$. We have found a discrete D-optimal design consisting of

i) a half-replicate of $SP(2,3)$ : 80 pairs,
ii) a half-replicate of $SP(0,0.5; w_1)$ : 16 pairs,
iii) a quarter-replicate of $SP_2(0,0.5; w_1)$ : 80 pairs,

In total : 176 pairs.

This number is smaller than 210, the number $m$ given in table 1.

We consider the case $n=4$. The design given in section 2 consists of $SP(2,2)$ with 48 pairs, $SP(0,0.4; w_1)$ with 16 pairs, $SP_2(0,0.4; w_1)$ with 48 pairs and $SP(1,2,1; w_2)$ with 192 pairs.

This set consists of the following sets.

First we consider the set $SP(1,2,1; w_2)$. This set consists of the following sets:

$S_1 = ((1,1,1,1),( 1,-1,-1, w_2))$,
$S_2 = ((1,1,1,1),(-1, 1,-1, w_2))$,
$S_3 = ((1,1,1,1),(-1,-1, 1, w_2))$,
$S_4 = ((1,1,1,1),( 1,-1, w_2,-1 ))$,
$S_5 = ((1,1,1,1), (-1, 1, w_2,-1))$,
$S_6 = ((1,1,1,1),( 1,-1, w_2, 1))$,
$S_7 = ((1,1,1,1),( w_2,-1, -1, 1))$,
$S_8 = ((1,1,1,1),(1, w_2, 1,-1))$,
$S_9 = ((1,1,1,1),(w_2, -1, 1,-1 ))$,
$S_{10} = ((1,1,1,1),(w_2, 1,-1,-1 ))$.

We will construct a half-replicate of $SP(1,2,1; w_2)$. A half-replicate of each set $S_i$ can be found by using the defining contrast ABCD. We find half-replicates of a $2^4$-factorial experiment:

(I): (1),ab,ac,bc,ad,bd,cd,abcd.
Defining contrasts I, ABCD.

(II): a,b,c,abc,d,abd,acd,bcd.
Defining contrasts I, -ABCD.

The confounded interactions are $BC = AD$, $AC = BD$, $AB = CD$. Therefore, in computing the information matrix $M_{(I)}$ and $M_{(II)}$, the following elements are important:

Table 6
The signs of $(M_{(i)})_{k,l}$

<table>
<thead>
<tr>
<th>(k,l)</th>
<th>i:</th>
<th>II</th>
<th>Confounded interactions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(11,12)</td>
<td>+</td>
<td>-</td>
<td>AD, BC</td>
</tr>
<tr>
<td>(10,13)</td>
<td>+</td>
<td>-</td>
<td>AC, BD</td>
</tr>
<tr>
<td>(9,14)</td>
<td>+</td>
<td>-</td>
<td>AB, CD</td>
</tr>
</tbody>
</table>

We can choose the following half-replicates of $S_i$: $I$ for $i=1,2,3,4,5,6$ and $II$ for $i=7,8,9,10,11,12$. This gives a half-replicate of $SP(1,2,1; w_2)$, for which the information
matrix is equal to $\frac{1}{2}MP(1,2,1; w_2)$. We consider the set $SP_2(0,0,4; w_1)$. It is not possible to construct a half-replicate of $SP_2(0,0,4; w_2)$ having an information matrix of type (2.4). Therefore, we consider the following D-optimal design.

i) the pairs of $SP(2,2)$ with weights $\nu_2 = 0.00711$,
ii) the pairs of $SP(0,0,4; w_1)$ with weights $\mu = 0.00186$,
iii) the pairs of $SP(0,0,4; w_1)$ with weights $\lambda = 0.00492$,
iv) the pairs of $SP(1,2,1; w_2)$ with weights $\rho = 0.00162$.

The number of pairs of $SP(0,0,4; w_1)$ is equal to 64, which is 16 more than the number of pairs of $SP_2(0,0,4; w_1)$. However, it is possible to construct a half-replicate of $SP_2(0,0,4; w_1)$, which consists of the sets

$T_1 = S((\begin{array}{cccc} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{array}),(-1,1,1,1))$,
$T_2 = S((\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{array}),(-1,1,1,1))$,
$T_3 = S((\begin{array}{cccc} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}),(-1,1,1,1))$.

Choosing the half-replicate (I) for each set $T_i$ we obtain a half-replicate of $SP_2(0,0,4; w_1)$. Finally we consider the set $S(2,2)$. This set consists of

$V_1 = S((\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{array}),(-1,1,1,1))$,
$V_2 = S((\begin{array}{cccc} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{array}),(-1,1,1,1))$,
$V_3 = S((\begin{array}{cccc} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}),(-1,1,1,1))$,
$V_4 = S((\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{array}),(-1,1,1,1))$.

In each set $V_i$ the pairs occur twice. Therefore, we need a quarter-replicate of $V_i$ to obtain a half-replicate of $SP(2,2)$. Consider the defining contrasts $I, D, ABC, ABCD$. They yield the following quarter-replicates of a $2^4$-factorial experiment.

<table>
<thead>
<tr>
<th>Defining contrasts</th>
<th>(I)</th>
<th>(II)</th>
<th>(III)</th>
<th>(IV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$I$, ab, ac, bc</td>
<td>$I$, -D, -ABC, ABCD</td>
<td>$I$, d, abd, acd, bcd</td>
<td>$I$, ad, bd, cd, abcd</td>
</tr>
<tr>
<td>$D$</td>
<td>$I$, -D, ABC, -ABCD</td>
<td>$I$, D, -ABC, -ABCD</td>
<td>$I$, D, ABC, ABCD</td>
<td>$I$, D, ABC, ABCD</td>
</tr>
</tbody>
</table>

By methods similar to the ones above it can be seen that a half-replicate of $SP(2,2)$ for which the information matrix is equal to $\frac{1}{2}MP(2,2)$, can be found by choosing the quarter-replicates given in table 7.
Table 7
Choice of quarter-replicate of $V_i$

<table>
<thead>
<tr>
<th>quarter-replicate (j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV</td>
<td>I</td>
<td>II</td>
<td>IV</td>
<td>II</td>
<td>II</td>
<td></td>
</tr>
</tbody>
</table>

A discrete D-optimal design has been constructed consisting of
i) a half-replicate of $SP(2,2)$ : 24 pairs,
ii) $SP(0,0,4; w_1)$ : 16 pairs,
iii) a half-replicate of $SP(0,0,4; w_1)$ : 32 pairs,
iv) a half-replicate of $SP(1,2,1; w_2)$ : 96 pairs,

In total : 168 pairs.

This number is larger than 105, the number $m$ given in table 1. Therefore, a further reduction can be achieved. However, this seems to entail many different weights, which is not attractive for practical applications.

4. Exact designs when $n = 2, 3, 4, 5$

In this section exact designs are constructed for $n = 2, 3, 4, 5$.

First we consider the case $n$ is odd.

If $n = 3$ we choose
i) $n_1$ times the pairs of $SP(1, 2)$,
ii) $n_2$ times the pairs of $SP(0, 0, 3; w_1)$,
iii) $n_3$ times the pairs of $SP(0, 0, 3; w_2)$.

If $n = 5$ we choose
i) $n_1$ times the pairs of a half-replicate of $SP(2, 3)$,
ii) $n_2$ times the pairs of a half-replicate of $SP(0, 0, 5; w_1)$,
iii) $n_3$ times the pairs of a quarter-replicate of $SP(2, 0, 5; w_2)$.

The information matrices of these designs have the structure of (2.4). An argument that can be used when choosing $n_1$, $n_2$ and $n_3$ is that the weights $n_j/(n_1 + n_2 + n_3)$ should have approximately the same values as the weights of the discrete designs. However the number of pairs of the design must be small for practical reasons. So the pairs in an exact design cannot have the same weights as in a discrete optimal design. This can be partly compensated by choosing other values for $w_1$ and $w_2$ than the ones of a discrete $D$-optimal design, where $w_1 = w_2 = -0.118$ for $n = 3$ and $w_1 = w_2 = -0.080$ for $n = 5$. A computer program has been written that determines the optimal value of $w_1$ and $w_2$ using the $D$-criterion or the $G$-criterion. For discrete $D$-($G$-)optimal designs it holds that $\delta = -45\%$. This is not true for exact designs. In some cases the values $w_1$ and $w_2$ are computed under the restriction $\delta = -45\%$, or under the restriction $w_1 = w_2$. Moreover, a design is given with $w_1 = w_2 = 0$, which might be useful in practice.
Results are given in tables 8-9.

The choices made are denoted as follows

(1) with restriction $\delta + 4\xi = 0$, using the $G$-criterion,
(2) no restriction; $G$-criterion,
(3) no restriction; $D$-criterion,
(4) with restriction $w_1 = w_2$; $G$-criterion,
(5) with restriction $w_1 = w_2$; $D$-criterion,
(6) $w_i = 0$ for all $i$.

$N$ is the number of pairs.

The results are satisfactory. The efficiency of the designs is good. The number of pairs is comparatively small and the information matrices have the structure of (2.4).

The relevant values of the covariance matrix are given in the tables. The efficiency of the designs is good.

Table 8

Constants determining exact designs given in section 4.

<table>
<thead>
<tr>
<th>$n = 3$</th>
<th>$n_1 = 1$ ; $n_2 = 1$ ; $n_3 = 1$ ; $N = 44$.</th>
<th>$n = 3$</th>
<th>$n_1 = 2$ ; $n_2 = 1$ ; $n_3 = 1$ ; $N = 56$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$\alpha$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>(1)</td>
<td>-0.47</td>
<td>-0.59</td>
<td>3.29</td>
</tr>
<tr>
<td>(2)</td>
<td>-0.10</td>
<td>-0.59</td>
<td>3.24</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.14</td>
<td>-0.20</td>
<td>1.49</td>
</tr>
<tr>
<td>(4)</td>
<td>-0.36</td>
<td></td>
<td>1.83</td>
</tr>
<tr>
<td>(5)</td>
<td>0.18</td>
<td></td>
<td>1.47</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
<td></td>
<td>1.38</td>
</tr>
</tbody>
</table>
Table 9

Constants determining exact designs given in section 4.

<table>
<thead>
<tr>
<th>$n = 5$</th>
<th>$n_1 = 1 ; n_2 = 1 ; n_3 = 1 ; N = 176.$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w_1$</td>
</tr>
<tr>
<td>(1)</td>
<td>-0.09</td>
</tr>
<tr>
<td>(2)</td>
<td>-0.11</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.09</td>
</tr>
<tr>
<td>(4)</td>
<td>-0.30</td>
</tr>
<tr>
<td>(5)</td>
<td>-0.12</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n = 5$</th>
<th>$n_1 = 2 ; n_2 = 1 ; n_3 = 1 ; N = 206.$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w_1$</td>
</tr>
<tr>
<td>(1)</td>
<td>-0.00</td>
</tr>
<tr>
<td>(2)</td>
<td>-0.00</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.07</td>
</tr>
<tr>
<td>(4)</td>
<td>-0.02</td>
</tr>
<tr>
<td>(5)</td>
<td>-0.01</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
</tr>
</tbody>
</table>

Now we consider the case $n$ is even.

If $n = 2$, we choose

i) $n_1$ times the pairs of $SP(1, 1)$,

ii) $n_2$ times the pairs of $SP(0, 0, 2; w_1)$,

iii) $n_3$ times the pairs of $SP(0, 0; w_2)$,

iv) $n_4$ times the pairs of $SP(0, 1, 1; w_3)$,

If $n = 4$, we choose

i) $n_1$ times the pairs of a half-replicate of $SP(2, 2)$,

ii) $n_2$ times the pairs of $SP(0, 0, 4; w_1)$,

iii) $n_3$ times the pairs of a half-replicate of $SP(0, 0, 4; w_2)$,

iv) $n_4$ times the pairs of a half-replicate of $SP(1, 2, 1; w_3)$.

Again for fixed $n_1, n_2, n_3$ and $n_4$ values of $w_1, w_2$ and $w_3$ have been computed according to the $G$-criterion or the $D$-criterion. Again designs are constructed under some restrictions ($\beta + 4\xi = 0$, or $w_1 = w_2$). Results are given in tables 10 and 11. The choices made are denoted in the same way as before.
Table 10
Constants determining exact designs given in section 4.

<table>
<thead>
<tr>
<th>n = 2</th>
<th>n₁ = 0 ; n₂ = 0 ; n₃ = 0 ; n₄ = 0 ; N = 8.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w₁</td>
</tr>
<tr>
<td>(2)</td>
<td>-</td>
</tr>
<tr>
<td>(3)</td>
<td>-</td>
</tr>
<tr>
<td>(6)</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 2</th>
<th>n₁ = 1 ; n₂ = 1 ; n₃ = 1 ; n₄ = 1 ; N = 20.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>-0.03</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.15</td>
</tr>
<tr>
<td>(4)</td>
<td>-0.08</td>
</tr>
<tr>
<td>(5)</td>
<td>-0.16</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 2</th>
<th>n₁ = 1 ; n₂ = 2 ; n₃ = 1 ; n₄ = 3 ; N = 40.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>-0.32</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.16</td>
</tr>
<tr>
<td>(4)</td>
<td>-0.14</td>
</tr>
<tr>
<td>(5)</td>
<td>-0.15</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 11
Constants determining exact designs given in section 4.

<table>
<thead>
<tr>
<th>n = 4</th>
<th>n₁ = 1 ; n₂ = 0 ; n₃ = 2 ; n₄ = 0 ; N = 88</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w₂</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.18</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 4</th>
<th>n₁ = 1 ; n₂ = 0 ; n₃ = 1 ; n₄ = 0 ; N = 56</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2)</td>
<td>-0.36</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.15</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 4</th>
<th>n₁ = 1 ; n₂ = 0 ; n₃ = 2 ; n₄ = 1 ; N = 184</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2)</td>
<td>-0.09</td>
</tr>
<tr>
<td>(3)</td>
<td>-0.12</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
</tr>
</tbody>
</table>
References


