From Lie Algebras to Geometry and Back

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Introduction

Around 450 BC, Euclid of Alexandria wrote his famous work *The Elements* [Euc56], which was the defining work of geometry for more than 2000 years. Starting from five postulates, Euclid derived many theorems on planar geometry. Later, it was commonly thought (though never proved) that one of the postulates, the so-called parallel postulate, could be derived from the four others. If this were true, then there could be no other, alternative “geometries” in which the first four of Euclid’s postulates hold, but the parallel postulate does not. The first signs (at least in western mathematics) that this belief was false showed when in 1733, an Italian named Giovanni Saccheri tried to prove that the parallel postulate followed from the others in a paper titled *Euclides ab Omni Naeco Vindicatus*, or *Euclid Freed from Every Flaw* [Sac33]: he assumed the parallel postulate was false and tried to derive a contradiction. He thought he had found one, but this was later found to be a mistake. Instead, he had unknowingly found many theorems of what we now know as hyperbolic geometry, a version of geometry in which the parallel postulate does not hold.

During the years that followed, the belief that there might be consistent alternative geometries to Euclid’s slowly grew. In 1823, Nikolai Lobachevsky from Russia and János Bolyai from Hungary independently wrote about such alternative geometries. In 1868, Eugenio Beltrami [Bel68], an Italian, finally proved that the parallel postulate was indeed independent by giving a concrete realization of Lobachevsky’s geometry. Three years later, Felix Klein [Kle71] also showed that Euclidean geometry is consistent if and only if a large family of non-Euclidean geometries is consistent.

Klein had conceived a method for unifying the different notions that were now called geometry by his *Erlanger Programm*. Written as the speech for his inauguration as a professor at Erlangen in 1872 (though not delivered on that occasion), it comprised the idea of studying a geometry through its group of transformations, and the properties of the geometry that are invariant under these transformations. The *Programm* was a result of a stay in Paris in 1870, together with Sophus Lie and strongly influenced by Camille Jordan. Jordan had earned some fame for his work on classifying groups of symmetries of Euclidean geometry. Around 1873-1874, also inspired by the stay in Paris and fitting into Klein’s *Programm*, Lie started developing what was to become his theory of continuous transformation groups, or Lie groups as we now call them. His
motivation was the study of differential equations. One of his important contributions was, that these groups could be better understood by considering the corresponding “infinitesimal group”. Today, these infinitesimal groups are known as Lie algebras.

Independently, Wilhelm Killing, a mathematician at a Prussian Jesuit gymnasium, also introduced Lie algebras in 1884 [Kil84], in an effort to study non-Euclidean geometry. He invented the notions of a Cartan subalgebra and a root system, and used these to restrict the set of possible semisimple finite-dimensional Lie algebras over the complex field to four infinite series and five exceptional cases. Later, Cartan finished the classification, showing that all of these algebras do indeed exist, by giving a realization for all of them in his 1894 thesis [Car94].

From around 1937, Ernst Witt, Nathan Jacobson [Jac37] and Hans Zassenhaus [Zas39] started developing the theory of finite-dimensional simple Lie algebras over fields of positive characteristic. Witt found an example of such a Lie algebra that was very unlike the algebras for characteristic zero, though somewhat similar to an infinite-dimensional simple Lie algebra for characteristic zero.

Since then, much research has been published on Lie algebras over fields of positive characteristic, also called modular Lie algebras. Over the next twenty years, analogues of the classical Lie algebras of Cartan were constructed for fields of positive characteristic. In the sixties and seventies, four infinite families of simple finite-dimensional modular Lie algebras generalizing Witt’s example were found and studied; they are known as Cartan type Lie algebras. Then in 1979, Hayk Melikyan [Mel79] found a family of new simple Lie algebras in characteristic 5.

Over many years, mathematicians have been trying to classify the finite-dimensional simple modular Lie algebras over algebraically closed fields of sufficiently big characteristic. The most recent development is a theorem by Alexander Premet and Helmut Strade stating that the Lie algebras from the preceding paragraph are an exhaustive list for characteristic at least 5, proven in a long series of papers [PS97, PS99, PS01, PS04, PS06, PS07]. For characteristic 3, a number of other simple finite-dimensional Lie algebras is known, to be found in e.g. Strade’s book [Str04]; in the same book, Strade claims that it could well be that this list is complete. Characteristic 2 is generally considered to be far out of reach of current methods; as recently as 2006, two new low-dimensional simple Lie algebras were discovered by Michael Vaughan-Lee [VL06].

This thesis is about Lie algebras and geometry, and links between them. In Chapter 2, we study some of the classical Lie algebras. We see that a certain class of elements, the extremal elements, carries a geometrical structure. This point of view has been explored earlier, for example by Cohen and Ivanyos in [CI06, CI]. We investigate this structure for the four infinite families of Chevalley type Lie algebras. Our main result is that we can, in the generic case, recognize the Lie algebra from certain aspects of this geometrical structure. In Chapter 3, we examine the Melikyan algebras. We implement it in the computer algebra system GAP [GAP06] and use its automorphism group to study a geometrical structure in these algebras and the subalgebra generated by its sandwich elements, a degenerate kind of extremal element. Using this subalgebra we find a list of 68 subspaces that are invariant under the automorphism group. In Chapter 4, we start with a certain class of so-called oriented partial linear spaces, and construct algebras from them. We determine the set of partial linear spaces for which the resulting algebra is a Lie algebra. We study relations between properties of the geometry and properties of the Lie algebra.

Finally, in the rest of this introductory chapter we present some material that is used in the other chapters. In Section 1.1, we present the nonstandard notation and conventions used in the remainder of this thesis. In Section 1.2 we review some general
preliminaries on algebra and geometry. We will state some results and definitions of the classical theory of simple Lie algebras in Section 1.3. Finally, in Section 1.4, we will give a short overview of some results in modular Lie theory.

Most biographical data in this introduction was taken from the MacTutor History of Mathematics Archive [OR07] from the University of St Andrews.

1.1. Conventions and notation

In this thesis, we let \( \mathbb{N}_0 = \{ n \in \mathbb{Z} \mid n \geq 0 \} \) and \( \mathbb{N}_+ = \{ n \in \mathbb{Z} \mid n > 0 \} \). For prime powers \( q \), the field with \( q \) elements will be denoted by \( \mathbb{F}_q \).

If a map \( \phi : A \rightarrow B \) is given, we denote its restriction to \( A' \subseteq A \) by \( \phi|_{A'} \).

In order to prevent confusion between different objects that can be generated by a given set \( X \) of objects, we will let

\[
\langle X \rangle_{\text{loop}} \quad \text{denote} \quad \text{the loop}^1 \text{ generated by } X,
\langle X \rangle_{\text{gp}} \quad \text{denote} \quad \text{the group generated by } X,
\langle X \rangle_{\text{F}} \quad \text{denote} \quad \text{the vector space over } \mathbb{F} \text{ spanned by } X,
\langle X \rangle_{\text{Lie}} \quad \text{denote} \quad \text{the Lie algebra generated by } X,
\langle X \rangle_{\text{Idl}} \quad \text{denote} \quad \text{the ideal generated by } X.
\]

The notation \( \langle X \rangle_{\text{Lie}} \) makes sense only if the field over which the Lie algebra is to be taken is understood; accordingly, we will only use that notation if it is clear which field is meant. Similarly, we will only write \( \langle X \rangle_{\text{Idl}} \) if it is clear in which ring this ideal is to be taken.

A vector space over a field \( \mathbb{F} \) spanned by a single vector \( v \) will often be denoted by \( \mathbb{F}v \) instead of \( \langle v \rangle_{\mathbb{F}} \). If \( V \) is a vector space, its dual will be denoted by \( V^* \). The vectors in \( V^* \) will usually be called functionals. For a field \( \mathbb{F} \), we will denote the set of invertible elements in \( \mathbb{F} \) by \( \mathbb{F}^* \).

When a set of generators of an algebra \( A \) is given, we will use the word monomial to signify elements of \( A \) that can be written as a product of these generators. A monomial written as the product of \( n \) generators will be said to be of degree \( n \).

If we discuss cosets of an algebraic structure \( H \) in another structure \( G \), we will always consider left cosets \( G/H \), so that the symbol \( \backslash \) can be reserved for the set minus operation.

1.2. General preliminaries

**Definition 1.1.** Let \( V \) be a vector space. A symplectic form \( B \) is a bilinear form satisfying \( B(v, v) = 0 \) for all \( v \in V \). The radical \( \text{Rad}(B) \) of \( B \) is defined as the subspace of \( V \), consisting of those vectors \( v \) for which \( B(v, w) = 0 \) for all \( w \in V \). If \( \text{Rad}(B) = 0 \), then \( B \) is said to be nondegenerate. If \( Q \) is a quadratic form on \( V \) (a map \( V \rightarrow \mathbb{F} \) with \( Q(av) = a^2Q(v) \) and such that \( (v, w) \mapsto Q(v + w) - Q(v) - Q(w) \) is a bilinear form \( B_Q \)), then its radical \( \text{Rad}(Q) \) is the subset of the radical of the associated bilinear form consisting of those vectors where \( Q \) is zero. \( Q \) is said to be nonsingular if \( \text{Rad}(Q) = 0 \).

If this is the case, then the radical of \( B_Q \) has at most dimension 1.

**Definition 1.2.** A graph \( \Gamma \) is a pair \((V, E)\) of a set \( V \) and a set \( E \) of subsets of \( V \) of cardinality 2. We call the elements of \( V \) vertices and those of \( E \) edges. Furthermore, we define \( V(\Gamma) = V \) and \( E(\Gamma) = E \).

---

1A loop is a not necessarily associative generalization of a group. Some loop theory is reviewed in Section 4.9.3.
1. INTRODUCTION

Definition 1.3. A derivation on a ring $R$ is a map $D: R \times R \to R$ such that for all $r, s \in R$, we have

$$D(r \cdot s) = D(r) \cdot s + r \cdot D(s).$$

Definition 1.4. An algebra $A$ over a field $F$ is a vector space over $F$ with a bilinear operation $\cdot : A \times A \to A$, called multiplication. If the multiplication operation is associative, we call $A$ an associative algebra. An ideal of $A$ is a subspace $I$ of $A$ such that $A \cdot I := \langle a \cdot i | a \in A, i \in I \rangle_F$ and $I \cdot A := \langle i \cdot a | a \in A, i \in I \rangle_F$ are contained in $A$. An algebra of dimension at least 2 in which 0 and 1 are the only ideals is called simple.

Example 1.5. The space $F^{n \times n}$ of $n \times n$ matrices over $F$ is an algebra with matrix multiplication. Since the multiplication is associative, we call $F^{n \times n}$ an associative algebra.

Definition 1.6. A Lie algebra is an algebra $L$ over a field $F$ satisfying:

$$x \cdot x = 0,$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + y \cdot (x \cdot z),$$

for all $x, y, z \in L$. The first property is called anticommutativity of $L$, the second identity is the Jacobi identity. Multiplication in a Lie algebra is usually denoted by $[,]$.

Example 1.7. If $A$ is an associative algebra with multiplication $\cdot$, then

$$[x, y] = x \cdot y - y \cdot x$$

defines a multiplication turning $A$ into a Lie algebra. We will denote $A$ with this multiplication as $A_{\text{Lie}}$.

Definition 1.8. Given an algebra $A$ over a field $F$, and two subspaces $S_1$ and $S_2$ of $A$, the conductor $\text{Cond}_A(S_1, S_2)$ in $A$ of $S_1$ into $S_2$ is the subspace $\langle a \in A | \forall s \in S_1, as \in S_2 \rangle_F$ of elements of $A$ mapping $S_1$ into $S_2$.

The main use of this concept in this thesis is in Section 3.5.1, where we use the fact that if $S_1$ and $S_2$ are invariant subspaces under the automorphism group of $A$, then so is $\text{Cond}_A(S_1, S_2)$.

1.3. Classical Lie theory

In this section, we will give a short introduction into the theory of simple Lie algebras over the complex field. There is a strong link with group theory. Most of the results we present here can be found in [Car89].

Given a linear algebraic group $G$ over a finite-dimensional space, we can define the Lie algebra of $G$ as the tangent space to $1 \in G$, as follows. Since $G$ is a linear group, its elements can be viewed as, say, $n \times n$-matrices. To find elements of the tangent space at $1$, one needs to find matrices $A$ such that for small $\epsilon$, the matrix $B := 1 + \epsilon A$ is “close to $G$”. We study polynomials in the entries of these matrices in $G$. Let $R$ be the ideal of all polynomials in matrix entries, that vanish at all elements of $G$. Since $G$ is algebraic, it is precisely the set of all zeroes of $R$. The phrase “close to $G$” is now made precise using the polynomials in $R$, as follows:

$$\text{Lie}(G) := \langle A \in F^{n \times n} | \forall \epsilon \in R, r(1 + \epsilon A) = 0 + O(\epsilon^2) \rangle_{\text{Lie}}.$$
This set forms a Lie algebra where the Lie bracket is given by the commutator. (There is a more rigorous, basis-free definition that also works for infinite-dimensional spaces, but we will only require the previous definition in this thesis.)

Example 1.9. Take \( G = \text{SL}(V) \), defined by \( \det B = 1 \). Note that \( \det(1 + \epsilon A) = 1 + \epsilon \text{tr} A + O(\epsilon^2) \); so the Lie algebra of \( G \) is \( \text{sl}(V) \), the Lie algebra of linear transformations with trace 0. If \( G \) is the subgroup of \( \text{SL}(V) \) respecting a form defined by a matrix \( M \), the computation is as follows: the defining equation of \( G \) states that \( (Bx)^T MBy = x^T My \) for all \( x \) and \( y \), so \( B^T MB = M \). Then we need

\[
(1 + \epsilon A)^T M(1 + \epsilon A) - M = \epsilon (A^T M + MA) + \epsilon^2 A^T MA
\]

to be a multiple of \( \epsilon^2 \); so \( A^T M = -MA \). This leads to the Lie algebras \( \text{sp}(V,B) \), if the form was symplectic, or \( \text{o}(V,B) \), if the form was orthogonal.

To turn these examples into full proofs of the fact that \( \text{sl}(V) \), \( \text{sp}(V,B) \) or \( \text{o}(V,B) \) is the Lie algebra of \( G \), we would have to show that the polynomial equations mentioned form a radical ideal. We also denote these Lie algebras by \( \text{sl}_n \), \( \text{sp}_n \) and \( \text{o}_n \) if \( \dim V = n \).

The Lie algebras \( \text{sl}_{n+1} \), \( \text{sp}_{2n+1} \), \( \text{sp}_{2n} \) and \( \text{o}_{2n} \) are, if the respective forms are nondegenerate, also known as the Lie algebras of type \( A_n \), \( B_n \), \( C_n \) and \( D_n \).

Definition 1.10. Let \( \mathcal{L} \) be a Lie algebra. For any subset \( S \) of \( \mathcal{L} \), we call the subalgebra \( C_L(S) = \langle a \in \mathcal{L} \mid (\forall b \in S \mid [a,b] = 0) \rangle \) the Lie centralizer (or simply the centralizer) of \( S \) in \( \mathcal{L} \); the fact that it is a subalgebra follows from the Jacobi identity. If \( S = \{s\} \) is a singleton, we will usually write \( C_L(s) \) for \( C_L(S) \).

Similarly, if \( S \) is a subspace of \( \mathcal{L} \), the subalgebra \( N_L(S) = \langle a \in \mathcal{L} \mid [a,S] \subseteq S \rangle \) is called the Lie normalizer (or simply the normalizer) of \( S \) in \( \mathcal{L} \).

Definition 1.11. For any subsets \( S \) and \( T \) of a Lie algebra \( \mathcal{L} \), we define \( [S,T] = \{[s,t] \mid s \in S, t \in T \} \). The derived subalgebra of \( \mathcal{L} \) is \([\mathcal{L},\mathcal{L}]\).

The classification of simple finite-dimensional Lie algebras over the complex field states that every such Lie algebra \( \mathcal{L} \) contains a maximal commutative subalgebra \( C \) that is its own normalizer. This subalgebra is called the Cartan subalgebra. It is unique up to automorphisms of \( \mathcal{L} \), so its dimension is an invariant of the Lie algebra called rank. All nontrivial simultaneous eigenspaces of \( C \) are 1-dimensional. The simultaneous eigenvalues, called roots, form a subset of Euclidean space that is closed under reflection along these roots. This subset is called the root system of \( \mathcal{L} \), and the classification of these root systems gives the classification of the Lie algebras. Beside the four infinite series \( A_n, B_n, C_n \) and \( D_n \), there are only 5 other isomorphism classes of finite-dimensional Lie algebras. These are referred to as types \( E_6, E_7, E_8, F_4 \) and \( G_2 \), or the exceptional types, whereas the four infinite series are known as classical types. This last term is somewhat unfortunate, however, for in literature on Lie algebras over positive characteristic, it is often used to describe all the simple Lie algebras that exist over the complex field, exceptional or classical. Hence we will not use it.

For all of these Lie algebras \( \mathcal{L} \), there is a basis \( B \) of \( \mathcal{L} \) (consisting of a basis of \( C \) and a vector in each of its nontrivial eigenspaces) such that the multiplication table of \( \mathcal{L} \) with respect to \( B \) is integral. This allows for defining a Lie ring \( \mathcal{L}_Z \) over \( \mathbb{Z} \) with the same multiplication table. Out of that we can define a Lie algebra \( \mathcal{L}_F = \mathcal{L}_Z \otimes _\mathbb{Z} F \) over a field \( F \) of prime characteristic. In most cases, this Lie algebra is also simple. In some cases it
information sending \( \sum X \) is the free Lie algebra \( \mathfrak{X} \). Over fields of characteristic 0, the map \( \exp(\mathfrak{ad} x) \) is an automorphism if it converges everywhere:

\[
\exp(t \mathfrak{ad} x)[y, z] = \sum_{i=0}^{\infty} \frac{t^i}{i!} [\mathfrak{ad} x]^i y, z] = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{t^i}{i! j!} [(\mathfrak{ad} x)^i y, (\mathfrak{ad} x)^j z] = \exp(t \mathfrak{ad} x)y, \exp(t \mathfrak{ad} x)z], \quad (1.1)
\]

where equality \((*)\) follows from the Jacobi identity. If \( L \) is defined over a field of positive characteristic \( p \) and \( \mathfrak{ad} x \) is nilpotent of degree \( k \), that is, if \( (\mathfrak{ad} x)^k \) is the zero map, then \( \exp(t \mathfrak{ad} x) \) is, by the above reasoning, also an automorphism if \( p \geq 2k - 2 \) then the terms \( \frac{t^i}{i!} [\mathfrak{ad} x]^i y, (\mathfrak{ad} x)^j z] \) make sense for \( i \leq 2k - 2 \), and hence we find terms \( \frac{t^i}{i!} [(\mathfrak{ad} x)^i y, (\mathfrak{ad} x)^j z] \) where both \( i \) and \( j \) run up to \( k - 1 \).

1.4. Modular Lie theory

Over fields of positive characteristic, the situation is rather different from the situation in characteristic 0: groups play a much smaller role and the theory is less finished. Most of the material in this section borrows heavily from Strade’s book [Str04].

Most of the known simple finite-dimensional Lie algebras come from the setting of divided power algebras, described below. Fix \( m \in \mathbb{N}_+ \). Consider the polynomial ring \( \mathbb{Q}[X_1, \ldots, X_m] \) and define \( X_i^{(r)} = \frac{1}{r} X_i^r \). We use multi-index notation; for \( a, b \in \mathbb{N}_0^m \), we
write
\[
X^{(a)} = \prod_{i=1}^{m} X^{(a)_i}, \quad \begin{pmatrix} a + b \\ a \end{pmatrix} = \prod_{i=1}^{m} \begin{pmatrix} a_i + b_i \\ a_i \end{pmatrix},
\]

\[a \leq b \text{ if and only if } \forall_{1 \leq i \leq m}(a_i \leq b_i), \quad |a| := \sum_{i=1}^{m} a_i.
\]

Then \(X^{(a)}X^{(b)} = \left(\frac{a+b}{a}\right)X^{(a+b)}\), and if we let \(\partial_i\) denote differentiation by \(X_i\), then \(\partial_iX^{(a)} = X^{(a-e_i)}\) where \(e_i\) is the \(i\)th unit vector (unless \(a_i = 0\), in which case \(\partial_iX^{(a)} = 0\)). Note that \(\mathbb{Q}[X_1, \ldots, X_m]\) is naturally \(\mathbb{Z}\)-graded with \(\mathbb{Q}[X_1, \ldots, X_m]_i = \langle X^a \mid |a| = i \rangle_{\mathbb{Q}}\). Now define a \(\mathbb{Z}\)-algebra \(O(m)\) as having a formal basis \(\{x^a \mid a \in \mathbb{N}_0^m\}\) and multiplication defined by
\[
x^a x^b = \left(\frac{a + b}{a}\right)x^{a+b}.
\]

For any given field \(\mathbb{F}\), we can now define \(O(m) = O(m) \otimes_{\mathbb{Z}} \mathbb{F}\). (Clearly for \(\mathbb{F} = \mathbb{Q}\), we would obtain for \(O(m)\) an isomorphic copy of \(\mathbb{Q}[X_1, \ldots, X_m]\).) This is an associative and commutative \(\mathbb{F}\)-algebra with unit element \(x^0 = 1\). If \(\mathbb{F}\) has positive characteristic \(p\), which we assume for the rest of this section, then some of the binomials involved are 0. Hence we find the following finite-dimensional subalgebra of \(O(m)\) for \(n \in \mathbb{N}_+^m\):
\[
O(m; n) = \langle x^a \mid 0 \leq a; < p^n \rangle_{\mathbb{F}}.
\]

Its dimension is \(p|n|\).

If we carry the definition of \(\partial_i\) over from \(\mathbb{Q}[X_1, \ldots, X_m]\) to \(O(m)\), then \(\partial_i \in \text{Der} O(m)\). Since \(O(m)\) is associative and commutative, we can obtain more derivations \(x^a \partial_i\) by letting \(\langle x^a \partial_i \rangle\): \(x^b \mapsto x^a(\partial_i x^b)\). These are also derivations of \(O(m; n)\). We define
\[
W(m) := \bigoplus_{i=1}^{m} O(m)\partial_i = \text{Der} O(m), \quad W(m; n) := \bigoplus_{i=1}^{m} O(m; n)\partial_i = \text{Der} O(m; n).
\]

These Lie algebras are called the Witt algebras. There is a natural \(O(m)\)-, resp. \(O(m; n)\)-action on them, determined by \(f \cdot (g\partial_i) = (fg)\partial_i\). The algebra \(W(m; n)\) is simple, unless \(m = n_1 = 1\) and \(p = 2\).

The \(\mathbb{Z}\)-grading of \(\mathbb{Q}[X_1, \ldots, X_m]\) also carries over to \(O(m)\). For \(i \in \mathbb{N}_0\), we define the following:
\[
O_i = \langle x^a \mid |a| = i \rangle_{\mathbb{F}}, \quad O_{\geq i} = \langle x^a \mid |a| \geq i \rangle_{\mathbb{F}};
\]
and with the inherited grading of \(\text{Der} O(m)\), for \(i + 1 \in \mathbb{N}_0\),
\[
W_i = \langle x^a \partial_j \mid |a| = i + 1 \rangle_{\mathbb{F}}, \quad W_{\geq i} = \langle x^a \partial_j \mid |a| \geq i + 1 \rangle_{\mathbb{F}}.
\]

\(O(m; n)\) and \(W(m; n)\) are homogeneous algebras with respect to this grading, in the sense that \(\mathcal{O}(m; n) = \bigoplus_{i=0}^{\infty} (O(m; n) \cap O_i)\) and similarly for \(W(m; n)\). Furthermore, \(O_{\geq 1}\) is a maximal ideal of \(\mathcal{O}(m)\) (and its intersection with \(O(m; n)\) is a maximal ideal of \(O(m; n)\)); \(W_{\geq 0}\) is a maximal ideal of \(W(m)\) (and its intersection with \(W(m; n)\) is a maximal ideal of \(W(m; n)\)).

To describe the automorphism group of \(W(m)\), we need the notion of a divided power automorphism.
Let \( W \) be a divided power automorphism of \( A \). A similar definition can be used to define \( \text{Aut}_W \) in this way. Similarly, all automorphisms of \( W \) are invariant under all automorphisms of \( A \).

These powers are \( (f \circ t) = f(t) \) for \( f \in \text{Aut}(A) \), \( (f \circ t)(x) = f(t(x)) \) for \( f \in \text{Aut}(W) \), and so forth. Let \( \Phi \) be a divided power automorphism of \( A \) respecting the system of divided powers, in the sense that \( (\Phi f)(x) = \Phi f(x) \).

Taking \( A = O(m) \) and \( M = O_2 \), we can let \( f \mapsto f^{(r)} \) be determined by \( x^{(r)} = x^r \). Let \( \mu \) be a divided power automorphism of \( O(m) \). Then

\[
\Phi \mu : W(m) \to W(m), \quad D \mapsto \mu \circ D \circ \mu^{-1}
\]

is an automorphism of \( W(m) \), and in fact every automorphism of \( W(m) \) can be obtained in this way. Similarly, all automorphisms of \( W(m; n) \) can be obtained from divided power automorphisms of \( O(m; n) \).

The ideal \( O_{21} \) is invariant under all divided power automorphisms. Therefore, its images under \( f \mapsto f^{(0)} \) are invariant under all divided power automorphisms as well. These powers are \( (O_{21})^{(r)} = O_{2r} \). From this it can be seen that the subalgebras \( W_{2r} \) are invariant under all \( W \)-automorphisms. Again, their intersections with \( W(m; n) \) are also invariant under all automorphisms of \( W(m; n) \).

This invariant filtration is a very strong property. We can use it to define a series of invariant subgroups of \( \text{Aut} W(m) \), as follows. For \( i \in \mathbb{N}_0 \), define

\[
\text{Aut}_i W(m) = \langle \Phi \mu \in \text{Aut} W(m) \mid \forall j \geq 1 \left( (\Phi \mu - 1)(W_{2j}) \subseteq W_{2j+i} \right) \rangle_{Gp}.
\]

A similar definition can be used to define \( \text{Aut}_i W(m; n) \). Since the filtration is invariant under all automorphisms, \( \text{Aut} W(m) = \text{Aut}_0 W(m) \). We see that the automorphism group has a large unipotent part, at least for \( W(m; n) \): every element of \( \text{Aut}_1 W(m; n) \) is unipotent.

\( W(m) \) is a simple Lie algebra, and so is \( W(m; n) \); unless \( p = 2 \) and \( m = 1 \), in which case \( W(m; n)^{(1)} := [W(m; n), W(m; n)] \) is simple. Some other subalgebras of \( W(m) \) are also simple.

**Definition 1.13.** Let \( A \) be a commutative unital ring and \( M \) a maximal ideal of \( A \). A **system of divided powers** is a sequence of maps \( M \to A, f \mapsto f^{(r)} \) for \( r \in \mathbb{N}_0 \), satisfying for \( r, s \in \mathbb{N}_0, t \in \mathbb{N}^+ \), \( f, g \in M, a \in A \):

\[
f^{(0)} = 1, \quad f^{(1)} = f, \quad f^{(r)} f^{(s)} = \frac{(r + s)!}{r! s!} f^{(r+s)}, \quad (f + g)^{(r)} = \sum_{l=0}^{r} f^{(l)} g^{(r-l)}, \quad (af)^{(r)} = a^r f^{(r)}, \quad (f^{(r)})^{(s)} = \frac{(s)!}{s!(r)!} f^{(rs)}.
\]

**A divided power automorphism** of \( A \) is an automorphism \( \phi \) respecting the system of divided powers, in the sense that \( (\phi f)^{(r)} = (f^{(r)})^{(r)} \).

Let \( \Phi \mu : W(m) \to W(m), \quad D \mapsto \mu \circ D \circ \mu^{-1} \) (1.2)

is an automorphism of \( W(m) \), and in fact every automorphism of \( W(m) \) can be obtained in this way. Similarly, all automorphisms of \( W(m; n) \) can be obtained from divided power automorphisms of \( O(m; n) \).

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\[
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\]

A similar definition can be used to define \( \text{Aut}_i W(m; n) \). Since the filtration is invariant under all automorphisms, \( \text{Aut} W(m) = \text{Aut}_0 W(m) \). We see that the automorphism group has a large unipotent part, at least for \( W(m; n) \): every element of \( \text{Aut}_1 W(m; n) \) is unipotent.

\( W(m) \) is a simple Lie algebra, and so is \( W(m; n) \); unless \( p = 2 \) and \( m = 1 \), in which case \( W(m; n)^{(1)} := [W(m; n), W(m; n)] \) is simple. Some other subalgebras of \( W(m) \) are also simple.

**Definition 1.14.** Define \( D_{ij}(f) = \partial_i(f) \partial_j - \partial_j(f) \partial_i \) for \( f \in O(m) \). Let

\[
S(m; n)^{(1)} := \langle D_{ij}(f) \mid 1 \leq i < j \leq m, f \in O(m; n) \rangle_F
\]

be the **special algebra**. It is simple of dimension \( (m - 1)(p^m - 1) \).

Suppose \( m > 1 \). Let \( r \in \mathbb{N}_0 \), such that \( m = 2r \) or \( m = 2r + 1 \) and define for \( 1 \leq j \leq 2r \):

\[
\sigma_j := \begin{cases} 1, & \text{if } j \leq r, \\ -1, & \text{if } r < j \leq 2r \\ \end{cases}, \quad j' := \begin{cases} j + r, & \text{if } j \leq r, \\ j - r, & \text{if } r < j \leq 2r. \\ \end{cases}
\]
If $m = 2r$, let

$$D_H(f) := \sum_{i=1}^{2r} \sigma_i \partial_i(f) \partial_i.$$ 

Then

$$H(2r;r)^{(2)} := \langle D_H(x^a) \mid 0 \leq a_i < p^{|a|}, \text{not all } a_i \text{ equal to } p^{|a|} - 1 \rangle_F$$

is known as the Hamiltonian algebra. It is simple of dimension $p^{|a|} - 2$.

If $m = 2r + 1$, then for $f \in \mathcal{O}(m)$ define

$$D_K(f) := \sum_{i=1}^{2r} (\sigma_i \partial_i(f) + x_r \partial_{2r+1}(f)) \partial_i + (2f - \sum_{i=1}^{2r} x_i \partial_i(f)) \partial_{2r+1}.$$ 

The Lie algebra

$$K(2r + 1; n) := D_K(\mathcal{O}(2r + 1; n))$$

is known as the contact algebra. It is simple of dimension $p^{|a|}$, unless $p \mid 2r - 2 = m - 3$.

In that case, its derived subalgebra

$$K(2r + 1; n)^{(1)} := \langle D_K(x^a) \mid 0 \leq a_i < p^{|a|}, \text{not all } a_i \text{ equal to } p^{|a|} - 1 \rangle_F$$

is simple and has dimension $p^{|a|} - 1$. This algebra is also referred to as the contact algebra. (If $p \nmid 2r - 2$, then $K(2r + 1; n)^{(1)} = K(2r + 1; n)$.)

These algebras are called the Cartan type Lie algebras. They have a natural definition by forms that are well-known from differential geometry. For example, the special algebra can also be defined as the subalgebra of $\mathcal{W}(m; n)$ consisting of the elements that vanish on the volume form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$.

The classification result that Premet and Strade will achieve in their series of papers referred to in the beginning of this section will state that every finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 5$ is either of Chevalley type, or of one of the four types referred to above, or it can be obtained from one of these types by a construction involving an automorphism of $\mathcal{O}(m)$ (we will not encounter this construction in the present thesis); and that for $p = 5$ there is one additional family of Lie algebras called the Melikyan Lie algebras. This family is the subject of Chapter 3.

The following notion often gives a lot of extra structure and information for the Cartan type Lie algebras.

**Definition 1.15.** A pair $(\mathcal{L}, [p])$ of a Lie algebra $\mathcal{L}$ over a field $F$ of characteristic $p > 0$ and a map $[p]: \mathcal{L} \to \mathcal{L}$, $x \mapsto x^{[p]}$ is called a restricted Lie algebra if

- $(\text{ad} a)^p = \text{ad} a^{[p]}$, for all $a \in \mathcal{L}$,
- $(\text{ad} a)^p = a^p a^{[p]}$, for all $a \in \mathcal{F}$ and $a \in \mathcal{L}$, and
- $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$, where $s_i(a, b)$ can be computed by the formula

$$\text{(ad}(a \otimes X + b \otimes 1)^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} is_i(a, b) \otimes X^{i-1}$$

in the Lie algebra $\mathcal{L} \otimes F[X]$ over the polynomial ring $F[X]$.

A Lie algebra $\mathcal{L}$ over $\mathcal{F}$ is called restrictive if for all $a \in \mathcal{L}$ we have $(\text{ad} a)^p \in \text{ad} \mathcal{L}$.

It turns out that for all restrictive Lie algebras $\mathcal{L}$ there exists a $[p]$ such that $(\mathcal{L}, [p])$ is restricted. This is a theorem by Jacobson [Jac37], who defined the notions of restricted-ness and restrictability.
Graphs for Classical Lie Algebras

2.1. Introduction

A nonzero element $x$ of a Lie algebra $L$ over a field $F$ is called extremal if $[x, [x, L]] \subseteq Fx$. Extremal elements are a well-studied class of elements in simple finite-dimensional Lie algebras of Chevalley type: they are the long root elements. In [CSUW01], Cohen, Steinbach, Ushirobira and Wales have studied Lie algebras generated by extremal elements, in particular those of Chevalley type. The authors also find the minimum size of a set of generating extremal elements for the Lie algebras of Chevalley type and find such minimal generating sets of extremal elements explicitly. In the present chapter, we also find such minimal generating sets of extremal elements explicitly for the four classical families of Lie algebras: those of type $A_n$, $B_n$, $C_n$ and $D_n$. We will do this in a more geometrical setting and will find criteria for sets of extremal elements to generate Lie algebras of this type.

By Lemma 2.5, each Lie algebra generated by a pair of linearly independent extremal elements is in one of only three isomorphism classes: either the two-dimensional commutative Lie algebra, or the so-called Heisenberg Lie algebra $\mathfrak{h}$, or $\mathfrak{sl}_2$. Given a generating set $S$ of extremal elements, we examine the subalgebras generated by pairs of these elements. These give rise to graphs: the vertices correspond to the elements of $S$, and two vertices are adjacent if the corresponding extremal elements generate a three-dimensional algebra and nonadjacent if they commute. We find one such graph for each Lie algebra of classical Chevalley type, depicted in Figures 2.1 up to 2.4, and show that if a Lie algebra realizes one of these graphs, then it is “almost always” isomorphic to the Lie algebra of the corresponding Chevalley type. To be somewhat more precise, given such a graph, we identify a parameter space such that the multiplication table of any Lie algebra realizing the graph is determined by an element of the parameter space. We show that the parameter values that lead to a Lie algebra of maximal dimension among those defined with the same graph, form an algebraic variety, and furthermore, that the subset of that space leading to Lie algebras isomorphic to those of Chevalley types contains an open dense subset of the full parameter space. In particular, there is a finite number of polynomial equations on the parameters such that if none of these equations is satisfied, the Lie algebra is isomorphic to the Chevalley type Lie algebra.
In the rest of this section we will introduce some conventions and notation that we will use in this chapter. In Section 2.2 we review some of the underlying theory. In Section 2.3 we show how to work with abstract Lie algebras that realize a given graph, which we apply to four proposed graphs for the classical Chevalley types in Section 2.4. In Section 2.5, we extend this study to the parameter space referred to above. In Section 2.6, we give concrete realizations of the Lie algebras of types $A_n$, $B_n$, $C_n$ and $D_n$ corresponding to the graphs of the previous section. Finally, in Section 2.7 we prove the main results of this chapter: Theorems 2.57, 2.59, 2.60 and 2.61, which state that the abstract Lie algebras from Section 2.4 are, in the general case, isomorphic to the concrete realizations from Section 2.6.

This chapter was inspired by the Masters thesis of Dan Roozemond [Roo05] and discussions with Arjeh Cohen, Hans Cuypers, Jan Draisma, Jos in 't panhuis and Dan Roozemond. Jos contributed to most of the proofs of Theorem 2.16 and Lemma 2.22 and their equivalents for type $B_n$.

2.1.1. Conventions and notation. For the rest of this chapter, $\mathbb{F}$ will be an algebraically closed field of characteristic not 2 and $L$ will be a Lie algebra over $\mathbb{F}$.

Since we approach the matter from the angle of the generating sets of abstract extremal elements, we let $n$ be the number of generating extremal elements. In Section 2.6, this will mean that we study, for example, the Lie algebra of type $C_{n/2}$, defined over a vector space of dimension $n$, where we have to assume that $n$ is even. In that section, it would be more convenient to study the Lie algebra of type $C_n$ instead, but we choose consistency over convenience and keep the meaning of $n$ as the number of extremal generators.

**Notation 2.1.** If no confusion is possible, we write $xy$ for $[x, y]$, and $xyz$ for $[x, [y, z]]$; we will write $(xy)z$ for $[[x, y], z]$. So, anticommutativity and the Jacobi identity will be written as

$$xx = 0 \quad \text{(AC)}$$

and

$$xyz + yzx + zxy = 0. \quad \text{(J)}$$

**Notation 2.2.** We will often work with long products of indexed elements. We use the following notation to make these products somewhat manageable. The general idea is that we put two numbers in the subscript with an operator consisting of one or two arrows in between, such as $x_{\uparrow 5 \downarrow 8 z}$; the first factor in the product is then indexed...
by the first number, after which we iterate adding (for up arrows) or subtracting (for down arrows) one (for single stroke arrows) or two (for double stroke arrows) to the index until we encounter the last number, where every step gives the next factor for this product. So the previous example $x_{5|3|2}$ is short for $x_5x_3x_4x_2$.

In particular, there are four operators that we use, defined more precisely as follows. If $i \leq j$, the notation $x_{ij}$ will mean $x_ix_{j+1}x_{j+2} \cdots x_{j-1}x_j$, and $x_{ji}$ will mean $x_jx_{j-1}x_{j-2} \cdots x_{i+1}x_i$. Furthermore, $x_{ij\bar i}$ will mean $x_jx_{j+1}x_{j-1}x_{j-2}x_{j-3} \cdots x_{i+1}x_i$, and similarly, $x_{ji\bar i}$ will mean $x_i x_{i-1} x_{i-2} \cdots x_{j+1}x_j x_{j+2}x_{j+3} \cdots x_{j-1}x_{j-2}$.

We will also use constructions such as $x_{3|7|6\underline {4|1}}$, which will mean the Lie product $x_3x_7x_6x_4x_5x_3x_4x_2x_3x_1$. Occasionally, it will be convenient to include in a set of monomials of the form, say, $x_{jj}x_{i-2}$ the case $j = i - 1$; this monomial will then simply be $x_{i-2}$. So in this case $x_{jj}$ cannot be seen as a separate monomial.

We extend the notation to cover the case where we have a sequence $i_1, \ldots, i_k$ of indices: then we write $x_{i_ki_1}$ for $x_{i_k}x_{i_{k-1}} \cdots x_{i_2}x_{i_1}$.

**Definition 2.3.** We say that a linearly independent set of Lie algebra elements $\{M_i \mid i \in V\}$ realizes a given graph $\Gamma = (V, E)$ if:
- each $M_i$ is an extremal element of $\langle M_j \mid j \in I \rangle_{\text{Lie}}$;
- vertices $i$ and $j$ are connected if and only if $M_i$ and $M_j$ do not commute.

We will sometimes also say that the Lie algebra $\langle M_i \rangle_{\text{Lie}}$ realizes $\Gamma$.

### 2.2. Preliminaries

In this section, we will introduce a bilinear form defined on all Lie algebras generated by extremal elements, and recall some of its properties. None of these results are new; most can be found in e.g. [CSUW01]. We will start by introducing a related family of linear functionals.

For extremal $x$, let $f_x : \mathcal{L} \to \mathbb{F}$ be the linear map defined by $xxy = f_x(y)x$. Since $\langle \cdot, \cdot \rangle$ is bilinear, this is indeed a linear map. We call $f_x$ the extremal functional on $x$.

**Lemma 2.4.** $f_x(y) = f_y(x)$ for all extremal $x$ and $y$.

**Proof.** If $x$ and $y$ commute, the statement is clearly true. Otherwise, $yxy = [y, f_x(y)x] = f_y(x)yx$ and $yxy = (yx)y + xyxy = -(yx)y - xyxy = -[x, f_y(y)x] = f_y(x)yx$. These two equalities imply the required equality if $xy \neq 0$. \hfill $\square$

**Lemma 2.5.** Let $\mathcal{L} = \langle x, y \rangle_{\text{Lie}}$ with $x$ and $y$ extremal and linearly independent. Then $\mathcal{L}$ is isomorphic to the two-dimensional commutative Lie algebra, the Heisenberg algebra, or $\mathfrak{sl}_2$.

**Proof.** If $x$ and $y$ commute, then $\mathcal{L}$ is two-dimensional. Otherwise, $xxy \in \mathbb{F}x$ and $yxy = -yxy \in \mathbb{F}y$, so dim $\mathcal{L} = 3$. Then it can be easily seen that the value of $f_x(y)$ determines the multiplication table of $\mathcal{L}$. If $f_x(y) = 0$, then $\mathcal{L}$ is isomorphic to the Heisenberg Lie algebra. Otherwise, mapping $\left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$ to $2f_x(y)^{-1}x$ and $\left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ to $y$ induces an isomorphism from $\mathfrak{sl}_2$ to $\mathcal{L}$, as can again be seen from the multiplication table. \hfill $\square$

**Lemma 2.6.** If $\mathcal{L}$ is generated by extremal elements, then it is linearly spanned by extremal elements.
Proof. If $x$ is an extremal element, then $\exp(t \text{ad } x)$ is an automorphism. This is clear by Eq. (1.1) for characteristic 0 or $p \geq 5$, but for characteristic 3 we need this computation:

$$\exp(t \text{ad } x)yz = \sum_{i=0}^{2} \frac{t^i}{i!} (\text{ad } x)^i yz = \sum_{i=0}^{2} \frac{t^i}{i!} \sum_{j=0}^{i} \left(\frac{i}{j}\right) [(\text{ad } x)^j y, (\text{ad } x)^{i-j} z]$$

$$= \sum_{i=0}^{2} \sum_{j=0}^{i} \frac{t^{i+j}}{i! j!} [(\text{ad } x)^j y, (\text{ad } x)^{i-j} z]$$

$$= [\exp(t \text{ad } x)y, \exp(t \text{ad } x)z]$$

$$- \frac{t^3}{2} [(\text{ad } x)^2 y, xz] + [xy, (\text{ad } x)^2 z] + \frac{t^2}{2} [(\text{ad } x)^2 y, (\text{ad } x)^2 z],$$

and that last sum of three terms is equal to

$$[f(x, y)x, xz] + [xy, f(x, z)x] + \frac{t}{2} [f(x, y)x, f(x, z)x]$$

$$= f(x, y)f(x, z)(x - x + [x, x]) = 0.$$  

Hence, if $x$ and $y$ are extremal, then $\exp(\text{ad } x)y = y + yx + \frac{t}{2} f_x(y)x$ is also an extremal element, so $xy$ is in the linear span of the extremal elements in $L$. This allows us to rewrite any product of extremal elements as a linear combination of extremal elements.  

Lemma 2.7. If $L$ is generated by extremal elements, the definition of $f_x(y)$ can be extended to a unique bilinear form $f(x, y)$ on $L$ with $f(x, y) = f_x(y)$ if $x$ is an extremal element.

Proof. We clearly have $f_{\alpha x}(y) = \alpha f_x(y)$ and linearity of $f_x(y)$ in $y$. By Lemma 2.6, $L$ has a basis consisting of extremal elements, say $\{x_i\}$, so there is at most one extension to a bilinear form. Let $\sum_{i \in I} y_i = \sum_{j \in J} y_j = x$ be different decompositions of $x$ as a sum of extremal elements. We need to show that $\sum_{i \in I} f_{y_i} = \sum_{j \in J} f_{y_j}$ or, in other words, $\sum_{i \in I \Delta J} f_{y_i} = 0$, where $\Delta$ denotes the symmetric difference operator, and $f_{y_j} = \pm f_{y_j}$ depending on whether $j \in J \setminus I$ or $j \in I \setminus J$. It suffices to show this for the case where $I = 0$. So let $\{y_i \mid i \in I\}$ be a set of extremal elements with $\sum y_i = 0$, then

$$\sum_i f_{y_i}(x_k) = \sum_i f_{x_k}(y_i) = f_{x_k}(0) = 0$$

for all $k$. Hence $\sum_i f_{y_i} = 0$.  

We call $f$ the extremal form. In this notation, Lemma 2.4 generalizes to the statement that if $L$ is generated by extremal elements, then $f$ is symmetric:

$$\forall x, y: f(x, y) = f(y, x).$$  

(SM)

We will use the following identities involving the extremal form, the first two of which go back to Premet and were first used in [Che89]:

Lemma 2.8. If $x$ is extremal and $y, z \in L$, then

$$2(xy)xz = f(x, yz)x + f(x, z)xy - f(x, y)xz,$$

(P1)

$$2xyz = f(x, yz)x + f(x, z)xy - f(x, y)xz,$$

(P2)

$$f(x, yxz) = -f(x, z)f(x, y).$$

(P3)

Proof. By the Jacobi identity,

$$(xy)xz = ((xy)x)z + x(xy)z = -f(x, y)xz + x(xy)z,$$
and similarly,

\[(xy)zx \overset{\text{(AC)}}{=} -(xz)xy \overset{\text{(I)}}{=} -((xz)x)y - x(xz)y \overset{\text{(I)}}{=} f(x,z)xy - xxz y - x(xy)z.\]

Adding these two equations and applying anti-commutativity a few times, we obtain Eq. (P1). Then we find Eq. (P2) as follows:

\[2xyz \overset{\text{(I)}}{=} 2(xy)xz + 2yxzx \overset{\text{(P1)}}{=} f(x,yz)x + f(x,z)xy - f(x,y)xz - 2f(x,z)xy.\]

For Eq. (P3), we need the next lemma. The equation is then easily obtained as follows:

\[f(x,yxz) \overset{\text{(AS)}}{=} f(xy,xz) \overset{\text{(AC)}}{=} -f(yx,xz) \overset{\text{(AS)}}{=} -f(y,xxz) = -f(x,y)f(x,z).\]

**Lemma 2.9.** If \(L\) is generated by extremal elements, then

\[f(x,yz) = f(xy,z)\]  \hspace{1cm} \text{(AS)}

for all \(x, y, z \in L\).

This property is often called *associativity*.

**Proof.** Let \(x, y\) and \(z\) be extremal elements of \(L\). On the one hand,

\[2(xz)zx \overset{\text{(AC)}}{=} -2(xz)xy \overset{\text{(P1)}}{=} -f(z,xxz)x - f(z,yx)zx + f(z,x)zxz \]

\[= -f(x,y)f(x,z)x + f(x,y)xz + f(z,x)zxz;\]

on the other,

\[2(xz)zx \overset{\text{(I)}}{=} 2((xz)z)x + 2z(xz)xy \overset{\text{(P1)}}{=} 2f(x,z)zx + z(f(x,zy)x + f(x,y)xz - f(x,z)xy) \]

\[= f(x,y)zx - f(x,y)f(x,z)x + f(x,z)zxz.\]

Hence \(f(xy,z)zx = f(x,yz)xz\). If \(xz \neq 0\), then we conclude that \(f(xy,z) = f(x,yz)\). Similarly, if both \(yz\) and \(xy\) are nonzero, then \(f(xy,z) = f(z,xy) = f(x,y) = f(y,zx) = f(yz,x) = f(x,yz)\), where we alternately use the symmetry of \(f\) and the rule just discovered. So we are left with the case where \(xz\) and, say, \(xy\) are both zero. Then \(f(xy,z) = 0\) and also \(f(x,y)z = zyxy \overset{\text{(I)}}{=} x(yz)z + xyxz = 0\). Lemma 2.6 finishes the proof. \(\square\)

**2.3. The general framework**

In this section, we will establish a framework for dealing with Lie algebras generated by extremal elements where we prescribe the values of the extremal form. In the end, we prove Theorem 2.14 which shows that a certain parameter space for these prescribed values is an algebraic variety. The main objective of this section is to introduce the techniques for proving that theorem. In Section 2.5 we will use those techniques to prove similar theorems for a substantially smaller parameter space, but then for specific Lie algebra families.

Let \(n \in \mathbb{N}_+\) be fixed. Let \(\Gamma\) be a graph on \(n\) numbered vertices. Let \(\mathcal{F}\) be the free Lie algebra over \(\mathbb{F}\) on \(n\) generators \(x_1, \ldots, x_n\) with the standard grading. We will construct a quotient of \(\mathcal{F}\) where the projections of \(x_i\) in the quotient are extremal generators. Let

\[\mathcal{F}_\Gamma = \mathcal{F}/\langle x_i x_j \mid [i,j] \notin E(\Gamma) \rangle_{\text{Id}}.\]

\(\mathcal{F}_\Gamma\) inherits the grading of \(\mathcal{F}\); this is possible because the ideal that is divided out is *homogeneous* with respect to the grading of \(\mathcal{F}\), in the sense that it is spanned by its intersections with the homogeneous components of \(\mathcal{F}\).
Let \( \mathfrak{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_n) \) be an element of \((\mathcal{F}_T)^n\), so it consists of \( n \) functionals in the dual of \( \mathcal{F}_T \); we will make sure that \( \mathfrak{f}_i \) is the extremal functional \( f_{x_i} \) in the Lie algebra we will construct. To that end, define the ideal

\[
I_{\mathfrak{f}, \mathfrak{f}} = \langle x_i x_j y - \mathfrak{f}_i(y) x_i | y, x_i \in \mathcal{F}_T, 1 \leq i \leq n \rangle_{\text{id}}.
\]

When taking \( \mathfrak{f} = 0 \), we see that \( I_{\mathfrak{f}, 0} \) is homogeneous with respect to the standard grading of \( \mathcal{F}_T \). Let \( \mathcal{L}_{\mathfrak{f}, \mathfrak{f}} = \mathcal{F}_T/I_{\mathfrak{f}, \mathfrak{f}} \) and let \( \xi_{\mathfrak{f}} : \mathcal{F}_T \to \mathcal{L}_{\mathfrak{f}, \mathfrak{f}} \) be the natural projection. We will sometimes omit \( \xi_{\mathfrak{f}} \) if that does not stand in the way of clarity. Clearly each \( x_i \) is either an extremal element of \( \mathcal{L}_{\mathfrak{f}, \mathfrak{f}} \) or zero, and \( \mathfrak{f}_i(y) = f(x_i, y) \). We find the following slight extension of Lemma 4.3 of [Zel80]:

**Lemma 2.10.** There is a finite list \( \mathcal{M}_F \) of monomials in \( x_1, \ldots, x_n \) satisfying the following properties:

1. \( \xi_0(\mathcal{M}_F) \) is a basis of \( \mathcal{L}_{\mathfrak{f}, 0} \).
2. If \( x, m \in \mathcal{M}_F \), then \( m \in \mathcal{M}_F \).
3. \( \mathcal{M}_F \) contains all generators \( x_i \), and
4. \( \mathcal{L}_{\mathfrak{f}, \mathfrak{f}} = \langle \xi_0(\mathcal{M}_F) \rangle_{\mathfrak{f}} \) for all \( \mathfrak{f} \in (\mathcal{F}_T)^n \).

**Proof.** \( \mathcal{L}_{\mathfrak{f}, 0} \) is a quotient of \( \mathcal{L}_n := \mathcal{L}_{K_n, 0} \), where \( K_n \) is the complete graph. By Theorem 1 of Zel’manov [Zel80] (or for characteristic 3, by Theorem 1 of Zel’manov and Kostrikin [ZK90]), we know that \( \mathcal{L}_n \) is finite-dimensional. Hence we can find a finite set \( \mathcal{M}_F \) of monomials in \( \mathcal{F}_T \) satisfying conditions 1, 2, and 3, by the following procedure. We start by setting \( \mathcal{M}_F \) equal to \( \{x_1, \ldots, x_n\} \). This set is linearly independent because the free Abelian Lie algebra on \( n \) generators is a quotient of \( \mathcal{L}_{\mathfrak{f}, 0} \), and the images of the \( x_i \) in it are linearly independent. Then we perform a number of rounds as follows. In each round we form the monomials \( x_i m \), where \( x_i \) iterates over the generators of \( \mathcal{F}_T \) and \( m \) iterates over the longest monomials in \( \mathcal{M}_F \) so far. We select a subset of these such that its images under \( \xi_0 \) in \( \mathcal{L}_{\mathfrak{f}, 0} \) are linearly independent of each other and of the images under \( \xi_0 \) of the elements in \( \mathcal{M}_F \) so far, and add it to \( \mathcal{M}_F \). Then we continue with the next round if we have added any new monomials this round. Since \( \mathcal{L}_{\mathfrak{f}, 0} \) is finite-dimensional, this procedure terminates after finitely many steps.

Let \( U = \langle \mathcal{M}_F \rangle_{\mathfrak{f}} \subset \mathcal{F}_T \). We now prove condition 4 by showing that \( \xi_{\mathfrak{f}}(U) = \mathcal{L}_{\mathfrak{f}, \mathfrak{f}} \) for all \( \mathfrak{f} \in (\mathcal{F}_T)^n \). Note that \( I_{\mathfrak{f}, 0} \) is spanned by elements of the form \( x_i^{m_{i1}} x_i u \), with \( u \) a monomial in \( \mathcal{F}_T \).

Clearly \( \xi_{\mathfrak{f}}(U) \subset \mathcal{L}_{\mathfrak{f}, \mathfrak{f}} \). Suppose that it is a proper subset; then there are monomials \( g \in \mathcal{F}_T \) such that \( \xi_{\mathfrak{f}}(g) \notin \xi_{\mathfrak{f}}(U) \), whence \( g \notin U \). Let \( g \) be such a monomial of lowest degree. Since \( \mathcal{F}_T = U + I_{\mathfrak{f}, 0} \), it is possible to express \( g \) as a linear combination of monomials in \( \mathcal{M}_F \) and monomials of the form \( x_{k_{i1}}^{m_{i1}} u \). All these monomials have the same degree, because only the Jacobi identity and anticommutativity can be used for rewriting, in addition to homogeneous elements being zero. Let \( T \) be a monomial of the form \( x_{k_{i1}}^{m_{i1}} u \) such that \( \xi_{\mathfrak{f}}(T) \notin \xi_{\mathfrak{f}}(U) \) and let \( T_0 = x_{k_{i1}}^{m_{i1}} u \). Then

\[
\mathfrak{f}_{i_1}(u) \xi_{\mathfrak{f}}(T_0) = \xi_{\mathfrak{f}}(T) \notin \xi_{\mathfrak{f}}(U),
\]

so \( \mathfrak{f}_{i_1}(u) \neq 0 \) and \( T_0 \notin U \). Since \( \deg T_0 < \deg T = \deg g \), we have a contradiction. Hence \( \xi_{\mathfrak{f}}(U) = \mathcal{L}_{\mathfrak{f}, \mathfrak{f}} \). □

Define \( U = \langle \mathcal{M}_F \rangle_{\mathfrak{f}} \) as in the preceding proof. Note that \( \mathcal{F}_T = U + I_{\mathfrak{f}, \mathfrak{f}} \) for all \( \mathfrak{f} \), not just for \( \mathfrak{f} = 0 \). Define \( I \) and \( m_i \) by letting \( \mathcal{M}_F = \{m_i | i \in I\} \).

**Lemma 2.11.** For every monomial \( m \in \mathcal{F}_T \), there exists a map \( n_m : (\mathcal{F}_T)^n \to U \), such that \( n_m(\mathfrak{f}) = m \pmod{I_{\mathfrak{f}, \mathfrak{f}}} \) for all \( \mathfrak{f} \in (\mathcal{F}_T)^n \) and the following property holds. If \( n_m(\mathfrak{f}) = \sum_{i \in I} \alpha_{m, i, \mathfrak{f}} m_i \),
Then \( \alpha_{m,i,j} \), when regarded as a function in \( i \) and \( j \), is a polynomial function in the values of \( f \) at monomials of degree less than \( \deg m \).

**Proof.** Let \( m = x_{i_1} \) be a monomial in \( F_\Gamma \) of degree \( k \). If \( k = 1 \), we put \( n_m(f) = m \). We proceed by induction on \( \deg m \). Since \( F_\Gamma = U + L_\Gamma,0 \), we can write \( m \) as the sum of an element \( u \) of \( U \) and an element \( w \) of \( L_\Gamma,0 \); all monomials involved have the same degree, because only the Jacobi identity and anticommutativity can be used for rewriting, in addition to homogeneous elements being 0. If we prove that \( n_u(f) = w \) (mod \( I_{\Gamma,f} \)) and that its coefficients \( \alpha_{m,i,j} \) satisfy the polynomiality condition, then setting

\[
n_m(f) = u + n_w(f)
\]

will be sufficient to show that the lemma holds for \( m \).

We may assume that \( w \) is a single monomial. So the proof obligation reduces to the case where \( m \in L_{\Gamma,0} \). Then there exist \( r, h \in \mathbb{N}_+ \) such that \( i_r = i_{r-1} = h \) and \( m \) is thus of the form \( x_{i_{r-1}} x_h x_{i_{r-2}1} \). Hence, by the induction hypothesis,

\[
m = f_h(x_{i_{r-2}1}) x_h x_{i_r} = f_h(x_{i_{r-2}1}) n_{x_{i_r}}(f) \pmod{I_{\Gamma,f}}.
\]

We choose

\[
n_m(f) = f_h(x_{i_{r-2}1}) n_{x_{i_r}}(f),
\]

so that

\[
\alpha_{m,i,j} = f_h(x_{i_{r-2}1}) \alpha_{x_{i_r},i,j}.
\]

The coefficients \( \alpha_{x_{i_r},i,j} \) are, by the induction hypothesis, polynomials in the values of \( f \) at monomials of degree less than \( k - r + 1 < k = \deg m \), so equations (2.1) and (2.2) define a map satisfying the conditions in the lemma. \( \square \)

Note that we do not claim that \( n_m \) is uniquely determined by these conditions. We choose a map \( n_r \) as above and extend it to general elements of \( F_\Gamma \) by linearity.

Let \( X_\Gamma = \{ f \mid \dim L_{\Gamma,f} = |M_{\Gamma}| \} \) and let \( R: X_\Gamma \to (U')^n \) be the map that restricts a functional to \( U \).

**Lemma 2.12.** The restriction map \( R \) is injective.

**Proof.** Let \( f \in X_\Gamma \). Then all \( m_i \in M_{\Gamma} \) are linearly independent in \( L_{\Gamma,i} \), so \( x_i \notin I_{\Gamma,j} \). Let \( m \) be a monomial in \( F_\Gamma \). We will show that \( f_i(m) \) can be expressed in the values of \( f_i \) on monomials in \( M \). If \( m \in M \), there is nothing to prove, so assume \( m \notin M \). Since

\[
x_i x_i m = f_i(m) x_i \pmod{I_{\Gamma,f}},
\]

and also

\[
x_i x_i m = x_i x_i n_m(f) = f_i(n_m(f)) x_i \pmod{I_{\Gamma,f}},
\]

we find that \( f_i(m) = f_i(n_m(f)) \). Since \( n_m(f) \) only depends on monomials of lower degree than \( m \), we see that \( f_i(m) \) can be expressed in the values of \( f_i \) at monomials of lower degree than \( m \). By induction on the degree of \( m \), it can therefore be expressed ultimately in the values of \( f_i \) on \( M \), as we set out to prove.

Let \( f, f' \in X_\Gamma \) with \( R(f) = R(f') \). Then \( f_i \) and \( f_i' \) agree on \( U \), and thus on \( F_\Gamma \), for all \( i \). Hence \( f = f' \). \( \square \)

**Lemma 2.13.** \( R(X_\Gamma) \) is a closed subset of \((U')^n\).

**Proof.** For all \( f \in (F_\Gamma^*)^n \), let the bilinear anticommutative map \([\cdot, \cdot]_f: U \times U \to U \) be determined by

\[
[v, w]_f = n_{[v,w]}(f).
\]

If \( f \in X_\Gamma \), then
(1) $[\cdot, \cdot]$ is a Lie multiplication (i.e. it satisfies the Jacobi identity),
(2) $[x_i, x_i, v]_f = f(v)x_i$ for all $v \in U$ and all $i$,
(3) $[x_i, x_j]_f = 0$ if nodes $i$ and $j$ are not connected by a line,
(4) the Lie algebra $(U, [\cdot, \cdot])$ is generated by $x_1, \ldots, x_n$.

On the other hand, if all of the above conditions hold for a multiplication map $\mu$, then $(U, \mu)$ is a quotient of $L_{\Gamma, f}$ of the same dimension, and hence isomorphic to $L_{\Gamma, f}$. But these conditions are all polynomial in the values of $f$ on $U$: the Jacobi identity and conditions 2 and 3 are polynomial in a straightforward way, and condition 4 is always satisfied: for every $w = x_{i_1} \cdots x_{i_\ell} \in M_{\Gamma}$, we have $w = [x_{i_1}, [x_{i_2}, \cdots, [x_{i_\ell}, x_{i_1}]_f \cdots ]_f]_f$ since $n_w(f) = w$. So $R(X_{\Gamma})$ is given as the zero set of a set of polynomial equations; thus it is closed. □

Theorem 2.14. $X_{\Gamma}$ carries a natural structure of an affine variety.

Proof. The restriction map $R$ is a continuous bijection of $X_{\Gamma}$ with a Zariski closed subset of $(U')^n$. Clearly the restriction map $R$ is continuous. The preceding two lemmas show that it is injective and that its image is closed. □

2.4. The monomials

In Figures 2.1 up to 2.4 we defined four graphs, $\Gamma_{\text{A}; n}$, $\Gamma_{\text{B}; n}$, $\Gamma_{\text{C}; n}$ and $\Gamma_{\text{D}; n}$, to be used for $\Gamma$ in the framework of Section 2.3. In this section, we will construct the basis $M_{\Gamma}$ of $U$ explicitly for each such $\Gamma$, resulting in Theorems 2.16, 2.19, 2.20 and 2.21.

Clearly, each of the algebras $F_{\Gamma}$ is defined by a subset of the four following relations.

$$x_i x_j = 0 \quad \text{for all } i, j \text{ with } |i - j| > 1, \{1, 3\} \neq \{i, j\} \neq \{n - 2, n\}, \quad \text{(R1)}$$
$$x_1 x_3 = 0, \quad \text{(R2)}$$
$$x_{n-2} x_n = 0, \quad \text{(R3)}$$
$$x_{n-1} x_n = 0. \quad \text{(R4)}$$

We will use the following technical lemma:

Lemma 2.15. Let $a, b, n \in \mathbb{N}_+, i, j, k, \ell, m, i_1, \ldots, i_a, j_1, \ldots, j_b \in \{1, \ldots, n\}$ and $t, u \in L$. Let $\Gamma$ be one of the graphs $\Gamma_{\text{A}; n}$, $\Gamma_{\text{B}; n}$, $\Gamma_{\text{C}; n}$ and $\Gamma_{\text{D}; n}$, let $f \in (F_{\Gamma})^n$ and let $x_i$ be the standard generators of $L_{\Gamma, f}$. Furthermore, let $x_i |_{\Gamma}$ commute with $x_k |_{\Gamma}$ for all $p$ and $q$ and let $x_i |_{\Gamma}$ commute with $x_j |_{\Gamma}$. For Eq. (Q2) only, assume that $i < n - 2$. Then:
\[ x_j x_i t = x_i x_j t, \quad (Q1) \]
\[ x_i x_{i+1} x_{i+2} x_i t = \frac{1}{2} (f(x_{i+1} x_{i+1} x_{i+2} t) x_i - f(x_{i+1} x_{i+2} t) x_i x_{i+1} - f(x_{i+1} x_{i+2} t) x_i x_{i+1}), \quad (Q2) \]
\[ x_k x_i x_{m} x_k t = \frac{1}{2} (f(x_k, x_m t) x_k x_k + f(x_k, t) x_k x_{m} x_k - f(x_k, x_m) x_k x_k t + f(x_k, x_m x_k) x_k - f(x_k, x_m) x_k x_m t - f(x_k, x_{m} x_k) x_k t + f(x_k, x_{m} x_k) x_k x_k t + f(x_k, x_m) x_k x_{m} t - f(x_k, x_{m} x_k) x_k t + f(x_k, x_{m} x_k) x_k x_k t + f(x_k, x_m) x_k x_{m} t - f(x_k, x_{m} x_k) x_k t + f(x_k, x_{m} x_k) x_k x_k t), \quad (Q3) \]
\[ f(u, x_k x_m x_k t) = \frac{1}{2} (f(x_k, x_m t) f(u, x_m) + f(x_k, t) f(u, x_m x_k t) - f(x_k, x_m) f(u, x_m) t + f(x_k, x_m x_k) x_k t, \quad (Q4) \]
\[ x_i x_j x_j \cdots x_j = 0, \quad (Q5) \]
\[ (x_i, x_j, \cdots, x_j) x_i x_j \cdots x_i = 0, \quad (Q6) \]
\[ (x_i, x_j, \cdots, x_j) x_i t = x_i (x_i, x_j, \cdots, x_j) t. \quad (Q7) \]

**Proof.**  \[ (Q1) \] \[ x_j x_i t = x_i x_j t + (x_j x_i) t \quad (R1) \]
\[ (Q2) \]
\[ x_i x_{i+1} x_{i+2} x_i t \equiv x_i x_{i+1} x_{i+2} x_i t = \frac{1}{2} (f(x_{i+1} x_{i+1} x_{i+2} t) x_i - f(x_{i+1} x_{i+2} t) x_i x_{i+1} - f(x_{i+1} x_{i+2} t) x_i x_{i+1}). \quad (Q3) \]
\[ x_k x_i x_m x_k t \equiv x_i x_k x_m x_k t = (x_k x_m x_k) t = x_i x_k x_m x_k t + (x_k x_m) x_k x_m t + (x_k x_m) (x_m x_k) t \]
\[ = x_i x_k x_m x_k t + (x_k x_m) x_k x_m t + (x_k x_m) (x_m x_k) t \quad (AC) \]
\[ = x_i x_k x_m x_k t + (x_k x_m) x_k x_m t - (x_k x_m) x_k x_m t + (x_k x_m) x_k x_m t - (x_k x_m) x_k x_m t = x_i x_k x_m x_k t. \quad (AC) \]

Applying Eqs. (P1) and (P2) gives the desired result.

**Proof.** This follows directly from \[ (Q3) \] .

\[ (Q5) \] \[ x_i x_j, x_j \cdots x_j, x_i = x_j x_j x_j \cdots x_j = \cdots = x_j x_j \cdots x_j \quad (Q1) \]
\[ x_i x_j, x_j \cdots x_j, x_i = x_i x_j x_j \cdots x_j. \quad (Q6) \]

\[ (x_i, x_j, \cdots, x_j) x_i x_j \cdots x_i \equiv x_i (x_i, x_j \cdots x_j) x_j x_j \cdots x_i + (x_i, x_j, x_j \cdots x_j) x_j x_i \cdots x_i \]
\[ = x_i (x_i, x_j \cdots x_j) x_i x_j \cdots x_i \equiv \cdots \equiv x_i x_j \cdots x_i x_j x_j \cdots x_j x_i \equiv (AC) \]
\[ = - x_i x_j \cdots x_i x_j x_j \cdots x_j x_i. \quad (Q5) \]
2.4.1. Monomials for $\Gamma_{D_n}$. In the rest of this section, we will be considering $\mathcal{F}_{D_n}$, that is, only the relations in (R1) from page 22 are divided out. In Theorem 2.16, we will give a list of $2n^2 - n$ monomials in $x_1, \ldots, x_n$ and prove that they span $\mathcal{F}_{D_n}$ linearly.

**Theorem 2.16.** Let $\mathcal{M}_{\Gamma_{D_n}}$ be the set consisting of the following monomials:

$$
y_{k,m}^1 = x_{k\mid m}, \quad n \geq k \geq m \geq 1, \\
y_{k,m}^2 = x_{k\mid n-2x_{n\mid m}}, \quad n - 2 \geq k > m \geq 1, \\
y_{k,m}^3 = x_{k\mid m+1x_{m-1}\mid 1}, \quad n \geq k \geq m \geq 3, \\
y_{k,m}^4 = x_{k\mid n-2x_{n\mid m+1x_{m-1}\mid 1}}, \quad n - 2 \geq k \geq m \geq 3, \\
y_{m}^5 = x_{n\mid n-2\mid m}, \quad n - 2 \geq k \geq m \geq 1, \\
y_{m}^6 = x_{n-1x_{n\mid n-2\mid m}}, \quad n - 2 \geq k \geq m \geq 1, \\
y_{k}^7 = x_{k\mid 3x_{1}}, \quad n \geq k \geq 3, \\
y_{k}^8 = x_{k\mid n-2x_{n\mid 3x_{1}}}, \quad n - 2 \geq k \geq 2, \\
y_{m}^9 = x_{n\mid n-2\mid m+1x_{m-1}\mid 1}, \quad n - 2 \geq k \geq m \geq 3, \\
y_{m}^{10} = x_{n-1x_{n\mid n-2\mid m+1x_{m-1}\mid 1}}, \quad n - 2 \geq k \geq m \geq 3, \\
y_{1}^{11} = x_{1x_{3\mid n-2x_{n\mid 1}}}, \\
y_{1}^{12} = x_{1x_{3\mid n-2x_{n\mid 2}1}}, \\
y_{1}^{13} = x_{n\mid n-2\mid 3x_{1}}, \\
y_{1}^{14} = x_{n-1x_{n\mid n-2\mid 3x_{1}}}, \\
y_{1}^{15} = x_{n-2x_{n\mid n-3\mid 1}}, \\
y_{1}^{16} = x_{n-1x_{n-2\mid x_{n\mid n-3\mid 1}}}, \\
y_{1}^{17} = x_{n\mid n-1x_{n-2\mid x_{n\mid n-3\mid 1}}}.
$$

Let $x$ be a monomial in $x_1, \ldots, x_n$ of length $\ell$. Then $x$ is a linear combination of monomials $y \in \mathcal{M}_{\Gamma_{D_n}}$ with length($y$) $\leq \ell$.

For convenience in the proof of Theorem 2.16, we extend the above definition: we define $y_{n-1,m}^2 = y_{1,m}^1$ and $y_{n-1,m}^4 = y_{n,m}^3$, for $m \leq n - 1$; and $y_{2,2}^3 = x_1$, $y_{2}^7 = x_1$, and $y_{n-1}^8 = y_{n}^7$. Before giving the proof however, we will first prove two technical lemmas.

**Lemma 2.17.** Modulo shorter monomials the following holds for all $n > 5$ and all $i \in \{3, \ldots, n - 4\}$:

$$
v_i = (-1)^{i+1}y_{n-i-1,n-i-1}^4 \quad (2.3)
$$

$$
w_i = (-1)^{i+1}y_{n-i,n-i}^4 \quad (2.4)
$$

with:

$$
v_i = x_{n-i-1}(x_{n-i+1\mid n-2x_{n-3x_{n\mid n-2x_{n-1}}}y_{n-i,n-i-1}^3}), \\
w_i = x_{n-i}(x_{n-i-1\mid n-2x_{n-3x_{n\mid n-2x_{n-1}}}y_{n-i-1,n-i-1}^3}).$$
Proof. We will work modulo monomials shorter than $2n - 4$ (which is the length of all monomials in the equations to be proven) throughout this proof.

Let $i \in \{3, \ldots, n - 4\}$. For $v_i$ we have:

$$v_i = x_{n-i-1}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i,n-i-1}^3$$

(I) $v_i = x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i,n-i-1}^3$

$$- x_{n-i-1}x_{n-i+1}(x_{n-i}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i,n-i-1}^3$$

(P2) $v_i = x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i,n-i-1}^3$

(I) $v_i = x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i,n-i-1}^3$

now, by applying Eq. (Q6) to the part from its first parenthesis to its end, the first term is zero. Hence,

$$v_i = - x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})x_{n-i}y_{n-i-2,n-i-2}^3$$

(I) $v_i = - x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i-2,n-i-2}^3$

$$+ x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i-2,n-i-2}^3$$

(Q5) $v_i = - x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})x_{n-i}y_{n-i-3,n-i-3}$

(I) $v_i = - x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i-3,n-i-3}$

$$+ x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i-3,n-i-3}$$

(P2) $v_i = x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})y_{n-i-3,n-i-3}$

Applying these last four rewritings another $n - i - 5$ times using Eqs. (I), (Q5) and (P2) we get:

$$v_i = (-1)^{n-i-5}x_{n-i-1}x_{n-i}(x_{n-i+2}||n-2x_{n-3}x_{n-2}x_{n-1})x_1$$

$$= (-1)^{n-i}x_{n-i-1}x_{n-i}(x_{n-i+1}||n-2x_{n-3}x_{n-2}x_{n-1})x_1.$$
and if we do this $i - 3$ more times, we find, using Eq. (Q1) till the end, that

$$w_i = (-1)^{n-i} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

$$= (-1)^{n-i+1} x_{n-i-1} x_{n-i-2} x_{n-i-3} x_{n-i-4} x_{n-1}$$

Now consider $w_i$, for $i \in \{1, \ldots, n-5\}$:

$$w_i = x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

$$= x_{n-i} x_{n-i+1} x_{n-i+2} x_{n-i+3} x_{n-i+4} x_{n-1} y_{n-i-1} y_{n-i-2} y_{n-i-3} y_{n-i-4}$$

Applying this procedure another $n-i-4$ times using Eqs. (j), (P1) and (P2) we have:

$$w_i = (−1)^{n−i−3} x_{n−i−1} x_{n−i−2} x_{n−i−3} x_{n−i−4} x_{n−1} x_{1}$$

$$= (−1)^{n−i−3} x_{n−i−1} x_{n−i−2} x_{n−i−3} x_{n−i−4} x_{n−1} x_{1}$$
Now, using Eq. (Q1) many times, we see that

\[ w_i = (-1)^{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]
\[ = (-1)^{n-i}x_{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]
\[ = (-1)^{n-i}x_{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]
\[ = (-1)^{n-i+1}x_{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]
\[ = (-1)^{n-i+2}x_{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]
\[ = (-1)^{n-i+1}x_{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]
\[ = (-1)^{n-i+1}x_{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]

Finally,

\[ w_i = (-1)^i x_{n-i}x_{n-i+1}x_nx_{i-1}x_{n-i+2}x_{n-i}x_{n-i+1}x_n-3x_{n-2}x_{n-1}x_n-2x_{n-1} \]
\[ \overset{[P2]}{=} (-1)^{i+1}y_{n-i}^{4}. \]

**Lemma 2.18.** Modulo shorter monomials the following holds:

\[ z_0 = (-1)^{n+1}(z_1 + z_2 + z_3) + z_4 + z_5 - z_6 - 2 \sum_{i=7}^{n+1} (-1)^{n-i}z_i, \]

where:

\[ z_0 = x_{n-2}x_{n-1}x_ny_{n-2,n-2}^3, \]
\[ z_1 = y_{12}^i, \]
\[ z_2 = y_{2}^i, \]
\[ z_3 = y_{3,1}^i, \]
\[ z_4 = y_{16}^i, \]
\[ z_5 = y_{n,n-1}^i, \]
\[ z_6 = y_{n-2,n-2}^i. \]

and for all \( i \in \{7, \ldots, n+1\} \):

\[ z_i = y_{i-4,i-4}^4. \]

**Proof.** The proof for \( n = 5 \) is a simplification of the proof for \( n > 5 \). Therefore, we will assume \( n > 5 \).
Again, we will work modulo monomials of length less than $2n - 4$ throughout this proof.

$$z_0 - z_4 \equiv (x_n x_{n-1}) x_n y_{n-2}^3, \quad (x_n x_{n-1}) x_n y_{n-3}^3$$

$$- (x_n x_{n-2} x_n) x_n y_{n-2}^2 = (x_n x_{n-1}) x_n y_{n-3}^3$$

$$x_n (x_n x_{n-1}) x_n y_{n-3}^3 - (x_n x_{n-2} x_n) x_n y_{n-2}^2$$

$$x_n x_{n-3} (x_n x_{n-1}) y_{n-2}^3 - x_n (x_n x_{n-2} x_n) y_{n-2}^2$$

$$- x_n x_{n-3} (x_n x_{n-2} x_n) y_{n-2}^2 + (x_n x_{n-3} x_n x_{n-1}) y_{n-2}^2$$

$$= x_n (x_n x_{n-3} x_n x_{n-2} x_n) y_{n-3}^3 - x_n (x_n x_{n-3} x_n x_{n-2} x_n) y_{n-3}^3$$

$$+ x_n x_{n-3} (x_n x_{n-2} x_n) y_{n-3}^3 + x_n (x_n x_{n-3} x_n x_{n-1}) y_{n-3}^3$$

$$- x_n x_{n-3} (x_n x_{n-2} x_n) y_{n-3}^3 - (x_n x_{n-3} x_n x_{n-1}) y_{n-3}^3$$

$$= - t_1 - t_2 + t_3 - t_4.$$

We will prove the following:

$$t_1 = - z_{5r},$$

$$t_2 = - z_{n+1},$$

$$t_3 = - z_{6r},$$

$$t_4 = (-1)^n (z_1 + z_2 + z_3 + 2 \sum_{i=7}^{n} (-1)^{n-i} z_i) - z_{n+1}.$$  

For $t_1$ we have

$$t_1 = x_n x_{n-2} (x_n x_{n-3} x_n x_{n-2} x_n) y_{n-3}^3 - x_n x_{n-2} x_n y_{n-4}^3$$

$$= x_n x_{n-2} (x_n x_{n-3} x_n x_{n-2} x_n) y_{n-3}^3 - x_n x_{n-2} x_n y_{n-4}^3$$

$$= x_n x_{n-2} (x_n x_{n-3} x_n x_{n-2} x_n) y_{n-3}^3 - x_n x_{n-2} x_n y_{n-4}^3.$$

Applying this procedure another $n - 6$ times using Eqs. (J), (P1) and (P2) we obtain

$$t_1 = (-1)^{n-5} x_n x_{n-2} x_{n-1} x_{3n-1}.$$
We use Eq. (Q1) a number of times:

\[ t_1 = (-1)^n x_n x_{n-2} x_1 x_{n-1} \]

\[ t_2 = x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-3} = x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]

Finally, we see that

\[ t_1 = -x_n x_{n-2} x_1 + x_n x_{n-2} x_3 x_2 x_1 \]

\[ t_2 = -z_5. \]

Now consider \( t_2 \).

\[ t_2 = x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-3} = x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]

Applying this procedure another \( n - 6 \) times using Eqs. (J), (Q5) and (Q6) we obtain:

\[ t_2 = (-1)^{n-5} x_{n-3} x_{n-2} (x_{n-3} x_n x_{n-2} x_{n-1}) x_1. \]

Now we can show \( t_2 = -z_{n+1} \), using Eq. (Q1).

\[ t_2 = (-1)^n x_{n-3} x_{n-2} x_1 x_{n-1} x_{n-2} x_{n-1} \]

\[ t_2 = (-1)^n y_{n-3} x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-3} = x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]

For \( t_3 \) we have the following:

\[ t_3 = x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-3} = x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]

\[ t_3 = -x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]

\[ t_3 = -x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]

\[ t_3 = -x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]

\[ t_3 = -x_n x_{n-2} x_{n-2} x_{n-1} y^3_{n-3, n-4} \]
Again, we apply this procedure another \( n - 6 \) times using Eqs. (J), (P1) and (P2). We obtain:

\[
t_3 = (-1)^{n-5}x_{n-2} \{ x_{2|n-3}x_{n-2}x_{n-1} \} x_1 \\
= (-1)^{n}x_{n-2} \{ x_{2|n-3}x_1x_{2|n-3}x_{n-2}x_{n-1} \}.
\]

We then use Eq. (Q1) a number of times to find that

\[
t_3 = (-1)^{n}x_{n-2} \{ x_{2|n-3}x_1x_{2|n-3}x_{n-2}x_{n-1} \} \\
= (-1)^{n}x_{n-2} \{ x_{n-3}x_1x_{2|n-3}x_{n-2}x_{n-1} \} \\
= (-1)^{n+1}x_{n-2} \{ x_{n-3}x_1x_{2|n-3}x_{n-1}x_{n-2} \} \\
= (-1)^{n+1}x_{n-2} \{ x_{n-3}x_1x_{2|n-2} \}x_{n-2}x_{n-3}x_{n-2}x_{n-1} \\
= (-1)^{n+1}x_{n-2} \{ x_{n-3}x_1x_{2|n-2} \}x_{n-2}x_{n-3}x_{n-2}x_{n-1}x_{2|n-3}x_{n-2}x_{n-1}.
\]

We finally see that

\[
t_3 \overset{(J)}{=} -x_{n-2}x_{n-1}x_{n-3}x_4x_2x_3 + x_{n-2}x_{n-1}x_{n-3}x_4x_3x_2x_1 \\
\overset{(P2)}{=} = -z_6.
\]

Now consider \( t_4 \). We will first prove that

\[
t_4 = \sum_{i=3}^{n-4}(v_i - w_i) + (x_{4|n-2}x_{n-3}x_{n-2}x_{n-1})y_{3,3}^2 =: t_5 + t_6.
\]

This is clear for \( n = 6 \) (in which case \( t_5 = 0 \)). If \( n > 6 \) we find that

\[
t_4 = (x_{n-2}x_{n-3}x_{n-2}x_{n-1})y_{3,3}^2 = (x_{n-2}x_{n-3}x_{n-2}x_{n-1}x_{n-4})y_{3,3}^2 \\
\overset{(J)}{=} x_{n-4} (x_{n-2}x_{n-3}x_{n-2}x_{n-1}y_{3,3}^2 - (x_{n-4}x_{n-2}x_{n-3}x_{n-2}x_{n-1})y_{3,3}^2 \\
= v_3 - (x_{n-4}x_{n-2}x_{n-3}x_{n-2}x_{n-1})y_{3,3}^2 \\
\overset{(J)}{=} v_3 - x_{n-3}(x_{n-4}x_{n-2}x_{n-3}x_{n-2}x_{n-1})y_{3,3}^2 + (x_{n-3}x_{n-4}x_{n-2}x_{n-3}x_{n-2}x_{n-1})y_{3,3}^2 \\
= v_3 - v_3 + (x_{n-3}x_{n-4}x_{n-2}x_{n-3}x_{n-2}x_{n-1})y_{3,3}^2.
\]

Applying this procedure another \( n - 7 \) times using Eq. (J) we do indeed find that \( t_4 = t_5 + t_6 \). It is now sufficient to prove the following:

\[
t_5 = (-1)^{n+1}z_7 + 2 \sum_{i=7}^{n}(-1)^{n-i}z_i - z_{n+1},
\]

\[
t_6 = (-1)^{n}(z_1 + z_2 + z_3) + (-1)^{n+1}z_7.
\]
We use Lemma 2.17 to prove the first of these.

\[ t_5 = (2.3, 2.4) \sum_{i=3}^{n-4} (-1)^{i+1} (y_{n-i+1,n-i+1} - y_{n-i,n-i}) = \sum_{i=3}^{n-4} (-1)^{i+1} (z_{n-i+3} - z_{n-i+4}) \]

\[ = \sum_{i=3}^{n-4} (-1)^{i+1} z_{n-i+3} + \sum_{i=3}^{n-4} (-1)^{i} z_{n-i+4} = \sum_{i=7}^{n} (-1)^{n-i} z_i + \sum_{i=8}^{n+1} (-1)^{n-i+4} z_i \]

\[ = (-1)^{n+1} z_7 + 2 \sum_{i=7}^{n} (-1)^{n-i} z_i - z_{n+1}. \]

For \( t_6 \) we have:

\[ t_6 = (x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_2 x_3 x_1 \]

\[ = x_2 (x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_3 x_1 - (x_2 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_3 x_1 \]

\[ = x_2 x_3 (x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 - x_2 (x_3 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 \]

\[ - x_3 (x_2 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 + (x_3 x_2 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 \]

\[ = x_2 x_3 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 - x_2 (x_3 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 \]

\[ + (x_3 x_2 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 \]

\[ = - x_2 x_3 x_4 \| n-2x_n-3x_n x_n-2x_n-1) x_1 + x_3 x_1 x_2 x_4 \| n-2x_n-3x_n x_n-2x_n-1 \]

\[ - x_1 x_3 x_2 x_4 \| n-2x_n-3x_n x_n-2x_n-1 =: -t_7 + f_8 - t_9. \]

We will prove:

\[ t_7 = (-1)^{n+1} z_2, \]

\[ f_8 = (-1)^{n} z_3 + (-1)^{n+1} z_7, \]

\[ t_9 = (-1)^{n+1} z_1. \]

For \( t_7 \) we apply Eq. (Q1) a number of times:

\[ t_7 = x_2 x_3 x_1 x_4 \| n-2x_n-3x_n x_n-2x_n-1 \]

\[ = x_2 x_3 x_1 x_4 \| n-3x_n-2x_n-4x_n-3x_n x_n-2x_n-1 \]

\[ = x_2 x_3 x_1 x_4 \| n-4x_n-3x_n-2x_n-5x_n-4x_n-3x_n x_n-2x_n-1 \]

\[ = x_2 x_3 x_1 x_4 \| n-3x_n-2x_n-4x_n x_n-2x_n-1 \]

\[ = x_2 \| n-2x_n x_n-2x_n-1 \]

\[ = x_2 \| n-2x_n x_n-2x_n-1 \]

\[ = - x_2 \| n-2x_n x_n-2x_n-1 \]

\[ = - x_2 \| n-2x_n x_n-2x_n-1 \]

\[ = (1)^{n+4} x_2 \| n-2x_n x_n-2x_n-1 = (-1)^{n+1} y_2^8 = (-1)^{n+1} z_2. \]
For $t_8$ we also apply Eq. (Q1) a few times.

\[
t_8 = x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1
\]

= $x_3x_4x_1x_2x_3x_5y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $-x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^{n-4}x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

We now see that

\[
t_8 = (-1)^n x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1
\]

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $(-1)^n y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

Finally, consider $t_9$. Again, we only need Eq. (Q1).

\[
t_9 = x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1
\]

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

= $x_1x_3x_1x_2x_4y_{n, n-2}x_nx_{n-3}x_nx_{n-2}x_n-1$

Now we are ready to prove Theorem 2.16.

**Proof of Theorem 2.16.** Let $x$ be a monomial in $x_1, \ldots, x_n$ of length $\ell$. If $\ell = 1$ then $x \in M_{D_n}$. If $\ell = 2$, then either $x = 0$ or $x \in M_{D_n}$ or $-x \in M_{D_n}$ (using Eqs. (AC) and (R1)). We use induction on $\ell$ and may assume $\ell > 2$.

We have an $i \in \{1, \ldots, n\}$ and a monomial $y$ of length $\ell-1$ such that $x = y_i$. Moreover, because of the induction hypothesis we can assume $y \in M_{D_n}$. We will consider $x = y$ for each of the seventeen classes in $M_{D_n}$ separately. In each case we will write $x$ as a linear combination of monomials of length at most $\ell$, where all monomials in the linear combination of length $\ell$ are members of $M_{D_n}$. By the induction hypothesis, this suffices to prove the theorem.

We will work modulo monomials of length at most $\ell - 1$, so because of extremality of $x_j$, $x_j x_\ell = 0$ (XT) whenever the left hand side occurs in a monomial of length $\ell$.

**Case 1:** $j = 1$, $k \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, k\}$. Since $\ell > 2$, we know that $m < k$. We distinguish the following sub-cases:

- If $j > k + 1$ and $k = n - 2$, then $x = y_{m}^5$.
- If $j > k + 1$ and $k \neq n - 2$, then $x = 0$ by Eq. (Q5).
- If $j = k + 1$, then $x = y_{k+1}^1$.
- If $j = k$, then extremality of $x_i$ shows that $x = 0$. 


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• If $i = k - 1$, then:
  \[
  x = \begin{cases} 
  x_i x_{i+1} x_i (AC) & - x_i x_{i+1} (XT) \iff 0, \text{ if } m = k - 1, \\
  x_i x_{i+1} y_{i-1,m} (P2) \iff 0, \text{ otherwise.}
  \end{cases}
  \]

• If $i < k - 1$ and $i = n - 2$, then $x = y_{n-2,m}^2$.

• If $i < k - 1$ and $i \neq n - 2$, then $x = x_i x_{k|m}$. Applying Eq. (Q1) a sufficient number of times leads to one of the following situations:
  - If $i > m$, then $x = x_{k|j+2} x_{i+1} x_i (P2) = 0$.
  - If $i = m$ and $i \neq 1$, then $x = x_{k|m+2} x_{m} x_{m+1} x_{m} (AC) (XT) \equiv 0$.
  - If $i = m$ and $i = 1$, then $x = x_{k|4} x_{1} x_3 x_1 x_1 (P2) = 0$.

Case 2: $j = 2, k \in \{2, \ldots, n-2\}$ and $m \in \{1, \ldots, k - 1\}$.

• If $i = n$, then
  \[
  x = x_n x_k y_{k+1,m}^2 (Q1) = x_k x_n y_{k+1,m}^2 (Q1) = x_k x_{n-2} x_{n} x_{n-2} y_{n-1,m}^1 (P2) = 0.
  \]

• If $i = n - 1$, then
  \[
  x = x_{n-1} x_k y_{k+1,m}^2 (Q1) = x_k x_{n-1} y_{k+1,m}^2 (Q1) = x_k x_{n-2} x_{n-1} x_{n-2} y_{n-3,m}^1 (Q3) = x_k x_{n-3} x_{n-2} x_{n-1} x_{n-2} y_{n-3,m}^1 (P2) = 0.
  \]

• If $k + 1 < i < n - 1$, then $x = x_i y_{k,m}^2 (Q1) = x_k x_{i-1} x_i y_{i+1,m}^2 (P2) = 0$.

• If $i = k$, then $x = x_k y_{k+1,m}^2 (XT) = 0$.

• If $m < i = k - 1$, then $x = y_{k-1,m}^2$.

• If $m = i = k - 1 > 1$, then
  \[
  x = x_{k-1} x_{k} x_{k-1} x_{k+1} x_{n} (AC) \equiv - x_{k-1} x_{k} x_{k-1} x_{k+1} x_{n}^k (Q1) \equiv - x_{k-1} x_{k} x_{k-1} x_{k+1} x_{n} (P2) = 0.
  \]

• If $m = i = k - 1 = 1$, then
  \[
  x = x_{1} x_{n-2} x_{n} x_{1} (Q1) \equiv x_{1} x_{n-2} x_{n} x_{1} x_{2} x_{3} + x_{1} x_{n-2} x_{n} x_{1} x_{2} x_{3} \equiv 0.
  \]

(AC), (XT), (Q1), (P2), (Q3)
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- If \( i < k - 1 \), then
  \[
  x = x_i x_{k|n-2} x_{n|m+1} x_m.
  \]

Applying Eq. (Q1) a sufficient number of times leads to one of the following situations:

- If \( i > m \), then
  \[
  x = x_{k|n-2} x_{n|1} x_{i+1} x_{i|m} \tag{P2}
  \]
- If \( i = m \) and \( i > 1 \), then:
  \[
  x = x_{k|n-2} x_{n|1} x_{m+1} x_m \tag{AC}
  \]
  \[
  x = x_{k|n-2} x_{n|1} x_{m+1} x_m \tag{XT}
  \]
- If \( i = m \) and \( k \neq 3 \), then:
  \[
  x = x_{k|n-2} x_{n|4} x_1 x_3 x_2 x_1 \tag{AC}
  \]
  \[
  x = x_{k|n-2} x_{n|4} x_1 x_3 x_2 \tag{P2}
  \]
- If \( i = m = 1 \) and \( k \neq 3 \), then:
  \[
  x = x_{k|n-2} x_{n|1} x_{m-1} x_m \tag{AC}
  \]
  \[
  x = -x_{k|n-2} x_{n|4} x_1 x_3 x_2 \tag{XT}
  \]
- If \( i = m-1 \) and \( k = 3 \), then \( x = y_{11} \).
- If \( i = m-1 \) and \( m \neq 3 \), then:
  \[
  x = x_{k|n-2} x_{n|1} x_{m-1} x_m \tag{Q1}
  \]
  \[
  x = -y_{k,m-1} \tag{R1}
  \]
- If \( i < m-1 \) and \( m = 3 \), then
  \[
  x = x_{k|n-2} x_{n|4} x_1 x_3 \tag{AC}
  \]
  \[
  x = -y_k \tag{R1}
  \]

**Case 3:** \( j = 3, k \in \{3, \ldots, n\} \) and \( m \in \{3, \ldots, k\} \).

- If \( i > k+1 \) and \( k = n-2 \), then \( x = y_m^9 \).
- If \( i > k+1 \) and \( k < n-2 \), then
  \[
  x = x_i x_{k|n-1} x_{m-1} x_m \tag{Q1}
  \]
  \[
  x = x_{k|n-1} x_{m-1} x_m \tag{R1}
  \]
- If \( i = k+1 \), then \( x = y_{i,m}^3 \).
- If \( i = k \), then
  \[
  x = x_i y_{i,m}^3 = \begin{cases} x_i x_{i-1} x_i y_{i-1,i-1}^3 & \text{if } k = m, \\ x_i y_{i-1,i-1}^3 & \text{otherwise.} \end{cases}
  \]
- If \( i = k-1 \), then we have
  \[
  x = x_i y_{i+1,m}^3 = \begin{cases} x_i x_{i+1} y_{i+1,i}^3 & \text{if } i+1 = m, \\ x_i x_{i+1} y_{i+1,i}^3 & \text{if } i+1 = m+1, \\ x_i x_{i+1} y_{i+1,i}^3 & \text{otherwise.} \end{cases}
  \]
- If \( i < k-1 \) and \( i = n-2 \), then \( x = y_{n-2,m}^4 \).
• If \( i < k - 1 \) and \( i \neq n - 2 \), then

\[
x = x_i x_{k|m+1} x_{m-1} x_1. 
\]

Repeated application of Eq. (Q1) leads to one of the following situations:

- If \( i > m \), then \( x = x_{k|j+2} x_{i+1} x_j y_{j-1, m}^3 \) (P2) \( \equiv 0 \).
- If \( i = m \), then \( x = x_{k|m+2} x_m x_{m+1} x_{m-1} x_1 = y_{k,m+1}^3 \).
- If \( i = m - 1 \), then \( x = x_{k|m+1} x_{m-1} x_{m-1} y_{k,m-1}^3 \) (XT) \( \equiv 0 \).
- If \( 1 \neq i < m - 1 \neq k \), then \( x = x_{k|m+1} x_{m-1} x_{m-1} x_{i+2} x_i x_{i+1} x_{i+1} x_j y_{j+1, i}^3 \) (Q2) \( \equiv 0 \).
- If \( 1 \leq i < m - 1 \neq 2 \), then:

\[
x = x_{k|m+1} x_{m-1} x_{m-1} x_1 x_3 x_4 x_2 x_3 x_1 = 0. 
\]

This last identity follows from:

\[
x_1 x_3 x_4 x_2 x_3 x_1 (Q3), (P2) = x_1 x_3 x_4 x_3 x_2 (P2) = 0. \tag{2.5}
\]

- If \( i = 1 \) and \( m = 3 \), then \( x = x_{k|4} x_1 x_2 x_3 x_1 \) (AC) \( \equiv -x_{k|4} x_1 x_2 x_1 x_3 = 0 \).

**Case 4:** \( j = 4 \), \( k \in \{3, \ldots, n-2 \} \) and \( m \in \{3, \ldots, k \} \). Note that \( y_{k,m}^4 = x_{k|n-2} x_{n|m+1} x_{m-1} x_1 \):

- If \( i = n > k \), then \( x = x_{k|n-3} x_n x_{n-2} x_n y_{n-2, m}^3 \) (P2).
- If \( i = n - 1 > k \), then:

\[
x = \begin{cases} 
x_{n-1} x_{n-2} x_n x_n x_{n-1} x_{n-2} y_{n-3, m}^3 & \text{if } m = n - 2, \\
x_{k|n-3} x_n x_{n-2} x_n x_n x_{n-2} x_{n-3} y_{n-3, m}^3 & \text{if } m = n - 3.
\end{cases} \tag{Q3}
\]

- If \( k < i < n - 1 \), then:

\[
x = x_i x_k x_{k+1} y_{k+2, m}^4 \equiv 0. \tag{Q1}
\]

- If \( i = k \), then \( x = x_i x_k y_{k+1, m}^4 \equiv 0. \tag{XT}
\]

- If \( i = k - 1 = m - 1 \), then:

\[
x = x_i x_k x_{i+1} x_{i+2} y_{i+2, i+1} \equiv 0. \tag{Q1}
\]

- If \( i = k - 1 \) and \( m < k \), then \( x = y_{k-1, m}^3 \).

- If \( i < k - 1 \), then:

\[
x = x_i x_{k|n-2} x_{n|m+1} x_{m-1} x_1 \equiv 0. \tag{Q1}
\]

Repeated application of Eq. (Q1) leads to one of the following situations:

- If \( i > m \), then:

\[
x = x_{k|n-2} x_{n|j+2} x_i x_{i+1} x_{i|j+m+1} x_{m-1} x_1 \equiv 0. \tag{P2}
\]

- If \( i = m \), then:

\[
x = x_{k|n-2} x_{n|m+2} x_m x_{m+1} x_{m-1} x_1 = y_{k,m+1}^4. \tag{XT}
\]

- If \( i = m - 1 \), then:

\[
x = x_{k|n-2} x_{n|m+1} x_{m-1} x_{m-1} x_1 = 0. \tag{XT}
\]
If 1 < i < m − 1, then:
\[ x = x_{k\mid n−2x_n\mid m+1x_{m−1}\mid i+2x_i+3x_i} \quad (Q2) \]
if i = m − 2 then the piece \( x_{m−1\mid j+2x_i+3} \) is missing but the computation is the same.
If i = 1 and \( m \neq 3 \), then:
\[ x = x_{k\mid n−2x_n\mid m+1x_{m−1}\mid i\mid 4x_3x_1x_3} \quad (2.5) \]
and
If i = 1 and \( m = 3 \), then:
\[ x = x_{k\mid n−2x_n\mid m+1x_{m−1}\mid i\mid 4x_1x_2x_3} \quad (P2) \]

Case 5: \( j = 5 \) and \( m \in \{1, \ldots, n−2\} \). Note that \( y_{m}^{5} = x_{n}x_{n−2}x_{m} \).
- If \( i = n \), then we apply extremality of \( x_{n} \).
- If \( i = n−1 \), then \( x = y_{m}^{6} \).
- If \( i = n−2 \), then:
\[ x = \begin{cases} 
  x_{n−2x_n\mid n−2} \quad (AC) & \text{if } m = n−2, \\
  x_{n−2x_n\mid n−3,m} \quad (P2) & \text{otherwise}.
\end{cases} \]
- If \( i < n−2 \), then \( x = x_{i}x_{n}x_{n−2}x_{m} \). Repeated application of (Q1) again leads to one of the following situations:
  - If \( i > m \), then:
\[ x = x_{i}x_{n−2j+i+2x_i}x_1x_1y_{i−1,m}^{1} \quad (P2) \]
  - If \( 1 < i = m \), then:
\[ x = x_{i}x_{n−2j+i+2x_i}x_1x_1x_1 \quad (AC) \]
  and
\[ x = x_{i}x_{n−2j+i+2x_i}x_1x_1 \quad (XT) \]
  - If \( i = m = 1 \), then:
\[ x = x_{n}x_{n−2}x_1x_1x_2x_3x_1 \quad (AC) \]
  - If \( i < m \), then:
\[ x = x_{n}x_{n−2}x_1x_1x_2x_3x_1 \quad (AC, R1) \]
\[ = \begin{cases} 
  y_{m−1}^{5} & \text{if } i = m−1, \\
  y_{13}^{13} & \text{if } (i, m) = (1, 3), \\
  0 & \text{otherwise}.
\end{cases} \]

Case 6: \( j = 6 \) and \( m \in \{1, \ldots, n−2\} \). Note that \( y_{m}^{6} = x_{n}x_{n−2}x_{m} \).
- If \( i = n \), then:
\[ x = x_{n}x_{n−1}x_1y_{n−2,m}^{1} \quad (P2) \]
- If \( i = n−1 \), then:
\[ x = x_{n−1}x_{n−1}y_{m}^{5} \quad (XT) \]
- If \( i = n−2 \), then we have:
\[ x = x_{n−2}x_{n−1}x_1y_{n−2,m}^{1} \quad (AC) \]
\[ = \begin{cases} 
  x_{n−2}x_{n−1}x_{n−2}x_1 \quad (P2) & \text{if } m = i, \\
  x_{n−2}x_{n−1}x_1x_2y_{n−3,m}^{1} \quad (Q3) & \text{otherwise}.
\end{cases} \]
- If \( i < n−2 \), then repeated application of Eq. (Q1) leads to one of the following situations:
Case 7: $j = 7$ and $k \in \{3,\ldots,n\}$. Note that $y_k^7 = x_{k\downarrow 3}x_1$.
- If $i > k + 1 = n - 1$, then $x = y_k^{13}$.
- If $i > k + 1 < n - 1$, then
  $$x = x_i x_{k\downarrow 3}x_1 \overset{(Q1)}{=} x_{k\downarrow 3}x_1 \overset{(R1)}{=} 0.$$
  - If $i = k + 1$, then $x = y_k^{7+1}$.
  - If $i = k$, then
    $$x = x_i y_{i-1} \overset{(XT)}{=} 0.$$
- If $i < k$, then:
  $$x = x_i x_{k\downarrow 3}x_1 \overset{(Q1)}{=} \begin{cases} 
  x_{k\downarrow 4}x_1x_3x_1 \overset{(AC)}{=} -x_k x_{k-\downarrow 4}x_1x_3 \overset{(XT)}{=} 0, & \text{if } i = 1, \\
  x_{k\downarrow 4}x_2x_3x_1 \overset{(Q3)}{=} y_{6,3}^3, & \text{if } i = 2, \\
  x_{k\downarrow i+2}x_i x_{i+1} x_1 y_{i-1}^{7} \overset{(P2)}{=} 0, & \text{otherwise.}
  \end{cases}$$

Case 8: $j = 8$ and $k \in \{2,\ldots,n-2\}$. Note that $y_k^8 = x_{k\uparrow n-2}x_{n\downarrow 3}x_1$.
- If $i > k$, then:
  $$x = x_i x_{k\uparrow n-2} x_n y_{n-1}^{7} \overset{(Q1)}{=} \begin{cases} 
  x_{k\uparrow n-3}x_n x_{n-2} x_n y_{n-1}^{7} \overset{(P2)}{=} 0, & \text{if } i = n, \\
  x_{k\uparrow n-3} x_{n-1} x_n x_{n-1} x_n x_{n-2} y_{n-3}^{7} \overset{(Q3)}{=} x_{k\uparrow n-3} x_{n-1} x_n x_{n-2} x_{n-1} x_n x_{n-2} y_{n-3}^{7} \overset{(P2)}{=} 0, & \text{if } i = n-1, \\
  x_{k\uparrow i-2} x_i x_{i-1} x_1 y_{i+1}^{8} \overset{(P2)}{=} 0, & \text{otherwise.}
  \end{cases}$$
  - If $i = k$, then we apply extremality of $x_i$ to obtain $x = 0$.
  - If $1 \neq i = k - 1$, then $x = y_k^{8}$.
  - If $1 = i = k - 1$, then:
    $$x = x_{1\uparrow n-2} x_{n\downarrow 3} x_1 \overset{(AC)}{=} -x_{1\downarrow n-2} x_{n\downarrow 4} x_1 x_3 \overset{(Q1)}{=} -x_1 x_2 x_3 x_1 x_4 \uparrow n-2 x_{n\downarrow 3}
    \overset{(Q3)}{=} -x_1 x_3 x_2 x_1 x_4 \uparrow n-2 x_{n\downarrow 3} \overset{(I)}{=} -x_1 x_3 x_2 x_1 x_4 \uparrow n-2 x_{n\downarrow 3} - x_1 x_3 x_1 x_2 x_4 \uparrow n-2 x_{n\downarrow 3}
    \overset{(Q7),(P2)}{=} -x_1 x_3 x_{n-2} x_{n\downarrow 4} (x_2 x_1) x_3 \overset{(AC)}{=} y_{11}^1.$$
If $1 < i < k - 1$, then
\[
x = x_i x_{k \uparrow n-2} x_{n \uparrow 3} \quad (Q1) = \begin{cases} x_k x_{n-2} x_{n \uparrow 4} x_2 x_3 x_1 = y_{k,3}^2, & \text{if } i = 2, \\ x_k x_{n-2} x_{n \uparrow i+2} x_i x_{i+1} x_i y_{i-1}^2 = 0, & \text{otherwise.} \end{cases}
\]

If $1 = i < k - 1$, then
\[
x = x_1 x_k x_{n-2} x_{n \uparrow 3} x_1 \quad (AC) = -x_1 x_k x_{n-2} x_{n \uparrow 4} x_3 x_1 \quad (Q1) = \begin{cases} -x_1 x_3 x_1 x_{4 \uparrow n-2} x_{n \uparrow 3} = 0, & \text{if } k = 3, \\ -x_1 x_1 x_k x_{n-2} x_{n \uparrow 3} = 0, & \text{otherwise.} \end{cases}
\]

**Case 9:** $j = 9$ and $m \in \{3, \ldots, n-2\}$. Remember that $y_m^9 = x_n x_{n-2 \downarrow m+1} x_{m-1} x_{m-1} x_{m-1}$.
- If $i = n$, then extremality of $x_n$ shows that $x = 0$.
- If $i = n - 1$, then $x = y_m^0$.
- If $i = n - 2$, then:
\[
x = \begin{cases} x_{n-2} x_n y_{n-2}^3 = y_{15}^1, & \text{if } m = n - 2, \\ x_{n-2} x_n x_{n-2} y_{n-3,m}^3 = 0, & \text{otherwise.} \end{cases}
\]

If $i < n - 2$, then
\[
x = x_i x_n x_{n-2 \downarrow m+1} x_{m-1} x_{m-1} \quad (XT).
\]

Repeated application of Eq. (Q1) leads to one of the following situations:
- If $i > m$, then
\[
x = x_n x_{n-2 \downarrow i+2} x_i x_{i+1} x_i y_{i-1,m}^3 = 0.
\]
- If $i = m$, then
\[
x = x_n x_{n-2 \downarrow i+2} x_i x_{m+1} x_{m-1} x_{m-1} = y_m^9.
\]
- If $i = m - 1$, then:
\[
x = x_n x_{n-2 \downarrow i+2} x_i x_{m+1} x_{m-1} x_{m-1} y_{m,m-1}^3 = 0.
\]
- If $1 < i < m - 1$, then
\[
x = x_n x_{n-2 \downarrow i+1} x_{m-1} x_{m-1} y_{m,m-1}^3 = 0.
\]
- If $i = 1$ and $m > 3$, then:
\[
x = x_n x_{n-2 \downarrow i+1} x_{m-1} x_{m-1} x_{m-1} x_{m-1} x_{m-1} x_{m-1} x_{m-1} = 0;
\]
if $m = 4$ the piece $x_{m-1 \downarrow i} x_{m-1}$ should be omitted.
- If $i = 1$ and $m = 3$, then:
\[
x = x_n x_{n-2 \downarrow i+1} x_1 x_{3 \downarrow 3} x_3 x_1 = 0.
\]

**Case 10:** $j = 10$. Then $m \in \{3, \ldots, n-2\}$. Note that $y_m^{10} = x_{n-1} x_n x_{n-2 \downarrow m+1} x_{m-1} x_{m-1}$.
- If $i = n$, then
\[
x = x_i x_n x_{n-1} x_{n-2,m} = x_n x_{n-1} x_n y_{n-2,m}^3 = 0.
\]
- If $i = n - 1$, then extremality of $x_{n-1}$ implies that $x = 0$.
- If $i = n - 2 > m$, then we have:
\[
x = x_n x_{n-2} x_{n-2} x_{n-2} y_{n-3,m}^3 = x_n x_{n-1} x_{n-2} y_{n-3,m}^3 = y_{n-2,m}^4.
\]
- If $i = n - 2 = m$, then Lemma 2.18 says that $x$ can be written as a linear combination of the following monomials:
\[
y_{3,1}^2, y_{n,n-1}^3, y_{n-2,m-2}^4, y_{n-3,m-3}^4, \ldots, y_{3,3}^4, y_2^8, y_2^8, y_2^{12} \text{ and } y_2^{16}.
\]
• If $i < n - 2$, then

$$x = x_{n-1}x_n x_{n-2} \downarrow m+1 x_{n-1} \uparrow 1$$

Repeated application of Eq. (Q1) leads to one of the following situations:

- If $i > m$, then:

$$x = x_{n-1}x_n x_{n-2} \downarrow j+2 x_i x_i+1 x_i y_{i,j}^3 \quad (P2) \equiv 0.$$

- If $i = m$, then:

$$x = x_{n-1}x_n x_{n-2} \downarrow m+2 x_m x_m+1 x_m-1 x_m x_m-2 \uparrow 1 = y_{m+1}^{10}.$$

- If $i = m - 1$, then:

$$x = x_{n-1}x_n x_{n-2} \downarrow m+1 x_m x_m-1 x_m \uparrow 3 \quad (Q2) \equiv 0.$$

- If $1 < i < m - 1$, then:

$$x = x_{n-1}x_n x_{n-2} \downarrow j+2 x_i x_i+1 x_i x_i+2 x_i y_{i,j+1}^3 \quad (XT) \equiv 0.$$

- If $1 = i$ and $m \neq 3$, then:

$$x = x_{n-1}x_n x_{n-2} \downarrow m+1 x_m x_m-1 \uparrow 4 x_3 x_3 x_2 x_3 x_1 \quad (2.5) \equiv 0;$$

if $m = 4$ then the piece $x_{m-1} \uparrow 4 x_3$ should be omitted.

- If $i = 1$ and $m = 3$, then:

$$x = x_{n-1}x_n x_{n-2} \downarrow 4 x_1 x_2 x_3 x_1 \quad (AC) \equiv 0.$$

**Case 11:** $j = 11$. Note that $y_{11} = x_1 x_3 \uparrow n-2 x_{n-1}^1$.

- If $i = n$, then:

$$x = x_{1,3} \uparrow n-3 x_n x_{n-2} x_n y_{n-1,1}^1 \quad (P2) \equiv 0.$$

- If $i = n - 1$, then:

$$x = x_{1,3} \uparrow n-3 x_n-1 x_n x_{n-1} \downarrow 1 = x_{1,3} \uparrow n-3 x_n-1 x_n x_{n-2} x_{n-1} \downarrow 1 \quad (P2) \equiv 0.$$

- If $3 < i < n - 1$, then:

$$x = x_{1,3} \uparrow i-2 x_i x_i-1 x_i y_{i+1,1}^2 \quad (P2) \equiv 0.$$

- If $i = 3$, then:

$$x = x_3 x_3 y_{3,1}^2 \quad (P2) \equiv 0.$$

- If $i = 2$, then:

$$x = x_2 x_1 x_3 \uparrow n-2 x_n x_{n-1} \downarrow 3 x_2 x_1 \equiv x_2 x_1 x_3 \uparrow n-2 x_n x_n-1 \downarrow 4 x_2 x_3 - x_2 x_1 x_3 \uparrow n-2 x_n x_{n-1} \downarrow 4 x_2 x_3 x_1 x_3 \quad (P2)(Q3) \equiv 0.$$

- If $i = 1$, then extremality of $x_1$ shows that $x = 0$.

**Case 12:** $j = 12$. Remember that $y_{12} = x_1 x_3 \uparrow n-2 x_{n-1}^2$.

- If $i = n$, then $x \equiv x_{1,3} \uparrow n-3 x_n x_{n-2} x_n \downarrow 2 \quad (P2) \equiv 0.$

- If $i = n - 1$, then:

$$x = x_{1,3} \uparrow n-3 x_n-1 x_n-2 x_n x_{n-1} \downarrow 2 = x_{1,3} \uparrow n-3 x_n-1 x_n-2 x_n x_n-1 x_n-2 \downarrow 2 \quad (P2) \equiv 0.$$

- If $3 < i < n - 1$, then:

$$x = x_{1,3} \uparrow i-2 x_i x_i-1 x_i y_{i+1,2}^2 \quad (P2) \equiv 0.$$
If \( i = 3 \), then:
\[
x = x_3 x_1 x_3 y_{4,2}^2 \equiv 0.
\]

If \( i = 2 \), then:
\[
x = x_2 x_1 x_3 x_{n-2} x_{n,2} \equiv -x_2 x_1 x_3 x_{n-4} x_2 x_3 \equiv -x_2 x_1 x_3 x_{n-2} x_{n,3}^{(Q1)} \equiv -x_2 x_1 x_2 x_{4,1} x_{n-2} x_{n,3}^{(Q3)} \equiv -x_2 x_3 x_{1,2} x_{4,1} x_{n-2} x_{n,3} \equiv -x_2 x_3 x_{1} x_{n,4} (x_1, x_2) x_3 = -y_{2,1}^2.
\]

If \( i = 1 \), then extremality of \( x_1 \) shows that \( x = 0 \).

**Case 13:** \( j = 13 \). Remember that \( y_{13} = x_{n-2} x_{n,3} x_1 \).

If \( i = n \), then extremality of \( x_n \) shows that \( x = 0 \).

If \( i = n - 1 \), then \( x = y_{14} \).

If \( i = n - 2 \), then:
\[
x = x_{n-2} x_n x_{n-2} y_{n-3}^2 \equiv 0.
\]

If \( i < n - 2 \), then:
\[
x = x_{n,2} x_{n-1} x_{n-1} x_{n-2} y_{n-3}^2 \equiv x_{n-2} x_n x_{n-1} x_{n-2} y_{n-3}^2 = y_{n-2}^2.
\]

**Case 14:** \( j = 14 \). Note that \( y_{14} = x_{n-1} x_n x_{n-2} x_{n,1} \).

If \( i = n \), then \( x = x_n x_{n-1} x_n y_{n,2}^2 \equiv 0 \).

If \( i = n - 1 \), then extremality of \( x_{n-1} \) shows that \( x = 0 \).

If \( i = n - 2 \), then:
\[
x = x_{n-2} x_{n-1} x_{n-2} y_{n-3}^2 \equiv x_{n-2} x_n x_{n-1} x_{n-2} y_{n-3}^2 = y_{n-2}^2.
\]

If \( i < n - 2 \), then:
\[
x = x_{n-2} x_n x_{n-2} x_{n-1} x_{n-1} x_n x_{n-2} y_{n-3}^2 \equiv 0, \quad \text{if } i > 2,
x = x_{n-1} x_n x_{n-2} x_{n-1} x_{n-2} y_{n-3}^2 \equiv y_{n-2}^2, \quad \text{if } i = 2,
x = x_{n-1} x_n x_{n-2} x_{n-1} x_{n-2} x_{n-3} x_{n-1} x_n x_{n-2} \equiv 0, \quad \text{if } i = 1.
\]

**Case 15:** \( j = 15 \). Remember that \( y_{15} = x_{n-2} x_n x_{n-3} x_{n,1} \).

If \( i = n \), then \( x = x_n x_{n-2} x_n y_{n,2}^2 \equiv 0 \).

If \( i = n - 1 \), then \( x = x_{n-1} x_n x_{n-2} x_n y_{n-2}^2 \equiv y_{16} \).

If \( i = n - 2 \), then applying extremality of \( x_{n-2} \) gives \( x = 0 \).

If \( i = n - 3 \), then:
\[
x = x_{n-3} x_{n-2} x_{n-3} x_{n-1} \equiv x_{n-3} x_{n-2} x_{n-3} x_{n} y_{n-2}^2 \equiv 0.
\]

If \( i < n - 3 \) and \( n > 5 \), then:
\[
x = x_{n-2} x_n x_{n-3} x_{n-1} x_{n-2} x_n y_{n-3}^2 \equiv 0, \quad \text{if } i > 1,
x = x_{n-3} x_{n-2} x_n x_{n-3} x_{n-4} x_{n-3} x_{n-4} x_{n-3} x_{n-4} x_{n-3} x_{n-1} \equiv 0, \quad \text{if } i = 1.
\]

If \( i < n - 3 \) and \( n = 5 \), then \( i = 1 \) and
\[
x = x_1 x_3 x_5 x_3 x_2 x_1 = x_1 x_3 x_5 x_3 x_2 x_1 + x_1 x_3 x_5 x_3 x_2 x_1 \equiv x_1 x_3 x_5 x_3 x_2 x_1 = 0.
\]

**Case 16:** \( j = 16 \). Note that \( y_{16} = x_{n-1} x_n x_{n-2} x_{n,1} \).

If \( i = n \), then \( x = y_{17} \).

If \( i = n - 1 \), then extremality of \( x_{n-1} \) gives us \( x = 0 \).
• If $i = n - 2$, then:
  \[ x = x_{n-2}x_{n-1}x_ny_{n-2,n-2}^3 \]  
  \[ (P2) \implies 0. \]

• If $i = n - 3$, then:
  \[ x = x_{n-3}x_{n-1}x_ny_{n-2,n-3}^3 \]  
  \[ (Q1) \implies x_{n-1}x_ny_{n-2,n-3}^3 \]  
  \[ (P2) \implies 0. \]

• If $i < n - 3$ and $n > 5$, then:
  \[ x \overset{(Q1)}{=} \begin{cases} 
  x_{n-1}x_ny_{n-2,n-3}^3 & \text{if } i > 1, \\
  x_{n-1}x_ny_{n-3}4x_5x_3x_2x_3x_1 & \text{if } i = 1.
  \end{cases} \]  

• If $i < n - 3$ and $n = 5$, then $i = 1$

\[ x = x_1x_4x_3x_5x_2x_3x_1 \]  
  \[ (Q1) \implies x_4x_1x_3x_5x_2x_3x_1 \]  
  \[ (P2),(Q1) \implies x_4x_3x_1x_3x_1x_3x_2 = 0. \]

Case 17: $j = 17$. Note that $y^{17} = x_nx_{n-1}x_{n-2}x_{n-3} \| 1$.

• If $i = n$, then extremality of $x_n$ shows that $x = 0$.

• If $i = n - 1$, then $x = x_{n-1}x_ny_{n-2,n-2}^3 \implies 0$.

• If $i = n - 2$, then:
  \[ x = x_{n-2}x_{n-1}x_ny_{n-2,n-2}^3 \]  
  \[ (Q3) \implies x_{n-2}x_{n-1}x_ny_{n-2,n-2}^3 \]  
  \[ (P2) \implies 0. \]

• If $i = n - 3$, then:
  \[ x \overset{(Q1)}{=} x_{n-1}x_ny_{n-3}3x_{n-3}x_{n-2}x_{n-3}y_{n-2,n-3}^3 \]  
  \[ (P2) \implies 0. \]

• If $i < n - 3$ and $n > 5$, then:
  \[ x \overset{(Q1)}{=} \begin{cases} 
  x_{n-1}x_ny_{n-2,n-3}^3 & \text{if } i > 1, \\
  x_{n-1}x_ny_{n-3}4x_5x_3x_2x_3x_1 & \text{if } i = 1.
  \end{cases} \]  

• If $i < n - 3$ and $n = 5$, then $i = 1$

\[ x = x_1x_5x_4x_3x_5x_2x_3x_1 \]  
  \[ (Q1) \implies x_5x_4x_1x_3x_5x_2x_3x_1 \]  
  \[ (P2),(Q1) \implies x_5x_4x_1x_3x_1x_3x_2 \]  
  \[ (P2) \implies x_5x_4x_1x_3x_1x_3x_2 = 0. \]

2.4.2. Monomials for $\Gamma_B$. Recall that in $\mathcal{F}_{\Gamma_B}$, we divide out relations (R1) and (R3) from page 22. In Theorem 2.19, we will give a list of $2n^2 - 3n + 1$ monomials in $x_1, \ldots, x_n$ and show that these monomials span $\mathcal{F}_{\Gamma_B}$ linearly. We will prove this using Theorem 2.16.
Theorem 2.19. Let $\mathcal{M}_{n,m}$ be the set consisting of the following monomials:

\[
\begin{align*}
    y^1_{k,m} &= x_{k|m}, & n \geq k \geq m \geq 1 & \text{and } (k, m) \neq (n, n - 1), \\
y^2_{k,m} &= x_{k|n-2|x_{n|m}}, & n - 2 \geq k & \geq m \geq 1, \\
y^3_{k,m} &= x_{k|1|m+1|x_{m-1}|1}, & n \geq k \geq m \geq 3 & \text{and } (k, m) \neq (n, n), \\
y^4_{k,m} &= x_{k|n-2|x_{n|m+1}|x_{m-1}|1}, & n - 2 \geq k \geq m & \geq 3, \\
y^5_{k,m} &= x_{n|x_{n-2}|m}, & n - 2 \geq m & \geq 1, \\
y^6_{k,m} &= x_{k|3|x_{1}}, & n \geq k & \geq 3, \\
y^7_{k,m} &= x_{k|2|x_{3}|x_{1}}, & n - 2 \geq k & \geq 2, \\
y^8_{k,m} &= x_{n|x_{n-2}|m+1|x_{m-1}|1}, & n - 2 \geq m & \geq 3, \\
y^9_{k,m} &= x_{1|x_{3}|n-2|x_{n-1}|1}, \\
y^{10}_{k,m} &= x_{n|x_{n-2}|3|x_{1}}, \\
y^{11}_{k,m} &= x_{n|x_{n-2}|3|x_{n-3}|1}, \\
y^{12}_{k,m} &= x_{n-1|x_{n-2}|x_{n}|x_{n-3}|1}. \\
\end{align*}
\]

Let $x$ be a monomial in $x_1, \ldots, x_n$ of length $\ell$. Then $x$ is a linear combination of monomials $y \in \mathcal{M}_{n,m}$ with length($y$) $\leq \ell$.

Proof. Because of Theorem 2.16 it suffices to prove that the following $2n - 1$ monomials can be written as a linear combination of shorter monomials and monomials from $\mathcal{M}_{n,m}$:

1. $y^1_{n,n-1} = x_nx_{n-1}$
2. $y^2_{3,n} = x_{n-1}|1|1$
3. $y^3_{3,n} = x_{n-1}|x_n|x_{n-2}|2|m$ with $n - 2 \geq m \geq 1$
4. $y^4_{10,m} = x_{n-1}|x_n|x_{n-2}|m+1|x_{m-1}|1$ with $n - 2 \geq m \geq 3$
5. $y^5_{12} = x_1|x_{3}|n-2|x_{n-3}|x_{1}$
6. $y^{14} = x_{n-1}|x_n|x_{n-2}|3|x_{1}$
7. $y^{17} = x_{n-1}|x_n|x_{n-2}|x_{n-3}|2|m|1$

We will consider these seven classes of monomials separately and work modulo shorter monomials.

Case 1: $y = y^1_{n,n-1} = x_nx_{n-1}$. This is 0 because of identity $\text{(R3)}$.

Case 2: $y = y^2_{3,n}$. We find $y = x_{n-1}|x_n|x_{n-2}|2|m|1 \equiv x_{n-1}|x_n|x_{n-2}|x_{n-3}|3|m|1 \equiv 0$. (P2)

Case 3: $y = y^3_{6,m}$. We find $y = x_{n-1}|x_n|x_{n-2}|2|m|1 \equiv x_{n-1}|x_n|x_{n-2}|x_{n-3}|3|m|1 \equiv 0$. (R3)

Case 4: $y = y^4_{10,m}$. We find $y = x_{n-1}|x_n|x_{n-2}|m+1|x_{m-1}|1 \equiv x_{n-1}|x_n|x_{n-2}|m+1|x_{m-1}|1 \equiv 0$. (R3)

Case 5: $y = y^{12}$. Lemma 2.18 says that $y^{12}$ can be written as a linear combination of the following monomials:

$y^2_{3,1}, y^3_{n,n-1}, y^4_{3,3}, y^4_{4,4}, \ldots, y^4_{n-2,n-2}, y^8, y^{12}, y^{16}$ and $x_{n-2}|x_{n-1}|x_n|x_{n-3}|1 =: t$.

All of these except $t$ are in $\mathcal{M}_{n,m}$, so we only need to analyze $t$.

$t = x_{n-2}|x_{n-1}|x_n|x_{n-3}|1 \equiv x_{n-2}|x_{n-1}|x_n|x_{n-3}|1 \equiv y^4_{n-2,n-2}$. (Q1)
2.4. THE MONOMIALS

Case 6: $y = y^{14}$. We find $y = x_{n-1}x_{n-2}x_1 \in M_{\ell}$ such that $x_{n-1}x_{n-2}x_1 = y_n^{7}$.

Case 7: $y = y^{17}$. We find $y = x_{n}x_{n-1}x_{n-2}x_{n-3} \in M_{\ell}$ such that $x_{n}x_{n-1}x_{n-2}x_{n-3} = 0$.

2.4.3. Monomials for $\Gamma_{\ell,m}$. Recall that in $F_{\Gamma_{\ell,m}}$, we modded out relations (R1) and (R4) from page 22. We will give a list of $n^2 - 1$ monomials in $x_1, \ldots, x_m$, and prove in Theorem 2.20 that these monomials span $F_{\Gamma_{\ell,m}}$ linearly.

**Theorem 2.20.** Let $M_{\ell,m}$ be the set consisting of the following monomials:

\[
\begin{align*}
    y_{k,m}^1 &= x_{k+1}x_{m}, & n \geq k &\geq m \geq 1, \\
    y_{k,m}^3 &= x_{k+1}x_{m+1}x_{m-1}x_{1}, & n \geq k &\geq m \geq 3, \\
    y_{k,m}^7 &= x_{k+1}x_{m}, & n \geq k &\geq 3.
\end{align*}
\]

Let $y$ be a monomial in $x_1, \ldots, x_m$ of length $\ell$. Then $y$ is a linear combination of monomials $y \in M_{\Gamma_{\ell,m}}$ with length($y$) $\leq \ell$.

**Proof.** Because of Theorem 2.16 it suffices to prove that the following $n^2 - n + 1$ monomials can be written as a linear combination of shorter monomials and monomials from $M_{\Gamma_{\ell,m}}$:

1. $y_{k,m}^1 = x_{k+1}x_{m}$,
2. $y_{k,m}^3 = x_{k+1}x_{m+1}x_{m-1}x_{1}$,
3. $y_{k,m}^5 = x_{n}x_{n-2}$,
4. $y_{k,m}^7 = x_{n}x_{n-2}x_{n-3}$,
5. $y_{k,m}^b = x_{k+1}x_{m}$,
6. $y_{m}^5 = x_{n}x_{n-2}x_{n-1}x_{1}$,
7. $y_{m}^{10} = x_{n-1}x_{n-2}x_{n-3}x_{1}$,
8. $y_{11} = x_{1}x_{m}$,
9. $y_{12} = x_{1}x_{m-2}x_{m}$,
10. $y_{13} = x_{n}x_{n-2}x_{1}$,
11. $y_{14} = x_{n-1}x_{n-2}x_{3}x_{1}$,
12. $y_{15} = x_{n-2}x_{m-3}$,
13. $y_{16} = x_{n-1}x_{n-2}x_{n-3}x_{1}$,
14. $y_{17} = x_{n}x_{n-1}x_{n-2}x_{n-3}$.\]

We will show that $y_{k,m}^1, y_{k,m}^5, y_{k,m}^b$ and $y_{15}$ are all zero, so we will not need to treat $y_{m}^{5}, y_{m}^{10}, y_{11}, y_{12}, y_{14}, y_{16}$ and $y_{17}$. We will consider the seven remaining classes of monomials separately, as before modulo shorter monomials.

**Case 1:** $y = y_{k,m}^2$. We find $y_{k,m}^2 = x_{n+1}x_{n-2}x_{n-1}x_{1} \in M_{\ell}$ such that $x_{n+1}x_{n-2}x_{n-1}x_{1} = y_{n}^{7} = 0$.

**Case 2:** $y = y_{k,m}^4$. We find $y_{k,m}^4 = x_{n+1}x_{m-1}x_{m-1} \in M_{\ell}$ such that $x_{n+1}x_{m-1}x_{m-1} = y_{n}^{3} = 0$.

**Case 3:** $y = y_{m}^5$. We find $y_{m}^5 = x_{n}x_{n-2} \in M_{\ell}$ such that $x_{n}x_{n-2} = y_{n}^{7} = 0$.

**Case 4:** $y = y_{k}^6$. We find $y_{k}^6 = x_{k+1}x_{3}x_{1} \in M_{\ell}$ such that $x_{k+1}x_{3}x_{1} = y_{n}^{7} = 0$.

**Case 5:** $y = y_{m}^9$. We find $y_{m}^9 = x_{n}x_{n-2}x_{n-1}x_{1} \in M_{\ell}$ such that $x_{n}x_{n-2}x_{n-1}x_{1} = y_{n}^{7} = 0$.

**Case 6:** $y = y_{13}$. We find $y_{13} = x_{n}x_{n-2}x_{1} \in M_{\ell}$ such that $x_{n}x_{n-2}x_{1} = y_{n}^{7} = 0$.
Case 7: \( y = y^{15}. \) We find \( y^{15} = x_{n-2}x_n x_{n-3} \) \((\Omega)\) \( = x_n x_{n-2} x_{n-3} x_3 x_n x_1 = 0. \)  

2.4.4. Monomials for \( \Gamma_{C_n}. \) Recall that in \( \mathcal{F}_{\Gamma_{C_n}}, \) we modded out relations \((R1), (R4)\) and \((R2)\) from page 22. We will give a list of \((n^2 + n)/2\) monomials in \( x_1, \ldots, x_n, \) and prove in Theorem 2.21 that these monomials span \( \mathcal{F}_{\Gamma_{C_n}} \) linearly.

**Theorem 2.21.** Let 
\[
\mathcal{M}_{\Gamma_{C_n}} = \{ y_{k,m}^j = x_{k+m} | n \geq k \geq m \geq 1 \}.
\]
Let \( x \) be a monomial in \( x_1, \ldots, x_n \) of length \( \ell. \) Then \( x \) is a linear combination of monomials \( y \in \mathcal{M}_{\Gamma_{C_n}} \) with length \( y \leq \ell. \)

**Proof.** Because of Theorem 2.20 it suffices to show that the following monomials are equal to zero:
\[
\begin{align*}
&\cdot y_{k,m}^3 = x_{k+m+1} x_{m-1}, \\
&\cdot y_{k,m}^2 = x_{k+1} x_1.
\end{align*}
\]
This follows from Eq. \((R2).\)  

2.5. The parameter space

Recall that \( X_\Gamma = \{ \mathfrak{f} \mid \dim \mathcal{L}_{\Gamma, \mathfrak{f}} = |\mathcal{M}_\Gamma| \}. \) For \( \Gamma \in \{ \Gamma_{A_n}, \Gamma_{B_n}, \Gamma_{C_n}, \Gamma_{D_n} \}, \) we will find bijections \( \psi_\Gamma \) from \( X_\Gamma \) to a vector space, such that \( \mathcal{F}_\Gamma \) is isomorphic to the Lie algebra of the corresponding Chevalley type if \( \psi_\Gamma(\mathfrak{f}) \) is in a certain open dense subset of that vector space. In this section, we will be constructing the bijection. It will take some work to show that the map is injective; this will be the content of Lemmas 2.22, 2.23, 2.24 and 2.25. We will find the open dense subset in Section 2.7.

2.5.1. Parameters for \( \Gamma_{D_n}. \)

**Lemma 2.22.** Let 
\[
\psi_{\Gamma_{D_n}} : X_{\Gamma_{D_n}} \to \mathbb{P}^{n+4}, \mathfrak{f} \mapsto (f_1(x_1), f_2(x_2), \ldots, f_{n-1}(x_{n-1}), f_1(x_1^2), f_1(x_2 x_3), f_1(x_3), f_1(x_2 x_4), f_1(x_4), f_1(x_3 x_4), f_1(x_5), f_1(x_3 x_5)).
\]
Then \( \psi_{\Gamma_{D_n}} \) is injective.

**Proof.** The values of all \( f_i \) together determine the values of the extremal bilinear form on all of \( \mathcal{F}_{\Gamma_{D_n}}, \) since \( f(x_i, y, z) = f(x_i, yz) = f_i(yz). \) We will show that each value \( f_i(y) \) for \( y \in \mathcal{M}_{\Gamma_{D_n}} \) can be expressed in the values \( f_i(y) \) in the theorem. To make this notationally convenient, let \( \mathbb{F}_{\Gamma_{D_n}} \) be the rational function field obtained from \( \mathbb{F} \) by extending it with \( n + 2 \) symbols as follows. For every \( f_i(y) \) in the lemma, we extend \( \mathbb{F} \) with the symbol \( f_i(y) \) and assume that evaluating \( f_i \) at \( y \) yields this symbol \( f_i(y) \in \mathbb{F}_{\Gamma_{D_n}}. \) We will show that each value \( f_i(y) \in \mathbb{F}_{\Gamma_{D_n}} \) for all \( i \) and all \( y \in \mathcal{M}_{\Gamma_{D_n}}. \)

Let \( y = y_{i,j}^l \in \mathcal{M}_{\Gamma_{D_n}} \) of length \( \ell \) and \( i \in \{1, \ldots, n\}. \) We will use induction on \( \ell. \) We consider the seventeen classes of monomials in \( \mathcal{M}_{\Gamma_{D_n}} \) separately. Let \( x = x_i x_j y = f(x_i, y) x_j. \)

**Case 1:** \( j = 1, k \in \{1, \ldots, n\}, m \in \{1, \ldots, k\}. \) Note that \( y_{k,m}^1 = x_{k+m}. \) If \( m = k \) then \( y \) has length \( 1 \) and \( f(x_i, y) \) is either 0 or in the list of values in the theorem. So assume that \( k > m. \)
• If \( i > k + 1 \) and \( k = n - 2 \), then:
  \[
  x \overset{(AC)}{=} -x_n x_n x_{n-2} x_{m+1} x_{m+1} \overset{(Q1)}{=} -x_m x_n x_{n-2} \overset{(R1)}{=} 0.
  \]

• If \( i > k + 1 \) and \( k < n - 2 \), then
  \[
  x \overset{(Q1)}{=} x_k x_i x_{i+1} x_{i-1,m} \overset{(XT)}{=} f(x_i, y_{i+1}) x_k \overset{(R1)}{=} 0.
  \]

• If \( i = k + 1 = n \) and \( m = k - 1 \), then \( f(x_i, y_i) = f(x_n, x_{n-1}, x_{n-2}) \in \mathbb{F}_{D_n} \).

• If \( i = k + 1, m = k - 1 \) and \( 3 < i < n \), then
  \[
  f(x_i, y_i) = f(x_{i-1}, x_{i-2}) \overset{(AS)}{=} -f(x_{i-1}, x_{i-2}) \overset{(R1)}{=} 0.
  \]

• If \( i = k + 1 = 3 \) and \( m = k - 1 \) then \( f(x_i, y_i) = f(x_3, x_2 x_1) \in \mathbb{F}_{D_n} \).

• If \( i = k + 1 \) and \( 1 = m < k - 1 \), then
  \[
  x \overset{(1)}{=} x_i x_{i+1} x_{i-1,3} + x_i x_{i+1} x_{i-2} x_3 \overset{(Q1)}{=} 2 x_i x_{i+1} x_{i-1} x_{i-2} x_3 \overset{(R1)}{=} 0.
  \]

• If \( i = k \), then \( x = x_i x_{i+1} = 0 \).

• If \( i = k - 1 = m \), then
  \[
  x = x_i x_{i+1} x_{i-1} \overset{(AC)}{=} -x_i x_{i+1} = 0.
  \]

• If \( i = k - 1 > m \), then
  \[
  f(x_i, y_i) = f(x_i, x_{i+1} y_{i-1,m} \overset{(P3)}{=} -f(x_i, x_{i+1}) f(x_i, y_{i-1,m}) \in \mathbb{F}_{D_n}.
  \]

Here we use the induction hypothesis for \( f(x_i, y_{i-1,m}) \).

• If \( i < k - 1, i = m = n - 2 \), then
  \[
  f(x_i, y_i) = f(x_n, x_{n-2}) \overset{(AS, SM)}{=} f(x_n, x_{n-1}, x_n - 2) \in \mathbb{F}_{D_n}.
  \]

• If \( i < k - 1 \) and \( i = m = n - 2 \), then
  \[
  f(x_i, y_i) = f(x_n, x_{n-2}) \overset{(AC)}{=} -f(x_n, x_{n-1}, x_n - 2) \overset{(P3)}{=} f(x_n, x_{n-2}, x_{n-1}).
  \]

• If \( n - 2 = i < k - 1 \), then
  \[
  f(x_i, y_i) = f(x_n, x_{n-2}) \overset{(AS)}{=} -f(x_n, x_{n-1}, x_n - 2) \overset{(P2)}{=} f(x_n, x_{n-2}, x_{n-1}) \in \mathbb{F}_{D_n}.
  \]

• If \( n - 2 > i < k - 1 \) and \( k > 3 \), then
  \[
  x = x_i x_{i+1} x_{i-1,m} \overset{(Q1)}{=} x_k x_i x_{i-1,m} \overset{(XT)}{=} f(x_i, y_{i-1,m}) x_k \overset{(R1)}{=} 0.
  \]

• If \( i < k - 1 = 2 \) and \( m = 2 \), then \( f(x_i, y_i) = f(x_1, x_3 x_2) \in \mathbb{F}_{D_n} \).

• If \( i < k - 1 \) and \( i = n - 2 \), then \( i = 3 \) and \( m = 1 \), then we have:
  \[
  f(x_i, y_i) = f(x_1, x_3 x_2) \overset{(AS)}{=} f(x_1, x_3 x_2) \overset{(AC)}{=} f(x_1 x_3, x_1 x_2) \overset{(AS)}{=} f(x_3, x_1 x_2) \in \mathbb{F}_{D_n}.
  \]

Case 2: \( j = 2, k \in \{2, \ldots, n - 2\} \) and \( m \in \{1, \ldots, k - 1\} \). Remember that \( y_{m,k}^2 = x_{k+1-n} x_{k+1-m} \).

• If \( i > k + 1 \) and \( k = n - 2 \), then \( f(x_i, y_i) = f(x_n, x_{n-2} x_{n-1} y_{n-1,m}^\prime) \overset{(P3)}{=} \mathbb{F}_{D_n} \).
\( \text{Case 3: } j = 3, k \in \{3, \ldots, n\}, m \in \{3, \ldots, k\}. \) Remember that \( y_{k,m}^3 = x_{k|m+1}x_{m-1|1} \).

- If \( i > k + 1 \) and \( k \neq n - 2 \), then
  \( x = x_i x_i x_{k|k+1+m}^2 \quad (Q1) \quad x_k x_i y_{k+1,m}^2 \quad (XT) \quad f(x_i y_{k+1,m}^2)x_k x_i \quad (R1) = 0. \)

- If \( i = k + 1 \) and \( k = n - 2 \), then:
  \[ f(x_i, y) = f(x_{n-1}, x_{n-2} x_n x_{n-1} x_{n-2} y_{n-3,m}^1) \quad (Q4) \quad \in F_{D|m} + f(x_{n-1}, x_{n-2} x_n x_{n-2} y_{n-3,m}^1) \quad (P3) = F_{D|m}. \]

- If \( i = k + 1 \) and \( k \neq n - 2 \), then \( f(x_i, y) = f(x_i x_{i-1} x_{i+2} x_n) \in F_{D|m}. \)

- If \( i = k \), then \( x = x_i x_i x_{i-2} x_n = 0. \)

- If \( i = k - 1 = m = 1 \), then:
  \[ f(x_i, y) = f(x_i x_{i+1} x_{i+2} x_n) \quad (AC) \quad f(x_i x_{i+1} x_{i+2} x_n) \quad (P3) \in F_{D|m}. \]

- If \( i = k - 1 > m \), then:
  \[ f(x_i, y) = f(x_i x_{i+1} x_{i+2} x_n) \quad (AS) \quad f(x_i x_{i+1} x_{i+2} x_n) \quad (AC) \quad f(x_i x_{i+1} x_{i+2} x_n) \quad (AC) \quad f(x_i x_{i+1} x_{i+2} x_n) \quad (P3) \in F_{D|m}. \]

- If \( k < k - 1 \) and \( i > 1 \) or \( k > 3 \), then:
  \[ x = x_{k|1} x_{i} x_{k+1|2} x_n \quad (R1) \quad f(x_i, x_{k+1|2} x_n) x_k x_i \quad (R1) = 0. \]

- If \((i, k, m) = (1, 3, 2)\), then \( f(x_i, y) = f(x_i, x_{3|2} x_n) \in F_{D|m}. \)

- If \((i, k, m) = (1, 3, 1)\), then:
  \[ f(x_i, y) = f(x_i, x_{3|2} x_n) \quad (AS) \quad f(x_i, x_{3|2} x_n) \quad (AC) \quad f(x_i, x_{3|2} x_n) \quad (AC) \quad f(x_i, x_{3|2} x_n) \quad (Q1) \quad f(x_i, x_{3|2} x_n) \quad (P2) \quad \in F_{D|m}. \]
• If \( n = i = k + 1 \) and \( m = k \), then
\[
f(x_i, y) = f(x_{12}, x_{23}) \quad (AS) = f(x_{12}, x_{23}) \quad (AC) = -f(x_{12}, x_{23}) \quad (Q1) = -f(x_{12}, x_{23}) \quad (AS, AC) = f(x_{12}, x_{23}) \quad (R1).
\]

Now we repeat the steps in the last line \( n - 6 \) times to obtain
\[
f(x_i, y) = (-1)^n f(x_3, x_4|x_n|2x_3x_4x_1).
\]

We will treat these terms separately.

\[
f(x_3, x_4|x_n|2x_3x_4x_1) = f(x_3, x_4|x_n|2x_3x_4x_1) \quad (R1) = f(x_3, x_4|x_n|2x_3x_4x_1) \quad (Q1) = f(x_3, x_4|x_n|2x_3x_4x_1) \quad (AS, AC) = f(x_3, x_4|x_n|2x_3x_4x_1) \quad (P1).
\]

• If \( n > i = k + 1 \) and \( m = k \), then
\[
x = x_{i1}x_{i2}y_{i1} \quad (Q1) = x_{i1}x_{i2}y_{i1} \quad (XT) = x_{i1}x_{i2}y_{i1} \quad (R1).
\]

• If \( i = k + 1 \) and \( m < k \), then
\[
x = x_{i1}x_{i2}y_{i1} \quad (Q1) = x_{i1}x_{i2}y_{i1} \quad (XT) = x_{i1}x_{i2}y_{i1} \quad (R1).
\]

• If \( i = k \) and \( m = k \), then
\[
f(x_i, y) = f(x_i, x_{i-1}x_{i-1}y_{i-1}^3) \quad (P1) = f(x_i, x_{i-1}x_{i-1}y_{i-1}^3) \quad (R1).
\]

• If \( i = k \) and \( m < k \), then
\[
f(x_i, y) = f(x_i, x_{i-1}x_{i-1}y_{i-1}^3) \quad (AS) = f(x_i, x_{i-1}x_{i-1}y_{i-1}^3) \quad (AC) = 0.
\]

• If \( i = k - 1 \) and \( m = k \), then
\[
f(x_i, y) = f(x_i, x_{i+1}y_{i+1}^3) \quad (AS) = f(x_i, x_{i+1}y_{i+1}^3) \quad (AC) = 0.
\]

• If \( i = k - 1 \) and \( m = k - 1 \), then
\[
f(x_i, y) = f(x_i, x_{i+1}x_{i+1}x_{i+2}y_{i+1}^3) \quad (AS) = f(x_i, x_{i+1}x_{i+1}x_{i+2}y_{i+1}^3) \quad (AC) = -f(x_i, x_{i+1}x_{i+1}x_{i+2}y_{i+1}^3) \quad (P2) = f(x_i, x_{i+1}x_{i+1}x_{i+2}y_{i+1}^3) \quad (R1).
\]

• If \( i = k - 1 > m \), then
\[
f(x_i, y) = f(x_i, x_{i+1}x_{i+1}x_{i+2}y_{i+1}^3) \quad (P3) = f(x_i, x_{i+1}x_{i+1}x_{i+2}y_{i+1}^3) \quad (R1).
\]

• If \( i < k - 1, m < k \) and \( i \neq n - 2 \), then
\[
x = x_{i1}x_{i2}y_{i1} \quad (Q1) = x_{i1}x_{i2}y_{i1} \quad (XT) = x_{i1}x_{i2}y_{i1} \quad (R1) = 0.
\]
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- If \( i < k - 1, m < k, i = n - 2 \) and \( m = n - 1 \), then
  \[
  f(x_i, y) = f(x_{n-2}, x_n x_{n-2} y_{n-1,n-2}^3) \in \mathbb{F}_{D_{2n}} .
  \]

- If \( i < k - 1, m < k, i = m = n - 2 \), then
  \[
  f(x_i, y) = f(x_{n-2}, x_n x_{n-1} x_{n-3} y_{11}^1) \equiv -f(x_n, x_{n-2} x_{n-1} x_{n-3} y_{11}^1) = -f(x_n, y_{n-1,n-1}^3) \in \mathbb{F}_{D_{2n}},
  \]
  as proven earlier.

- If \( i < k - 1, m < k, i = n - 2 \) and \( m < n - 2 \), then
  \[
  x = x_{n-2} x_n y_{n-1,m+1} x_{n-2} y_{n,m-1}^3 \quad \text{(Q1)} \quad \Rightarrow \quad x = x_{n-2} x_n x_{n-2} x_{n,m} y_{n,m-1}^3 \quad \text{(Q1)} \quad \Rightarrow \quad f(x_{n-2}, y_{n,m-1}^3) x_{n-1} x_{n-2} \equiv 0 .
  \]

- If \( i = k - 2 > 1 \) and \( m = k \), then
  \[
  f(x_i, y) = f(x_{i+1}, x_i x_{i+1} x_{i-1} y_{11}^3) \in \mathbb{F}_{D_{2n}} + f(x_{i+1}, x_i x_{i+1} y_{11}^3) = \mathbb{F}_{D_{2n}} .
  \]

- If \( i = k - 2 = 1 \) and \( m = k \), then
  \[
  f(x_i, y) = f(x_1, x_2 x_3 y_{11}^3) \quad \text{(AC)} \quad \Rightarrow \quad f(x_1, x_2 x_1) \in \mathbb{F}_{D_{2n}} .
  \]

- If \( i = k - 3 > 1 \) or \( i < k - 3, \) and \( m = k \), then
  \[
  x = x_{i} x_{i-1} x_{i+1} y_{11}^3 \quad \text{(Q1)} \quad \Rightarrow \quad x = x_{i} x_{i} y_{11}^3 \quad \text{(Q1)} \quad \Rightarrow \quad x_{i} y_{11}^3 x_{i-1} \equiv 0 .
  \]

- If \( i = k - 3 = 1 \) and \( m = k \), then
  \[
  f(x_i, y) = f(x_1, x_3 x_4 x_2 x_3 y_{11}^3) \quad \text{(Q1)} \quad \Rightarrow \quad f(x_1, x_3 x_4 x_2 x_3 x_1) + f(x_1, x_3 x_4 x_1 x_3 x_2) \quad \text{(Q1)} \quad \Rightarrow \quad f(x_1, x_3 x_4 x_2 x_3) \quad \text{(P2)(P3)} \quad \in \mathbb{F}_{D_{2n}} .
  \]

Case 4: \( j = 4, k \in [3, \ldots, n-2], m \in [3, \ldots, k] \). Remember that \( y_{k,m}^4 = x_{k,n-2} x_{n,m} + 1 x_{n-1} y_{11}^1 \).

- If \( i > k + 1 \) and \( k = n - 2 \), then
  \[
  f(x_i, y) = f(x_{i}, x_{n-2} x_{n} y_{n-1,n-1}^3) \quad \text{(P5)} \quad \in \mathbb{F}_{D_{2n}} .
  \]

- If \( i > k + 1 \) and \( k \neq n - 2 \), then
  \[
  x = x_{i} x_{i} y_{k+1,m}^4 \quad \text{(Q1)} \quad \Rightarrow \quad x_{i} y_{k+1,m}^4 x_{i} \equiv f(x_{i} y_{k+1,m}^4) x_{i} \equiv 0 .
  \]

- If \( i = k + 1 = n - 1 \), then
  \[
  f(x_i, y) = f(x_{n-1}, x_{n-2} x_{n} x_{n-1} y_{2m}^2) \equiv -f(x_{n-2}, x_{n-1} x_{n} y_{2m}^2) \in \mathbb{F}_{D_{2n}} .
  \]

- If \( i = k + 1 \) and \( i < n - 1 \), then
  \[
  f(x_i, y) = f(x_{i-1} x_{i-2} x_{n,m} + 1 x_{n-1} y_{11}^3) \quad \text{(P2)} \quad \in \mathbb{F}_{D_{2n}} .
  \]

- If \( i = k \), then
  \[
  f(x_i, y) = f(x_{i}, x_{i+1} x_{i+2} x_{i+1} y_{11}^3) \equiv f(x_{i}, x_{i+1} x_{i+2} x_{i+1} y_{11}^3) \equiv 0 .
  \]

- If \( i = k - 1 \) and \( m = k \), then
  \[
  f(x_i, y) = f(x_{i+1} x_{i+1} x_{i+2} x_{i+1} y_{11}^3) \quad \text{(Q1)} \quad \Rightarrow \quad f(x_{i+1} x_{i+1} x_{i+2} x_{i+1} y_{11}^3) \quad \text{(P3)} \quad \in \mathbb{F}_{D_{2n}} .
  \]
Case 5: \( j = 5, \) \( m \in \{1, \ldots, n - 2\} \). Remember that \( y_m^5 = x_n x_{n-2m} \).

- If \( i = n \), then
  \[
  f(x_i, y)^{i} = f(x_n, x_n y_{n-2m}^1) \overset{\text{(AS)}}{=} f(x_n x_n, y_{n-2m}^1) \overset{\text{(AC)}}{=} 0.
  \]

- If \( i = n - 1 \), then
  \[
  f(x_i, y) = f(x_{n-1}, x_n x_{n-2m}) \overset{\text{(AS)}}{=} f(x_{n-1}, x_{n-1}) = f(x_n, y_{n-1,m}^1),
  \]
  which we have treated earlier in this proof (with the same value for the length of \( y \)).

- If \( i = n - 2 \) and \( m = n - 2 \), then
  \[
  x = x_i x_{i} x_{n} x_{n} \overset{\text{(AC)}}{=} -x_i x_{n} x_{n} \overset{\text{(XT)}}{=} -f(x_i, x_n) x_i x_i \overset{\text{(AC)}}{=} 0.
  \]

- If \( i = n - 2 \) and \( m < n - 2 \), then
  \[
  f(x_i, y) = f(x_{n-2}, x_n x_{n-2} y_{n-3m}^1) \overset{\text{(P3)}}{\in} F_{D,m}.
  \]

- If \( i < n - 2 \), then
  \[
  x = x_i x_{n} x_{n} y_{n-2m}^1 \overset{\text{(Q2)}}{=} x_i x_{n} y_{n-2m} \overset{\text{(XT)}}{=} f(x_i, y_{n-2m}^1) x_i x_i \overset{\text{(R1)}}{=} 0.
  \]
If $i = n$, then
\[ f(x_i, y) = f(x_n, x_{n-1}x_n y_{n-2}^1) \quad \text{(P3)} \in \mathbb{F}_{D_n}. \]

If $i = n - 1$, then
\[ f(x_i, y) = f(x_{n-1}, x_{n-1}x_n y_{n-2}^1) \quad \text{(AS)} = f(x_{n-1}x_n y_{n-2}^1) \quad \text{(AC)} = 0. \]

If $i = n - 2$, then
\[ f(x_i, y) = f(x_{n-2}, x_{n-1}x_n y_{n-2}^1) \quad \text{(P3)} = f(x_{n-2}, x_{n-1}x_n y_{n-2}^1) + f(x_{n-2}, x_{n-1}x_n y_{n-2}^1). \]
The first of these terms can be reduced by Eq. (P3) if $m < n - 2$, otherwise it is $f(x_{n-2}x_{n-2}, x_{n-1}x_{n-1}) = 0$. The second term is $f(x_{n-2}, y_{n-1}^1)$, which we have treated earlier.

If $i < n - 2$, then
\[ x = x_i x_{n-1}x_n y_{n-2} \quad \text{(Q1)} = x_{n-1}x_i x_n y_{n-2}^1 \quad \text{(XT)} = f(x_i, y_{n-1}^1) \quad \text{(R1)} = 0. \]

**Case 7:** $j = 7$ and $k \in \{3, \ldots, n\}$. Note that $y_k^2 = x_k x_3 x_1$.

If $i > 3$, then
\[ x = x_i x_k x_3 x_1 \quad \text{(AC)} = x_i x_k x_3 x_1 \quad \text{(Q1)} = x_i x_k x_3 x_1 \quad \text{(XT)} = f(x_i, y_{k-1}^1) x_k x_1 \quad \text{(R1)} = 0. \]

If $i \leq 3$ and $k > 5$ or if $(i, k) \in \{(1, 4), (1, 5), (2, 4), (2, 5)\}$ or if $(i, k) = (3, 5)$ with $n > 5$, then
\[ x = x_i x_k y_{k-1}^2 \quad \text{(Q1)} = x_i x_k y_{k-1}^2 \quad \text{(XT)} = f(x_i, y_{k-1}^1) x_k x_1 \quad \text{(R1)} = 0. \]

If $i = 3$, $k = 5$ and $n = 5$, then
\[ f(x_i, y) = f(x_3, x_5 x_4 x_3 x_1) \quad \text{(AS)} \quad f(x_3, x_5 x_4 x_3 x_1) \quad \text{(AC)} \quad f(x_3, x_5 x_4 x_3 x_1) \quad \text{(P3)} \in \mathbb{F}_{D_5}. \]

If $i = 3$ and $k = 4$, then
\[ f(x_i, y_k^1) = f(x_3, x_4 x_3 x_1) \quad \text{(P3)} \in \mathbb{F}_{D_5}. \]

If $i = 3$ and $k = 3$, then
\[ f(x_i, y_k^1) = f(x_3, x_3 x_1) \quad \text{(AS)} 
\]

If $i = 2$ and $k = 3$, then
\[ f(x_i, y_k^1) = f(x_3, x_3 x_1) \quad \text{(AS)} \quad f(x_3, x_3 x_1) \quad \text{(AC)} = 0. \]

If $i = 1$ and $k = 3$, then
\[ f(x_i, y_k^1) = f(x_3, x_3 x_1) \quad \text{(AC)} \quad f(x_3, x_3 x_1) \quad \text{(AS)} \quad f(x_3, x_3 x_1) \quad \text{(AC)} = 0. \]

**Case 8:** $j = 8$ and $k \in \{2, \ldots, n - 2\}$. Remember that $y_k^3 = x_k x_{n-2} x_{n-3} x_1$.

If $i > 3$ and $k > 3$, then
\[ x = x_i x_k x_{n-2} x_{n-3} x_1 \quad \text{(AC)} = x_i x_k x_{n-2} x_{n-3} x_1 \quad \text{(Q1)} = x_i x_k x_{n-2} x_{n-3} x_1 \quad \text{(XT)} = f(x_i, x_k x_{n-2} x_{n-3} x_1) x_k x_1 \quad \text{(R1)} = 0. \]

If $x_i$ and $x_k$ commute, that is, if $|k - i| > 2$ or $(i, k) \in \{(2, 2), (2, 4), (3, 3), (3, 5), (4, 2)\}$ (note that $k = 5$ does not occur if $n = 5$, whence we can include $(3, 5)$ in the previous list), or if $(i, k) = (5, 3)$ and $n > 5$, then
\[ f(x_i, y_k^1) = f(x_i, x_k x_{n-2} x_{n-3} x_1) \quad \text{(AS)} 
\]

\[ f(x_i, x_k x_{n-2} x_{n-3} x_1) \quad \text{(AC)} = 0. \]
Now we have left the cases where \((i, k) \in \{(1, 2), (1, 3), (2, 3), (3, 2), (3, 4), (4, 3)\}\) and where \((i, k, n) = (5, 3, 5)\).

- If \(i = 5, k = 3\) and \(n = 5\), then
  \[
  f(x_i, y) = f(x_5, x_3x_5x_4x_3x_1) \in \mathbb{F}_{D^5}.
  \]

- If \(i = 4, k = 3\) and \(n = 5\), then
  \[
  f(x_i, y) = f(x_4, x_3x_5x_4x_3x_1) = f(x_4, x_3(x_5x_4)x_3x_1) + f(x_4, x_3x_4x_5x_3x_1) \in \mathbb{F}_{D^5}.
  \]

- If \(i = k + 1 < n - 1\), then
  \[
  f(x_i, y) = f(x_i, x_i-1_{i\uparrow n-2}x_{n\downarrow 3}x_1) \in \mathbb{F}_{D^5}.
  \]

- If \(i = 3\) and \(k = 4\), then
  \[
  f(x_i, y) = f(x_3, x_4\uparrow n-2x_{n\downarrow 3}x_1) = -f(x_3, x_4\uparrow n-2x_{n\downarrow 3}x_1)
  \]

- If \(i = 2\) and \(k = 3\), then
  \[
  f(x_i, y) = f(x_2, x_3\uparrow n-2x_{n\downarrow 3}x_1) = -f(x_3, x_4\uparrow n-2x_{n\downarrow 3}x_1)
  \]

and since the first term is a defining element of \(\mathbb{F}_{D^5}\) and the second has been treated earlier, we find that \(f(x_i, y) \in \mathbb{F}_{D^5}\).

- If \(i = 1\) and \(k = 3\), then
  \[
  f(x_i, y) = f(x_1, x_3\uparrow n-2x_{n\downarrow 3}x_1) = -f(x_1, x_3\uparrow n-2x_{n\downarrow 3}x_1)
  \]

- If \(i = 1\) and \(k = 2\), then
  \[
  f(x_i, y) = f(x_1, x_2\uparrow n-2x_{n\downarrow 3}x_1) = f(x_1, x_2\uparrow n-2x_{n\downarrow 3}x_1)
  \]

Case 9: \(j = 9\) and \(m \in \{3, \ldots, n - 2\}\). Note that \(y_m^3 = x_nx_{n-2}\) \(m+1x_{m-1\uparrow 1}\).

- If \(i = n\), then
  \[
  f(x_i, y) = f(x_n, x_ny_{n-2}^3) = f(x_n, x_ny_{n-2}^3).
  \]
If $i = n - 1$, then
\[
x = x_{n-1}x_{n-1}x_nx_{n-2}x_{m+1}x_{m-1} \in \mathcal{I}_{D,n}
\]
\[
\begin{align*}
&= x_{n-1}x_{n-1}x_nx_{n-2}x_{m+1}x_{m-1} \\
\Rightarrow & f(x_{n-1}, x_nx_{n-2}x_{m+1}x_{m-1})x_{n-1}x_{n-1} = 0. 
\end{align*}
\]

If $i = n - 2$ and $m = n - 2$, then
\[
f(x_i, y) = f(x_{n-2}, x_nx_{n-3} \in \mathcal{I}_{D,n} \quad (P_2)
\]

If $i = n - 2$ and $m < n - 2$, then
\[
f(x_i, y) = f(x_{n-2}, x_nx_{n-2}y_{n-3,m} \in \mathcal{I}_{D,n} \quad (P_3)
\]

If $i < n - 2$, then
\[
x_i^n = x_i^n \in \mathcal{I}_{D,n} \quad (Q_1)
\]
\[
x_i^n = x_i^n \in \mathcal{I}_{D,n} \quad (R_1)
\]

Case 10: $j = 10$ and $m \in \{3, \ldots, n - 2\}$. Note that $y_{10}^m = x_{n-1}x_nx_{n-2}x_{m+1}x_{m-1} \in \mathcal{I}_{D,n}$.

If $i = n$, then
\[
f(x_i, y) = f(x_{n-1}, x_{n-1}x_{n-2}y_{n-3,m} \in \mathcal{I}_{D,n} \quad (P_2)
\]

If $i = n - 1$, then
\[
f(x_i, y) = f(x_{n-1}, x_{n-1}x_{n-2}y_{n-2,m} \in \mathcal{I}_{D,n} \quad (P_2)
\]

If $i = n - 2$ and $m = n - 2$, then
\[
f(x_i, y) = f(x_{n-2}, x_{n-1}x_{n-3}x_{m+1}x_{m-1} \in \mathcal{I}_{D,n} \quad (Q_1)
\]
\[
f(x_{n-3}, y_{n-2,n-3} \in \mathcal{I}_{D,n} \quad (Q_4)
\]

We saw that $f(x_{n-3}, y_{n-2,n-3} \in \mathcal{I}_{D,n})$ earlier in this proof.

If $i = n - 2$ and $m < n - 2$, then
\[
x = x_{n-2}x_{n-2}x_{n-1}x_nx_{n-2}x_{m+1}x_{m-1} \in \mathcal{I}_{D,n} \quad (Q_1)
\]
\[
x = x_{n-1}x_nx_{n-2}x_{m+1}x_{m-1} \in \mathcal{I}_{D,n} \quad (R_1)
\]

Case 11: $j = 11$. Note that $y_{11}^m = x_1x_2x_{m-2}x_{n-1}$.

If $i > 3$, then
\[
x = x_i x_{n-1}y_{3,1}^2 \in \mathcal{I}_{D,n} \quad (Q_1)
\]
\[
x = x_i x_{n-1}y_{3,1}^2 \in \mathcal{I}_{D,n} \quad (R_1)
\]

If $i = 3$, then
\[
f(x_i, y) = f(x_3, x_1x_{n-2}x_{n-1} \in \mathcal{I}_{D,n} \quad (P_3)
\]

If $i = 2$, then
\[
f(x_i, y) = f(x_2, x_1y_{3,1}^2) \in \mathcal{I}_{D,n} \quad (P_3)
\]

We treated this case in part 2 of this proof.
• If $i = 1$, then
\[ f(x_i, y) = f(x_1, x_1 y_{3,1}^2) = f(x_1 x_1, y_{3,1}^2) = 0. \]

**Case 12:** $j = 12$. Note that $y^{12} = x_1 x_{3|n-2} x_{n|2}$.

- If $i > 3$, then
  \[ x = x_i x_i x_1 y_{3,2}^2 = x_i x_i x_1 y_{3,2}^2 = f(x_i, y_{3,2}) x_1 x_i = 0. \]

- If $i = 3$, then
  \[ f(x_i, y) = f(x_3, x_1 x_3 y_{3,2}^2) \in \mathbb{F}_{D,n}. \]

- If $i = 2$, then
  \[ f(x_i, y) = f(x_2, x_1 y_{3,2}^2) = -f(x_1, x_2 x_{3|n-2} x_{n|2}) \]
  \[ = f(x_1, x_2 x_{3|n-2} x_{n|4} x_2 x_3) = f(x_1, x_2 x_3 x_2 x_{3|n-2} x_{n|3}) \in \mathbb{F}_{D,n}. \]

- If $i = 1$, then
  \[ f(x_i, y) = f(x_1, x_1 y_{3,2}^2) = f(x_1 x_1, y_{3,2}^2) = 0. \]

**Case 13:** $j = 13$. Remember that $y^{13} = x_n x_{n-2|3} x_1$.

- If $i = n$, then
  \[ f(x_i, y) = f(x_n, x_n y_{n-2}^7) = f(x_n x_n, y_{n-2}^7) = 0. \]

- If $i = n - 1$, then
  \[ f(x_i, y) = f(x_{n-1}, x_n x_{n-2|3} x_1) = -f(x_n, x_{n-1|3} x_1) = -f(x_n, y_{n-1}). \]

This case was treated earlier in this proof.

- If $i = n - 2$, then
  \[ f(x_i, y) = f(x_{n-2}, x_n x_{n-2|3} x_1) \in \mathbb{F}_{D,n}. \]

- If $i < n - 2$, then
  \[ x = x_i x_i x_n x_{n-2|3} x_1 = x_i x_i x_n x_{n-2|3} x_1 = f(x_i, y_{n-2}) x_n x_i = 0. \]

**Case 14:** $j = 14$. Note that $y^{14} = x_{n-1} x_n x_{n-2|3} x_1$.

- If $i = n$, then
  \[ f(x_i, y) = f(x_n, x_{n-1} x_n x_{n-2|3} x_1) \in \mathbb{F}_{D,n}. \]

- If $i = n - 1$, then
  \[ f(x_i, y) = f(x_{n-1}, x_n x_{n-1|3} x_1) = f(x_{n-1} x_{n-1}, x_n x_{n-2|3} x_1) = 0. \]

- If $i = n - 2$ and $n = 5$, then
  \[ f(x_i, y) = f(x_5, x_4 x_5 x_3 x_1) = -f(x_4, x_3 x_5 x_3 x_1) \in \mathbb{F}_{D,n}. \]

- If $i = n - 2$ and $n > 5$, then
  \[ x = x_{n-2} x_{n-2|2} x_{n-1} x_n x_{n-2|3} x_1 = x_{n-2} x_{n-2|2} x_{n-1} x_n x_{n-2|2} x_{n-1} x_3 \]
  \[ = f(x_{n-2}, x_{n-1} x_n x_{n-2|3} x_1) x_1 x_{n-2} \]
  \[ = f(x_{n-2}, x_{n-1} x_n x_{n-2|3} x_1) x_1 x_{n-2} = 0. \]

- If $i < n - 2$, then
  \[ x^i = x_i x_{n-1} x_n y_{n-2}^7 = x_{n-1} x_i x_n y_{n-2}^7 = f(x_i, y_{n-2}) x_{n-1} x_i = 0. \]

**Case 15:** $j = 15$. Note that $y^{15} = x_{n-2} x_n x_{n-3|3} x_1$. 

• If \( i = n \), then
\[
f(x_i, y) = f(x_n, x_{n-2}x_ny_{n-2,n-2}^3) \in \mathbb{F}_{D,n}.
\]

• If \( i = n - 1 \), then
\[
f(x_i, y) = f(x_{n-1}, x_{n-2}x_ny_{n-2,n-2}^3) = f(x_{n-2}, x_{n-1}x_ny_{n-2,n-2}^3) = -f(x_{n-2}, y_{n-2,n-2}^3).
\]

We treated this case earlier in this proof.

• If \( i = n - 2 \), then
\[
f(x_i, y) = f(x_{n-2}, x_{n-2}x_ny_{n-2,n-2}^3) = f(x_{n-2}x_{n-2}, x_ny_{n-2,n-2}^3) = 0.
\]

• If \( i = n - 3 \), then
\[
f(x_i, y) = f(x_{n-3}, x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = f(x_{n-2}x_{n-3}, x_{n-2}x_{n-2}x_nx_{n-4}y_{n-2,n-2}^3) \in \mathbb{F}_{D,n}.
\]

• If \( i < n - 3 \), then
\[
x = x_ix_{n-2}x_ny_{n-2,n-2}^3 = f(x_i, x_{n-2}x_ny_{n-2,n-2}^3) = 0.
\]

Case 16: \( j = 16 \). Remember that \( y_{16} = x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3 \).

• If \( i = n \), then
\[
f(x_i, y) = f(x_n, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = f(x_{n-1}, x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) \in \mathbb{F}_{D,n}.
\]

• If \( i = n - 1 \), then
\[
f(x_i, y) = f(x_{n-1}, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = f(x_{n-1}, x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = 0.
\]

• If \( i = n - 2 \), then
\[
f(x_i, y) = f(x_{n-2}, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = f(x_{n-2}, x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = 0.
\]

• If \( i < n - 2 \), then
\[
x = x_ix_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3 = f(x_i, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = 0.
\]

Case 17: \( j = 17 \). Remember that \( y_{17} = x_nx_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3 \).

• If \( i = n \), then
\[
f(x_i, y) = f(x_n, x_nx_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = f(x_{n-1}, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = 0.
\]

• If \( i = n - 1 \), then
\[
f(x_i, y) = f(x_{n-1}, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = f(x_{n-2}, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = 0.
\]

• If \( i = n - 2 \), then
\[
f(x_i, y) = f(x_{n-2}, x_nx_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = f(x_n, x_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = 0.
\]

• If \( i < n - 2 \), then
\[
x = x_ix_nx_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3 = f(x_i, x_nx_{n-1}x_{n-2}x_nx_{n-3}y_{n-2,n-2}^3) = 0.
\]
2.5.2. Parameters for $\Gamma_{B,n}$.

**Lemma 2.23.** Let
$$
\psi_{\Gamma_{B,n}} : X_{\Gamma_{B,n}} \to \mathbb{F}^{n+2}, \{i\} \mapsto (f_1(x_2), f_2(x_3), \ldots, f_{n-2}(x_{n-1}); f_1(x_3), f_1(x_2x_3), f_{n-2}(x_n), f_1(x_{\Delta-1}x_{n+2}).
$$
Then $\psi_{\Gamma_{B,n}}$ is injective.

**Proof.** Remember that $\mathcal{F}_{\Gamma_{B,n}}$ is a quotient of $\mathcal{F}_{\Gamma_{D,n}}$. Hence relations between values of $f_i$ that hold in $\mathcal{F}_{\Gamma_{D,n}}$ hold in $\mathcal{F}_{\Gamma_{B,n}}$, as well. This allows us to express $f_i(y)$ in the values of Lemma 2.22. It then suffices to prove that $f(x_n, x_{n-1})$ and $f(x_n, x_{n-1}x_{n-2})$ are zero:

$$
x_n x_{n-1} x_{n-2} \quad \text{(R3)}
$$

$$
f(x_n, x_{n-1}x_{n-2}) = f(x_n, x_{n-1}x_{n-1}) \quad \text{(AS)}
$$

$$
f(x_n, x_{n-1}x_{n-2}) = f(x_n, x_{n-1}x_{n-2}) \quad \text{(R3)}
$$

Then $\psi_{\Gamma_{A,n}}$ is injective.

2.5.3. Parameters for $\Gamma_{A,n}$.

**Lemma 2.24.** Let
$$
\psi_{\Gamma_{A,n}} : X_{\Gamma_{A,n}} \to \mathbb{F}^{n+1}, \{i\} \mapsto (f_1(x_2), f_2(x_3), \ldots, f_{n-1}(x_n); f_1(x_3), f_1(x_2x_3)).
$$
Then $\psi_{\Gamma_{A,n}}$ is injective.

2.5.4. Parameters for $\Gamma_{C,n}$.

**Lemma 2.25.** Let
$$
\psi_{\Gamma_{C,n}} : X_{\Gamma_{C,n}} \to \mathbb{F}^{n-1}, \{i\} \mapsto (f_1(x_2), f_2(x_3), \ldots, f_{n-1}(x_n)).
$$
Then $\psi_{\Gamma_{C,n}}$ is injective.

2.5.6. $\psi_{\Gamma}(X_{\Gamma})$ is a closed subset of the respective vector spaces in the above lemmas, for all $\Gamma \in \{\Gamma_{A,n}, \Gamma_{B,n}, \Gamma_{C,n}, \Gamma_{D,n}\}$.

**Proof.** By Lemma 2.13, $X_{\Gamma}$ is a closed set; since $\psi_{\Gamma}$ is continuous, its image is closed as well.

**Corollary 2.27.** $\psi_{\Gamma}(X_{\Gamma})$ is an algebraic variety for all $\Gamma \in \{\Gamma_{A,n}, \Gamma_{B,n}, \Gamma_{C,n}, \Gamma_{D,n}\}$.
2.6. Realizations of the four classical families

In this section, we will find generating sets of extremal elements for the four classical families of Lie algebras, where these generating sets realize the graphs $\Gamma_{A_n}$, $\Gamma_{B_n}$, $\Gamma_{C_n}$, and $\Gamma_{D_n}$. We keep $n$ as the number of extremal generators and will see that these graphs correspond to the Lie algebras of type $A_{n-1}$, $B_{n-1}$, $C_{n/2}$, and $D_n$. In particular, the objective of this section will be the formulating and proving of Theorems 2.37, 2.38, 2.44 and 2.45, giving these explicit generators.

The extremal elements in Lie algebras of type $A_{n-1}$ or $C_{n/2}$ correspond to the so-called transvections, which will be discussed in Section 2.6.1. In the orthogonal Lie algebras, the extremal elements correspond to the Siegel transvections or Siegel transformations. These are examined in Section 2.6.2. In both sections, we first explore these elements in a general setting, and then discuss generators for the two series of Lie algebras specifically.

2.6.1. Transvections. Let $n \in \mathbb{N}_+$, $x \in V = \mathbb{F}^n$ and $h \in V^*$. In Lemma 2.30, we will see that the linear transformation $x \otimes h : v \mapsto h(v) x$ is an extremal element of $\mathfrak{sl}(V)$ if $h(x) = 0$ and $x, h$ nonzero. A transvection is a linear transformation of the form $1 + x \otimes h$ where $h(x) = 0$ and $x, h$ nonzero. We call $x$ the centre of the transvection and $h$ the axis. Then call $x \otimes h$ an infinitesimal transvection.

A transvection group is a group $\{1 + tx \otimes h \mid t \in \mathbb{F}\}$. The Lie algebra of a transvection group, as defined near Example 1.9 in Chapter 1, consists of the transvections $tx \otimes h$. We will use a result of McLaughlin [McL67] which classifies groups generated by transvection subgroups. This is a weaker version of a reformulation by Cameron and Hall, Theorem 2 from [CH91]:

**Theorem 2.28.** Let $G$ be a nontrivial group of linear transformations of the finite-dimensional $\mathbb{F}$-vector space $V$, which is generated by $\mathbb{F}$-transvection subgroups. If $V$ is spanned by a $G$-orbit on centres of these transvection subgroups, and $V^*$ is spanned by the axes, then one of the following holds:

1. $G = \text{SL}(V)$;
2. $G = \text{Sp}(V, B)$ for some symplectic form $B$.

To obtain this weaker version, we have used Lemma 3 of [McL67].

We will need a tool to distinguish between $\text{SL}(V)$ and $\text{Sp}(V, B)$. This tool will be provided by analysis of the occurrence of Heisenberg subalgebras generated by pairs of extremal elements, further detailed in Lemma 2.32 and Corollary 2.36.

An infinitesimal transvection $x \otimes h$ is a linear transformation with trace 0, since $(\text{Tr} x \otimes h)^2 = \text{Tr}(x \otimes h)^2 = \text{Tr} 0 = 0$. Hence infinitesimal transvections are in $\mathfrak{sl}(V)$. It is easy to see that every matrix with a single nonzero entry that is off-diagonal corresponds to an infinitesimal transvection. This can be extended to a spanning set of $\mathfrak{sl}(V)$ by taking into account the transformations of the form $(e_i - e_j) \otimes (f_i + f_j)$, where $(e_i)_{i=1}^n$ is a basis of $V$ and $(f_i)_{i=1}^n$ is its dual basis. This leads to the following lemma.

**Lemma 2.29.** $\mathfrak{sl}(V)$ is spanned by infinitesimal transvections.

Since

$$[x \otimes h, y \otimes k](v) = h(y)k(v)x - k(x)h(v)y = (h(y)x \otimes k - k(x)y \otimes h)(v),$$

the multiplication is given by

$$[x \otimes h, y \otimes k] = h(y)x \otimes k - k(x)y \otimes h.$$  \hspace{1cm} (2.6)

Hence

$$[x \otimes h, [x \otimes h, y \otimes k]] = [x \otimes h, h(y)x \otimes k - k(x)y \otimes h] = -2h(y)k(x)x \otimes h.$$  \hspace{1cm} (2.7)
We find the following lemma.

**Lemma 2.30.** Infinitesimal transvections are extremal elements of $\mathfrak{sl}(V)$, and $f(x \otimes h, y \otimes k) = -2h(y)k(x)$. Furthermore,

$$\exp(t \text{ad } x \otimes h)(y \otimes k) = (y + th(y)x) \otimes (k - tk(x))h).$$

**Lemma 2.31.** All extremal elements in $\mathfrak{sl}(V)$ are of the form $x \otimes h$, with $h(x) = 0$.

**Proof.** Let $M$ be the matrix of an extremal element of $\mathfrak{sl}(V)$. Then for any traceless matrix $A$, we have that

$$[M, [M, A]] = M^2 A - 2MAM + AM^2 \in \mathbb{F}M. \quad (2.8)$$

Since the statement is vacuously true if we substitute $A = 1$ and since all terms are linear in $A$, it holds for every matrix $A$ regardless of trace. We set $M$ equal to the block matrix

$$\left( \begin{array}{cc} a & b^T \\ \frac{1}{c} & D \end{array} \right),$$

with blocks of size $1$ and $n-1$, respectively, for both rows and columns. Now substitute for $A$ the block matrix $\left( \begin{array}{c} 1 \\ 0 \end{array} \right)$, then Eq. (2.8) gives that

$$M' = \left( \begin{array}{cc} 2b^Tc & b^T(D - aI) \\ (D-aI)c & -2cb^T \end{array} \right)$$

is a multiple of $M$. In $M'$, every $2 \times 2$-subdeterminant not involving the $(1,1)$-entry is 0, since for every row, the last $n-1$ entries are a multiple of $b^T$ and for every column, the last $n-1$ entries are a multiple of $c$. If $M' \neq 0$, the same property holds for $M$; otherwise $b$ and $c$ are 0. In other words, if $M$ has off-diagonal nonzero entries in the first row or column, then any $2 \times 2$-subdeterminant not involving the $(1,1)$-entry is zero.

Since extremality is a property of a linear transformation rather than of a matrix itself, the conjugate of $M$ by an invertible matrix is also extremal. Using the permutation matrix of the permutation interchanging 1 and $i$, we find that if $M$ has off-diagonal nonzero entries in the $i$th row or column, then any $2 \times 2$-subdeterminant not involving the $(i,i)$-entry is zero.

If $M$ has an off-diagonal nonzero entry at position $(i, j)$, then there is an off-diagonal nonzero entry in the $k$th row or column for $k$ equal to either $i$ or $j$. Thus any $2 \times 2$-subdeterminant is 0. Then $M$ has rank 1. So $M$ can be written as $x \otimes h$. But then the trace of $M$ is $h(x)$, if we take a basis containing $x$ and a suitable cobasis containing $h$. So $h(x) = 0$. We see that $M$ is an infinitesimal transvection.

If $M$ has no off-diagonal nonzero entries, then it is a diagonal matrix, say with diagonal $(m_1, \ldots, m_n)$. Then substitute the matrix $E_{ij}$ for $x$ in Eq. (2.8), which is the matrix that is zero everywhere except at position $(i, j)$ where it is one. This leads to the conclusion that $(m_i - m_j)^2 E_{ij}$ is a multiple of $M$; hence, all $m_i$ are equal. Thus $M$ is a multiple of $I$. Then it is 0.

**Lemma 2.32.** Let $x \otimes h$ and $y \otimes k$ be two infinitesimal transvections and let $L = \langle x \otimes h, y \otimes k \rangle_{\text{Lie}}$. Then the isomorphism class of $L$ depends on the geometrical configuration of $\mathbb{F}x, \mathbb{F}y, \text{Ker } h$ and $\text{Ker } k$, as follows:

- If $\mathbb{F}x = \mathbb{F}y$ and $\text{Ker } h = \text{Ker } k$, then $L$ is one-dimensional.
- If either $\mathbb{F}x = \mathbb{F}y$ or $\text{Ker } h = \text{Ker } k$ but not both, then $L$ is two-dimensional.
- Assume for the other cases that $\mathbb{F}x \neq \mathbb{F}y$ and $\text{Ker } h \neq \text{Ker } k$.
- If $\mathbb{F}x \subset \text{Ker } k$ and $\mathbb{F}y \subset \text{Ker } h$, then $L$ is two-dimensional as in the preceding case.
- If either $\mathbb{F}x \subset \text{Ker } k$ or $\mathbb{F}y \subset \text{Ker } h$ but not both, then $L$ is isomorphic to the Heisenberg algebra.
- If $\mathbb{F}x \not\subset \text{Ker } k$ and $\mathbb{F}y \not\subset \text{Ker } h$, then $L$ is isomorphic to $sl_2$. 


2. GRAPHS FOR CLASSICAL LIE ALGEBRAS

The first three cases are clear either immediately or after using Eq. (2.6). For the last two, we see that $x \otimes h$ and $y \otimes k$ do not commute and that the extremal form is zero and nonzero, respectively. □

If $x \otimes h$ and $y \otimes k$ generate a Heisenberg algebra, we say that they form an extraspecial pair.

We will also realize $\text{sp}(V)$, the Lie algebra of type $C_{n/2}$, using infinitesimal transvections. In order to do this, let us assume that $n$ is even and that we have a nondegenerate symplectic form $B$. We will denote the matrix of $B$ by $B$ as well. We write

$$u(y) := y \otimes (v \mapsto B(y, v)): V \to V, v \mapsto B(y, v)y.$$ 

Hence $u(y)$ is an infinitesimal transvection. We will prove that all infinitesimal transvections are of this form:

Lemma 2.33. The infinitesimal transvections in the Lie algebra of type $C_{n/2}$ can all be written as $u(y)$ for some $y$.

Proof. If we view $B$ as a matrix, then the algebra consists of those matrices $A$ for which $A^T B = -BA$. If this should hold for the matrix of the infinitesimal transvection $x \otimes h$, then for all $v, w \in V$:

$$h(v)B(x, w) = v^T (x \otimes h)^T Bw = -v^T B(x \otimes h)w = -h(w)B(v, x) = h(w)B(x, v).$$

So if $B(x, v) \neq 0$, we see that $h(v)/B(x, v)$ is independent of $v$, and if $B(x, v) = 0$ then $h(v) = 0$ as well. So $h$ is a multiple of $B(x, \cdot)$, say $h(v) = aB(x, v)$. If $a = b^2$, then $x \otimes h = u(bx)$. □

Since $u(y)$ is an extremal element in $\text{sl}(V)$, it is also an extremal element in $\text{sp}(V)$. The following two lemmas are equivalents of Lemmas 2.29 and 2.31 for $\text{sp}(V)$.

Lemma 2.34. $\text{sp}(V)$ is spanned by its infinitesimal transvections.

Proof. We may assume that $B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Then all matrices $A$ of elements of $\text{sp}(V)$ are of the form $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & -A_{10}^T \end{pmatrix}$ with $A_{01}$ and $A_{10}$ symmetric; this follows from the identity $A^T B = -BA$. Let $y \in V$ and write $y = (y_0 \ y_1)$ to denote the decomposition of $y$ into its first set of $n$ coordinates and its second set of $n$ coordinates. The matrix of the infinitesimal transvection $u(y)$ is then

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & -A_{10}^T \end{pmatrix} = y(By)^T = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \begin{pmatrix} y_0^T & y_1^T \\ -y_1 & -y_0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} -y_0y_1^T \\ -y_1y_0^T \end{pmatrix} = \begin{pmatrix} y_0y_1^T \\ y_1y_0^T \end{pmatrix}.$$ 

By taking $y_0$ zero and $y_1$ equal to a standard basis vector, we find matrices with only one nonzero entry, on the diagonal of $A_{10}$. If instead $y_1$ is the sum of two different standard basis vectors, then modulo those diagonal elements we obtain for $A_{10}$ the symmetric matrix with two nonzero off-diagonal elements. Together these constitute a basis for the space of all elements of $\text{sp}(V)$ that are zero outside $A_{10}$. We find a basis for $A_{01}$ similarly. To complete this to a basis of $\text{sp}(V)$ we may proceed modulo entries in $A_{01}$ and $A_{10}$. Hence we can choose $y_0$ and $y_1$ to both have a single nonzero entry. □

Lemma 2.35. All extremal elements in $\text{sp}(V)$ are infinitesimal transvections.

Proof. Let $M$ be the matrix of an extremal element of $\text{sp}(V)$, then $M$ can be written as a block matrix $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & -M_{11}^T \end{pmatrix}$, where $M_{12}$ and $M_{21}$ are symmetric. For every matrix $A$ also in $\text{sp}(V)$, Eq. (2.8) holds. In general we will take for $A$ a block matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with
$A_{12}$ and $A_{21}$ symmetric and $A_{22} = -A_{11}^T$. We alternate between different matrices $A$ and restrict the possibilities for $M$ at every step.

If we take $A_{11}$ and $A_{12}$ (and thus $A_{22}$) equal to zero, then Eq. (2.8) gives, among others, that $M_{12}A_{21}M_{12}$ is a multiple of $M_{12}$. Suppose that $M_{12}$ has a nonzero entry in position $(i, j)$. Taking for $A_{21}$ the matrix $E_{ij}$ with a one in position $(j, i)$ and zeroes elsewhere, $M_{12}A_{21}M_{12}$ is the matrix with as $(k, \ell)$-entry the product of the $(k, j)$- and the $(i, \ell)$-entries of $M_{12}$. This matrix certainly has a nonzero $(i, j)$-entry, so it is a nonzero multiple of $M_{12}$. Furthermore, it is a rank one matrix, so $M_{12}$ is also a rank one matrix, and since it is symmetric, we can write $M_{12} = y_0y_0^T$ (remembering that $F$ was algebraically closed).

If, on the other hand, $M_{12}$ is the zero matrix, this equality can clearly also be attained by setting $y_0$ equal to the null vector. A similar reasoning shows that we can find $y_1$ with $M_{21} = -y_1y_1^T$. We now have

$$M = \begin{pmatrix} \; M_{11} & y_0y_0^T \\ -y_1y_1^T & -M_{11}^T \end{pmatrix}$$

and wish to show that $M_{11} = -y_0y_1^T$ as in the proof of Lemma 2.34.

Like in the proof of Lemma 2.31, we use the fact that substituting the identity matrix for $A$ yields a tautology. This gives us the possibility to make $A_{11}$ different from $A_{22}$, in particular, we take $A_{11} = I$ and set $A_{12}, A_{21}$ and $A_{22}$ to zero. Then we obtain that

$$M' := \begin{pmatrix} 2M_{12}M_{21} & -M_{11}M_{12} - M_{12}M_{11}^T \\ -M_{21}M_{11} - M_{11}^T M_{21} & -2M_{21}M_{12} \end{pmatrix}$$

is a multiple of $M$. (2.9)

We proceed by case distinction.

**Case 1:** If $M'$ is nonzero, then $M_{12}M_{21} = -y_0y_0^Ty_1y_1^T$ is a multiple of $M_{11}$ by a nonzero factor; this is $-y_0^Ty_1$ (a scalar) times $y_0y_1^T$. Thus, for some $\alpha \in F$, we can write $M$ as

$$\begin{pmatrix} \alpha y_0y_1^T & y_0y_0^T \\ -y_1y_1^T & -\alpha y_1y_0^T \end{pmatrix}.$$

Then $M'_{11}$ from observation (2.9) is equal to

$$\begin{pmatrix} -2(y_0^Ty_1)y_0y_1^T & -2\alpha(y_0^Ty_1)y_0y_0^T \\ 2\alpha(y_0^Ty_1)y_1y_1^T & 2(y_0^Ty_1)y_1y_0^T \end{pmatrix}.$$

At the upper right block, we see that this matrix is $-2\alpha(y_0^Ty_1)$ times $M_1$; the upper left block then shows that $\alpha^2 = 1$. For $\alpha = 1$, we have $M_{11} = -y_0y_1^T$; for $\alpha = -1$, substitute $-y_1$ for $y_1$, then we have the same.

**Case 2:** If the matrix $M'$ of observation (2.9) is the zero matrix, then $y_0^Ty_1 = 0$ (whence $M_{12}M_{21} = M_{21}M_{12} = 0$) and $M_{11}y_0y_0^T = -y_0y_0^T M_{11}^T$. Hence $M_{11}y_0y_0^T$ is an antisymmetric matrix of rank at most one; the only matrix that satisfies those criteria is the null matrix. So $M_{11}y_0$ and $y_0^TM_{11}$ are zero, and by a similar reasoning, so are $M_{11}y_1$ and $y_1^TM_{11}$. Hence all products of distinct elements of $[M_{11}, M_{12}, M_{21}]$ are zero.

We take a different matrix $A$, with $A_{12} = A_{21} = 0$ and as before $A_{22} = -A_{11}^T$. Using the fact that some products are zero, we find that

$$\begin{pmatrix} [M_{11}, [M_{11}, A_{11}]] + 2M_{12}A^T_{11}M_{21} & -2(M_{11}A_{11}M_{12} + M_{12}A^T_{11}M_{11}^T) \\ -2(M_{21}A_{11}M_{11} + M_{11}^T A^T_{11}M_{21}) & -[M_{11}, [M_{11}, A_{11}]] - 2M_{21}A_{11}M_{21} \end{pmatrix} =: M''$$

is a multiple of $M$. (2.10)

Take $A_{11} = E_{ij}$, the matrix with as only nonzero entry a one at position $(i, j)$, then the upper right block of $M''$ is the matrix with $(k, \ell)$-entry $-2(y_{0k})(y_{0j})(y_{0\ell}) + (M_{11})_{k, j}(y_{0\ell})$; this should be a multiple of $(y_0)_k(y_0)_\ell$, by a factor that does not depend on $k$ and $\ell$, say a
factor of \(-2\alpha_{i,j}\). If \(y_0\) has a nonzero entry in coordinate \(j\), then we find that \(\alpha_{i,j} = 2(M_{11})_{(i,j)}\). By then examining the \((k, k)\)-entries of \(M''\) and \(M\), we see that

\[(M_{11})_{(k,k)}(y_0)_j = (M_{11})_{(i,j)}(y_0)_k.\]  

(2.11)

if \((y_0)_k\) is also nonzero. If, on the other hand, \((y_0)_k = 0\), then consider the \((j, k)\)-entries; they show that \(M_{(k,j)} = 0\). So if \(y_0\) has a nonzero entry, then Eq. (2.11) holds for all \(i, j\) and \(k\). Similarly, if \(y_1\) has a nonzero entry, then

\[(M_{11})_{(i,k)}(y_0)_j = (M_{11})_{(i,j)}(y_0)_k\]

for all \(i, j\) and \(k\). We perform case distinction again.

**Case 2a:** If \(y_0\) and \(y_1\) both have nonzero entries, say at coordinates \(i\) and \(j\), respectively, then \(M_{11} = \alpha y_0 y_1^T\). Then if we substitute \(E_{ij}\) for \(A\) in the definition of \(M''\) from observation (2.10), we find that

\[\begin{pmatrix}
-2\alpha^2 - 2(y_0)_i(y_1)_j y_0 y_1^T \\
4\alpha(y_0)_i(y_1)_j y_1 y_1^T
\end{pmatrix}
\]

-4\alpha \begin{pmatrix}
y_0 y_1^T \\
y_0 y_1^T
\end{pmatrix}

\[2(\alpha^2 + 2)(y_0)_i(y_1)_j y_1 y_1^T\]

is a multiple of \(M\).

The upper right block shows that the factor is \(-4\alpha(y_0)_i(y_1)_j\); the upper left block then gives us that \(\alpha^2 = 1\). By the same reasoning as in case 1, \(M\) is the matrix of an infinitesimal transvection.

**Case 2b:** If \(y_0\) or \(y_1\) is the zero vector, then we need to show that \(M_{11} = 0\). We will assume that \(M_{11}\) is nonzero and derive a contradiction. The upper left block of \(M''\) in observation (2.10) reduces to \([M_{11}, [M_{11}, A_{11}]\]). Decompose \(M_{11}\) into a part \(M'_{11}\) of trace 0 and a multiple \(aI\) of the identity matrix. Since

\[\left[M'_{11} + aI, [M'_{11} + aI, A_{11}]\right] = [M'_{11}, [M'_{11}, A_{11}]\]

is in \(\langle M'_{11}, I \rangle\) and traceless, it is a multiple of \(M'_{11}\). Hence \(a = 0\) and \(M_{11} = M'_{11}\) is an infinitesimal transvection by Lemma 2.31.

If, say, \(y_0 = 0\), then write \(M_{11} = xh^T\) for some nonzero vectors \(x\) and \(h\) with \(h^T x = 0\). Let \(g\) be such that \(g^T x \neq 0\), let \(A_{11} = A_{21} = A_{22} = 0\) and let \(A_{12} = gg^T\). Then

\[M_{11} = \begin{pmatrix}
0 & 0 \\
2hx^T g g^T x h^T & 0
\end{pmatrix} = 2(\alpha^T x)^2 \begin{pmatrix}
0 & 0 \\
h h^T & 0
\end{pmatrix}\]

is a nonzero multiple of \(M\).

This gives a contradiction with \(M_{11} \neq 0\).

If \(y_1\) has nonzero entries but \(y_0\) does not, a similar reasoning as above applies.

**Corollary 2.36.** \(sw(V)\) does not contain an extraspecial pair.

**Proof.** Let \(x \otimes h\) and \(y \otimes k\) form an extraspecial pair. Then we may assume that \(h(y) = 0 \neq k(x)\). But \(h(y) = B(x, y) = -B(y, x) = -k(x)\).

We now proceed with the theorems giving the generating extremal elements.

**Theorem 2.37.** Suppose that \(n\) is even. Let \(B\) be the nondegenerate symplectic form determined by the matrix \(\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}\). The transformations

\[u(x_i) : v \mapsto B(x_i, v)x_i, \quad i = 1, \ldots, n,\]

realize the graph \(\Gamma_{C_n}\) for \(n \geq 2\) if we take these values for \(x_i:\)

\[x_{2\ell-1} = \epsilon \ell \quad \text{for } 1 \leq \ell \leq n/2,\]

\[x_{2\ell} = \epsilon \ell + n/2 + \epsilon \ell + n/2 + 1 \quad \text{for } 1 \leq \ell < n/2,\]

\[x_n = \epsilon n.\]

The Lie algebra \(L = \langle u(x_i) \rangle_{\text{Lie}}\) is \(sw_n\).
Proof. It is easy to check that the transformations \( u(x) \) realize the graph. In order to prove that they generate \( \mathfrak{sp}_n \), consider the group \( G = \langle 1 + t u(x) \mid t \in F, 1 \leq i \leq n \rangle_{G_p} \), of which \( L \) is the Lie algebra. The action of \( G \) on \( L \) is such that

\[
u(x)^{1+t u(y)} = (1 + t u(y)) u(x) (1 - t u(y)) = u(x^{1+t u(y)}),
\]
as can be immediately seen by inspection. So if \( B(x, y) \neq 0 \), then

\[
(1 + B(x, y)^{-1} u(y))(1 + B(x, y)^{-1} u(x)) = x.
\]

This shows that the orbit of \( G \) on \( x_1 \) spans \( V \). Hence, using Theorem 2.28, either \( G = \text{Sp}(V, B) \) or \( G = \text{SL}(V) \). Thus \( L \) is either \( \mathfrak{sp}_n \) or \( \mathfrak{sl}_n \). But all given transformations are in \( \mathfrak{sp}_n \). This can be verified by examining the matrices \( A_i^T M + MA_i \), where \( M \) is the matrix of \( B \) and \( A_i \) is the matrix of \( u(x_i) \): if we view elements of \( V \) as column vectors, then \( A_i = x_i x_i^T M \), so \( A_i^T = -M x_i x_i^T \). Thus \( A_i^T M + MA_i = 0 \), so \( L = \mathfrak{sp}_n \).

**Theorem 2.38.** The transformations \( x_i \otimes h_i \) realize the graph \( \Gamma_{A_{n}} \) for \( n \geq 2 \) if we take these values for \( x \) and \( h \):

\[
\begin{align*}
x_1 &= e_1 - e_2 & h_1 &= f_1 + f_2, \\
x_i &= e_{i-1} + e_i & h_i &= f_{i-1} - f_i & \text{for } 1 < i \leq n.
\end{align*}
\]

The Lie algebra \( L = \langle x_i \otimes h_i \rangle_{\text{Lie}} \) is \( \mathfrak{sl}_n \).

Proof. It is easy to check that the transformations \( x_i \otimes h_i \) realize the graph. In order to prove that they generate \( \mathfrak{sl}_n \), consider the group \( G = \langle 1 + t(x_i \otimes h_i) \mid t \in F, 1 \leq i \leq n \rangle_{G_p} \), of which \( L \) is the Lie algebra. It is clear that the orbit of \( G \) on \( x_1 \) spans \( V \). Hence, using Theorem 2.28, either \( G = \text{Sp}(V, B) \) or \( G = \text{SL}(V) \). Hence \( L \) is either \( \mathfrak{sp}_n \) or \( \mathfrak{sl}_n \). Now consider \( \exp(2 \text{ad} x_3 \otimes h_3)(x_1 \otimes h_1) = (x_1 - 2x_3) \otimes (h_1 - 2h_3) \). It forms an extraspecial pair with \( x_2 \otimes h_2 \), because \( (h_1 - 2h_3)(x_2) = 0 \neq h_2(x_1 - 2x_3) \). By Corollary 2.36, \( L = \mathfrak{sl}_n \).

### 2.6.2. Siegel transvections

In order to realize the two orthogonal types of algebras, for graphs \( \Gamma_{B_m} \) and \( \Gamma_{D_m} \), we will need a different kind of linear transformations: so-called **Siegel transvections**. As an equivalent of Theorem 2.28 we will use the main theorem of Steinbach’s paper [Ste97], dealing with Siegel transvection groups in a similar way to how Theorem 2.28 deals with transvection groups. Steinbach’s main theorem is reprinted here in a weaker form as Theorem 2.42.

Let the dimension of \( V \) be \( 2n \) or \( 2n - 1 \). Let \( B \) be a nondegenerate orthogonal bilinear form on \( V \); we will denote the corresponding matrix by \( B \) as well. We may assume that

\[
\begin{align*}
\text{if } d &= 2n, \quad &B &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\text{if } d &= 2n - 1, \quad &B &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\end{align*}
\]

Let \( u, v \in V \) be linearly independent with \( B(u, u) = B(u, v) = B(v, v) = 0 \); in other words, \( u \) and \( v \) span an **isotropic line**. When we speak of collinearity or of lines in an orthogonal space, we will mean this with respect only to isotropic lines. Then

\[
S_{u,v} : V \rightarrow V, \quad x \mapsto x + B(u, v)v - B(v, v)u
\]
is known as the **Siegel transvection** determined by \( u \) and \( v \). If \( S_{u,v} \) is a Siegel transvection, then we will call the map \( T_{u,v} = S_{u,v} - 1 \) an **infinitesimal Siegel transvection**. Note that \( T_{u,v} \) is determined up to scalar multiples by the projective line containing \( u \) and \( v \). We call the group \( \langle 1 + t T_{u,v} \mid t \in F \rangle_{G_p} \) a **Siegel transvection group**.
Lemma 2.39. o(V) is spanned by infinitesimal Siegel transvections.

Proof. We will assume the odd-dimensional case here; the even-dimensional case is proven similarly. The matrix $A$ of an element of $o(V)$, which satisfies $A^T B = -B A^T$, is then of the form

$$A = \begin{pmatrix} A_{00} & A_{01} & -2a_{21} \\ A_{10} & -A_{00} & -2a_{20} \\ a_{20}^T & a_{21}^T & 0 \end{pmatrix}$$

with $A_{ij} \in F^{(n-1) \times (n-1)}$ and $a_{ij} \in F^{n-1}$, and with $A_{01}$ and $A_{10}$ antisymmetric. The matrix of the linear transformation $T_{u,v}$ is proven similarly. The matrix $A$ of the linear transformation $T_{u,v}$ is $(uu^T - vv^T)B$, or if we write $u = (u_0^T \ u_1^T \ u_2)\!^T$ and $v = (v_0^T \ v_1^T \ v_2)\!^T$ with $u_0, u_1, v_0, v_1 \in F^{n-1}$ and $u_2, v_2 \in F$, then

$$T_{u,v} \quad \text{has matrix} \quad \begin{pmatrix} u_0v_0^T - v_0u_0^T & u_0v_1^T - v_0u_1^T & 2v_2u_0 - 2u_2v_0 \\ u_1v_0^T - v_1u_0^T & u_1v_1^T - v_1u_1^T & 2v_2u_1 - 2u_2v_1 \\ u_2v_0^T - v_2u_0^T & u_2v_1^T - v_2u_1^T & 0 \end{pmatrix}.$$ 

We can get matrices that are zero everywhere except at two (off-diagonal) entries in $A_{10}$ by taking $u_0 = v_0 = 0$ and $u_2 = v_2 = 0$ and taking for $u_1$ and $v_1$ two different unit vectors, and similarly for $A_{01}$. For the rest of this proof, we will therefore disregard entries in $A_{01}$ and $A_{10}$.

To obtain a matrix with only one nonzero entry in $A_{00}$, occurring in an off-diagonal position, take $u_0$ and $v_1$ equal to two different standard basis vectors and all others equal to 0. Diagonal entries are a bit more complicated. We can find a spanning set by considering two types of elements. Write $e_i$ for the $i$th standard basis vector. We take $u_0$ equal to some basis vector $e_i$ and take $u_1 = e_j$, then let $v_0 = -e_i$ and $v_1 = e_j$. This gives an entry of 1 at both the $i$th and the $j$th diagonal position. On the other hand, if we take $u_0 = e_i + e_j$ and $v_1 = e_i - e_j$, and the other entries equal to 0, then the $i$th diagonal entry is 1 and the $j$th is $-1$.

Finally, to obtain a single nonzero entry in $a_{21}$ or $a_{20}$, say in position $i$, one can take $u_0 = -u_1 = e_j$, $u_2 = 1$ and either $v_0$ or $v_1$ equal to $e_i$, the other components of $v$ equal to 0. \hfill \Box

Using

$$[T_{u,v} \ T_{w,x}] = B(u, x)T_{w,v} - B(v, w)T_{u,x} + B(v, x)T_{u,w} + B(u, w)T_{v,x},$$

(2.12)

it is easy to see that

$$[T_{u,v} \ [T_{u,v} \ T_{w,x}]] = 2(B(u, w)B(v, x) - B(u, x)B(v, w))T_{u,v}.$$ 

Hence the following lemma:

Lemma 2.40. Infinitesimal Siegel transvections are extremal elements of $o(V)$, and

$$f(T_{u,v} \ T_{w,x}) = 2(B(u, w)B(v, x) - B(u, x)B(v, w)).$$

Furthermore,

$$\exp(t \ \text{ad} \ T_{u,v} \ T_{w,x}) = T_{w + tT_{u,v} + tT_{w,x}}.$$ 

In the following lemma, we will let “point” refer to projective points and “line” to projective isotropic lines.

Lemma 2.41. Let $T_{u,v}$ and $T_{w,x}$ be two infinitesimal Siegel transvections and let $\mathcal{L} = \langle T_{u,v} \ T_{w,x} \rangle_{\text{Lie}}$. Then the isomorphism class of $\mathcal{L}$ depends on the geometrical configuration of the lines $\ell = \langle u, v \rangle_F$ and $m = \langle w, x \rangle_F$, as follows:

- If $\ell = m$, then $\mathcal{L}$ is one-dimensional.
- If $\ell \cap m$ is a point, then $\mathcal{L}$ is two-dimensional.
• If each point on \( \ell \) is collinear to each point on \( m \), then \( \mathcal{L} \) is two-dimensional.
• If exactly one point on \( \ell \) is collinear with all of \( m \) and exactly one point on \( m \) is collinear with all of \( \ell \), then \( \mathcal{L} \) is isomorphic to the Heisenberg algebra.
• If every point on \( \ell \) is collinear with exactly one point of \( m \) and every point on \( m \) is collinear with exactly one point of \( \ell \), then \( \mathcal{L} \) is isomorphic to \( \mathfrak{sl}_2 \).

There are no other cases.

In the case where \( \mathcal{L} \) is isomorphic to the Heisenberg algebra, we say that \( \ell \) and \( m \) form an *extraspecial pair*, just as in the section on transvections.

**Proof.** We first show that these are all the cases there are. The only a priori possibility for other cases is if \( \ell \) and \( m \) do not intersect. Note first that every point is collinear to either exactly one or all points on the other line: \( u \) is collinear to all of \( m \) if \( B(u, w) = B(u, x) = 0 \), and otherwise only to \( B(u, w)x - B(u, x)w \). Furthermore, if two points on \( \ell \), say \( u \) and \( v \), are collinear to all of \( m \), then \( B \) is identically 0 on \( \langle u, v, w, x \rangle_F \), so every point on \( \ell \) is collinear to every point on \( m \). If, on the other hand, \( u \) is the only point on \( \ell \) collinear to all of \( m \), then say \( v \) is collinear to \( w \). Then \( (v, x) \) is the only pair of elements out of \( \{u, v, w, x\} \) where \( B \) is nonzero, so \( w \) is collinear to all of \( m \) and it is the only point on \( m \) with that property. We have proven that there are no other cases.

The first case is immediately clear and the second and third are clear after using Eq. (2.12). For the fourth case we may assume that again \( (v, x) \) is the only pair of elements out of \( \{u, v, w, x\} \) where \( B \) is nonzero. Then \( [T_{u,v}, T_{w,x}] = B(v,x)T_{u,w} \neq 0 \), but \( f(T_{u,v}, T_{w,x}) = 0 \), so \( \mathcal{L} \) is isomorphic to the Heisenberg algebra. Finally, for the last case, we may assume that \( u \) is collinear to \( w \) and \( v \) to \( x \); then \( f(T_{u,v}, T_{w,x}) \) is nonzero. \( \Box \)

We focus our attention on the Siegel transvection groups now. Since the theorem of Steinbach is somewhat more involved than Theorem 2.28, we need to introduce some notation and terminology first.

Let \( Y = \Omega(V, B) \) be the commutator subgroup of the orthogonal group \( O(V, B) \). Then by [Ste97], section 1.1, \( Y \) is generated by all Siegel transvection subgroups of \( O(V, B) \). Let \( G \not= 1 \) be a subgroup of \( Y \) generated by some of the Siegel transvection groups. For each Siegel transvection group \( A \subseteq Y \), let \( A^0 = A \cap G \); then either \( A^0 = 1 \) or \( A^0 = A \) – this can easily be seen in the Lie algebra. This situation is a special case of the situation in the main theorem of [Ste97], reprinted here in a weaker version as Theorem 2.42, which tells us to which isomorphism classes such a group \( G \) can belong. Following [Ste97], we use the following abbreviations for different situations:

**O:** \( G \) is an orthogonal group over the same vector space, viewed as a vector space over a smaller field. Also, if \( B \) has maximal Witt index over a vector space of even dimension, then it contains the orthogonal group over a vector space of one dimension less where \( B \) has a one-dimensional radical.

**E:** \( G \) is a special case related to triality and \( \dim V = 8 \).

**ND:** \( G \) is the special linear group on a vector space \( V' \) such that \( V \) has double the dimension of \( V' \) and \( B \) has maximal Witt index. \( V \) is the direct sum of the space \( V' \) where \( G \) acts naturally and a space where it acts dually.

**I:** \( G \) is a unitary group. The unitary space is then regarded as orthogonal space over the fixed field of the involutory automorphism.

**IQ:** \( G \) is a unitary group over a quaternion division ring instead of a field.

**KC:** \( G \) is a special case related to the Klein correspondence and \( \dim V \in \{5, 6\} \).

**Theorem 2.42.** Let \( G \not= 1 \) be a subgroup of \( Y = \Omega(V, q) \) generated by Siegel transvection groups \( S_{u,v} \). Suppose these conditions are satisfied:
(H1') If $A, B$ are Siegel transvection subgroups of $Y$ that intersect $G$ nontrivially, and $\langle A, B \rangle_{G'} = SL_2(\mathbb{F})$, then $\langle A^0, B^0 \rangle_{G'} = SL_2(\mathbb{F})$.

(H2) All nilpotent normal subgroups of $G$ are contained in $Z$.

(H3) There are extraspecial pairs in $G$.

There is no $G$-invariant decomposition of $V$ into two equal-dimensional subspaces, such that each of these subspaces intersects every line corresponding to a Siegel transvection subgroup nontrivially.

Then one of the situations (O), (E), (ND), (I), (IQ) and (KC) holds.

**Theorem 2.43.** Suppose all conditions from Theorem 2.42 hold and the following extra conditions hold as well.

(X1) The dimension of $V$ is greater than 8.

(X2) There are extraspecial pairs in $G$.

(X3) There is no $G$-invariant decomposition of $V$ into two equal-dimensional subspaces, such that each of these subspaces intersects every line corresponding to a Siegel transvection subgroup nontrivially.

(X4) One of the Siegel transvection subgroups is isomorphic to $F^+$. 

(X5) $\langle u_i, v_i \rangle_F = V$.

Then $G = Y$.

**Proof.** Extra condition (X1) shows that we are not in situations (E) or (KC). By extra condition (X2), we must be in situation (ND) or (O). By extra condition (X3), we are not in situation (ND). Hence, we are in situation (O): $G$ is an orthogonal group. Then condition (X4) shows that the field is all of $F$. Finally, because of extra condition (X5), $G$ is all of $Y$.  

We define the basis of $V$ in an order corresponding to the matrix of $B$. Let $k = n$ for $D_n$ and $k = n - 1$ for $B_{n-1}$. The basis of $V$ consists first of vectors $\{e_i\}_{i=1}^k$ spanning a maximal isotropic subspace, then of vectors $\{f_i\}_{i=1}^k$ spanning a maximal isotropic complement to $\langle e_i \rangle_F$ and with $B(e_i, f_j) = \delta_{ij}$, and if $i$ is odd, finally a vector $g$. We can interpret the vectors and linear functionals from Section 2.6.1, and in particular from Theorem 2.38, in this context as well: $\{f_i\}$ is still the dual basis of $\{e_i\}$. Therefore we find an isomorphic copy of $\mathfrak{sl}(W)$ in $\mathfrak{o}(V)$, where $W$ is contained in a maximal isotropic subspace; the isomorphism is determined by sending $x \otimes h \in \mathfrak{sl}(W)$ to $T_{x,h} \in \mathfrak{o}(V)$. We use the generating elements of $\mathfrak{sl}(W)$ in finding those for $\mathfrak{o}(V)$, but with extra parameters for which the necessity will become apparent in Section 2.7.

**Theorem 2.44.** Let $\alpha, \beta \in \mathbb{F}$ and write $\kappa$ for $\sqrt{1 + \beta}$. Let

$$
\lambda = \begin{cases} 
\frac{\alpha}{\alpha + 2}, & \text{if } n \text{ is odd}, \\
\frac{\alpha \kappa}{(\alpha + 2)(1 + \beta + \kappa)}, & \text{if } n \text{ is even}.
\end{cases}
$$

Suppose that $(\alpha + 2)\beta(\beta + 1) \neq 0$ and that $\lambda(2 - \beta + \lambda \beta) \neq 1$. The transformations $T_{u_i, v_i}$ for $i \leq n$ realize the graph $\Gamma_{D_n}$ for $n \geq 5$ if we take these values for $u$ and $v$:

$$
u_1 = e_1 - e_2, \quad v_1 = f_1 + f_2 + \alpha f,
$$
$$
u_i = e_{i-1} + e_i, \quad v_i = f_{i-1} - f_i, \quad \text{for } 1 < i < n,
$$
$$
u_n = e_{n-2} + \beta f_{n-1} + e_n, \quad v_n = f_{n-2} + e_{n-1} - (1 + \beta) f_n.$$
where

\[\hat{f} = (0, 0, 1, \ldots, 1, 0 \mid 0, 0, 1, -\ldots, -1 - \beta),\]

\[\hat{f} = \frac{1}{1 + \beta + \kappa} (0, 0, -\kappa, -\kappa, \ldots, -\kappa, 1 | 0, 1 + \beta + \kappa, -1 - \beta - \kappa, \ldots, 1 + \beta + \kappa, (\beta + 1)(\kappa + 1)),\]

if \(n\) is odd;

where we first write the coefficients of \(e_i\), then a bar (\(\overline{\phantom{a}}\)), then those of \(f_j\). Then \(\mathcal{L} = \langle T_{u, v}\rangle_{\text{Lie}}\) is \(\mathfrak{sl}_n\).

Note that \(B(\hat{f}, f) = 0\), and

\[B(\hat{f}, u_i) = \begin{cases} 0, & \text{if } i \neq 3, \\ 1, & \text{if } i = 3; \end{cases}\]

\[B(\hat{f}, v_i) = \begin{cases} 0, & \text{if } i \neq 3, \\ -1, & \text{if } i = 3 \text{ and } n \text{ is odd}, \\ \frac{\kappa}{1 + \beta + \kappa}, & \text{if } i = 3 \text{ and } n \text{ is even}. \end{cases}\]

Hence \(\lambda = -\frac{d}{n} B(\hat{f}, v_3)\).

We will prove this theorem using Theorem 2.43. We will first state a similar theorem for the Lie algebra of type \(B_{n-1}\), Theorem 2.45, then state and prove some additional lemmas necessary to show that the conditions of Theorem 2.43 hold, and finally prove Theorems 2.44 and 2.45 on page 68.

**Theorem 2.45.** Let \(\gamma \in \mathbb{F}\) be such that \(\gamma(\gamma + 1) \neq 0\). The transformations \(T_{u, v}\) for \(i \in I = \{1, \ldots, n\}\) realize the graph \(\Gamma_{B_n}\) for \(n \geq 5\) if we take these values for \(u\) and \(v\):

\[u_1 = e_1 - e_2, \quad v_1 = f_1 + f_2,\]

\[u_i = e_{i-1} + e_i, \quad v_i = f_{i-1} - f_i, \quad \text{for } 1 < i \leq n,\]

\[u_n = \gamma e_{n-2} + f_{n-2} + \gamma e_{n-1} - f_n - f_{n-1}, \quad v_n = e_{n-2} - f_{n-2} + (1 - \gamma) e_{n-1} + \gamma.\]

Then \(\mathcal{L} = \langle T_{u, v}\rangle_{\text{Lie}}\) is \(\mathfrak{so}_{2n-1}\).

The proof will be similar for \(D_n\) and \(B_n\), so let \(\mathcal{L}\) be one of the two algebras defined in Theorems 2.44 and 2.45 and let \(u_i\) and \(v_i\) be the corresponding vectors. Let \(G\) be the group generated by the Siegel transvection groups \(S_{u, v}\). We denote by \(\Sigma\) the orbit of \(G\) on the lines \(\langle u_i, v_i \rangle_\mathbb{F}\), or alternatively, on the projective infinitesimal Siegel transvections \(\mathbb{P}T_{u, v}\), or alternatively, on the Siegel transvection subgroups \(\langle 1 + \mu T_{u, v}\rangle_{\mathbb{P}G}\), there are obvious bijections between these three sets. Before we start with the conditions of Theorem 2.43, we will first see that the action of \(G\) on these three classes of objects is equivalent and that \(\mathfrak{g}\) is transitive on \(\Sigma\).

**Lemma 2.46.** The action of \(1 + \mu T_{w, x}\) on \(\sigma(V)\) by conjugation from the left is the same as the natural action of \(\exp(\mu \text{ ad } T_{w, x})\). Furthermore, the bijections sending \(\mathbb{P}T_{u, v}\) to \(\langle u, v \rangle_\mathbb{F}\) and \(\langle 1 + \mu T_{u, v}\rangle_{\mathbb{P}G}\) commute with the action of \(G\).

**Proof.** Since \(\sigma(V)\) is spanned by Siegel transvections, we need only check the following to prove the first assertion:

\[
\exp(\mu \text{ ad } T_{w, x})T_{u, v} = T_{u, v} + \mu [T_{w, x}, T_{u, v}] + \frac{\mu^2}{2} [T_{w, x}, [T_{w, x}, T_{u, v}]] = T_{u, v} + \mu T_{w, x} T_{u, v} - \mu T_{u, v} T_{w, x} - \mu^2 T_{w, x} T_{u, v} T_{w, x} = (1 + \mu T_{w, x})T_{u, v}(1 - \mu T_{w, x}).
\]
For the second assertion, we see that the expression above is equal to
\[
T_{u,v} + \mu (B(v,v)T_{u,x} - B(u,x)T_{v,v} + B(u,v)T_{x,x} - B(v,x)T_{u,v})
\]
\[
+ \mu^2 (B(u,v)B(v,x) - B(u,x)B(v,v))T_{w,x}
\]
\[
= T_{u,v} + \mu (B(u,w) - B(v,w))T_{w,v} + \mu B(v,v)T_{x,x} - B(v,x)T_{u,v}
\]
\[
= T_{u,v} + \mu T_{u,v} + \mu T_{w,v}.
\]

This corresponds to \(\langle u + \mu T_{w,v}, u + \mu T_{w,v}\rangle\), the image of \(\langle u, v\rangle\) under \(1 + \mu T_{w,v}\). Finally, since the action is linear,
\[
(1 + \mu T_{w,v})(1 + \beta T_{w,v})(1 - \mu T_{w,v}) = 1 + (1 + \mu T_{w,v})T_{u,v}(1 - \mu T_{w,v}).
\]

**Lemma 2.47.** All \(T_{u,v}\) are in one orbit under \(G\).

**Proof.** Let \(T_{u,v}\) and \(T_{w,x}\) correspond to adjacent vertices. Then it suffices to show a group element mapping \(T_{u,v}\) to \(T_{w,x}\). Since \(f(T_{u,v}, T_{w,x}) \neq 0\), we can define \(\mu = 1/(B(u,v)B(v,x) - B(u,x)B(v,v))\); then
\[
\exp(\text{ad } T_{u,v})\exp(\mu \text{ ad } T_{w,x}) = \mu T_{B(u,v)B(v,x) - B(u,x)B(v,v)} = T_{w,x}.
\]

**Lemma 2.48.** \(\Sigma\) contains an extraspecial pair.

**Proof.** We take \(\langle u_2, v_2\rangle\) as the first line. For \(D_n\), the second line is the line corresponding to
\[
\exp(2 \text{ ad } T_{u_3,v_3}) T_{u_1,v_1} = T_{u_1 + 2u_3, v_1 + 2(1 + \alpha)\beta_3 + 2(\alpha + 2)\lambda u_3};
\]
the line in special position is \(\langle u_1 + 2u_3, v_2 + 2\lambda u_2\rangle\). For \(B_n-1\), we specialize from these values by setting \(\alpha = \lambda = 0\) and obtain as second line (the line corresponding to)
\[
\exp(2 \text{ ad } T_{u_3,v_3}) T_{u_1,v_1} = T_{u_1 + 2u_3, v_1 + 2v_3},
\]
with line in special position \(\langle u_1 + 2u_3, v_2\rangle\).

**Lemma 2.49.** \(\Sigma\) contains a copy of the set of infinitesimal transvections in \(\mathfrak{sl}_{n-1}\).

**Proof.** We will exhibit two sets of generators of different isomorphic copies of \(\mathfrak{sl}_{n-1}\). Let \(U\) be the vector space spanned by \(e_i\) for \(i < n\). If we are studying \(D_n\), then define
\[
v'_i = \begin{cases} v_i - \lambda u_i, & i \in \{1, n\} \\ v_i + \lambda u_i, & 1 < i < n; \end{cases} h_i: U \to F, x \mapsto -B(v_i, x);
\]
for \(B_{n-1}\), substitute \(\lambda = 0\) (whence \(v'_i = v_i\), but let \(v'_n = u_n - \gamma v_n\). Additionally, for both \(B_{n-1}\) and \(D_n\) define \(u'_i = u_i\), except that \(u'_n = \frac{1}{1 + \gamma}(u_n + \gamma v_n)\) for \(B_{n-1}\). Then \(B(u'_i, u'_j) = B(v'_i, v'_j) = 0\) for all \(i, j \neq n-1, n\). Furthermore, \(u_i \otimes h_i = T_{u_i, v_i} = T_{u'_i, v'_i}\) as linear transformations, and \(U^* := \langle h_i \mid i < n\rangle\) is a vector space dual to \(U\) where the duality is provided by the form \(B\). The infinitesimal transvections \(u_i \otimes h_i\) generate \(\mathfrak{sl}(U)\) because of the same arguments that prove Theorem 2.38: the centres and axes of the transvections span \(U\) and its dual, respectively, and there is an extraspecial pair (both lines from Lemma 2.48 are in \(\mathfrak{sl}(U)\)). The group generated by all of the corresponding transvections is a subgroup of \(G\), and it is transitive on all transvections in \(\mathfrak{sl}(U)\). So in particular, the group of Siegel transvections is transitive on all infinitesimal Siegel transvections in \(\mathfrak{sl}(U)\).

Similarly, we can define \(\tilde{U} = \langle u'_1, u'_2, \ldots, u'_{n-2}, u'_n\rangle_F\) and \(\tilde{U}^* = \langle v'_1, v'_2, \ldots, v'_{n-2}, v'_n\rangle_F\), on which the transvections generate \(\mathfrak{sl}(\tilde{U})\).
For $B_{n-1}$, there is a nontrivial intersection between $U + U'$ and $U^* + \bar{U}^*$: since $\gamma(1 + \gamma)u_n' + v_n' = (1 + \gamma)u_n = (1 + \gamma)(\gamma u_{n-1} + v_{n-1}) = \gamma(1 + \gamma)u_{n-1}' + (1 + \gamma)v_n'$, the intersection is spanned by $u_n' - u_{n-1}'$, which is $\gamma(1 + \gamma)$ times $(1 + \gamma)v_n'$.

**Lemma 2.50.** $\Sigma$ contains a 4-tuple of projectively distinct infinitesimal Siegel transvections $(T_a, T_b, T_c, T_d)$ such that $T_c$ and $T_d$ do not commute, but every other pair does.

**Proof.** Lemma 2.49 shows that $\Sigma$ contains all infinitesimal Siegel transvections in an isomorphic copy of $\mathfrak{sl}_4$, so we can find a tuple of infinitesimal Siegel transvections corresponding to $(e_1 \otimes f_2, e_1 \otimes f_3, e_1 \otimes f_4, e_2 \otimes f_2)$. This tuple satisfies the requirements. \(\square\)

**Lemma 2.51.** Let $T_a = T_{u_a, v_a} \in \Sigma$ and $T_b = T_{u_b, v_b} \in \Sigma$ satisfy $C_L(T_a) = C_L(T_b)$. Then $T_a = T_b$.

**Proof.** Since $G$ is transitive on $\Sigma$, we may assume that $T_a = T_{u_2, v_2} = T_{u_2', v_2'}$. Let us denote the subspace of $V$ perpendicular with respect to the bilinear form $B$ to a vector $u \in V$ by $u^\perp$, and similarly, let us denote the subspace of $V$ perpendicular to a subspace $S$ of $V$ by $S^\perp$.

Recall from Lemma 2.49 the definitions of $U$ and $U'$. Pick a nonzero vector $u \in U$ which is perpendicular to $v_2$, and let nonzero $v \in U'$ be perpendicular to $u$ and to $u_2'$. Then the infinitesimal transvection $u \otimes v = T_{u,v}$ is in $\mathfrak{sl}(U)$, hence in $\Sigma$, and it commutes with $T_{u_2', v_2'}$. So $T_b$ should also commute with it. Then $\langle u_b, v_b \rangle_F$ either intersects $\langle u, v \rangle_F$, or $u_b$ and $v_b$ are both perpendicular to $u$ and $v$. So if $\langle u_b, v_b \rangle_F$ does not intersect all $\langle u, v \rangle_F$ for fixed $u$ and all nonzero $v \in S := u^\perp \cap u_2'^\perp \cap U'$, then $u_b$ and $v_b$ are perpendicular to $u$. Let us consider the case where $\langle u_b, v_b \rangle_F$ intersects every such $\langle u, v \rangle_F$ we will show that $u_b$ and $u_b$ are perpendicular to $u$ in this case as well. $S$ has codimension 1 or 2 in $U'$ of dimension $n - 1$, so its dimension is at least 2. If dim $S > 2$, then $\langle u_b, v_b \rangle_F$ must contain $u$ to intersect every $\langle u, v \rangle_F$. In that case, $u_b$ and $v_b$ are certainly perpendicular to $u$. Hence assume dim $S = 2$ and $u \not\in \langle u_b, v_b \rangle_F$. Then for different lines $\langle u, v \rangle_F$, the intersection with $\langle u_b, v_b \rangle_F$ is different. So the intersections span all of $\langle u_b, v_b \rangle_F$. In particular, $u_b$ and $v_b$ themselves are on lines $\langle u, v \rangle_F$. Thus both are perpendicular to $u$.

We see that $u_b$ and $v_b$ are perpendicular to all of $v_2^\perp \cap U$. Similarly, they are perpendicular to $v_2^\perp \cap U'$; that is, they are perpendicular to $S_0 := v_2^\perp \cap (U + \bar{U})$. Dually, we see that $u_b$ and $v_b$ are perpendicular to $S_u := u_2^\perp \cap (U^* + \bar{U}^*)$.

For $B_{n-1}$, the intersection of $U + \bar{U}$ and $U' + \bar{U}'$ is spanned by $u_n' - u_{n-1}'$, which is perpendicular to both $u_2'$ and $v_2'$; for $D_n$, the intersection of $U + \bar{U}$ with $U' + \bar{U}'$ is empty. So $S_u + S_v$ is a $2n - 2$-dimensional space for $D_n$ and to a $2n - 3$-dimensional space for $B_{n-1}$. Since the form is nondegenerate, there is in both cases only a 2-dimensional space of vectors perpendicular to $S_u + S_v$. This space is $\langle u_2', v_2' \rangle_F$. Hence $u_b$ and $v_b$ are in $\langle u_2', v_2' \rangle_F$. \(\square\)

**Lemma 2.52.** The graph $F(\Sigma)$ with vertex set $\Sigma$ and where two infinitesimal Siegel transvections are adjacent if they generate an algebra isomorphic to $\mathfrak{sl}_2$, is connected.

**Proof.** According to Lemma 2.13 of [Tim01], if $|\Sigma| > 1$, then $F(\Sigma)$ is connected if and only if $\Sigma$ is a conjugacy class in $G$ and $F(\Sigma)$ has an edge. Both of these conditions are fulfilled. \(\square\)

Recall that a group $E$ is called *quasisimple* if $E = [E, E]$ and $E/Z(E)$ is simple, where $Z(E)$ is the centre of $E$.

**Lemma 2.53.** $G$ is a quasisimple group.
Proof. We use Lemma 2.14 of [Tim01]. A weaker version of it states that if the following conditions are satisfied:

- $|\Sigma| > 1$;
- the graph $F(\Sigma)$ from Lemma 2.52 is connected;
- there exists no pair $A \neq C \in \Sigma$ with $C_{\Sigma}(A) = C_{\Sigma}(B)$;
- $\Sigma$ contains an extraspecial pair;
- the elements of $\Sigma$ correspond to extremal Lie algebra elements;

then $G$ is a quasisimple group. Lemmas 2.48, 2.51 and 2.52 show that the three nontrivial conditions are fulfilled.

\[\square\]

Lemma 2.54. $G$ satisfies condition (H2) from Theorem 2.42: All nilpotent normal subgroups of $G$ are contained in $Z(G)$ and in the commutator group $G'$; it is not possible to decompose $\Sigma$ into two nonempty parts $\Sigma_1$, $\Sigma_2$ such that all groups from $\Sigma_1$ commute with all groups from $\Sigma_2$.

Proof. $G$ is quasisimple by Lemma 2.53, so $G' = G$. Let $Z = Z(G)$ and $S = G/Z(G)$ be the simple quotient. Let $N \triangleleft G$. Since $(NZ)/Z$ is normal in $S$, either $(NZ)/Z = 1$ or $(NZ)/Z = S$. If $(NZ)/Z = 1$, then $N \subseteq Z$, so we may assume that $(NZ)/Z = S$, whence $NZ = G$. Since $Z$ contains only central elements, $G = [G, G] = [NZ, NZ] \subseteq N$.

Now suppose that $\Sigma$ is the disjoint union of $\Sigma_1$ and $\Sigma_2$, and every element of $\Sigma_1$ commutes with every element of $\Sigma_2$. Then all generators are in one of the two parts, say in $\Sigma_1$. Elements commuting with all generators commute with all of $G$, so $\Sigma_2$ can only contain subgroups of $Z$. Let $I + T_{u,v} \in Z$. For all $\langle w, x \rangle_F \in \Sigma$, there is either an intersection between $\langle u, v \rangle_F$ and $\langle w, x \rangle_F$ or $u$ and $v$ are both perpendicular to both $w$ and $x$. The vectors perpendicular to $u$ and $v$ form a proper subspace of $V$ since the form is nondegenerate, so $\langle u, v \rangle_F$ intersects every line in the complement. This is a contradiction.

\[\square\]

Lemma 2.55. There is no decomposition of $V$ into two equal-dimensional $G$-invariant subspaces $W$ and $W'$, such that every line in $\Sigma$ intersects $W$ and $W'$ in a projective point.

Proof. The statement is clearly satisfied for $B_{n-1}$, since $\dim V$ is odd in that case.

Suppose that there is such a decomposition $V = W \oplus W'$ for $D_n$. Then $W$ or $W'$ contains $v_3 + \mu u_3$ for some $\mu$; say that $W$ is the space containing it. For every $1 + T \in G$ and every $w \in W$, clearly $(1 + T)w = w + Tw$ is also in $W$, hence also $Tw$. So $W$ contains

$$T_{u_1,v_3}T_{u,v_3}T_{u_2,v_3}(v_3 + \mu u_3) = (2 + 2\alpha)v_3 - 2(\mu - (\alpha + 2)\lambda)u_3,$$

which must be a multiple of $v_3 + \mu u_3$. Hence $\mu = \lambda$. Since

$$T_{u_{i+1},v_i}(v_i + \lambda u_i) = -(v_{i+1} + \lambda u_{i+1})$$

for $1 < i < n - 1$, we find that $v_i + \lambda u_i \in W$ for $1 < i < n$. However,

$$T_{u_{i+1},v_i}(v_{i+1} + \lambda u_{i+1}) = -(v_{i+1} + \lambda u_{i+1}),$$

$$T_{u_{i+1},v_i}(v_{i+2} + \lambda u_{i+2}) = -(v_{i+2} + \lambda u_{i+2}),$$

$$T_{u_{i+1},v_i}(v_{i+3} + \lambda u_{i+3}) = -(v_{i+3} + \lambda u_{i+3}).$$

These two vectors are linearly dependent if $\lambda(2 - \beta + \lambda\beta) = 1$, which is not true by one of the assumptions in Theorem 2.44.

\[\square\]

Proof of Theorems 2.44 and 2.45. We intend to apply Theorem 2.43, so we will need to show that its conditions hold.

- Condition (H1') follows from the fact that for every Siegel transvection subgroup $A$, we have $A^0 = 1$ or $A^0 = A$.
- Condition (H2) follows from Lemma 2.54.
- Condition (H3) follows from Lemma 2.50.
• Condition (X1) is clearly satisfied.
• Condition (X2) follows from Lemma 2.48.
• Condition (X3) follows from Lemma 2.55.
• Condition (X4) is true for every Siegel transvection subgroup.
• Condition (X5) is clearly satisfied. □

2.7. The algebras nearly always correspond to these realizations

In this section, we show that a Lie algebra $\mathcal{L}$ realizing one of the graphs $\Gamma_{A_n}$, $\Gamma_{B_n}$, $\Gamma_{C_n}$, and $\Gamma_{D_n}$, is in the general case a quotient of the realization $\mathcal{M}$ we found in the previous section. In order to see this, we find different sets of generators. For types $B_{n−1}$ and $D_n$, we will need the parameters $\alpha$, $\beta$ and $\gamma$ from Theorems 2.44 and 2.45 to have sufficient degrees of freedom to be able to make the sets of generators of $\mathcal{L}$ and $\mathcal{M}$ match up.

Since $\mathcal{M}$ is simple in most cases, it will follow that $\mathcal{L}$ and $\mathcal{M}$ are isomorphic. The only exception is $A_{n−1}$ if $p | n$, as follows from the following Theorem (recall that in our case, the characteristic of the field is not 2):

**Theorem 2.56.** The Lie algebras of Chevalley type are all simple over any algebraically closed field, with these exceptions:

• the Lie algebra of type $A_{n−1}$ if the characteristic of the field is a prime dividing $n$;
• the Lie algebras of type $G_2$ and $E_6$ if the characteristic of the field is $3$;
• the Lie algebras of type $B_n$, $C_n$, $D_n$, $E_7$ and $F_4$ if the characteristic of the field is $2$.

For characteristic 0, these are classical results, found in e.g. Humphreys [Hum72]. For characteristic $p \geq 5$, the results are in Section 4.1 of Strade [Str04]. Section 4.4 of the same book gives the results for characteristic 3 and 2.

**Theorem 2.57.** Let $\mathcal{L}$ be any Lie algebra realizing the graph $\Gamma_{C_n}$ from Figure 2.1. Suppose that $n$ is even and that $\psi_{\Gamma_{C_n}}(f)$ has no zeroes. Then $\mathcal{L}$ is isomorphic to $sp_n$.

**Proof.** Let $\mathcal{M}$ be the realization from Theorem 2.37. By that Theorem, $\mathcal{M} = sp_n$. Denote the generators of $\mathcal{L}$ realizing $\Gamma_{C_n}$ by $x_i$ and let $z_i$ be those of $\mathcal{M}$. Scale $x_i$ such that the extremal form is identical on both sets of generators. Then the map $\sigma : \mathcal{M} \to \mathcal{L}$ mapping each of the monomials $y_{k,m}$ in $z_i$ to the same monomial in $x_i$, is a Lie algebra homomorphism by Lemma 2.25. Hence $\mathcal{L}$ is a quotient of $\mathcal{M}$. But since $\mathcal{M}$ is simple, $\mathcal{L} \cong \mathcal{M}$. □

Already for Lie algebras of type $A_{n−1}$, we need more degrees of freedom than can be obtained by just scaling the generators. The following lemma, which is based on Section 5 of [CSUW01], will be sufficient.

**Lemma 2.58.** Let $\pi, \rho, \sigma \in \mathbb{F}$ all be nonzero. Let $x$, $y$, $z$ be extremal elements of $\mathcal{L}$ such that $f(x, yz)^2 \neq 2f(x, y)f(x, z)f(y, z)$ and $f(x, y) \neq 0 \neq f(y, z)$, and such that $x$ and $y$ commute with a set $S$ of elements of $\mathcal{L}$. We can find extremal elements $x'$, $y'$ and $z'$ with the following properties:

• $\langle x, y, z \rangle_{\text{Lie}} = \langle x', y', z' \rangle_{\text{Lie}}$,
• $x'$ and $y'$ commute with $S$,
• $(f(x', y'), f(x', z'), f(y', z'), f(x', y'z')) = (\pi, \rho, \sigma, 0)$. 


Proof. Let \( s = f(x, yz)/(f(x, y)f(y, z)) \). Let \( \hat{x} = \exp(s \text{ad } y)(x) \). Then \( \langle x, y, z \rangle_{\text{Lie}} = \langle \hat{x}, y, z \rangle_{\text{Lie}} \) (since \( \exp(-s \text{ad } y)(\hat{x}) = x \)) and

\[
\begin{align*}
f(\hat{x}, y) &= f(x, y), \\
f(\hat{x}, z) &= f(x, z) - \frac{f(x, yz)^2}{2f(x, y)f(y, z)}, \\
f(\hat{x}, yz) &= 0.
\end{align*}
\]

Note that \( f(\hat{x}, z) \neq 0 \). We drop the circumflex from now on and use \( x \) to denote \( \hat{x} \). Scale \( x, y \) and \( z \) to obtain \( \tilde{x}, \tilde{y} \) and \( \tilde{z} \):

\[
\begin{align*}
\tilde{x} &= \sqrt[\pi \rho f(y, z)]{\sigma f(x, y)f(x, z)} x, \\
\tilde{y} &= \sqrt[\pi \alpha f(x, z)]{\rho f(x, y)f(y, z)} y, \\
\tilde{z} &= \sqrt[\pi \alpha f(x, y)]{\pi f(x, z)f(y, z)} z.
\end{align*}
\]

Now all values of \( f \) are as intended, possibly up to a factor of \(-1\), and \( f(\tilde{x}, \tilde{y})f(\tilde{x}, \tilde{z})f(\tilde{y}, \tilde{z}) = \pi \rho \sigma \). So either all values of \( f \) are exactly as intended (including sign), in which case we are done, or exactly two of them need their sign changed; say \( f(\tilde{x}, \tilde{y}) \) and \( f(\tilde{x}, \tilde{z}) \). Then let \( x' = -\tilde{x}, y' = \tilde{y} \) and \( z' = \tilde{z} \).

\( y' \) commutes with \( S \), since it is merely a scaled version of \( y \). By the Jacobi identity, \( [x, y] \) commutes with \( S \), as well; hence \( x' \in \langle [x, y], y \rangle_F \) commutes with \( S \). \( \square \)

Theorem 2.59. Let \( \mathcal{L} \) be a realization of the graph \( \Gamma_{A_n} \) in Figure 2.2. Then there exists an open dense subset \( S \) of \( \psi_{\Gamma_{A_n}}(X_{\Gamma_{A_n}}) \) such that if \( \psi_{\Gamma_{A_n}}(f) \in S \), then:

- if \( \text{char } F = p > 0 \) and \( p \mid n \), then \( \mathcal{L} \) is isomorphic to either \( sl_n \) or its simple subalgebra of codimension 1;
- otherwise, \( \mathcal{L} \) is isomorphic to \( sl_n \).

Proof. Let \( \mathcal{M} \) be the realization from Theorem 2.38. By that theorem, \( \mathcal{M} = \psi_{\Gamma_{A_n}}(X_{\Gamma_{A_n}}) \). Denote the generators of \( \mathcal{L} \) realizing \( \Gamma_{A_n} \) by \( L_i \) and let \( M_i \) be those of \( \mathcal{M} \). We will exhibit a Lie algebra homomorphism from \( \mathcal{M} \) to \( \mathcal{L} \), showing that \( \mathcal{L} \) is a quotient of \( \mathcal{M} \). Since \( \mathcal{L} \) cannot be one-dimensional, we then have the desired result.

Suppose that \( f(x_{1i}, x_{2i}x_3) \neq 2f(x_{1i}, x_2)f(x_1, x_3)f(x_2, x_3) \) and that \( f(x_{1i}, x_{i+1}) \neq 0 \) for all \( i \). Apply Lemma 2.58 with \( (\pi, \rho, \sigma) = (1, 1, 1) \) and \( (x, y, z) = (x_1, x_2, x_3) \). We obtain a new set of generators \( x'_i \) of \( \mathcal{L} \) that still realize \( \Gamma_{A_n} \). Also apply Lemma 2.58 with \( (x, y, z) = (y_1, y_2, y_3) \) and with the same values for \( \pi, \rho \) and \( \sigma \), obtaining new generators \( y'_i \). Now for any pair of monomials in \( y'_1, y'_2 \) and \( y'_3 \), the extremal form on that pair is equal to the extremal form on the corresponding pair of monomials in \( x'_1, x'_2 \) and \( x'_3 \). For \( i > 3 \), inductively define \( x'_i = f(y_{i-1}, y_i)f(x'_{i-1}, x_i)^{-1} x_i \) and \( y'_i = y_i \). Now the extremal form is equal on all pairs of monomials given by \( \psi_{\Gamma_{A_n}}(f) \), so the extremal form is identical. Then the desired Lie algebra homomorphism can be obtained as mapping \( x'_i \) to \( y'_i \). \( \square \)

Theorem 2.60. Let \( \mathcal{L} \) be a realization of the graph \( \Gamma_{D_n} \) in Figure 2.3. Then \( \mathcal{L} \) is isomorphic to \( \psi_{\Gamma_{D_n}}(f) \) is in a certain open dense subset of \( \psi_{\Gamma_{D_n}}(X_{\Gamma_{D_n}}) \).

Proof. Let \( \mathcal{M} \) be the realization we defined in Theorem 2.44, for some values of \( \alpha \) and \( \beta \) which we will choose later. We will make some changes to the sets of generating
elements that do not change the algebra generated by these elements and then claim that \( f \) is identical on the two. This shows that \( L \) is isomorphic to a quotient of \( D_n \). Since \( D_n \) is simple, this quotient is \( D_n \) itself.

Let \( x_i \) be the \( i \)-th extremal generator of \( L \) (so the value of \( x_i \) will change during the rest of this proof). First, we perform the procedure of Lemma 2.58 above on \( x_1, x_2 \) and \( x_3 \) with \((\pi, \rho, \sigma) = (\pm, 1, 2)\). (The condition \( f(x, y)^2 \neq 2f(x, y)f(x, z)f(y, z) \) from the lemma is one of the polynomial conditions defining the open dense subset.) Then we do the same on \( x_{n-2}, x_{n-1} \) and \( x_n \), this time with \((\pi, \rho, \sigma) = (2, 2, 1)\). Now there are \( n - 5 \) pairs of elements left (on the “connecting line between the two triangles”), on which the value of \( f \) has not yet been adjusted, and additionally \( f(x_1, x_{3n-2}) \). We assume that \( f(x_{i-1}, x_i) \neq 0 \) for \( e \leq i \leq n - 3 \) and scale \( x_4 \) up to \( x_{n-3} \) such that \( f(x_{i-1}, x_i) = 2 \). This leaves \( f(x_{n-3}, x_{n-2}) \) and \( f(x_1, x_{3n-2}) \). The values of \( f \) other than \( f(x_1, x_{3n-2}) \) are as given in Figure 2.5.

We now perform a similar procedure on \( M \). Call the \( i \)-th extremal generator \( z_i = T_{u_1,v_1} \). The values of \( f \) prior to any changes are as given in Figure 2.6. We perform the procedure of Lemma 2.58 on \( z_1, z_2 \), and \( z_3 \), with \((\pi, \rho, \sigma) = (\pm, 1, 2)\). Note that \( s = \alpha/4 \). In the first step, \( z_1 \) becomes

\[
z_1 = \alpha/4 [z_1, z_2] - \alpha^2/4 z_2 = T_{u_1,-\frac{\alpha}{4}u_2,\frac{\alpha}{4}v_2}.
\]

Now

\[
f(z_1, z_2) = -8, \quad f(z_1, z_3) = (\alpha + 2)^2/2, \quad f(z_2, z_3) = 2, \quad f(z_1, z_2 z_3) = 0,
\]

so \( z_1 \) and \( z_3 \) are divided by \((\alpha + 2)/\sqrt{2}\) and \( z_2 \) is multiplied with that same constant.

At the other end, we find that applying the procedure of Lemma 2.58 only entails scaling the generators by a factor; in particular, for \((\pi, \rho, \sigma) = (2, 2, 1)\), we divide \( z_{n-1} \) and \( z_n \) by \(
\sqrt{2 - 2\beta}\) and multiply \( z_{n-2} \) by the same factor. Finally, to obtain a 2 for the value of \( f(z_{i-1}, z_i) \) where \( 4 \leq i \leq n - 3 \), we multiply \( z_i \) by \((\alpha + 2)/\sqrt{2}\) for even \( i \) and divide it by that factor for odd \( i \). Hence

\[
f(z_{n-3}, z_{n-2}) = (2\alpha + 4)^{ (-1)^{n+1} \sqrt{1 - \sqrt{2 - 2\beta}}}.
\]  

(2.14)

Now consider \( z = z_{3n-2}\). The generators involved have been scaled, but not changed otherwise. For the generators that occur twice, viz. \( z_3, z_4, \ldots, z_{n-2} \), their scaling
factor affects \( z \) twice; the scaling factors of the other three (\( z_2, z_{n-1} \), and \( z_n \)) have an effect only once. In total, \( z \) was multiplied by \((\alpha + 2)/\sqrt{2}\) by all these scalings, if \( n \) is odd, and divided by that constant if \( n \) is even. We will now compute the value of \( z \) explicitly. This is easier if we temporarily forget all the scaling that occurred; so until further notice, we will use the values before scaling of the \( z_i \).

With induction it is easy to see that \( z_{k|2} = (-1)^k(T_{u_i,v_2} + T_{u_2,v_0}) \) for \( 3 \leq k < n \). Then

\[
z_{n|2} = (-1)^{n-1}( [z_n, T_{u_{n-1},v_2}] + [z_n, T_{u_2,v_{n-1}}] )
= (-1)^{n}(T_{u_2,v_2} - \beta T_{u_2,v_n} + T_{u_2,v_2} = (-1)^{n}(T_{u_2,v_2} - \beta T_{u_2,v_0}).
\]

We can again use induction to see that \( z_{k|n-2|n|2} = (-1)^n(T_{u_i,v_2} - T_{u_2-\beta u_2,v_2}) \) for \( 4 \leq k \leq n - 2 \). Finally, we compute

\[
z = z_{3|n-2|n|2} = (-1)^n( [z_3, T_{u_2,v_2} + T_{u_2-\beta u_2,v_4}] )
= (-1)^n(-T_{u_2,v_3} + T_{u_2,v_2} + T_{u_2,v_2} - T_{u_2-\beta u_2,v_3} + T_{u_2,v_4} - \beta T_{v_3,v_4} )
\]

\[( -1)^n(T_{u_2,v_2} - T_{u_2-\beta u_2,v_4} + T_{u_2,v_4} - \beta T_{v_3,v_4} = ( -1)^n(T_{u_2,v_2} + T_{u_2-\beta u_2,v_4} - T_{u_2-\beta u_2,v_4} )).
\]

We re-remember the scaling factors and find that

\[
f(z_1,z)z_1 = [z_1, [z_1, z] ] = (-1)^n \frac{2}{(\alpha + 2)^2} \left[ \frac{\alpha + 2}{\sqrt{2}} \right]^{(-1)^{n+1}} [T_{u_1-\frac{\alpha}{2},v_1} + \frac{\alpha}{2} v_2, [T_{u_1-\frac{\alpha}{2},v_1} + \frac{\alpha}{2} v_2, T_{u_3,v_2} + v_2 - T_{u_2-\beta u_2,v_3}] ]
\]

\[
= \begin{cases} 
  f(T_{u_1-\frac{\alpha}{2},v_1} + \frac{\alpha}{2} v_2, T_{u_2-\beta u_2+v_4-\beta v_4,v_3} - T_{u_3,v_2} + v_2 - T_{u_2-\beta u_2+v_4-\beta v_4,v_3} )z_1, & \text{if } n \text{ is odd}, \\
  \frac{2}{(\alpha + 2)^2} f(T_{u_1-\frac{\alpha}{2},v_1} + \frac{\alpha}{2} v_2, T_{u_3,v_2} + v_2 - T_{u_2-\beta u_2+v_4-\beta v_4,v_3} )z_1, & \text{if } n \text{ is even}.
\end{cases}
\]

A straightforward computation shows that

\[
f(T_{u_1-\frac{\alpha}{2},v_1} + \frac{\alpha}{2} v_2, T_{u_2-\beta u_2+v_4-\beta v_4,v_3} - T_{u_3,v_2} + v_2 - T_{u_2-\beta u_2+v_4-\beta v_4,v_3} ) = 4\alpha(1 + \beta) + 8.
\]

We finish the proof by case distinction.

**Case 1:** Suppose that \( n \) is odd. Taking the previous equations together with Eq. (2.14), we need to choose \( \alpha \) and \( \beta \) such that

\[
f(x_1,x_{3|n-2|n|2}) = 4\alpha(1 + \beta) + 8 \tag{2.15}
\]

and

\[
f(x_{n-3},x_{n-2}) = (2\alpha + 4) \sqrt{-1 - \beta}. \tag{2.16}
\]

If \( f(x_1,x_{3|n-2|n|2}) = 8 \), then we choose \( \alpha = 0 \) and \( \beta = -1 - f(x_{n-3},x_{n-2})^2/16 \), otherwise we use Eq. (2.15) to obtain

\[
1 + \beta = \frac{f(x_1,x_{3|n-2|n|2}) - 8}{4\alpha}; \tag{2.17}
\]

substituting this into Eq. (2.16), we obtain

\[
(\alpha + 2) \sqrt{\frac{8 - f(x_1,x_{3|n-2|n|2})}{\alpha}} = f(x_{n-3},x_{n-2}),
\]

or

\[
\sqrt{\alpha} + \frac{2}{\sqrt{\alpha}} = \frac{f(x_{n-3},x_{n-2})}{\sqrt{8 - f(x_1,x_{3|n-2|n|2})}}.
\]
This is a quadratic equation in $\sqrt[\gamma]{\alpha}$ that can easily be solved to
\[
\alpha = \pm \frac{f(x_{n-3}, x_{n-2}) \pm \sqrt{f(x_{n-3}, x_{n-2})^2 + 8f(x_1, x_{3\uparrow n-2}x_{1\downarrow 2}) - 64}}{32 - 4f(x_1, x_{3\uparrow n-2}x_{1\downarrow 2})},
\]
four solutions (some of which may coincide). Then $\beta$ can be found from Eq. (2.17). Now all values of $f$ are the same and hence $L$ is a quotient of $M$. Since $M$ is simple, $L = M$.

**Case 2:** Suppose that $n$ is even. We need to choose $\alpha$ and $\beta$ such that
\[
f(x_1, x_{3\uparrow n-2}x_{1\downarrow 2}) = -\frac{8\alpha(1 + \beta)}{(\alpha + 2)^2} + 16
\]
and
\[
f(x_{n-3}, x_{n-2}) = (2\alpha + 4) \sqrt{-1 - \beta}.
\]
The last equation can be written as
\[
\beta = -\frac{f(x_{n-3}, x_{n-2})^2}{2(\alpha + 2)} - 1.
\]
If we substitute this into Eq. (2.18), we get
\[
f(x_1, x_{3\uparrow n-2}x_{1\downarrow 2})(\alpha + 2)^3 = 8\alpha f(x_{n-3}, x_{n-2})^2 - 16\alpha - 32,
\]
which we can solve for $\alpha$ and thus find an explicit value for $\beta$. We find an isomorphism again.

**Theorem 2.61.** Let $L$ be a realization of the graph $\Gamma_{B\alpha}$ in Figure 2.4. Then $L$ is isomorphic to $B_{n-1}$ if $\psi_{\Gamma_{B\alpha}}(f)$ is in a certain open dense subset of $\psi_{\Gamma_{B\alpha}}(X_{B\alpha})$.

**Proof.** Let again $M$ be the realization defined in Theorem 2.45; we will specify the value of $\gamma$ later. Let $x_i$ be the $i$th extremal generator of $L$ and $z_i = T_{u_i,v_i}$ the $i$th extremal generator of $M$. In the same manner as in the proof of Theorem 2.60, we change $x_j$ such that the values of $f(x_i, x_j)$ are equal to those of $f(z_i, z_j)$ for $i, j \leq 3$, and such that $f(x_1, x_2x_3) = f(z_1, z_2z_3)$. By scaling $x_i$ we can assure that $f(x_{i-1}, x_i) = f(z_{i-1}, z_i)$ for $i < n$, and by scaling $x_n$ we can assure that $f(x_{n-2}, x_n) = f(z_{n-2}, z_n)$. Then what remains is assuring that $f(x_1, x_{3\uparrow n-2}x_{1\downarrow 2}) = f(z_1, z_{3\uparrow n-2}z_{1\downarrow 2})$.

Like in the proof of Theorem 2.60, we will do this by choosing the value of $\gamma$ appropriately. This requires explicitly constructing $z_{3\uparrow n-2}z_{1\downarrow 2}$. Again like before, for $3 \leq k < n$ we find with induction that $z_{k|2} = (1)^{n-k}(T_{u_k,v_k} + T_{u_k,v_k})$. Then
\[
z_{n|2} = (-1)^n(z_{n|2} - [z_{n|2}, T_{u_{n-1}, v_{n-1}}]) = (-1)^{n-1}T_{u_{n-1}, v_{n-1} + v_2}.
\]
With induction we show that $z_{k|n-2}z_{n|2} = (1)^{n-k}T_{u_k, v_k + v_3, u_2 + v_2}$ for $3 \leq k \leq n - 2$. Then
\[
z_{3|n-2}z_{n|2} = (1)^{n-3}T_{u_3, v_3, u_4 + v_4, u_2 + v_2}.
\]
Then
\[
f(z_1, z_{3\uparrow n-2}z_{1\downarrow 2}) = 2(B(u_1, u_3 + v_3)B(v_1, u_2 + v_2 + u_4 + v_4) - B(u_1, u_2 + v_2 + u_4 + v_4)B(v_1, u_3 + v_3)) = (-1)^n8\alpha.
\]
So we choose $\alpha$ to be $(-1)^n/8$ times the value of $f(x_1, x_{3\uparrow n-2}x_{1\downarrow 2})$. Then $f$ is identical on $L$ and $M$, so $L$ is a quotient of $M$; since $M$ is simple, they are isomorphic.

For each of the four infinite families of Chevalley type Lie algebras, we have shown a family of graphs such that a generic Lie algebra generated by a set of extremal elements realizing such a graph, is isomorphic to the corresponding Lie algebra.
3.1. Introduction

The Melikyan algebras are a two-parameter family of simple Lie algebras over fields of characteristic 5. They were discovered around 1980 by Melikyan [Mel79]. The underlying objective of this chapter will be to try and find a geometrical description of properties of the algebra.

We start by giving an overview of literature on the algebra in Section 3.2: we review a construction of the algebra and a proof that it defines a simple Lie algebra. In Section 3.3, we present an explicit construction of the automorphism group, which we have implemented in the computer algebra system GAP [GAP06]. Then in Section 3.4, we study an incidence geometry on some of the nilpotent elements of the Melikyan algebra. As part of this study we examine the orbits of the automorphism group on some quotients of the algebra. Finally in Section 3.5, we show that, using the computer-implemented automorphism group, it is fairly easy to determine the subalgebra of the Melikyan algebra generated by its sandwich elements, see Theorem 3.31; that is, its elements $x$ satisfying $(\text{ad } x)^2 = 0$. We determine a list of 68 invariant subalgebras derived from the algebra of sandwich elements. We consider the Hasse diagram of these subspaces and view the quotients as modules for the automorphism group of the Melikyan algebra.

3.2. Construction

We present the construction by Kuznetsov [Kuz91]. With this construction, the filtration of the algebra that is invariant under all automorphisms is easily found. In Melikyan’s original construction [Mel79], this was less apparent.

Recall from Chapter 1 the definition of the Witt algebra $W(2)$. Let $\tilde{W}(2)$ be a copy of $W(2)$ and endow the vector space $M = W(2) \oplus O(2) \oplus \tilde{W}(2)$ with an antisymmetric bilinear product $[\cdot, \cdot]$ defined as follows. On $W(2) \times W(2)$, the product is the usual multiplication;
on the rest of the space, it is determined by the following equations:

\[
[D, \tilde{E}] = [\tilde{D}, \tilde{E}] + 2 \text{div}(D)\tilde{E},
\]
\[
[D, f] = D(f) - 2 \text{div}(D)f,
\]
\[
[f_1 \tilde{\partial}_1 + f_2 \tilde{\partial}_2, g_1 \tilde{\partial}_1 + g_2 \tilde{\partial}_2] = f_1 g_2 - f_2 g_1,
\]
\[
[f, \tilde{E}] = fE,
\]
\[
[f, g] = 2(g \tilde{\partial}_2(f) - f \tilde{\partial}_2(g))\tilde{\partial}_1 + 2(f \tilde{\partial}_1(g) - g \tilde{\partial}_1(f))\tilde{\partial}_2,
\]

for \(D, E \in W(2), f, f_1, f_2, g, g_1, g_2 \in O(2)\). In this definition, \(\text{div}\) is the linear map determined by \(\text{div}(\delta i) = \delta i f\).

\(M\) is an \(O(2)\)-module: the \(O(2)\)-action on \(O(2)\) is the standard multiplication of \(O\); the action on \(W(2)\) and \(\hat{W}(2)\) is the standard \(O\)-action on \(W\). The first two lines of (3.1) make \(\hat{W}(2)\) and \(O(2)\) into \(W(2)\)-modules in a non-obvious way: one might more generally define

\[W(2)_{(\alpha_1, \text{div})} = \sigma_{\alpha_1}(D)E = [D, E] + \alpha_1 \text{div}(D)E,
\]
\[O(2)_{(\alpha_2, \text{div})} = \rho_{\alpha_2}(D)f = D(f) + \alpha_2 \text{div}(D)f,
\]

where the current multiplication has \((\alpha_1, \alpha_2) = (2, -2)\). These are \(W(2)\)-modules for all values of \(\alpha\). If we take different values or a field of a characteristic different from 5 and extend this to a multiplication on all of the algebra by the last three lines of Eq. (3.1), however, we do not get a Lie algebra: the resulting algebra does not satisfy the Jacobi identity. Indeed, the Jacobi identity for the triples \((1, x_2 \tilde{\partial}_2, \tilde{\partial}_1), (1, x_2, \tilde{\partial}_2)\) and \((1, \tilde{\partial}_1, x_2 \tilde{\partial}_2)\) leads to the equations \(\alpha_1 + \alpha_2 = 0, \alpha_2 = 3\) and \(\alpha_1 = 2\), respectively.

In the case \((\alpha_1, \alpha_2) = (2, -2)\) and \(F\) has characteristic 5, \(M\) is called the Melikyan algebra. From now on, we assume that this is the case. In Theorem 3.4, we will prove that it is actually a Lie algebra.

Finite-dimensional subalgebras of \(M\) can be obtained in a similar way to how \(O(m; n)\) and \(W(m; n)\) are obtained from \(O(m)\) and \(W(m)\): we let \(M(n_1, n_2)\) be the subalgebra of \(M\) consisting of \(W(2; (n_1, n_2)) \oplus O(2; (n_1, n_2)) \oplus W(2; (n_1, n_2))\). Its dimension is \(5^{n_1 + n_2 + 1}\).

### 3.2.1. Geometric construction

The following is what Kuznetsov calls the geometric interpretation of multiplication in \(M\), as introduced in [Kuz91]. It gives some motivation for the definition of the multiplication by Eq. (3.1) from the viewpoint of differential geometry. We will see parallels with the situation where \(O\) is the space of smooth scalar functions on \(\mathbb{R}^2\) and where \(W\) is the space of smooth sections of the tangent bundle of that manifold. In that situation, \(W\) is a free \(O\)-module of rank 2 with basis, \(\{\partial_1, \partial_2\}\). The elements of \(W\) act as derivations on \(O\). The cotangent bundle, dual to \(W\), is also a free \(O\)-module of rank 2 with basis, say, \([dx_1, dx_2]\) dual to \(\{\partial_1, \partial_2\}\).

With this in mind, let us return to the divided power algebra \(O(2)\) and the Witt algebra \(W(2)\). In the rest of this section, let \(O\) stand for \(O(2)\) and \(W\) for \(W(2)\). Then \(W\) is a free \(O\)-module of rank 2. Hence \(\wedge^2_O W\) is a free \(O\)-module of rank 1, generated by \(\partial_1 \wedge \partial_2\). The Witt algebra acts as derivations on \(O\). The \(O\)-module \(\text{Hom}_O(W, O)\) is free of rank 2 and dual to \(W\): it is generated by \(O\)-maps \(dx_1\) and \(dx_2\), determined by \(dx_i(f_1 \partial_1 + f_2 \partial_2) = f_i\). The second exterior power \(\wedge^2_O \text{Hom}_O(W, O)\) is again a free \(O\)-module of rank one, in duality with \(\wedge^2_O W\) by the rule \(\langle f \wedge g, D \wedge E \rangle = f(D)g(E) - f(E)g(D)\) for \(f, g \in \text{Hom}_O(W, O)\) and \(D, E \in W\).

We now formally define \(\omega = \omega^1 = dx_1 \wedge dx_2\), \(\omega^0 = 1 \in O\) and \(\omega^{-1} = \partial_1 \wedge \partial_2\). The natural actions of \(W\) on the free \(O\)-modules \(\wedge^2_O \text{Hom}_O(W, O)\), \(O\) and \(\wedge^2_O W\) are such that \(D(f \omega^i) = (D(f) + if(\text{div } D))\omega^i\) for \(i \in \{-1, 0, 1\}\).
Let $\mathbb{Z}_{(5)}$ be the ring of fractions in $\mathbb{Q}$ of which the denominator is not divisible by 5 (the localization of $\mathbb{Z}$ at $5\mathbb{Z}$). We then define for $\lambda \in \mathbb{Z}_{(5)}$ the free $O$-module $V_{\lambda} = O\omega^\lambda$, with the $W$-action following the same rule as above:

$$D(f\omega^\lambda) = (D(f) + \lambda f(\text{div } D))\omega^\lambda;$$

that is, $D$ acts as a derivation on the ‘product’ $f\omega^\lambda$ where $D(\omega^\lambda) = \lambda(\text{div } D)\omega^\lambda$. Furthermore, we define $A = \bigoplus_{\lambda \in \mathbb{Z}_{(5)}} V_{\lambda}$, and introduce the structure of a commutative $O$-algebra on $A$ by letting $\omega^\lambda \cdot \omega^\mu = \omega^{\lambda+\mu}$. This in turn determines an action of $W$ by derivations on $A$ by the isomorphism of $W$-modules $V_{\lambda} \otimes O V_{\mu} \cong V_{\lambda+\mu}$.

We will now identify $M$ with an algebra related to $A$, partially using the multiplication from $A$, partially using a different multiplication. This identification is built up by identifying

\[
\begin{align*}
W \subset M & \quad \text{with} \quad W \equiv V_0 \otimes O W \subset A \otimes O W, \\
O \subset M & \quad \text{with} \quad V_{-\frac{1}{2}} \subset A, \\
\overline{W} \subset M & \quad \text{with} \quad V_{\frac{1}{2}} \otimes O W \subset A \otimes O W.
\end{align*}
\]

The multiplication is defined as follows.

- On $W \times W$, the multiplication is the ordinary product in $W$, which is also the ordinary product of $A \otimes O W$.
- On $W \times V_{-\frac{1}{2}}$ and $W \times (V_{\frac{1}{2}} \otimes O W)$, the product is given by the standard action of $W$ on these two modules.
- On $(V_{\frac{1}{2}} \otimes O W) \times (V_{\frac{1}{2}} \otimes O W)$, the product is given by the commutative diagram below:

\[
\begin{array}{ccc}
(V_{\frac{1}{2}} \otimes O W) \times (V_{\frac{1}{2}} \otimes O W) & \xrightarrow{[\cdot, \cdot]} & V_{-\frac{1}{2}} \\
\wedge O & \downarrow & \mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu) \\
(W \wedge O W) \otimes O V_{\frac{1}{2}} \otimes O V_{\frac{1}{2}} & \cong & V_{-1} \otimes O V_{\frac{1}{2}} \otimes O V_{\frac{1}{2}}
\end{array}
\]

where $\mu$ is the product in $A$.

- On $V_{-\frac{1}{2}} \times (V_{\frac{1}{2}} \otimes O W)$ the product is given by this commutative diagram.

\[
\begin{array}{ccc}
V_{-\frac{1}{2}} \times (V_{\frac{1}{2}} \otimes O W) & \xrightarrow{[\cdot, \cdot]} & W \\
\mu \otimes 1 & \downarrow & \equiv \\
O \otimes O W & & \end{array}
\]

- On $V_{-\frac{1}{2}} \times V_{-\frac{1}{2}}$ the product is given by the following formula:

\[
[f \omega^{-\frac{1}{2}}, g \omega^{-\frac{1}{2}}] = \omega^{-\frac{1}{2}} d_f(g \omega^{-\frac{1}{2}}) - \omega^{-\frac{1}{2}} d_g(f \omega^{-\frac{1}{2}}),
\]

where $d_f$ is a derivation of $A$ into the $A$-module $A \otimes O W$, defined by

\[
d_f(g \omega^\lambda) = -\lambda g \omega^{\lambda+1} \otimes (\partial_1 f \partial_2 - \partial_2 f \partial_1).
\]

It is easy to see that these multiplication rules are the same as those in Eq. (3.1). One can even use these rules to define a multiplication on $W(2) \oplus O(2) \oplus W(2)$ for arbitrary characteristic (other than 3), but the Jacobi identity will not necessarily hold for the resulting algebra. It is not too difficult to see that the Jacobi identity

\[
[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0
\]
does hold in this setting if one of \(a, b\) and \(c\) is in \(W(2)\). However, this is not very easy to extend to a full proof of the Jacobi identity in characteristic 5.

3.2.2. Gradings and filtrations. Recall from the introduction that the grading and filtration play an important role in the Cartan type Lie algebras. The Melikyan algebra also has a grading, by the group \(\mathbb{Z}^2\). It is given by the following (inductive) table, where \(u\) is a homogeneous element of which the degree has been determined earlier:

<table>
<thead>
<tr>
<th>(f)</th>
<th>(\deg f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\partial_1)</td>
<td>((-2, -1))</td>
</tr>
<tr>
<td>(\partial_2)</td>
<td>((-1, -2))</td>
</tr>
<tr>
<td>(1)</td>
<td>((-1, -1))</td>
</tr>
<tr>
<td>(\tilde{\partial}_1)</td>
<td>((-1, 0))</td>
</tr>
<tr>
<td>(\tilde{\partial}_2)</td>
<td>((0, -1))</td>
</tr>
<tr>
<td>(x_1^i u)</td>
<td>((2i, i) + \deg u)</td>
</tr>
<tr>
<td>(x_2^i u)</td>
<td>((i, 2i) + \deg u)</td>
</tr>
</tbody>
</table>

Here \(x_1^i u\) means the action of \(O(2)\) on \(M\), not the bracket defined in Eq. (3.1). Out of this grading, a grading over \(\mathbb{Z}/3\mathbb{Z}\) follows by mapping \((1, 0)\) and \((0, 1)\) both to \(1 \in \mathbb{Z}/3\mathbb{Z}\); the homogeneous spaces with respect to this grading are \(W(2)\) (degree 0), \(O(2)\) (degree 1) and \(W(2)\) (degree 2).

Another useful grading follows by mapping \((1, 0)\) and \((0, 1)\) both to \(1 \in \mathbb{Z}\); so we can define \(M_i = \bigoplus_{r+s=i} M_{(r,s)}\). We denote by \(M_{\mathbb{Z}_1} = \bigoplus_{j \in \mathbb{Z}_1} M_j\) the terms of the corresponding decreasing filtration and by \(M_{\mathbb{Z}_2} = \bigoplus_{j \in \mathbb{Z}_2} M_j\) the terms of the corresponding increasing filtration. By Kuznetsov [Kuz91], the decreasing filtration \((M_{\mathbb{Z}_1})\) is invariant under all automorphisms of \(M\). We will give part of the proof in Section 3.3.1. If we write \(\deg x\) we will usually mean the degree with respect to this \(\mathbb{Z}\)-grading.

3.2.3. Jacobi identity and simplicity.

Lemma 3.1. The linear map sending

\[
\begin{align*}
x_1^i x_2^j \tilde{\partial}_1 & \quad \to \quad (-1)^{i+1} x_1^i x_2^j \tilde{\partial}_2, \\
x_1^i x_2^j \partial_1 & \quad \to \quad (-1)^i x_1^i x_2^j \tilde{\partial}_1, \\
x_1^i x_2^j \tilde{\partial}_2 & \quad \to \quad (-1)^i x_1^i x_2^j \partial_2, \\
x_1^i x_2^j \partial_2 & \quad \to \quad (-1)^{i+1} x_1^i x_2^j \partial_1
\end{align*}
\]

is an automorphism of \(M\).

Proof. This can easily be seen by manual checking. \(\square\)

This automorphism will be called “the automorphism interchanging \(x_1\) and \(x_2\)” in the rest of this chapter.

Lemma 3.2.

\[C_M(\partial_1) = \langle x_2^k \partial_1, x_2^k x_2^k \partial_1 : 0 \leq k < 5^{m_i}, i \in \{1, 2\} \rangle_F.\]

Proof. The space given in the lemma is clearly contained in the centralizer. To show the other inclusion, let \(u \in C_M(\partial_1)\). Put \(u = \sum u_v\) where \(u_v\) is homogeneous with respect to the \(\mathbb{Z}^2\)-grading, of degree \(v \in \mathbb{Z}^2\). Since \(\partial_1\) is a homogeneous element, each \(u_v\) is itself an element of \(C_M(\partial_1)\). So we may assume that \(u\) is homogeneous. The only homogeneous elements centralized by \(\partial_1\) are those in the assertion. \(\square\)
Corollary 3.3. \( C_M(\partial_1) \cap C_M(\partial_2) = M_{<0}. \)

Proof: We have determined \( C_M(\partial_1) \) in Lemma 3.2. Lemma 3.1 then shows that \( C_M(\partial_2) = \langle x_1^j \partial_i, x_1^j x_i^k \tilde{\partial}_i | 0 \leq k < 5^n, i \in \{1, 2\} \rangle_F. \) The intersection of the two is \( M_{<0}. \)

Theorem 3.4. \( M(n_1, n_2) \) is a Lie algebra for all \( n_1, n_2. \)

Proof sketch. \( M \) is clearly an anti-commutative algebra. We need to prove the Jacobi identity. We use the proof of Strade [Str04].

By a long computation \( \text{ad} \tilde{\partial}_1 \in \text{Der} M \) (see pages 201-202 of Strade [Str04]). By Lemma 3.1, \( \text{ad} \tilde{\partial}_2 \in \text{Der} M \) as well. Then \( \text{Der} M \) contains the algebra generated by these two, which is \( \text{ad} M_{\leq -1}. \)

Now let \( J(A, B, C) = [A, [B, C]] + [B, [C, A]] + [C, [A, B]]. \) We consider the value of \( J \) on triples of homogeneous elements with respect to the \( Z \)-grading.

If any of \( \text{ad} A, \text{ad} B \) and \( \text{ad} C \) are in \( \text{Der} M, \) then this expression vanishes. If \( \deg J(A, B, C) < 0, \) then at least one of \( A, B \) and \( C \) has negative degree, whence \( J(A, B, C) = 0. \) Next we apply induction: suppose \( \deg J(A, B, C) \geq 0, \) then

\[
\partial_i, J(A, B, C) = J([\partial_i, A], B, C) + J(A, [\partial_i, B], C) + J(A, B, [\partial_i, C])
\]

which has smaller intersection and is therefore 0. So \( J(A, B, C) \in C_M(\partial_1) \cap C_M(\partial_2). \) But that has empty intersection with \( M_{\geq 0} \) by Corollary 3.3. So indeed \( J = 0 \) on triples of homogeneous elements. Therefore it is 0 on \( M. \) Since \( M(n_1, n_2) \) is a subalgebra of \( M, \) it is a Lie algebra.

Lemma 3.5. \( O(2; n_1, n_2) \) and \( \widetilde{W(2; \widetilde{n_1, n_2})} \) are irreducible \( W(2; n_1, n_2) \)-modules.

Proof. Every nontrivial \( W(2; n_1, n_2) \)-submodule of \( O(2; n_1, n_2) \) contains \( F \cdot 1: \) just apply \( \partial_1 \) often enough. If we let \( \tau = (5^{n_1} - 1, 5^{n_2} - 1), \) then

\[
[x_1^j \partial_1, 1] = x_1^j,
\]

\[
[x^\tau \partial_1, x_1] = x_1^\tau - 2x_1^{\tau - e_1} \cdot x_1 = (1 - 2(5^{n_1} - 1))x_1^\tau,
\]

where \( e_1 \) is the first standard unit vector. From this element all other basis elements can be obtained by application of \( \partial_1 \) and \( \partial_2. \)

Let \( S \) be a nontrivial \( W(2; n_1, n_2) \)-submodule of \( \widetilde{W(2; n_1, n_2)}. \) By applying \( \partial_1 \) and \( \partial_2, \) we know that \( S \) contains at least some \( F(\alpha \tilde{\partial}_1 + \beta \tilde{\partial}_2) \) (where \( \alpha \) and \( \beta \) are not both equal to zero). Since \( (x_1 \partial_2, x_2 \partial_1)_{\text{Lie}} \) acts irreducibly on \( (\tilde{\partial}_1, \tilde{\partial}_2)_F, \) that space is contained in \( S. \)

Since for \( i \neq j \) we have

\[
[x_1^j \partial_i, \tilde{\partial}_i] = -x_1^i \tilde{\partial}_j
\]

\[
[x^\tau \partial_i, x_1 \tilde{\partial}] = x^\tau \tilde{\partial}_j - 0 \tilde{\partial}_j + 2x_1^{\tau - e_1} x_i \tilde{\partial}_j = (1 + 2(5^{n_1} - 1))x^\tau \tilde{\partial}_j = -x_1^i \tilde{\partial}_j.
\]

Again, all other standard basis elements can be obtained from this element by application of \( \partial_1 \) and \( \partial_2. \)

Theorem 3.6. \( M \) and \( M(n_1, n_2) \) are simple Lie algebras.

The proof below follows a number of similar proofs by Strade [Str04].

Proof. When considering \( M(n_1, n_2), \) one should interpret \( W, O, \tilde{W} \) and \( M \) as having the suffix \( (n_1, n_2) \) in this proof.

The \( Z \)-grading satisfies the following properties:

(P-1) For \( i \geq 0 \) and \( 0 \neq x \in M_i, \) there is \( y \in M_{-1} \) such that \( [x, y] \neq 0. \)

(P-2) If \( i < 0, \) then \( M_i = (M_{-1})^{-i}. \)

(P-3) \( M_{-1} \) is a faithful irreducible \( M_0 \)-module.
In property (P-2), the power is defined by letting $M_i^0 = M_i$ and $M_i^{j+1} = [M_i, M_i^j]$ for $j > 0$. Property (P-3) is the only one that is not immediately clear by a simple computation, but it follows by seeing that $M_0$ acts as $\mathfrak{g}_2$. These properties have been introduced by Weisfeiler in [Wei78] and were used by Strade in [Str04] for several simplicity proofs.

Let $J$ be a nonzero ideal. By property (P-4), $J \not\subseteq M_0$. So we can take $x \in J$, $x = \sum_{j=1}^k x_j$, $x_j \in M_j$, $x_k \neq 0$ and $k \geq 0$. By property (P-1), we can multiply this element by the right elements of $M_{-1}$ and end up with $x \in J$ with $k = -1$.

Now using property (P-3), $M_{-1} \subseteq J + \sum_{j<-1} M_j$; say $y \in M_{-1}$, then for some $z \in M_0$, we have $zx_{-1} = y$ so

$$y = z(x - \sum_{j<-1} x_j) \in J + \sum_{j<-1} M_j.$$

Using property (P-2), we see inductively that for $i < -1$, we have $M_i = [M_{-1}, M_{i+1}] \subseteq J + \sum_{j<i} M_j$; in particular, $M_{-3} \subseteq J$. From this we see inductively that $M_i \subseteq J$ for $i < 0$. So $M_0 \subseteq J$. In particular, each of $W$, $O$ and $\bar{W}$ intersect $J$ nontrivially. But they all are irreducible $W$-modules (by Lemma 3.5 and because $W$ is simple). So $J = M$.  

\[\square\]

3.3. Automorphisms

In this section, we will consider the automorphism group of $M(n_1, n_2)$, mainly for the case where $n_1 = n_2$. We will prove the following theorem (which can be found in e.g. [KM01]) in Section 3.3.2:

**Theorem 3.7.** The automorphism group Aut $M(n, n)$ of $M(n, n)$ is of the form $M^+_{\geq 1} \rtimes \text{GL}_2(\mathbb{F})$, where $(M^+_{\geq i})_{i=1}^\infty$ is a descending chain of groups such that $M^+_{\geq i}/M^+_{\geq i+1}$ is the additive group of $M_i$.

The study of automorphisms of the Melikyan algebra first requires an excursion into the study of its maximal subalgebras.

**3.3.1. Maximal subalgebras.** In [Kuz91], Kuznetsov showed that the filtration $(M_{2i})_i$ is invariant under the automorphism group. We will review this result here. The proof uses the notion of the depth of a maximal subalgebra, which is defined as follows. Let $L_0$ be a maximal subalgebra of $L$. For any $L_0$-module $L_{-1} \supset L_0$ such that $L_{-1}/L_0$ is an irreducible $L_0$-module, define $q(L_{-1}) = \min \{ s \in \mathbb{N} \cup \{ \infty \} \mid L = L_{-1} + L_{-1}^2 + \cdots + L_{-1}^s \}$. Then the depth of $L_0$ is $d(L_0) = \min_{L_{-1} \supset L_0} q(L_{-1})$. This is clearly a property that is invariant under automorphisms; that is, for an automorphism $\phi \in \text{Aut} L$, we find $d(L_0) = d(L_0^\phi)$.

Kuznetsov first shows that every proper subalgebra of $M(n_1, n_2)$ of codimension at most 5 has codimension exactly 5 and is therefore maximal, and it has one of these two forms:

- $M_{\geq 0}$, of depth 3. This is the only subalgebra of codimension 5 and depth 3, so it is invariant under automorphisms of $M$. An $M_{\geq 0}$-module realizing this depth is $M_{2,-1}$.
- $M^\circ$ for some projective point $\mathbb{F}v$ in $M_{-1}$, of depth 2 and defined as follows. $M^\circ$ is spanned by $v$, the intersection of the normalizer of $\mathbb{F}v$ with $M_0$, and $M_{\geq 1}$. For example, $M^\circ_{\hat{1}} = \langle \partial_1, x_1\partial_1, x_2\partial_1, x_2\partial_2 \rangle_{\mathbb{F}} \oplus M_{\geq 1}$ is the subalgebra generated by those standard basis elements having a positive second coordinate in the $\mathbb{Z}^2$-grading. An $M^\circ$-module realizing this depth is $\mathbb{F}\partial_1 \oplus M_{\geq 2}$, the subspace
spanned by those standard basis elements having a second coordinate in the \( \mathbb{Z}^2 \)-grading that is at least \(-1\). It can be shown that the \( \text{ad} M^\delta \)-action is irreducible by considering the action of \( \text{ad} \partial_1, \text{ad} x_1 \in \text{ad} M^\delta \).

If \( n_1 = n_2 \), then all these subalgebras are in one orbit under the automorphism group (so they are all isomorphic to \( M^\delta \)); if (say) \( n_1 < n_2 \), there are two orbits, one containing only \( M^\delta \) and the other containing \( M^\nu \) for \( \nu \) in other projective points of \( M_{-1} \).

This result is obtained as follows. Kuznetsov first studies homogeneous subalgebras of \( W(2; n_1, n_2) \) of codimension at most 3 and finds three isomorphism classes:

- \( W_{\geq 0} \) of codimension 2,
- \( W_{\geq 1} \oplus \mathfrak{sl}_2 \) of codimension 3, and
- \( W_{\geq 1} \oplus B \) where \( B \) is a Borel subalgebra of \( \mathfrak{gl}_2 = W_0 \); the codimension of this homogeneous subalgebra is also 3.

Then he considers a homogeneous subalgebra \( N \) of \( M(n_1, n_2) \). For \( i \in \mathbb{Z}/3\mathbb{Z} \), let \( N_i \) be the intersection of \( N \) with \( M_i \); that is, with \( W(2) \) for \( i = 0 \), with \( O(2) \) for \( i = 1 \) and with \( \tilde{W}(2) \) for \( i = 2 \). Kuznetsov then determines the tuples \( (n_0, n_1, n_2) \), where \( n_i \) is the codimension of \( N_i \) in \( M_i \), for \( n_0 + n_1 + n_2 \leq 5 \); these turn out to be \((2, 1, 2)\) and \( (3, 1, 1) \), corresponding to the two cases \( M_{\geq 0} \) and \( M^\nu \) above, respectively. Now let \( N \) be a not necessarily homogeneous subalgebra of \( M(n_1, n_2) \) of codimension 5. Let \( N \) be the homogeneous subalgebra associated to \( N \); that is, for all \( j \), let \( N_j \subset M_j \) such that \( N_j + M_{2j+1} = (N \cap M_{2j}) + M_{2j+1} \), and define \( N = \bigoplus N_j \). Then \( N \) is closed under multiplication and is thus a homogeneous subalgebra of \( M \) of the same codimension as \( N \). Hence \( N \) is either \( M_{\geq 0} \) or \( M^\nu \) for some \( \nu \in M_{-1} \). Finally, it is seen that \( N = N \), proving the result.

This result implies that \( M_{\geq 0} \) is an invariant subalgebra, for it is the only subalgebra of codimension 5 and depth 3. Furthermore, the union of all subalgebras of codimension 5 is the subspace \( M_{-1} \). We find the other subspaces in the filtration as follows. Recall the definition of the conductor from Section 1.2. We will see that \( \text{Cond}_M(M_{\geq 0}, M_{2i}) = M_{2i+1} \). For \( i \geq 0 \) and \( x \in M_{i+1} \), we have \( 0 \neq [x, M_{-1}] \subset M_{i} \), so \( M_{i+1} \cap \text{Cond}_M(M_{\geq 0}, M_{2i}) = \emptyset \). We see that \( \text{Cond}_M(M_{\geq 0}, M_{2i}) \subset M_{2i+1} \). On the other hand, clearly \( M_{2i} \subset \text{Cond}_M(M_{\geq 0}, M_{2i}) \). Hence all \( M_{2i} \) for \( i \geq 0 \) are invariant subspaces of \( M \). Similarly \( \text{Cond}(M_{\geq 0}, M_{2i}) = M_{2i-1} \) for \( i \leq 0 \). This shows that \( M_{2i} \) is invariant for all \( i \).

### 3.3.2. The structure of the automorphism group

In this section, we will prove Theorem 3.7, using well-known techniques and lemmas that can be found in e.g. [Skr01, KM01]. In the process of doing so, we will heavily use the invariance of the filtration of \( M \). Recall from Section 1.4 the corresponding series of subgroups \( \text{Aut} M = \text{Aut}_0 M \supset \text{Aut}_1 M \supset \cdots \) with \( \cap_{i=0}^{\infty} \text{Aut}_i M = 1 \), defined as follows:

\[
\text{Aut}_i M = \{ \phi \in \text{Aut} M \mid \forall j \in \mathbb{Z} \ [(\phi - 1)(M_{2j}) \subset M_{2j+1}] \}.
\]

Note that the equality of \( \text{Aut} M \) and \( \text{Aut}_0 M \) is equivalent to the invariance of the filtration. Furthermore, \( \text{Aut}_i M \) is a normal subgroup of \( \text{Aut} M \): it is the kernel of the natural action on \( \bigoplus_{j \in \mathbb{Z}} (M_{2j}/M_{2j+1}) \).

**Lemma 3.8.** Let \( j \geq 0 \) and let \( \phi \in \text{Aut} M \) satisfy \( (\phi - 1)(M_{-1}) \subset M_{-1} \). Then \( \phi \in \text{Aut}_j M \).
Proof. Let \( \psi = \phi - 1 \). We will show that \( \psi(M_i) \subseteq M_{2i+j} \) for all \( i \). Let \( x \in M_i, y \in M_j \). If \( \psi(x) \in M_{2i+j} \) and \( \psi(y) \in M_{2j+i+j} \), then

\[
[x, y]_\psi = [x^\phi, y^\psi] - [x, y] = [x + x^\psi, y + y^\psi] = [x, y] + [x^\phi, y^\psi] + [x^\phi, y^\psi],
\]

so \( [x^\psi, y] = [x, y^\phi] - [x^\phi, y^\psi] \). Both of these terms are in \( M_{2i+j} \), the first by the induction hypothesis and the second because \( \phi(M_{2i}) = M_{2i} \). So \( [x^\psi, y] \in M_{2i+j} \). If \( x^\phi \) has a homogeneous component \( x_k \) of degree \( k \leq i + j \), then \( x_k, M_{-1} = 0 \), and by property (P-1) from the proof of Theorem 3.6, \( x_k = 0 \). So \( x^\phi \in M_{2i+j+1} \).

Corollary 3.9. Let \( \phi \in \text{Aut} M \). Suppose \( \phi|_{M_{-1}} = 1 \). Then \( \phi = 1 \).

Proof. By Lemma 3.8, such a \( \phi \) would be in \( \text{Aut}_s M \) for every \( s \geq 0 \). Hence \( \phi \in \mathfrak{g}^\infty_{s=0} \text{Aut}_s M = 1 \).

Lemma 3.10. The map

\[
\xi: \text{Aut} M / \text{Aut}_1 M \to \text{GL}(M_{2-1}/M_{20}) \cong \text{GL}(M_{-1})
\]

defined by the action of \( \text{Aut} M \) on \( M_{2-1}/M_{20} \), is injective.

Proof. We first show that the map is well-defined. Let \( \phi, \psi \in \text{Aut} M \). Suppose \( \phi \psi^{-1} \in \text{Aut}_1 M \). Then for all \( x \in M_{-1} \), we have \( x^\phi - x^\psi \in M_{20} \). Hence \( \phi \) and \( \psi \) define the same map on \( M_{2-1}/M_{20} \cong M_{-1} \). Since this map is linear, we have shown that \( \xi \) is well-defined.

Now assume that \( \phi, \psi \in \text{Aut} M \) and \( \phi|_{M_{2-1}/M_{20}} = \psi|_{M_{2-1}/M_{20}} \). Let \( x \in M_{-1} \). Then \( x^\phi - x^\psi \in M_{20} \), so \( x^\phi x^\psi - x \in M_{20} \). Hence \( \phi \psi^{-1} \in \text{Aut}_1 M \) and \( \xi \) is injective.

Lemma 3.11. \( \text{Aut}_j M / \text{Aut}_{j+1} M \) embeds into the additive group of \( M_{j} \) for \( j > 0 \).

Proof. Let \( \phi \in \text{Aut}_j M \). Then \( \phi - 1 \) induces a linear transformation \( \text{gr} M \to \text{gr} M \) of degree \( j \). In particular, let \( x_i^\phi = x_i + x_i' \) for some \( x_i \in M_{k_i}, x_i' \in M_{k_{i+j}}, i = 1, 2 \). Then

\[
[x_1, x_2]^\phi = [x_1 + x_1', x_2 + x_2'] = [x_1, x_2] + [x_1, x_2'] + [x_1', x_2] + [x_1', x_2'],
\]

whence \( [x_1, x_2]_\phi^{-1} = [x_1, x_2]_\phi^{-1} + [x_1', x_2] \) in \( \text{gr} M \). So \( \phi - 1 \in \text{Der}_j \text{gr} M \cong M_j \).

Now let \( \phi, \psi \in \text{Aut}_j M \) such that \( \phi - 1 \) and \( \psi - 1 \) give rise to the same maps in \( \text{Der}_j \text{gr} M \). So for \( x \in M_{j} \), we find that \( x^\phi - x^\psi = x^\phi - x^\psi \in M_{2i+j+1} \). Then also \( x^\phi x^\psi - x \in M_{2i+j+1} \), so \( \phi \psi^{-1} \in \text{Aut}_{j+1} M \).

We have proven part of Theorem 3.7, in particular, we have shown that \( \text{Aut} M \) can be embedded into a group of the form \( M_{+}^j \rtimes \text{GL}_2(\mathbb{F}) \), where \( M_{+}^j \) is as in the theorem. In order to show that the full theorem holds, we invoke a result from literature. The following proposition is Corollary 3 from Kuznetsov and Mulyar [KM01], applied to the Melikyan algebra.

Proposition 3.12.

\[
\text{Lie}(\text{Aut} M(n_1, n_2)) \cong \text{Lie}(\text{Aut} W(2; n_1, n_2)) \oplus (O(2; n_1, n_2) \cap M_{20}) \oplus (W(2; n_1, n_2) \cap M_{20}).
\]
We see that the embeddings of Lemma 3.11 are isomorphisms if $M_j$ corresponds to the subspaces $O$ or $W$; that is, if $j \equiv 0 \pmod{3}$. For the cases where $j \equiv 0 \pmod{3}$ and for the embedding of Lemma 3.10, we need to examine $\text{Aut}(W; n_1, n_2)$. We have already seen that this group consists of automorphisms $\Phi_{\mu}$ such as in Eq. (1.2) from Section 1.4, determined by divided power automorphisms $\mu$ of $O(n_1, n_2)$. In the following lemma we characterize these automorphisms.

**Lemma 3.13.** Let $m \in \mathbb{N}_+$ and $n_1 = n_2 = \cdots = n_m$. The divided power automorphisms of $O(n_1, n_2, \ldots, n_m)$ over the field $\mathbb{F}$ of positive characteristic $p$ are determined by their values on the generators $x_i$. There is an automorphism $\mu$ mapping $x_i$ to $u_i + f_i$ for $1 \leq i \leq m$, such that $u_i \in O_1$ and $f_i \in O_{\geq 2}$, if and only if $(u_i | i \leq m)_\mathbb{F} = O_1$.

**Proof.** In this proof, we will explicitly use parentheses to indicate divided powers, while we otherwise often omit them. Furthermore, we use some of the multi-index notation from Section 1.4. Let $\mu$ be a divided power automorphism and let $i \leq m$. Then for all $k \leq p^m$, we have $\mu(x_i^{(k)}) = (\mu(x_i))^{(k)}$. Now since for $a \in \mathbb{N}_0^m$ we have $\mu(x^{(a)}) = \mu(x_1^{(a_1)})\mu(x_2^{(a_2)})\cdots\mu(x_m^{(a_m)})$, the value of $\mu$ on the generators $x_i$ determines the value of $\mu$ on all monomials, and hence determines $\mu$.

Let $\mu$ be a divided power automorphism. Since all divided power automorphisms fix $O_{\geq 1}$, we have $\mu(x_i) \in O_{\geq 1}$. Hence we can decompose $\mu(x_i)$ as $u_i + f_i$ with $u_i \in O_1$ and $f_i \in O_{\geq 2}$. Consider the action of $\mu$ on $O_{\geq 1}/O_{\geq 2} =: V$. Since $O_{\geq 1}$ is an invariant subspace, $\mu$ must map it to itself, not to a proper subspace. Hence $(\mu(x_i) | i \leq m)_\mathbb{F}/O_{\geq 2}$ cannot be a proper subspace of $V$. Thus $(u_i | i \leq m)_\mathbb{F} = O_1$.

For the other implication, let $\mu$ be defined by $\mu(x_i) = u_i + f_i$ and for $a \in \mathbb{N}_0^m$ with $a_i < p^{m_i}$, by $\mu(x^{(a)}) = \prod_{i=1}^m (\mu(x_i))^{(a_i)}$. Suppose $(u_i | i \leq m)_\mathbb{F} = O_1$ and that all $f_i$ are in $O_{\geq 2}$. If $a$ is as above and $b$ satisfies the same properties, then

$$
\mu(x^{(a)})\mu(x^{(b)}) = \prod_{i=1}^m \mu(x_i^{(a_i)})\mu(x_i^{(b_i)}) = \prod_{i=1}^m \left(\frac{a_i + b_i}{a_i}\right)^{\mu(x_i^{(a_i+b_i)})} = \left(\frac{a + b}{a}\right)^{\mu(x^{(a+b)})} = \mu\left(\frac{a + b}{a}\right)^{\mu(x^{(a+b)})} = \mu(x^{(a)}x^{(b)}),
$$

and since multiplication is bilinear, $\mu$ respects multiplication.

Since every $u_i$ is a nonzero element of $O_1$, we see that $\mu(x_i^{(0)}) = 0$ if and only if $x_i^{(0)} = 0$. Now suppose that $r \geq 0$ and $f \in O$ satisfies $\mu(f^{(r)}) = \mu(f)^{(r)}$. Let $\lambda \in \mathbb{F}$ and $a$ as above. We will show that $\mu((\lambda x^{(a)} + f)^{(r)}) = \mu(\lambda x^{(a)} + f)^{(r)}$. This shows that $\mu$ respects the divided power structure.

$$
\mu((\lambda x^{(a)} + f)^{(r)}) = \mu\left(\sum_{\ell=0}^r (\lambda x^{(a)})^{(\ell)} f^{(r-\ell)}\right) = \sum_{\ell=0}^r \mu((\lambda x^{(a)})^{(\ell)} f^{(r-\ell)}) = \mu((\lambda x^{(a)}) + f)^{(r)} = \mu(\lambda x^{(a)} + f)^{(r)}.
$$

Now let us prove that $\mu$ is invertible. This will finish the proof of the lemma. If all $f_i$ are zero, then $\mu$ is clearly invertible: in this case, $\mu$ is an invertible linear transformation on $O_1$ and it acts on $O_2$ by the automorphism with the $k$th symmetric tensor of $O_1$; this action preserves the subspace $O_k \cap O(m; n_1, \ldots, n_m)$.

It is easy to see that every automorphism can be written as a product of an automorphism where every $f_i$ is zero and an automorphism where every $u_i = x_i$; so we need only show that $\mu$ is invertible assuming that $u_i = x_i$ for all $i$. Let $\lambda_\alpha \in \mathbb{F}$ be nonzero
for $a$ in some index set $A \subset \mathbb{N}_0^n$, with every $a_i < p^{n_i}$ for all $a \in A$. We will show that 
\[ \mu(\sum_{a \in A} \lambda_a x^{(a)}) \neq 0. \]
Note that
\[ \mu(\sum_{a \in A} \lambda_a x^{(a)}) = \sum_{a \in A} \lambda_a \prod_{i=1}^m \mu(x_j^{(a)}). \]
Since $\mu(x_i) \in O_{\geq 1} \setminus O_{= 2}$, its divided powers are such that $\mu(x_i)^{(k)} \in O_{= k} \setminus O_{= k+1}$ (unless $\mu(x_i)^{(k)} = 0$). If we decompose $\mu(\sum_{a \in A} \lambda_a x^{(a)})$ according to the grading of $O$, then the lowest-degree part that is potentially nonzero is that of degree $\min_{a \in A} |a|$. We will show that this component is nonzero. Hence we may assume that all $a \in A$ have the same value for $|a|$, say $|a| = k < p^{n_i}$. The degree-$k$ component of $\mu(\sum_{a \in A} \lambda_a x^{(a)})$ is equal to
\[ \sum_{a \in A} \lambda_a \prod_{i=1}^m a_j^{(a)} = \sum_{a \in A} \lambda_a x^{(a)} \neq 0. \]
This finishes the proof. \[ \square \]

**Lemma 3.14.** Let $m \in \mathbb{N}_+$ and $n_1 = n_2 = \cdots = n_m$. Let $1 \leq j \leq m$ and $a \in \mathbb{N}_0^m$ with every $a_i < p^{n_i}$ and with $|a| > 1$. Let $\mu$ be the automorphism determined by
\[ \mu: x_i \mapsto \begin{cases} x_i, & \text{if } i \neq j, \\ x_i + x^a, & \text{if } i = j. \end{cases} \]
Then $\Phi_\mu = \mathbb{I} - \text{ad}(x^a \partial_j) + \phi$, where $\phi$ is a linear map with $\phi(W_{\geq i}) \subseteq W_{\geq i+|a|}$.

**Proof.** First note that for all $i$, we have $\mu^{-1}(x_i) = x_i - \delta_{ij} x^a + O_{\geq |a|+1}$, where $\delta_{ij}$ is Kronecker’s symbol. Hence,
\[ (\partial_k^{D^\mu})(x_i) = (\mu^{-1} \circ \partial_k)(x_i + \delta_{ij} x^a) = \mu^{-1} (\delta_{jk} + \delta_{ij} x^{a-e_k}) = \delta_{jk} + \delta_{ij} x^{a-e_k} + O_{\geq |a|} \]
where we write $e_k$ for the $k$th unit vector and where $x^{a-e_k}$ should be interpreted as zero if $a_k = 0$. With these conventions, we see that
\[ \partial_k^{D^\mu} \in \partial_k + x^{a-e_k} \partial_j + W_{\geq |a|-1} = (\mathbb{I} - \text{ad}(x^a \partial_j))\partial_k + W_{\geq |a|-1}. \]
Thus $\Phi_\mu - (\mathbb{I} - \text{ad}(x^a \partial_j))$ maps each $\partial_k$ into $W_{\geq |a|-1}$. From this it is easy to see that $\Phi_\mu - (\mathbb{I} - \text{ad}(x^a \partial_j))$ maps $W_{\geq i}$ into $W_{\geq i+1}$. \[ \square \]

**Proof of Theorem 3.7.** We see from Lemma 3.13 that for all $\phi \in \text{GL}_2$, there is an automorphism of $M$ acting as $\phi$ on $\langle x_1, x_2 \rangle_F$. This group acts dually on $\langle \partial_1, \partial_2 \rangle_F = M_{-3}$. Since $[\partial_1, [\partial_2, \partial_1]] = \partial_1$, an automorphism acting by a matrix $A$ on $M_{-1}$, acts on $M_{-3}$ as $A \text{det} A$. Hence, an automorphism acting by a matrix $B$ on $M_{-3}$ acts on $M_{-1}$ as $(\text{det} B)^{-1/3} B$. This is not necessarily a surjective map into $\text{GL}(M_{-1})$; however, the “missing” automorphisms can be found by composing with the automorphisms $\phi_\lambda$ for $\lambda \in \mathbb{F}$, which send each homogeneous $x \in M_i$ to $\lambda^i x$.

Hence the embeddings of Lemma 3.10 are isomorphisms. The maps referred to in Lemma 3.13 where each $u_i = x_i$ show that the embeddings of Lemma 3.11 are isomorphisms by Lemma 3.14. This finishes the proof. \[ \square \]

**3.3.3. Constructing concrete automorphisms.** For the rest of Section 3.3, assume that $n_1 = n_2 = n$ except where we explicitly make a remark for the other case. Given that the structure of the automorphism group is as in Theorem 3.7, we will present a procedure for finding concrete elements of $\text{GL}(M(n, n))$ generating the full automorphism group. We will find a subgroup isomorphic to $\text{GL}_2(F)$. Furthermore, for the case where $n = 1$, we will find for all $i > 0$ a spanning set of homogeneous elements $x$ of $M_i$ and
corresponding group elements acting as $1 + \text{ad } x + \phi$ where $\phi$ is a map of higher degree than $\text{ad } x$. These procedures are implemented in a program in the computer algebra system GAP [GAP06].

To determine a set of elements $\phi \in \text{Aut } M$ that generate a subgroup isomorphic to $\text{GL}_2(\mathbb{F})$, we proceed as follows. In the proof of Theorem 3.7, we have found that the subgroup of $\text{Aut } W$ fixing the grading of $W$ consists of all invertible transformations on $\langle x_1, \ldots, x_m \rangle_F$. So for the Melikyan algebra, these elements generate a subgroup isomorphic to $\text{GL}_2$. The $\text{GL}_2(\mathbb{F})$ group acting naturally on $\langle x_1, x_2 \rangle_F$ acts dually on $\langle \partial_1, \partial_2 \rangle_F = M_{-3}$. Since $[\partial_i, [\partial_2, \partial_1]] = \partial_i$, an automorphism acting by a matrix $A$ on $M_{-1}$, acts on $M_{-3}$ as $A \det A$. Hence, an automorphism acting by a matrix $B$ on $M_{-3}$ acts on $M_{-1}$ as $(\det B)^{-1/3}B$. This is not necessarily a surjective map into $\text{GL}(M_{-1})$; however, the “missing” automorphisms can be found by composing with the automorphisms $\phi_\lambda$ for $\lambda \in \mathbb{F}$, which send each homogeneous $x \in M_i$ to $\lambda^i x$. This shows

Let $\mu \in \text{Aut } M$. Given the images under $\mu$ of $\partial_1$ and $\partial_2$, we will determine the matrix of $\mu$ as acting on all of $M$ by the following procedure. We write $N$ for $5^n - 1$. Observe that $M$ is generated by the four elements $\partial_i$ and $x_i^N x_j^N \partial_i$ for $i = 1, 2$; this is easily seen by manual checking. So it suffices to determine the elements that $x_i^N x_j^N \partial_i$ are mapped to. (Note that these are elements of the innermost nonzero component of the invariant filtration, which has dimension 2.)

Let
\[
\mu: \begin{cases} 
\partial_1 &\mapsto m_{11} \partial_1 + m_{12} \partial_2 + f_1, \\
\partial_2 &\mapsto m_{21} \partial_1 + m_{22} \partial_2 + f_2,
\end{cases}
\]
with $f_1, f_2 \in M_{\geq 0}$. It is easily seen that $1^\mu = (m_{11} m_{22} - m_{12} m_{21}) \cdot 1 \pmod{M_{\geq -1}}$.

Let
\[
D_1 = (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2,
\]
\[
D_2 = (\text{ad } \partial_1) \circ (\text{ad } \partial_2)^2.
\]

Then for any $f \in O$, we find that $D_1(f) = -\partial_1 f$ and $D_2(f) = \partial_2 f$. So in particular,
\[
D_1^N \circ D_2^N = (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2 \circ (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2 = (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2 \circ (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2 = (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2 \circ (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2 = (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2 \circ (\text{ad } \partial_2) \circ (\text{ad } \partial_1)^2.
\]

Let $D_1^\mu = (\text{ad } \partial_2^\mu) \circ (\text{ad } \partial_1^\mu)^2$ and $D_2^\mu = (\text{ad } \partial_1^\mu) \circ (\text{ad } \partial_2^\mu)^2$. It is easily checked that if $f \in M_{3i-2}$ (whence its $O$-components are in $O_{2i}$), then
\[
D_1^\mu(f) = (m_{12} m_{21} - m_{11} m_{22})(m_{11} \partial_1 f + m_{12} \partial_2 f) \pmod{M_{3i-4}},
\]
\[
D_2^\mu(f) = (m_{12} m_{21} - m_{11} m_{22})(m_{21} \partial_1 f + m_{22} \partial_2 f) \pmod{M_{2i-4}}.
\]

Therefore, we have
\[
(D_1^\mu(x_1^N x_2^N) = (m_{12} m_{21} - m_{11} m_{22}) \sum_{k=0}^\ell \binom{\ell}{k} m_{11}^{\ell-k} m_{12}^k x_1^N m_{21}^{\ell-k} m_{22}^k x_2^N \pmod{M_{23(i+j-\ell)-1}}
\]
for all $\ell$, where the terms with negative powers of $x_1$ or $x_2$ are to be omitted; and similarly,
\[
(D_2^\mu(x_1^N x_2^N) = (m_{12} m_{21} - m_{11} m_{22}) \sum_{k=0}^N \binom{N}{k} m_{21}^{N-k} m_{22}^k x_1^N m_{21}^{k} x_2^N \pmod{M_{23N-1}}.
\]
Hence,

\[(D_1^N \circ D_2^N)^\mu : x_1^N x_2^N \mapsto (m_{12} m_{21} - m_{11} m_{22})^{2N} \sum_{k=0}^{N} \binom{N}{k} \binom{N-k}{k} m_{11}^{m_{12}^N - m_{21} m_{12}^{N-k} - m_{22}} \cdot 1 =: \Xi \cdot 1 \pmod{M_{\Xi-1}}.\]  

(3.3)

Note that \( \binom{n}{k} \) is always 0 mod \( p \) for all \( i \), unless \( k \in \{0, p'\} \); since \( \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \), we see that \( \binom{n}{k} = (-1)^k \pmod{5} \) and therefore \( \binom{N}{k}^2 = 1 \). Since \( \mu \) necessarily preserves \( M_{\Xi-1} = M_{\Xi-1} = (x_1^N x_2^N d_1, x_1 x_2 d_2) \), we can find \( \ell_{ij} \) with \( \mu: \{ x_1^N x_2^N d_1 \mapsto \ell_{11} x_1^N x_2^N d_2 + \ell_{12} x_1 x_2 d_1, x_1 x_2 d_2 \mapsto \ell_{21} x_1 x_2 d_1 + \ell_{22} x_1^N x_2^N d_2 \}. \)

We can now evaluate \( \mu((D_1^N \circ D_2^N \circ (ad \tilde{d}_i))(x_1^N x_2^N d_2)) \) in two ways. Firstly, by Eq. (3.2),

\[\mu((D_1^N \circ D_2^N \circ (ad \tilde{d}_i))(x_1^N x_2^N d_2)) = (j - i) \mu(1) = (j - i)(m_{11} m_{22} - m_{12} m_{21}) \cdot 1 \pmod{M_{\Xi-1}}.\]

Secondly, using Eq. (3.3),

\[\mu((D_1^N \circ D_2^N \circ (ad \tilde{d}_i))(x_1^N x_2^N d_2)) = (D_1^N \circ D_2^N)^\mu((m_{11} \tilde{d}_1 + m_{12} \tilde{d}_2, \ell_{11} x_1^N x_2^N \tilde{d}_1 + \ell_{12} x_1 x_2 \tilde{d}_1, \ell_{21} x_1 x_2 \tilde{d}_1 + \ell_{22} x_1^N x_2^N \tilde{d}_2)) = (D_1^N \circ D_2^N)^\mu((m_{11} \ell_1 - m_{12} \ell_1) x_1^N x_2^N) = (m_{11} \ell_2 - m_{12} \ell_2) \Xi \cdot 1 \pmod{M_{\Xi-1}}.\]

From this it is easy to see that \( \ell_{ij} = \Xi^{-1} m_{ij} \) for all \( i, j \in \{1, 2\} \). We have now determined the images of a generating set of elements. Since every standard basis element can be obtained by applying \( ad \tilde{d}_1 \) and \( ad \tilde{d}_2 \) to \( x_1^N x_2^N \tilde{d}_1 \) or \( x_1 x_2 \tilde{d}_2 \) sufficiently many times, we can easily read off the images of the standard basis elements under \( \mu \).

We have seen that there is a group of automorphisms of \( M \) isomorphic to GL\(_2(\mathbb{F})\), preserving the grading and acting faithfully on \( M_{-1} \). Hence for a set of generators \( g \) of GL\(_2(\mathbb{F})\), we let \( g \) act on \( M_{-1} \) and then perform the procedure above to obtain generators of the subgroup of Aut\(_M\) isomorphic to GL\(_2(\mathbb{F})\). Automorphisms in Aut\(_1 M\) can be obtained even more easily, since they fix \( \langle x_1^N x_2^N \tilde{d}_1 | i \in \{1, 2\} \rangle \) pointwise. In fact, for most standard basis elements \( x \) of positive degree, exp(ad \( x \)) is a well-defined automorphism. A simple check using GAP [GAP06] reveals that for \( n = 1 \), the only standard basis elements of positive degree where this does not work are \( x_1, x_2, x_1 x_2 \tilde{d}_1, x_1 x_2 \tilde{d}_2 \) and \( x_1 x_2 \). (This check can be performed by running the function check_1 from page 147.) For the last three, it is sufficient to add exp(ad \( x \)) for these values of \( x \) to the automorphisms we already have of that degree:

\[
x_1^2 + x_1 x_2 + x_2^2, \quad x_1^2 \tilde{d}_1 + x_1 x_2 \tilde{d}_2 + x_2^2 \tilde{d}_1, \quad x_1^2 \tilde{d}_2 + x_1 x_2 \tilde{d}_2 + x_2^2 \tilde{d}_2.
\]

Finally consider the component Aut\(_1 M\). To find an automorphism of the form 1 + ad \( x_i + \phi_i \) for \( i = 1, 2 \), we need only find suitable images of \( \tilde{d}_1 \) and \( \tilde{d}_2 \) and then apply the procedure above. It turns out that these two maps can be extended to automorphisms, which can be verified in GAP:

\[
\tilde{d}_1 \mapsto \exp(\text{ad} x_2)(\tilde{d}_1) + 2 x_2^3 \tilde{d}_1 \quad \text{and} \quad \tilde{d}_2 \mapsto \exp(\text{ad} x_2)(\tilde{d}_2) + 2 x_2^3 \tilde{d}_2;
\]

\[
\tilde{d}_1 \mapsto \exp(\text{ad} x_1)(\tilde{d}_1) + 3 x_1^3 \tilde{d}_1 \quad \text{and} \quad \tilde{d}_2 \mapsto \exp(\text{ad} x_1)(\tilde{d}_2) + 3 x_1^3 \tilde{d}_2.
\]

This verification is implemented in the function check_2 on page 148.

It seems that the methods of this section generalize fairly well to the case where \( n_1 \neq n_2 \), except that in that case Aut\(_M\)/Aut\(_1 M\) is not GL\(_2(\mathbb{F})\) but a Borel subgroup (stabilizing the \( x_i \) for which \( n_i < n_{-i} \)).
3.3.4. The automorphism group as an algebraic group.

**Lemma 3.15.** The automorphism group of $M(n_1, n_2) \otimes \mathbb{F}$ is an algebraic group over $\mathbb{F}$, where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}$.

**Proof.** The automorphism group is a subgroup of $\text{GL}(M)$ and the conditions for elements to preserve multiplication are polynomial equations of degree 2. □

**Lemma 3.16.** For $i \geq 0$, we have $\dim \text{Aut}_i M(n_1, n_2) = \dim M_{2i}(n_1, n_2)$.

**Proof.** By induction. The lemma holds if $i = 0$. Now for the induction step, assume that the lemma holds for some $i+1 > 1$. As $x$ iterates over a basis, automorphisms $1 + ad x + \theta(M_{2j}) \subseteq M_{2i+1}$ are clearly linearly independent of each other and of $\text{Aut} M_{i+1}$. The embedding of $\text{Aut}_i M / \text{Aut}_{i+1} M$ into $\text{M}_i$ shows that the lemma holds for $i > 0$. Finally, since $\dim \text{GL}_2 = \dim M_{0}$, the lemma holds in general. □

We see that the automorphism group has dimension $5^{2n+1} - 5$. Let $S$ denote the stabilizer of the subspace $\mathbb{F}\partial_1$ in $\text{Aut}_M$.

**Lemma 3.17.** For $i \geq 0$, we have $\dim (S \cap \text{Aut}_i M) = \dim C_{M_{2i}}(\partial_1)$.

**Proof.** Like in the proof of Lemma 3.16, this is obvious if $i$ is such that $M_i = \{0\}$. So let us assume that for some $i > 0$, we have $S \cap \text{Aut}_{i+1} M$ is a $\dim C_{M_{2i+1}}(\partial_1)$-dimensional (normal) subgroup of $\text{Aut}_i M$.

Let $B$ be a cobasis of $C_{M_{2i}}(\partial_1)$ with respect to $C_{M_{2i+1}}(\partial_1)$. For $\lambda \in \mathbb{F}$ and $b \in B$, let $\phi_{\lambda,b} = 1 + \lambda \text{ad} b + \theta_{\lambda,b} \in S \cap \text{Aut}_i M$, with $\theta_{\lambda,b}(M_j) \subseteq M_{2i+1}$, $\lambda \in \mathbb{F}$, $b \in B \cup \{0\}$. Furthermore, let $x \in M_i$ and let $\phi = 1 + \text{ad} x + \lambda \theta_{\lambda,b} \in S$ with $\theta(M_{2j}) \subseteq M_{2i+1}$. Then $x \in C_{M_{2i}}(\partial_1)$, for otherwise $\phi$ does not fix $\partial_1$. So $x$ can be written as a linear combination of elements of $B$, say $\sum \lambda_i b_i$. Then $x(\prod \phi_{\lambda_i,b_i})^{-1} \in S \cap \text{Aut}_{i+1} M$. So $S \cap \text{Aut}_i M \subseteq (S \cap \text{Aut}_{i+1} M) \times G$. Clearly the other inclusion holds too. The dimension of $(S \cap \text{Aut}_i M)/(S \cap \text{Aut}_{i+1} M)$ is precisely that of $C_{M_{2i}}(\partial_1)/C_{M_{2i+1}}(\partial_1)$, which shows that the lemma holds for $i > 0$.

Since both the stabilizer of $\mathbb{F}\partial_1$ in $\text{Aut}_M/\text{Aut}_1 M$, and the centralizer of $\partial_1$ in $M$, have dimension 3, the lemma holds for $i = 0$ as well. □

This implies that $\dim S = 5^{2n+1} - 5$.

Since the dimension of the automorphism group is five less than the dimension of the algebra, there are approximately $q^5$ orbits of the group on the algebra, where $q$ is the size of the field. This is much more than what one would expect to find with, for example, the Chevalley type algebras, where the dimension of the automorphism group is always at least the dimension of the algebra. We can also approximate the number of orbits that $S$ has on the $\text{Aut}_M$-orbit $O$ containing $\mathbb{F}\partial_1$. By Lagrange’s theorem, there is a one to one correspondence between $O$ and $\text{Aut}_M/S$. It is easily seen that there is a similar correspondence between $S$-orbits on $O$ and double cosets $S \phi S$ for $\phi \in \text{Aut}_M$. There are at least $|\text{Aut}_M|/|S|^2$ of those, which is approximately $q^{5^{2n+1} - 2 \cdot 5^{n+1} + 5}$.

### 3.4. Geometry

In this section, we investigate an incidence geometry occurring in the Melikyan algebra $M(1, 1)$. Such geometries are found in the Chevalley type Lie algebras by taking as point set $P$ the projective points corresponding to long root elements, and as line set $L$ those projective lines consisting only of such projective points. In the Chevalley type Lie algebras, the points are nilpotent elements of low degree (degree 3, in particular). Such elements are studied in Chapter 2 of this thesis. Classes of geometries can often be
characterized by a set of axioms, and this characterization of the geometries leads to more insight into the underlying algebraic structure, see e.g. Cohen and Ivanyos [CI06, CI].

The automorphism group $G$ of the underlying algebraic structure also acts on the geometry, typically transitively on the incident point-line pairs. If this is the case, then $P$ – the orbit of $G$ on any single point $p$ – can be identified with the set of cosets of the point stabilizer $G_p$ in $G$, by Lagrange’s theorem. Similarly $L$ can be identified with the cosets of the line stabilizer $G_\ell$ in $G$. Two cosets $G_p g$ and $G_\ell h$ correspond to an incident point-line pair if and only if $G_p g \cap G_\ell h$ is nonempty. A quotient of this geometry by a normal subgroup $N$ of $G$ is a smaller, simpler geometry. Its point set consists of the cosets of $G_p N$ in $G$, the lines are cosets of $G_\ell N$ and incidence is, like in the original geometry, given by nonempty intersection. Equivalently, the point set of the quotient consists of the $N$-orbits on $P$ and the line set consists of the $N$-orbits on $L$.

The next step is typically to find a set of objects corresponding to three-dimensional subspaces in the algebraic structure, called planes. One may choose to distinguish different types of these subspaces, or add still bigger types of objects. In this way one ends up with a set of objects, partitioned into different types, and an incidence relation given by e.g. symmetrized containment. One then studies flags: sets of objects that are mutually incident. If no two objects of the same type are incident, and every flag can be extended to have an object of every type, then the ensemble of the objects, the types and the incidence relation is called a geometry. Ideally, one would like the automorphism group to be transitive on the maximal flags.

Now fix a pair of types. Take a flag $F$ containing objects of all but these two types, and consider the geometry consisting of all objects incident with all of $F$ (called the residue of $F$). Suppose that these residues are all in a certain class of geometries, e.g. projective planes, independent of $F$ (but possibly depending on the pair of types fixed). If this holds for every pair of types, then we typically illustrate this in a diagram where the nodes are the types and two nodes are connected by an edge signifying the class of geometries occurring in the residues of that type.

Our object is to construct such a diagram geometry in the Melikyan algebra. We will propose a point set and a line set in Section 3.4.1. There are several notions of a plane that make sense in that context. One is to simply take the points and lines occurring inside a subspace spanned by two intersecting lines; we will call this a subspace plane. This point of view is taken in Section 3.4.2: we list all the different subspace planes that we can find. We will see that the structure that we find is very complex. Therefore, in Section 3.4.3, we choose to regard only one of the isomorphism classes as planes and examine the geometry from that viewpoint. In order to understand this structure better, in Section 3.4.4 we study some of its quotients.

Starting in Section 3.4.2, we will let $q$ denote the size of the field, if that is finite; if the field is infinite, all expressions involving $q$ should be read as infinite as well.

### 3.4.1. Definition and basic properties.
We need to find a class of low-degree nilpotent elements that will serve as points of the geometry. If we would take a class of elements in a proper subspace $M_{\geq 2}$ of $M$, then the full structure of the Melikyan algebra could never be reflected in the geometry, since all points come from a proper subspace of $M$. So we choose a class of points outside $M_{\geq 2}$. For the rest of this section, we will write $H := M_{\geq 2}$.

Consider the projective points $F\partial_1$ and $F\partial_2$. We know that $(\text{ad } \partial_1)^5 = (\text{ad } \partial_2)^5 = 0$ and that $[\partial_1, \partial_2] = 0$. Thus, $(\partial_1, \partial_2)_F$ is a restricted subalgebra of $M$. The following is Theorem 2.1 of Skryabin [Skr01]:
Theorem 3.18. Let $A$ be a restricted subalgebra of $M$ such that $M = A \oplus H$. Then every embedding $i: A \rightarrow M$ of restricted Lie algebras such that $M = i(A) \oplus H$ can be extended to an automorphism $\theta$ of $M$.

In particular, for every ordered pair of linearly independent commuting elements $a$ of $M \setminus H$ with $(ad a)^3 = 0$, that differ modulo $H$, there is an automorphism that maps that pair to $(\partial_1, \partial_2)$. The projective points on such elements will form the points of the geometry, and the lines of the geometry will be the projective lines consisting of only such points. So let $P = \{([Fd]_\phi | \phi \in \text{Aut} M) \text{ and } L = \{[\partial_1, \partial_2]_\phi | \phi \in \text{Aut} M\}$. Theorem 3.18 shows that the automorphism group is transitive on the points. We determine the stabilizer below.

**Lemma 3.19.** The stabilizer of $[\partial_1, \partial_2]_F$ in Aut $M$ is GL$_2(F)$.

**Proof.** The subgroup of Aut $M$ stabilizing the grading clearly stabilizes $[\partial_1, \partial_2]_F$ and is of the form GL$_2(F)$. We will denote this group by $G$ and show that it is the full stabilizer.

Let $\phi \in \text{Aut}_1 M$ be different from 1. Then $\phi = 1 + ad x + \psi$, where for some $i$ and all $j$, we have $x \in M_{2i} \setminus M_{2i+1}$ and $\psi(M_{2j}) \subseteq M_{2j+i+1}$. By Corollary 3.3, $\phi$ then does not stabilize $[\partial_1, \partial_2]_F$.

Now let $\phi \in \text{Aut} M$ and suppose that $\phi$ stabilizes $[\partial_1, \partial_2]_F$. Since $G$ is a full complement to Aut$_1 M$ in Aut $M$, there is a $\psi \in G$ such that $\phi \psi^{-1} \in \text{Aut}_1 M$. This is an element of Aut$_1 M$ stabilizing $[\partial_1, \partial_2]_F$, so it is equal to $1$. Hence $\phi = \psi \in G$. □

**Corollary 3.20.** The automorphism group of $M(1, 1)$ is transitive on the incident point-line pairs.

**Proof.** The group $G$ determined above is clearly transitive on the points in $[\partial_1, \partial_2]_F$. □

A nice property of both points and lines is that, if we denote the corresponding space by $S$, we have that the centralizer of the centralizer of $S$ in $M$ is $S$ itself. (This is called the double centralizer of $S$.) This is the content of the following lemmas.

**Lemma 3.21.** For any point $Fd \in P$, the double centralizer $C_M(C_M(d))$ is equal to $Fd$.

**Proof.** Since the points are all in one orbit, we may assume that $d = \partial_1$. In Lemma 3.2, we found that $C_M(\partial_1)$ is spanned by all elements of the form $x^k d$, for $d \in \{\partial_1, \partial_2, 1, \partial_1, \partial_2\}$. Hence an element in the double centralizer commutes with every element of the form $x^k d$. In particular, it is an element of the centralizer of $\partial_1$. Let $u \subseteq C_M(C_M(\partial_1))$. We will show that $u = \partial_1$ (up to a scalar multiple).

$C_M(\partial_1)$ has a homogeneous basis relative to the $\mathbb{Z}^2$-grading, so we may again assume that $u$ is homogeneous as well. All homogeneous elements of $C_M(\partial_1)$ are of the form $x^k d$ (up to a scalar multiple).

If $u \neq \partial_1$, then we can easily find some $x^k d$ such that $[x^k d, u] \neq 0$:

- if $u = x^\ell \partial_2$, choose $x_2 \partial_1$;
- if $u = x^\ell \partial_2$, choose $\partial_1$;
- if $u = x^\ell \partial_1$ with $\ell > 0$, or $u = x^\ell$ or $x^\ell \partial_1$, choose $\partial_2$.

This proves that $C_M(C_M(\partial_1)) \subseteq Fd_\partial_1$. Clearly the other direction holds too. □

**Lemma 3.22.** For any line $\langle d_1, d_2 \rangle_F \in L$, the double centralizer $C_M(C_M(\langle d_1, d_2 \rangle_F))$ is equal to $\langle d_1, d_2 \rangle_F$. 
Proof. Since the lines are all in one orbit, we may assume that \( d_1 = \ell_1 \) and \( d_2 = \ell_2 \). In Corollary 3.35, we found that \( C_M(\langle \ell_1, \ell_2 \rangle) \) is \( M_{c_0} \). The centralizer of \( M_{c_0} \) must certainly centralize \( \ell_1 \) and \( \ell_2 \), so it is contained in \( M_{c_0} \). The only elements of \( M_{c_0} \) centralizing \( \ell_1 \) and \( \ell_2 \) are in \( \langle \ell_1, \ell_2 \rangle \).

3.4.2. Subspace planes. We give a partial list of the isomorphism classes of subspace planes occurring in this geometry; that is, given a pair \( \ell_1, \ell_2 \) of distinct intersecting lines in \( L \), we find the subspace of the algebra spanned by those lines and see what the subgeometry on that subspace is. We find at least the following possibilities.

- The plane is isomorphic to the plane spanned by \( \ell_1, \ell_2 \) and \( 1 \). This means that the plane has \( q^2 + q \) points (all projective points outside \( H \)). Any two points belonging to different equivalence classes modulo \( H \) are collinear; there are \( q^2 \) lines. Such a plane is called a dual affine plane: it can be obtained from the corresponding projective plane by removing a point (in this case, the one in \( H \)) and all incident lines, whereas an affine plane can be obtained from a projective plane by removing a line and all incident points. Since these projective planes are self-dual, the affine and dual affine planes are each others dual. The dual affine plane is depicted in Figure 3.1(a).

- The plane is isomorphic to the plane spanned by \( \ell_1, \ell_2 \) and \( x_2 \). In this case, the plane also has \( q^2 + q \) points, but only \( q \) lines. One point \( p \) is collinear to all points except for the ones in the same equivalence class modulo \( H \) as \( p \), and these are all the lines there are. In particular, \( q - 1 \) of the points are not collinear to any point in the plane. (In the example above, \( p = \ell_1 \).) We call this a pencil plane. See Figure 3.1(b).

- The plane is isomorphic to the plane spanned by \( \ell_1, \ell_2 \) and \( 2x_2 \ell_1 + x_2 \ell_2 + x_2^2 \ell_1 + 3x_2^2 \ell_2 \). In this case, there are only \( 2q + 1 \) points in the plane, on the two intersecting lines. The other projective points in the plane are in other orbits. We call this a 2-line plane. See Figure 3.1(c).

- The plane is isomorphic to the plane spanned by \( \ell_1, \ell_2 \) and

\[-\ell_1 + \ell_2 - x_2 \ell_1 + x_2 \ell_2 + 3x_2 \ell_1 + 2x_2 \ell_2 + 2x_2^2 \ell_1 - x_2^2 \ell_2 + 2x_2^2 + x_2^3 + x_2^4 \ell_2 - x_2^4 \ell_1 + 3x_2^4 \ell_2.\]

This plane contains an isomorphic copy of the previous one, and additionally contains the \( q - 1 \) points in the \( H \)-coset of the intersection point of the two lines, making the total number of points equal to \( 3q \). We call this an extended 2-line plane. See Figure 3.1(d).

- The plane is isomorphic to the plane spanned by \( \ell_1, \ell_2 \) and

\[x_2 \ell_1 + 3x_2 \ell_2 + 3x_2 + 3x_2 \ell_1 + 2x_2 \ell_2 + x_2^2 \ell_1 + x_2^3 + x_2^3 \ell_1 + 2x_2^3 \ell_2 + 3x_2^3 \ell_1 + x_2^4 \ell_2.\]

This plane contains exactly \( 3q + 1 \) points on 3 lines intersecting in one point. We call this a 3-line plane. See Figure 3.1(e).

In Proposition 3.25, we will see that if there are more different subspace planes than those in this list, then they are contained in the pencil plane, and contain the 2-line plane. The extended 2-line plane and the 3-line plane are examples of such geometries.

Lemma 3.23. Any subspace plane in the geometry that contains two nonintersecting lines, contains a triangle.

Proof. Let one line be \( \langle x, y \rangle \), and let the subspace contain \( h \in H \). Then we may assume that the other line is \( \langle x + h, y + h \rangle \). So \( [x, y] = 0 = [x + h, y + h] = [x - y, h] \). Then \( x - y \) is collinear with both \( x + h \) and \( y + h \), but the three are not on a line (they span the plane). 

\[\Box\]
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(a) The full plane.  (b) The pencil plane.  (c) The 2-line plane.  
(d) The extended 2-line plane.  (e) The 3-line plane.

Figure 3.1: The known planes occurring in the geometry.

Lemma 3.24. Any subspace plane in the geometry that contains a triangle is a dual affine plane.

Proof. Let \( x, y \) and \( z \) form the triangle. Then every pair of them spans a line: they commute and they are points. Since there is only one orbit on the lines, all projective points on these lines are points of the geometry. Any pair of points on the triangle commutes, since these points are linear combinations of commuting points. So they span a line, unless they are the same modulo \( H \). But such lines cover the dual affine plane. □

Proposition 3.25. Any subspace plane in the geometry is either the dual affine plane or contained in the pencil plane.

Proof. Let \( P \) be a plane. Suppose it is not the dual affine plane. By Lemmas 3.23 and 3.24, it cannot contain a pair of nonintersecting lines or a triangle. So any triple of lines is convergent. Hence at most one point is on more than one line. □

3.4.3. Dual affine planes. In the previous section, we have found a number of isomorphism classes of subspace planes not containing triangles. These are, in a sense, degenerate; so it is meaningful to consider only the other planes. Because of Lemma 3.24 this leaves only the dual affine planes. So now we have a geometry containing three types of objects: points, lines and planes, where every plane is dual affine. We call this geometry the Melikyan geometry.

We would now like to describe the residues in this geometry. We already know the residue of a plane: the points and lines incident with a given plane form a dual affine plane. This is indicated in the diagram of this geometry by the label \( \text{Af}^* \) along the edge between points and lines. The points and planes incident with a given line are all incident with all objects of the other type; this situation is called a generalized digon and indicated in the diagram by omitting the edge between points and planes. However, it is not so easy to describe the residue of a point. One reason is given by the following lemma.
Lemma 3.26. For every dual affine plane, there is an automorphism mapping it to the plane in either \( \langle \partial_1, \partial_2, \partial_1 \rangle_{\mathbb{F}} \) or \( \langle \partial_1, \partial_2, 1 + \alpha \partial_1 \rangle_{\mathbb{F}} \), for some \( \alpha \in \mathbb{F} \). These two classes of planes are in different orbits.

Proof. Take a dual affine plane \( \Pi \). Let \( \ell \) be a line in \( \Pi \). We may assume that \( \ell = \langle \partial_1, \partial_2 \rangle_{\mathbb{F}} \), since the automorphism group is transitive on the lines. Then \( \Pi \) is spanned by \( \ell \) and a point collinear to both \( \mathbb{F} \partial_1 \) and \( \mathbb{F} \partial_2 \); in particular, this point must be in \( C_M(\ell) = M_{\geq 0} \). We can then apply, if necessary, an automorphism from the \( GL_2(\mathbb{F}) \)-group respecting the grading, in order to make the \( M_{-1} \)-component a multiple of \( \partial_1 \). Then \( \Pi \) is of the given form.

To show that the two given classes of planes cannot be in the same orbit, it suffices to see that the first intersects \( M_{\geq -1} \) nontrivially, whereas the second does not. \( \Box \)

To find the residue of a point, one would need to have an understanding of the lines and planes incident with a point, say \( \mathbb{F} \partial_1 \). This is quite a challenge: the lines contain all points in \( C_M(\partial_1) \) of dimension 25 (except those equal to \( \partial_1 \) modulo \( H \)). With our current methods it is not feasible to describe this residue accurately. In the diagram of Figure 3.2, where we have written \( P \) for the points, \( L \) for the lines and \( \Pi \) for the planes, the residue has been indicated by a question mark.

3.4.4. Quotients. As indicated in the introduction to this section on page 88, we have found that the structure of the geometry is hard to describe completely, and so we examine some of the quotients of the geometry. We choose a normal subgroup \( N \) of \( Aut M \) and study the incidence structure on the \( N \)-orbits on the point and line sets of our geometry, and the dual affine planes occurring in that geometry. (Since the automorphism group is not transitive on the dual affine planes and certainly not on the maximal flags, we cannot necessarily obtain these dual affine planes as the \( N \)-orbits on the planes.) The natural normal subgroups of \( Aut M \) are the subgroups \( Aut_i M \), so these are the subgroups that we will use. Hence let \( i > 0 \) and let us study the points and lines in this quotient geometry. They are the \( Aut_i M \)-orbits of the points in \( P \) and the lines in \( L \), respectively. We can identify these with cosets \( p + M_{\geq i-3} \) of points \( p \in P \), and cosets \( \ell + M_{\geq i-3} \) of lines \( \ell \in L \). (Some projective points \( q + M_{\geq i-3} \) in \( M/M_{\geq i-3} \) may not be points of the quotient geometry, because there are no \( p \in P \) with \( p + M_{\geq i-3} = q + M_{\geq i-3} \); for example, if \( q \in H \).) This naturally leads us to studying the orbits of \( Aut M/ Aut_i M \) on \( M/M_{\geq i-3} \).

3.4.4.1. \( i = 1 \). We obtain the geometry consisting of \( q + 1 \) points on one line. The automorphism group is transitive on the two-dimensional space \( M/M_{\geq -2} \). There are no dual affine planes.

3.4.4.2. \( i = 2 \). The geometry consists exactly of a single dual affine plane. (The diagram is that of Figure 3.2 with the node \( \Pi \) and the incident edge removed.)

The automorphism group acts on the ordered basis \( \partial_1, \partial_2, 1 \) of \( M/M_{\geq -1} \) as follows. If an automorphism fixing the grading acts on \( M_{-1} \) as a matrix \( A \), then it acts on the ordered basis above by a block matrix (with block sizes 2 and 1, respectively) of the form \( \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} \), acting from the left. An automorphism in \( Aut_1 M \) acts as \( \begin{pmatrix} a \\ 0 \end{pmatrix} \), for some \( a \in \mathbb{F}^2 \).
3.4.4.3. $i = 3$. The points in this quotient geometry are projective points in a 5-dimensional space with a 3-dimensional space cut out (the intersection with $H$). In order to study the lines and planes, let us examine the automorphism group action again. We claim that for an invertible $2 \times 2$-matrix $A$, for $v \in \mathbb{F}^2$ and for $j, k \in \{1, 2\}$, there are automorphisms acting on the ordered basis $\partial_1, \partial_2, 1, \bar{\partial}_1, \bar{\partial}_2$ of $M/M_{\geq 2}$ by the following block matrices (each with block sizes 2, 1 and 2, respectively):

$$
\begin{pmatrix}
A \det A & 0 & 0 \\
0 & \det A & 0 \\
0 & 0 & A
\end{pmatrix},
\begin{pmatrix}
I & 0 & 0 \\
v & 1 & 0 \\
0 & M^T & I
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
I & 0 & 0 \\
0 & 1 & 0 \\
E_{jk} & 0 & 1
\end{pmatrix},
$$

(3.4)

where $M = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$ and $E_{jk}$ is the $2 \times 2$-matrix with zeroes everywhere except for a one at position $(j, k)$. The first of these three matrices comes from an element of the automorphism group stabilizing the $\mathbb{Z}$-grading and acting on $M_{-1}$ as $A$. The third can be found in Aut$_2 M$. Hence matrices for elements of Aut$_1 M$ can be changed to the form of the second matrix.

**Lemma 3.27.** Aut$_1 M$ is transitive on both the projective points and the projective lines of $M/M_{\geq 0}$ outside $H/M_{\geq 0}$.

**Proof.** Using the matrices of Eq. (3.4), it is trivial to see that the group is transitive on the projective points outside $H$. Now take a projective line, which we may assume to be $\langle \partial_1, \partial_2 + u \rangle_T$ for some $u \in M_{-1} \oplus M_{-2}$. The matrices of Eq. (3.4) stabilizing $\partial_1$ are still sufficient to clear $u$. □

Because of this lemma, every pair of points in the quotient geometry that differs modulo $H$ is collinear. This may seem counterintuitive at first, since for example $F(\partial_1 + \bar{\partial}_1)$ and $F(\partial_2 + \bar{\partial}_2)$ do not commute. However, $\partial_1 + \bar{\partial}_1$ does commute with $\partial_2 + \bar{\partial}_2 - x_1 - x_2^2 \partial_1 - x_2^2 \partial_1$, which, as a simple check in the computer algebra system GAP [GAP06] shows, also spans a point of the geometry.

The previous lemma implies that every triple of points that differ pairwise modulo $H$ are in a dual affine plane; in other words, every pair of intersecting lines is in a dual affine plane. Thus the residue of a point is a linear space. But we can be more precise.

**Lemma 3.28.** The residue of a point in the Melikyan geometry modulo Aut$_3 M$ forms the points and lines of the affine geometry of vector space dimension 4.

**Proof.** Let $p$ be a projective point outside $H$. We determine the residue of $p$ by dualizing $M/M_{\geq 2}$. This transforms $H$ into a vector space $H'$ of dimension 2. It sends $p$ to a hyperplane $p'$ intersecting $H'$ in a projective point. A line containing $p$ is mapped to a complement of $H'$ inside $p'$ and a plane containing $p$ is mapped to a subspace of $p'$ of dimension 2 disjoint from $H'$.

Now consider the dual of $p'$. It is a vector space of dimension 4 containing a hyperplane that is the dual of $p' \cap H'$. By this second dualization, the duals of the lines and planes of the Melikyan geometry modulo Aut$_3 M$ are mapped to the projective points and projective lines outside this hyperplane, respectively. Thus, we have found an affine geometry. □

3.4.4.4. $i = 4$. The points of this quotient geometry correspond to projective points in a 9-dimensional space with a 7-dimensional space cut out. Essentially the same arguments as before show that the automorphism group is transitive on $M/M_{\geq 3}$ minus $H$; in particular, the elements of Aut$_3 M$ act on the ordered basis $\partial_1, \partial_2, 1, \bar{\partial}_1, \bar{\partial}_2, x_1 \partial_1$, \ldots.
Every projective point in $M$ minus $H$ in $\text{di}_i$ we see that the automorphism group is transitive on the projective points in $M$ component of degree $j$.

We see that the geometry is well-behaved only if one takes sufficiently small quotients. The analysis will be much harder if one takes such quotients that not all projective points in the quotient outside $H$ are points of the Melikyan geometry. The following proposition states that this is true for $i \leq 12$.

**Proposition 3.29.** Every projective point in $M/M_{2i-3}$ outside $H$ is a point of the Melikyan geometry modulo $\text{Aut}_i M$ if and only if $i \leq 12$.

**Proof.** Let $i > 0$. Let $p$ be a projective point in $M/M_{2i-3}$ outside $H$. Using automorphisms respecting the $\mathbb{Z}$-grading one can make the component of $p$ of degree $-3$ equal to $\partial_1$. We now try to make the higher-degree components equal to zero. To make the component of degree $j$ equal to 0 if the part of degree lower than $j$ is equal to $\partial_1$, one needs an automorphism of the form $1 + \text{ad } x + \phi$ with $\deg x = j + 3$ and $\phi$ such that $\phi(M_{\geq k}) \subseteq M_{\geq k+j+4}$ for all $k$. The degree-$j$ part is cleared if $[\partial_1, x]$ is minus that degree-$j$ part. Hence $p$ is always in the same orbit as $\partial_1$ if and only if $\text{ad } \partial_1$ is surjective on all $M$ with $j < i - 3$. The lowest-degree element outside the codomain of $\text{ad } \partial_1$ is $x_i^4 \partial_1$ of degree 9.

**3.4.5. Conclusion.** We have seen that this geometrical approach for studying the Melikyan Lie algebras is not very fruitful. Even for fairly small quotients we did not determine the type of all residues. With some work, this could be in reach of the current methods, though, for sufficiently small quotients. The analysis will be much harder if one takes such quotients that not all projective points in the quotient outside $H$ are points of the geometry. This feature of the geometry, essential for understanding it, only happens for $i \geq 13$.

These difficulties are in stark contrast to the effectiveness of a similar approach for the Lie algebras of Chevalley type by Cohen and Ivanyos [CI06, CI]. The main reason seems to be that the automorphism group of the Melikyan algebra is “too small”, and consequently, that there are so many orbits on intersecting pairs of lines. In a sense, the geometry is well-behaved only if one takes sufficiently small quotients, that is, if

$$x_1 \partial_2, x_2 \partial_1, x_2 \partial_2$$ by block matrices of the following form, where the block sizes are 2, 1, 2 and 4:

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
E & 0 & 0
\end{pmatrix}$$

where $E$ is a matrix in the subspace of $\mathbb{F}^{4 \times 2}$ spanned by these six matrices:

$$\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
1 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.$$  

We see that the automorphism group is transitive on the projective points in $M/M_{\geq 1}$ minus $H$, but not on the lines. Hence there are pairs of points in the quotient geometry in different classes modulo $H$ that are not collinear. An example is the pair $\partial_1 + M_{\geq 1}$ and $\partial_2 + x_1 \partial_2 + M_{\geq 1}$:

$$[\partial_1 + M_{\geq 1}, \partial_2 + x_1 \partial_2 + M_{\geq 1}] = \partial_2 + M_{2-2},$$

so no two elements in $\partial_1 + M_{\geq 1}$ and $\partial_2 + x_1 \partial_2 + M_{\geq 1}$ will commute.

It seems likely that in this quotient, every point $p + M_{\geq 1}$ is collinear to every point in $C_M(p) + M_{\geq 1}$ (except those in the same class modulo $H$). It is certainly true that for all $q \in C_M(p) \setminus H$, there are elements $q'$ of $q + M_{\geq 1}$ with $(\text{ad } q')^5 = 0$ and thus such that $\mathbb{F}q'$ is a point of the geometry, but $\partial_1$ and $q'$ then do not necessarily commute.

3.4.4.5. $i > 4$. We have seen that for $i \leq 4$, all projective points in $M/M_{2i-3}$ outside $H$ are points of the quotient geometry. The following proposition states that this is true for $i \leq 12$.

**Proposition 3.29.** Every projective point in $M/M_{2i-3}$ outside $H$ is a point of the Melikyan geometry modulo $\text{Aut}_i M$ if and only if $i \leq 12$.

**Proof.** Let $i > 0$. Let $p$ be a projective point in $M/M_{2i-3}$ outside $H$. Using automorphisms respecting the $\mathbb{Z}$-grading one can make the component of $p$ of degree $-3$ equal to $\partial_1$. We now try to make the higher-degree components equal to zero. To make the component of degree $j$ equal to 0 if the part of degree lower than $j$ is equal to $\partial_1$, one needs an automorphism of the form $1 + \text{ad } x + \phi$ with $\deg x = j + 3$ and $\phi$ such that $\phi(M_{\geq k}) \subseteq M_{\geq k+j+4}$ for all $k$. The degree-$j$ part is cleared if $[\partial_1, x]$ is minus that degree-$j$ part. Hence $p$ is always in the same orbit as $\partial_1$ if and only if $\text{ad } \partial_1$ is surjective on all $M$ with $j < i - 3$. The lowest-degree element outside the codomain of $\text{ad } \partial_1$ is $x_i^4 \partial_1$ of degree 9.

**3.4.5. Conclusion.** We have seen that this geometrical approach for studying the Melikyan Lie algebras is not very fruitful. Even for fairly small quotients we did not determine the type of all residues. With some work, this could be in reach of the current methods, though, for sufficiently small quotients. The analysis will be much harder if one takes such quotients that not all projective points in the quotient outside $H$ are points of the geometry. This feature of the geometry, essential for understanding it, only happens for $i \geq 13$.

These difficulties are in stark contrast to the effectiveness of a similar approach for the Lie algebras of Chevalley type by Cohen and Ivanyos [CI06, CI]. The main reason seems to be that the automorphism group of the Melikyan algebra is “too small”, and consequently, that there are so many orbits on intersecting pairs of lines. In a sense, the geometry is well-behaved only if one takes sufficiently small quotients, that is, if
one stays sufficiently close to the simple group, $\text{SL}_2$; the long unipotent tail of the automorphism group is what makes the current approach ineffective.

## 3.5. Sandwich elements and their algebras

It follows from, e.g., Premet’s paper [Pre86] and from Tange’s Master’s thesis [Tan02] that if $\text{char } F > 5$ and $F$ is algebraically closed, then every finite-dimensional simple Lie algebra $L$ has an extremal element, say $x$. From the classification in [PS97, PS99, PS01, PS04, PS06, PS07], it follows that this also holds if $\text{char } F = 5$. Furthermore, if $(\text{ad } x)^2 = 0$ (in other words, $x$ is a sandwich element), then all extremal elements of $L$ are sandwich elements. If $L$ has these sandwich elements, the subalgebra generated by them (the sandwich algebra) is an invariant proper subalgebra. On the other hand, if $(\text{ad } x)^2 \neq 0$, then $L$ contains no sandwich elements; in this case, the Lie algebra is a Lie algebra of Chevalley type.

This dichotomy is one reason for our interest in the sandwich subalgebra of the Melikyan algebras. Another is the following. Sandwich elements were used heavily in the solution by Zel’manov and Kostrikin of the restricted Burnside problem, see e.g. [Kos86]. In a survey article [Kos73], Kostrikin conjectured that the normalizer of the sandwich subalgebra of $L$ is a maximal subalgebra of $L$ in any non-classical simple Lie algebra $L$ over a field of characteristic $p > 3$. This conjecture was proven first for special cases by Kostrikin and Šafarevič [KŠ69], then for a more general class by Ėl’sting [El’76a, El’76b, El’75], and finally for all Lie algebras of Cartan or Melikyan type by Kirillov; a listing of his results can be found in [Kir92]. (It was not known at the time that these are all the Lie algebras satisfying Kostrikin’s properties.) In particular, in [Kir91], Kirillov proves the following theorem. We use the natural action of $O$ on $W$:

**Theorem 3.30.** The sandwich subalgebra of $M(n_1, n_2)$ is the direct sum of the sandwich subalgebra of $W(2; n_1, n_2)$, of $O_{\geq 3}$ and of $O_{\geq 3} \cdot W(2; n_1, n_2)$.

The proof of this theorem in [Kir91] is fairly long (nearly 23 typewritten pages). We will see that, using the automorphism group as implemented in the computer algebra system GAP [GAP06], a much shorter but computer-dependent proof can be found for the case $n_1 = n_2 = 1$. In particular, we will prove the following special case of the above theorem.

**Theorem 3.31.** The sandwich subalgebra of $M(1, 1)$ is $S := \langle D \in M_6 \mid \text{div } D = 0 \rangle_F \oplus M_{\geq 7}$.

Our proof will make heavy use of the grading of $M$ by $\mathbb{Z}^2$, introduced in Section 3.2.2 and illustrated in Figure 3.3: the dots represent homogeneous components of $M$ with respect to the $\mathbb{Z}^2$-grading, the thick lines are the axes of $\mathbb{Z}^2$.

**Lemma 3.32.** No element of degree less than 6 that is homogeneous with respect to the $\mathbb{Z}^2$-grading is a sandwich.

**Proof.** This will be a proof by case distinction. We will prove it for degrees $(x, y)$ with $x \geq y$; the automorphism interchanging $x_1$ and $x_2$ then completes the proof. Figure 3.3 illustrates where these degrees are to be found. There are 14 degrees we need to examine; four of them correspond to a 2-dimensional subspace and the others to a one-dimensional subspace. We show that for every element $u$ of these spaces, $[u, [u, x_1^2 + x_2^2]] \neq 0$. We will sometimes write expressions like $(x_1^2)$ or $x_2^{-1}$ where $i$ may be zero; the expressions $(\frac{1}{2})$ and $x_2^{-1}$ should be interpreted as zero in those cases.
We first study the degrees corresponding to elements of $W$. For $i \leq 2$, consider $x_2^i \partial_1$ of degree $(2i - 1, i - 2)$. We have

$$[x_2^i \partial_1, [x_2^j \partial_1, x_1^2 + x_2^2]] = [x_2^i \partial_1, x_1 x_2^j] = x_2^{2j}.$$ 

For $i \leq 1$, let $u = \alpha x_1 x_2^i \partial_1 + \beta x_2^{i+1} \partial_2$ of degree $(2i, i)$. Then

$$[u, [u, x_1^2 + x_2^2]] = [u, 3\beta x_1^2 x_2^i - (i + 2)(i\beta + (i + 1)\alpha)x_2^{i+2}]$$

$$= \left(3 \binom{2i}{i+1} - \binom{2i}{i}\right) \beta^2 x_1^2 x_2^{2i} + (i + 2)(i\beta + (i + 1)\alpha) \left(2(\alpha + \beta) \binom{2i + 2}{i} - \beta \binom{2i + 2}{i+1}\right) x_2^{2i+2}.$$ 

The coefficients are $-\beta^2$ and $-2\alpha(-\alpha + 2\beta)$ for $i = 0$ and $\beta^2$ and $3(\beta + 2\alpha)(3\alpha + 2\beta)$ for $i = 1$. In both cases, if both coefficients are zero, then both $\alpha$ and $\beta$ are zero. This finishes the degrees belonging to $W$.

Now consider the degrees belonging to $O$; in particular, for $i \leq 2$, consider $x_2^i$ of degree $(2i - 1, i - 1)$. We have

$$[x_2^i, [x_2^j, x_1^2 + x_2^2]] = [x_2^i, 2(x_1^2 x_2^{i-1} + (3i - 1)(i + 1)x_2^{i+1}) \tilde{\partial}_1 + 2x_1 x_2^j \tilde{\partial}_2]$$

$$= 2 \left(\binom{i+1}{2} x_1 x_2^{i+1} + (3i - 1)(i + 1) \binom{2i + 1}{i} x_2^{2i+1}\right) \tilde{\partial}_1 + 2 \binom{2i}{i} x_1 x_2^j \tilde{\partial}_2.$$ 

For $x_1 x_2$ of degree $(2, 2)$, we see that

$$[x_1 x_2, [x_1 x_2, x_1^2 + x_2^2]] = [x_1 x_2, (x_1^3 + 3x_1 x_2^2) \tilde{\partial}_1 + (2x_1^2 x_2 - x_2^3) \tilde{\partial}_2]$$

$$= (-x_1^4 x_2 + 3x_1^2 x_2^3) \tilde{\partial}_1 + (2x_1^2 x_2^2 + x_1 x_2^4) \tilde{\partial}_2.$$
This was the last degree belonging to $O$. We turn our attention to $\tilde{W}$. For $i \leq 2$, consider $x_i^2 \partial_1$ of degree $(2i, i - 1)$. We have

$$[x_i^2 \partial_1, x_i^2 \partial_1, x_i^2 + x_i^2] = \left[ x_i^2 \partial_1, -x_i^2 x_i^2 \partial_1 - \left( i + \frac{2}{2} \right) x_i^2 x_i^2 \partial_1 \right] = \left( \frac{2i}{i} \right) x_i^2 \partial_1.$$ 

We will treat the last two degrees separately. First let $u = \alpha x_1 \partial_1 + \beta x_2 \partial_2$ of degree $(1, 1)$. Then

$$[u, [u, x_i^2 + x_i^2]] = [u, -\alpha(3x_i^2 + x_1 x_2) \partial_1 - \beta(x_i^2 x_2 + 3x_i^2) \partial_2] = \alpha((2\alpha + \beta)x_i^3 + (2\alpha - \beta)x_1 x_i^2) \partial_1 + \beta((2\beta - \alpha)x_i^2 x_2 + (2\beta + \alpha)x_i^2) \partial_2.$$ 

Finally, if $u = \alpha x_1 x_2 \partial_1 + \beta x_2^2 \partial_2$ of degree $(3, 2)$, then

$$[u, [u, x_i^2 + x_i^2]] = [u, -3\alpha(x_i^2 x_2 + x_1 x_i^2) \partial_1 - \beta(x_i^2 x_2 + x_i^2) \partial_2] = \alpha(2\beta - \alpha)x_1 x_2 \partial_1 + \alpha(\beta - \alpha)x_1 x_2 \partial_1 + \beta(2\beta + \alpha)x_i^2 x_i^2 \partial_2. \quad \Box$$

**Lemma 3.33.** The sandwich algebra is contained in $M_{26}$.

**Proof.** Let $s$ be a sandwich element not contained in $M_{26}$ (so in particular, $s$ is nonzero). Write $s = \sum_{v \in \mathbb{Z}^2} s_v$, where $s_v$ is the homogeneous component of $s$ of degree $v$ with respect to the $\mathbb{Z}^2$-grading. Let $u \in \mathbb{Z}^2$ and take $w \in \mathbb{Z}^2$ such that the standard inner product $u \cdot w$ is maximal subject to $s_w \neq 0$. For each homogeneous $t \in M$ of degree $v$, we have $[s, [s, t]] = 0$. We now examine the “leading term” of $[s, [s, t]]$, in the following sense: the component of $[s, [s, t]]$ of degree $v + 2w$ is $[s_{2w}, s_{wv}]$. This component is clearly also zero. Since this holds for all homogeneous $t$, we find that $s_w$ is a sandwich.

Now take a specific value for $u$: let $u = (-1, -1)$. This gives us an element $s_w = s'$ of homogeneous degree with respect to the $\mathbb{Z}$-grading. Since $s \notin M_{26}$, the degree of $s'$ is strictly less than 6. Then take $u = (0, -1)$ and perform the same procedure on $s'$. This gives a sandwich element that is homogeneous with respect to the $\mathbb{Z}^2$-grading, of which the degree is less than 6, a contradiction to Lemma 3.32. This finishes the proof. \quad \Box

**Lemma 3.34.** Let $S$ be the subalgebra of $M$ defined in Theorem 3.31. Then $S$ contains the sandwich algebra of $M$.

**Proof.** Lemma 3.33 shows that the only possible candidate sandwiches are in $M_{26}$. A nonzero sandwich outside $S$ has a nonzero component of degree 6, and therefore its homogeneous component of degree 6 is a sandwich itself, by the same arguments involving leading terms that prove Lemma 3.33.

Consider the following basis of $M_6$, which is homogeneous with respect to the $\mathbb{Z}^2$-grading:

<table>
<thead>
<tr>
<th>degree</th>
<th>(5, 1)</th>
<th>(4, 2)</th>
<th>(3, 3)</th>
<th>(2, 4)</th>
<th>(1, 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>$x_2^3 \partial_1$</td>
<td>$x_1 x_2 x_1^2 \partial_1 - x_1 x_2 x_2^2 \partial_2$</td>
<td>$x_1 x_2 x_1^2 x_2 \partial_1 - x_1 x_2 x_2^2 x_2 \partial_2$</td>
<td>$x_1 x_2 x_1^2 x_2^2 \partial_1 - x_1 x_2 x_2^2 x_2 \partial_2$</td>
<td>$-x_1 x_2 x_2^3 \partial_2$</td>
</tr>
<tr>
<td>$U$</td>
<td>$x_1 x_2 x_1^2 \partial_1 + 3x_2 x_1 x_2 \partial_2$</td>
<td>$x_1 x_2 x_1^2 x_2 \partial_1 + x_1 x_2 x_2^2 x_2 \partial_2$</td>
<td>$3x_2 x_1 x_2 x_1^2 x_2 \partial_1 + x_1 x_2 x_2^2 x_2 \partial_2$</td>
<td>$3x_2 x_1 x_2 x_1^2 x_2^2 \partial_1 + x_1 x_2 x_2^2 x_2 \partial_2$</td>
<td></td>
</tr>
</tbody>
</table>

The five elements in the upper row are in $S$, the three elements in the lower row aren’t, and the spaces generated by these two rows of elements are both closed under the action of $\text{Aut} M$ on $M_{26}/M_{27}$. Let $V$ and $U$ be the subspaces of $M_{26}/M_{27}$ spanned by the elements in the upper and lower row, respectively, and let $u$ be an element of $M_6$ outside $S$. Then the decomposition of $u$ with respect to $V$ and $U$ will have a nonzero part in $U$. We will show that $u$ is not a sandwich. In order to do this, we will let the automorphism group act on $u$, making it into a normal form. The vector space $U$ is, as a module of the automorphism group $\text{Aut} M$, isomorphic to the irreducible $\text{GL}_2$-module of dimension
three, consisting of homogeneous polynomials of degree 2 in two variables (say, $a$ and $b$); the action maps
\[
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
\lambda & 0 \\
0 & 1
\end{pmatrix}
to
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
\lambda^{-2} & 0 & 0 \\
0 & \lambda^{-3} & 0 \\
0 & 0 & \lambda^{-4}
\end{pmatrix}.
\]
This is a twisted version of the standard action on the module if we identify
\[
a^2 \quad \text{with} \quad x_1^2x_2^2\partial_1 + 3x_1^3\partial_2,
\]
\[
ab \quad \text{with} \quad x_1^2x_2\partial_1 + x_1x_2^2\partial_2,
\]
\[
b^2 \quad \text{with} \quad 3x_1^2\partial_1 + x_1^2x_2\partial_2.
\]
Hence $\text{Aut} M$ has two orbits on $V$ beside $\{0\}$: with coefficient $\lambda_{ij}$ for $a^2$ and similarly for the other coefficients, one orbit consists of the nonzero elements with $\lambda_{ab} = \lambda_{ab} = \lambda_{12}$, the second of the rest. So representatives of these two nonzero orbits are $a^2$ and $ab$, respectively; or $x_1x_2^2\partial_1 + 3x_2^3\partial_2$ and $x_1^2x_2\partial_1 + x_1x_2^2\partial_2$. Hence we may assume that $u$ is the sum of an element of $V$ and either $x_1x_2^2\partial_1 + 3x_2^3\partial_2$ or $x_1^2x_2\partial_1 + x_1x_2^2\partial_2$.

Let $v_{ij}$ be the algebra element of $\mathbb{Z}^2$-degree $(i, j)$ in $V$. Let us consider
\[
u = \lambda_{51}v_{51} + \lambda_{42}v_{42} + \lambda_{33}v_{33} + \lambda_{24}v_{24} + \lambda_{15}v_{15} + x_1^2x_2\partial_1 + 3x_2^3\partial_2. \quad (3.6)
\]
Consider $[u, [u, x_i^4]]$. Note that $x_i^4$ is an element of $\mathbb{Z}^2$-degree $(3, 7)$ and consider the $(11, 11)$-component of $[u, [u, x_i^4]]$. If we expand the product $[u, [u, x_i^4]]$ by substituting the value from Eq. (3.6) for $u$, then the only terms that could be nonzero need to have a total contribution of the two $u$-factors of $(11, 11) - (3, 7) = (8, 4)$. These terms are
\[
\lambda_{51}\lambda_{33}[v_{51}[v_{33}, x_i^4]] \quad \text{but} \quad [v_{33}, x_i^4] = 0,
\]
\[
\lambda_{51}\lambda_{33}[v_{33}[v_{51}, x_i^4]] \quad \text{but} \quad [v_{51}, x_i^4] = 0,
\]
\[
\lambda_{42}^2[v_{42}, [v_{42}, x_i^4]] \quad \text{but} \quad [v_{42}, [v_{42}, x_i^4]] = 0,
\]
\[
\lambda_{42}[v_{42}, [x_1x_2^2\partial_1 + 3x_2^3\partial_2, x_i^4]] \quad \text{but} \quad [v_{42}, [x_1x_2^2\partial_1 + 3x_2^3\partial_2, x_i^4]] = 0,
\]
\[
\lambda_{42}[x_1x_2^2\partial_1 + 3x_2^3\partial_2, [v_{42}, x_i^4]] = 2\lambda_{42}x_1x_4^4, 
\]
\[
[x_1x_2^2\partial_1 + 3x_2^3\partial_2, [x_1x_2^2\partial_1 + 3x_2^3\partial_2, x_i^4]] = 3x_4^4.
\]
Hence if $u$ is to be a sandwich, we need that $2\lambda_{42} + 3 = 0$ or $\lambda_{42} = 1$. Thus the component of $u$ of degree $(4, 2)$ is $2x_3^2\partial_2 + 2x_1x_2^2\partial_1$. A similar reasoning for the expansion of $[u, [u, x_i^4]]$, with $\mathbb{Z}^2$-degree of $x_i^4\partial_2$ equal to $(2, 7)$, shows that the terms that could give a contribution to degree $(10, 11)$ are
\[
\lambda_{51}\lambda_{33}[v_{51}[v_{33}, x_i^4\partial_2]] \quad \text{but} \quad [v_{33}, x_i^4\partial_2] = 0,
\]
\[
\lambda_{51}\lambda_{33}[v_{33}[v_{51}, x_i^4\partial_2]] \quad \text{but} \quad [v_{51}, x_i^4\partial_2] = 0,
\]
\[
[2x_3^2\partial_2 + 2x_1x_2^2\partial_1, [2x_3^2\partial_2 + 2x_1x_2^2\partial_1, x_i^4\partial_2]] = -x_1x_4^4\partial_2.
\]
Hence $u$ is not a sandwich.

For the other case, let
\[
u = \lambda_{51}v_{51} + \lambda_{42}v_{42} + \lambda_{33}v_{33} + \lambda_{24}v_{24} + \lambda_{15}v_{15} + x_1^2x_2\partial_1 + x_1x_2^2\partial_2. \quad (3.7)
\]
We now consider $[u, [u, x_i^4]]$, where $x_i^4$ is of degree $(5, 5)$. Consider the component of $[u, [u, x_i^4]]$ of degree $(11, 11)$. The terms that could give a contribution to degree
Each element of $M$ whose components in the $Z^2$-grading at all vectors $(x, y)$ with $2x - y \leq 8$ and $x + y \leq 13$ and $2y - x \leq 8$, are zero, is a sandwich.

The region that is the subject of this lemma is indicated in Figure 3.3.

**Proof.** The $Z^2$-grading has nontrivial components at exactly the integer vectors $(x, y)$ inside the hexagon with these corner points:

$$(-1, -2), (7/3, 2/3), (12, 11), (11, 12), (2, 7/3), (7/3, -2/3), (-2, -1).$$

If $u \in M$ is not a sandwich, then $[u, [u,v]] \neq 0$ for some $v \in M$. Then $u$ and $v$ have homogeneous components $u'$, $v'$ for which also $[u', [u',v']] = w$, where $w$ is a nonzero element of degree $2 \deg u' + \deg v'$. So $\deg u' = 1(\deg w - \deg v')$. Examining the convex hull of the nontrivial components of the grading, we see that this expression can only assume values satisfying the equations in the assertion.

**Lemma 3.35.** Each element of $M$ whose components in the $Z^2$-grading at all vectors $(x, y)$ with $2x - y \leq 8$ and $x + y \leq 13$ and $2y - x \leq 8$, are zero, is a sandwich. This finishes the proof.

**Lemma 3.36.** The automorphism of $M$ sending $\partial_1$ to $\partial_1 - \partial_2$ and fixing $\partial_2$ is the linear map $\phi$ determined by

$$\phi(v) = \begin{cases} 
\exp(\mathrm{ad} \, x_1 \partial_2) (x_3 \partial_1) + x_4 \partial_2, & \text{for } v = x_3 \partial_1, \\
\exp(\mathrm{ad} \, x_1 \partial_2) (x_3 \partial_1) + x_4 \partial_2, & \text{for } v = x_2 \partial_1, \\
\exp(\mathrm{ad} \, x_1 \partial_2) (v), & \text{if } v \text{ is a different homogeneous element with respect to the } Z^2-\text{grading}.
\end{cases}$$

Note that $\phi$ is well-defined, since $\mathrm{ad} \, x_1 \partial_2$ is nilpotent of class 5.

**Proof.** By Lemma 3.9, we need only check that $\phi$ is an automorphism. It is fairly easy to check that

$$[\phi(x), \phi(y)] = \phi([x, y])$$

(3.8)

for homogeneous $x$ and $y$ with respect to the $Z^2$-grading if neither $x$ nor $y$ nor $[x, y]$ are the exceptions in the definition of $\phi$. However, trying to take these exceptions into account would lead to heavy case distinction. Hence we simply check Eq. (3.8) in the computer algebra system GAP [GAP06] for all pairs of standard basis elements and see that it is true. This check is performed in the function check_3 on page 148.

**Lemma 3.37.** The elements $D \in M_6$ with $\text{div} \, D = 0$ are in the sandwich algebra.

**Proof.** By Lemma 3.35, $x_3 \partial_1$ is a sandwich element. We may apply $\phi$ as defined in Lemma 3.36 a number of times and again obtain a sandwich. It can be easily checked by hand that these sandwiches span the (5-dimensional) subspace of $M_6$ of elements with $\text{div} \, D = 0$.\[\square\]
**Lemma 3.38.** Let $S$ be the subalgebra of $M$ spanned by the elements in Theorem 3.31. Then $S$ is contained in the sandwich algebra of $M$.

**Proof.** We already know that $S \cap M_6$ is contained in the sandwich algebra, and we also know it for the part of $S$ spanned by the elements outside the boundaries of Lemma 3.35, in particular for $M_{\geq 14}$. Our status is illustrated by Figure 3.3: the leftmost vertical line indicates that $S$ does not have parts of degree less than 6, the next vertical line indicates that we have identified the intersection of $S$ with $M_6$, and the parts outside the shaded quadrangle, but right of the leftmost vertical line, contain only sandwiches by Lemma 3.35. So we only need to show that the piece inside the shaded quadrangle, right of the second vertical line, is spanned by sandwiches. Let $U$ be the vector space spanned by these vectors.

Let $V = S \cap M_6$. Using the GAP-function check_4 from page 149, we see that for almost all $\mathbb{Z}^2$-homogeneous elements $u \in U$, there is a homogeneous element $x$ of positive degree and an element $v \in V$ such that $[v, x] = u$. In particular, this holds for all degrees $(x, y)$ with $x + y > 8$.

We proceed with induction. Assume that $S \cap M_{2k} \oplus V$ is spanned by sandwiches. We know this for $k = 14$. Suppose $k > 9$ and let $u \in U \cap M_k$. By the previous paragraph, we find $x \in M_{k-7}$ and $v \in V$ such that $[v, x] = u$. Since $k - 7 > 2$, there exists an automorphism $\psi \in \text{Aut}_{k-7} M$ such that $\psi - 1$ acts as $\text{ad} x$ on $\text{gr} M$, as we have seen in Section 3.3.2. Then $v^\psi = v + u + w$, with $w \in M_{2k}$. But now $v^\psi$, $v$ and $w$ are all linear combinations of sandwiches, hence so is $u$.

For degrees 7 and 8, we use GAP again. The function check_5 from page 149 shows that applying elements of $\text{Aut}_1 M$ to the $\mathbb{Z}^2$-homogeneous elements of $V$ gives sufficiently many linearly independent elements to obtain the desired result. \hfill \square

We are now in a position to prove Theorem 3.31.

**Proof of Theorem 3.31.** Let $S$ again be the subalgebra of $M$ spanned by the elements in the theorem. We know that $S$ is contained in the sandwich algebra by Lemma 3.38. Suppose that there is a sandwich element $s$ outside $S$. Then by Lemmas 3.33 and 3.34 it is in $M_6$ and not an $F_5$-linear combination of the standard basis elements. In particular, it is in $W$, and hence a sandwich element of $W$. But by Kostrikin and Šafarevič [KŠ69], there are no sandwich elements in the Witt algebra other than those with $\text{div} D = 0$. This is a contradiction with our assumptions. \hfill \square

It is not extremely straightforward to extend this proof to a proof of Theorem 3.30. For example, if $[a, [a, b]] = 0$, then it is not necessarily true that $[a, [a, x^5 b]] = 0$; a counterexample is $a = b = \partial_1$. Thus if $a$ is a sandwich element of $M(1, 1)$, then it is not immediately clear if $a$ is also a sandwich element of other Melikyan algebras $M(n_1, n_2)$.

### 3.5.1. Other invariant subspaces.
Given two invariant subspaces, $L_1$ and $L_2$, of a Lie algebra $L$, the **conductor** of $L_1$ to $L_2$ in $L$ is again an invariant subspace of $L$. This is the subspace of elements $x$ such that $[x, L_1] \subseteq L_2$. We now know three invariant subspaces of $M_{11}$: the sandwich algebra, $M$ itself, and 0. With GAP [GAP06], we computed the closure of this set under taking arbitrary conductors. It has size 68,
Table 3.1: The dimensions of the quotient modules given by the invariant subspaces.

<table>
<thead>
<tr>
<th>Dimension Frequency</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>12</td>
<td>24</td>
<td>26</td>
<td>18</td>
<td>11</td>
</tr>
</tbody>
</table>

contains the full invariant filtration (28 spaces), and for each part of the grading in $W$ or $\hat{W}$, except those of dimension 2, there are two spaces in between the corresponding parts of the filtration (these are again 28 spaces); then there are 12 more spaces for which $M_{3i} \subset S \subset M_{3i+2}$ and no tighter inclusions can be given. Examining the 28 spaces in between filtration parts, we find that they are of the following form. For $0 \leq i \leq 3$, let

$$M_{\geq 3i}^- := M_{\geq 3i+1} \oplus \langle x_1^{i} x_2^{i-1} \partial_1, x_1^{i} x_2^{i-1} \partial_1 | 0 \leq j \leq i \rangle_F,$$

$$M_{\geq 3i+1}^+ := M_{\geq 3i+1} \oplus \langle (j+1)x_1^{i+1} x_2^{i-1} \partial_1 + (i-j+1)x_1^{i+1} x_2^{i-1} \partial_2 | 0 \leq j \leq i \rangle_F,$$

$$M_{\geq 3i+2}^- := M_{\geq 3i+3} \oplus \langle x_1^{i+1} x_2^{i-1} \partial_1, x_1^{i+1} x_2^{i-1} \partial_1 - x_1^{i+1} x_2^{i-1} \partial_2 | 0 \leq j \leq i \rangle_F,$$

$$M_{\geq 3i+3}^+ := M_{\geq 3i+3} \oplus \langle (j+1)x_1^{i+1} x_2^{i-1} \partial_1 + (i-j+1)x_1^{i+1} x_2^{i-1} \partial_2 | 0 \leq j \leq i \rangle_F,$$

and for $3 \leq i \leq 6$, let

$$M_{\geq 3i}^- := M_{\geq 3i+1} \oplus \langle x_1^{i-3} x_2^{i-3} \partial_1, (j+1)x_1^{i+1} x_2^{i-1} \partial_1 + (i-j+1)x_1^{i+1} x_2^{i-1} \partial_2 | 0 \leq j \leq i \rangle_F,$$

$$M_{\geq 3i+1}^+ := M_{\geq 3i+1} \oplus \langle x_1^{i-3} x_2^{i-3} \partial_1, (j+1)x_1^{i+1} x_2^{i-1} \partial_1 + (i-j+1)x_1^{i+1} x_2^{i-1} \partial_2 | 0 \leq j \leq i \rangle_F,$$

$$M_{\geq 3i+2}^- := M_{\geq 3i+3} \oplus \langle x_1^{i-3} x_2^{i-3} \partial_1, x_1^{i+1} x_2^{i-1} \partial_1 - x_1^{i+1} x_2^{i-1} \partial_2 | 0 \leq j \leq i \rangle_F,$$

$$M_{\geq 3i+3}^+ := M_{\geq 3i+3} \oplus \langle x_1^{i-3} x_2^{i-3} \partial_1, x_1^{i+1} x_2^{i-1} \partial_1 - x_1^{i+1} x_2^{i-1} \partial_2 | 0 \leq j \leq i \rangle_F.$$

Note that four of these spaces occur twice here: those with $i = 3$. Furthermore, whenever $M_{\geq 3i+1}^+$ and $M_{\geq 3i+1}^-$ exist, we have $M_{\geq 3i+1}^- \subset M_{\geq 3i+1}^+ \subset M_{\geq 3i+1}$ and $M_{\geq 3i+1}^+ \subset M_{\geq 3i+1}^-$. Additionally, $M_{\geq 3i+2}^- \subset M_{\geq 3i+1}$ and $M_{\geq 10}^+ \subset M_{\geq 3i+1}$. The spaces $M_{\geq 3i}$ and $M_{\geq 3i+2}$ for $0 \leq i \leq 3$, and $M_{\geq 3i+1}$ and $M_{\geq 3i+3}$ for $3 \leq i \leq 6$ can be described as follows:

$$M_{\geq 3i}^- = M_{\geq 3i+1} \oplus \langle D \in M_{3i} | \text{div } D = 0 \rangle_F,$$

$$M_{\geq 3i+1}^- = M_{\geq 3i+1} \oplus \langle D \in M_{3i} | \text{div } D = 0 \rangle_F,$$

$$M_{\geq 3i+2}^- = M_{\geq 3i+3} \oplus \langle D \in M_{3i+2} | \text{div } D = 0 \rangle_F,$$

$$M_{\geq 3i+3}^- = M_{\geq 3i+3} \oplus \langle D \in M_{3i+2} | \text{div } D = 0 \rangle_F.$$
The twelve remaining invariant spaces can be defined as follows.

\[ M'_{23} := M_{23} \oplus \langle x_2, x_1, x_2 d_2 + x_1 d_1 \rangle_F, \]

\[ M'_{24} := M_{24} \oplus \langle x_2 d_2 + x_1 d_1, 2x_2^2 d_2 + x_1 x_2 d_1, x_1 x_2 d_2 + 2x_1^2 d_1 \rangle_F, \]

\[ M'_{25} := M_{25} \oplus \langle x_2^2 d_2 + x_1 x_2 d_1, x_1 x_2 d_2 + 2x_1^2 d_1, \]

\[ 3x_2^2 d_2 + x_1 x_2^2 d_1, x_1 x_2^2 d_2 + x_1^2 x_2 d_1, x_1^2 x_2 d_2 - x_1^2 d_1, x_1^3 d_1, \]

\[ M'_{10} := M_{10} \oplus \langle 3x_2^2 d_2 + x_1 x_2^2 d_1, x_1 x_2^2 d_2 + x_1^2 x_2 d_1, x_1^2 x_2 d_2 - x_1^2 d_1, \]

\[ x_1^3 d_1, x_1 x_2^2 d_2 - x_1^2 x_2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

\[ M'_{13} := M_{13} \oplus \langle x_2^2 d_2 - x_1 x_2^2 d_1, x_1 x_2^2 d_2 - x_1^2 x_2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

\[ x_1 x_2^2 d_2 - x_1^2 x_2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

\[ M'_{16} := M_{16} \oplus \langle x_1 x_2^2 d_1, 2x_1 x_2^2 d_2 - x_1^2 x_2 d_2, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

\[ x_1 x_2^2 d_2 - x_1^2 x_2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

\[ M'_{18} := M_{18} \oplus \langle x_1 x_2^2 d_1, 3x_1 x_2^2 d_2 + x_1 x_2^2 d_2 + x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

\[ x_1 x_2^2 d_2 - x_1^2 x_2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

\[ M'_{20} := M_{20} \oplus \langle x_1 x_2^2 d_1, x_1 x_2^2 d_2 + x_1 x_2^2 d_2 - x_1^2 d_1, x_1 x_2^2 d_2 - x_1^2 d_1, \]

All the assertions so far in this section are checked by the GAP function check_6 on page 154. The function check_7 on page 154 shows that all of these invariant subspaces are closed under multiplication, except for \( M_{2-2} \) and \( M_{2-1} \).

As a subject for future research, we pose the question whether there are more invariant subspaces than the 68 found so far. As a possible way to answer this question, we consider the Hasse diagram of the invariant subspaces under inclusion, depicted in Figure 3.5.1 on page 104. For every proper inclusion \( H_1 \subsetneq H_2 \) for which there is no subspace \( H_3 \) with \( H_1 \subsetneq H_3 \subsetneq H_2 \), the Hasse diagram includes an edge between a node representing \( H_1 \) (below) and a node representing \( H_2 \) (above). We consider the action of the automorphism group on the quotients \( H_2/H_1 \). These quotients have dimensions from 1 to 5. The unipotent part of the automorphism group, \( \text{Aut}_1 M \), has trivial action on each of these quotients; only the \( \text{GL}_2(F) \)-part has a nontrivial action. The dimensions of these modules can be read off of Figure 3.5.1; the frequencies with which these dimensions occur are given in Table 3.1. A computation in GAP [GAP06], which we have not included in this thesis, shows that these modules are all irreducible.

Every “diamond” in the Hasse diagram represents a direct sum decomposition of the module in the top of the diamond modulo the module in the bottom of the diamond into two direct summands. A situation in which there would be an additional invariant subspace is if in one of these diamonds, the two direct summands would be isomorphic, say \( A \cong B \), identified by an isomorphism \( \phi \). Then diagonal modules could be formed, that is, modules of the form \( D_{\lambda} = \langle a + \lambda d \phi \mid a \in A \rangle_F \), which would be invariant if \( A \)
were invariant. However, all of the diamonds in the Hasse diagram represent sums of nonisomorphic modules, so this situation does not occur. This could be considered an indication that there are no other invariant subspaces.
Figure 3.4: The Hasse diagram of the 68 invariant subspaces. Each edge carries a label giving the dimension of the quotient represented by that label.
4.1. Introduction

In [Kap82], Kaplansky described how to construct a Lie algebra from a finite-dimensional symplectic space. In this construction, the set of projective points of the geometry forms a basis of the Lie algebra; the hyperbolic lines play a role in the multiplication. Then Rotman and Weichsel [Rot93, RW94] found an alternative proof for some theorems in that paper. This inspired Cuypers [Cuy05] to investigate a construction of Lie algebras over a field of characteristic 2 from partial linear spaces. Subsequently, J.I. Hall suggested that if one associates a cyclic orientation to the lines of the partial linear space, one might obtain Lie algebras over fields with arbitrary characteristic. In this chapter, we investigate the Lie algebras arising from this construction.

More specifically, the construction is as follows. Let \((P, L, \sigma)\) be an oriented partial linear space. Let \(L\) be the algebra over a field \(F\), linearly spanned by the formal basis \(P\), with the bilinear multiplication denoted by brackets and determined by:

\[
[p, q] = \begin{cases} 
0 & \text{if } p \perp q, \\
r & \text{if } \ell = \{p, q, r\} \in L \text{ and } \sigma(\ell) = q, \\
-r & \text{if } \ell = \{p, q, r\} \in L \text{ and } \sigma(\ell) = p.
\end{cases}
\]

We call \(L\) the Kaplansky algebra of the oriented partial linear space \((P, L, \sigma)\) over \(F\), and denote it by \(\mathcal{L}_F(P, L, \sigma)\). If \(\mathcal{L}_F(P, L, \sigma)\) is a Lie algebra for some field \(F\), then \((P, L, \sigma)\) is...
called a Lie oriented partial linear space and $\sigma$ is called a Lie orientation on $(P, L)$. If $(P, L, \sigma)$ is a Lie oriented partial linear space for some $\sigma$, then we call $(P, L)$ Lie orientable.

The multiplication on $L_\mathbb{F}(P, L, \sigma)$ is always bilinear and antisymmetric, but it does not necessarily satisfy the Jacobi identity. In this chapter we will investigate in which cases it does. Furthermore, if $\text{char } \mathbb{F} = 2$, then $\sigma$ is irrelevant for the multiplication.

If $(P, L)$ is disconnected, i.e. $P$ is the disjoint union of nonempty subsets $P_1$ and $P_2$, and every line is either contained in $P_1$ or in $P_2$, then its Kaplansky algebra will be the direct sum of the Kaplansky algebras of the connected components. Hence we will only study the case where $(P, L)$ is connected.

A subspace of a partial linear space $(P, L)$ is a pair of subsets $(P', L')$ of $P$ and $L$, such that $L'$ contains all lines of $L$ intersecting $P'$ in more than one point, and $P'$ contains the points of all lines in $L'$. The subspace generated by a set $\pi \subset P$ is the smallest subspace containing $\pi$; a subspace generated by the points on two intersecting lines is called a plane. In Section 4.3, we will define four families of partial linear spaces:

- a family of spaces $\mathcal{T}(\Omega, \Omega')$, related to a set of transpositions;
- a family of spaces $\text{Sp}(V, B)$, related to a symplectic space;
- a partial linear space $\mathcal{O}(V, Q)$, related to an orthogonal space;
- and a partial linear space $\text{PG}(\mathbb{F}_2^3) - \text{PG}(\mathbb{F}_2^3)$, related to projective geometry.

We will then prove this theorem, using results from Hale [Hal77] and Hall [Hal89]:

**Theorem 4.2.** If $(P, L, \sigma)$ is a Lie oriented partial linear space, then $(P, L)$ is one of the spaces $\mathcal{T}(\Omega, \Omega')$, $\text{Sp}(V, B)$, $\mathcal{O}(V, Q)$ and $\text{PG}(\mathbb{F}_2^3) - \text{PG}(\mathbb{F}_2^3)$ for $k \leq n - 3$.

Later in this chapter, in Theorems 4.12 and 4.30, we will see that there are Lie orientations for all partial linear spaces of the first three families, and for partial linear spaces of the last family if and only if $k = n - 3$.

An important tool in the proof of Theorem 4.2 will be the following lemma.

**Lemma 4.3.** If $(P, L)$ is generated by two intersecting lines, then there is a Lie orientation on $(P, L)$ if and only if:

- $(P, L)$ is the dual affine plane of order 2, or
- $(P, L)$ is the Fano plane; in this case, $L_\mathbb{F}(P, L, \sigma)$ is a Lie algebra only if $\text{char } \mathbb{F} = 3$.

This lemma will be proven in Section 4.2. As stated above, in Section 4.3 we will introduce the four families of partial linear spaces that support Lie orientations. Section 4.4 will introduce two general constructions of new oriented partial linear spaces from existing ones, that are useful for the rest of the chapter. In Section 4.5 we study the first three families of partial linear spaces mentioned above, and in Section 4.6 the last family. In these two sections we find Lie orientations if they exist, and prove that they don’t exist otherwise. In Section 4.7, we find geometrical indications for the partial linear spaces to give rise to simple Lie algebras. Then, in Section 4.8, we find the isomorphism types of the Lie algebras found in Sections 4.5 and 4.6. In Section 4.9, we study the conjecture that whenever two Lie algebras are constructed from the same partial linear space, they are isomorphic. Finally, in Section 4.10, we study a generalization of the construction. In Lemma 4.59 and Theorem 4.64, we will see that the equivalent of being Lie orientable for this generalization holds for all of the spaces of Theorem 4.2 and more.

### 4.2. Small cases

We study the Kaplansky algebras of some small partial linear spaces. For the smallest interesting case, take for $(P, L)$ the single oriented line $\ell$ of length 3. Label the points $i_i
4.2. SMALL CASES

Let $\pi : \mathcal{Q}(\mathbb{F}) \to \mathcal{Q}(\mathbb{F})$ be the projection fixing $i, j$ and $k$ and having 1 in its kernel. We will see that $\mathcal{L}_\mathbb{F}(P, L, \sigma) = \pi(\mathcal{Q}(\mathbb{F}))$. Indeed, the multiplication in $\mathcal{L}_\mathbb{F}(P, L, \sigma)$ can be obtained by identifying the elements $i, j$ and $k$ in $\mathcal{L}_\mathbb{F}(P, L, \sigma)$ with their namesakes in $\pi(\mathcal{Q}(\mathbb{F}))$ and defining $[a, b] = \pi(a \cdot b)$, where $\cdot$ denotes multiplication in $\mathcal{Q}(\mathbb{F})$. This quotient of $\mathcal{Q}(\mathbb{F})$ turns out to be a form of $\mathcal{L}_\mathbb{F}$.

The next smallest case after the single line is a plane. This case is handled in Lemma 4.3, which we prove here. It has a number of corollaries.

**Proof of Lemma 4.3.** We will set out to prove the “only if” part of the lemma. The “if” part will follow from the same proof.

Let $\{a, b, c\}$ and $\{a, d, e\}$ be two lines, of which we may assume that they are ordered alphabetically by $\sigma$. The Jacobi identity on $b, c$ and $d$ is the following:

$$[[b, c], d] + [[c, d], b] + [[d, b], c] = 0.$$

Any product of standard basis elements is either 0 or plus or minus a standard basis element, and therefore, so are the second and third terms of this sum. Hence, either one of the terms is $-c$ and the other is 0, or both are $c$ and $\mathcal{F} = 3$.

**Case 1:** Suppose $[[c, d], b] = -c$ and $[[d, b], c] = 0$ (the case where these values are switched is symmetric). Then $c$ and $d$ must be collinear; call the third point on that line $f = [c, d]$ (the case where $f = [c, d]$ leads to an isomorphic oriented partial linear space). Then $b, f$ and $e$ must be on a line as well, with $[b, f] = e$. The plane containing only the points and lines mentioned so far is the dual affine plane of order 2, and with this choice for $\sigma$ it satisfies the Jacobi identity. It is depicted in Figure 4.1(a).

If, in addition to the aforementioned points and lines, we have $b \sim d$, then the third point on that line (say $g = [b, d]$) cannot be collinear to $c$. The Jacobi identity on $g, d$ and $a$ then requires that $[a, g, f] = 0$ (the case where these values are switched is symmetric). Then $f$ and $e$ must be on a line as well, with $[b, f] = e$. The plane containing only the points and lines mentioned so far is the dual affine plane of order 2, and with this choice for $\sigma$ it satisfies the Jacobi identity. It is depicted in Figure 4.1(a).

**Case 2:** Now suppose $\text{char } \mathbb{F} = 3$ and $[[c, d], b] = [[d, b], c] = 0$. We cannot have that $[c, d] = \pm[d, b]$; for, if that would be the case, then by the axiom of partial linear spaces, $c = b$. So let $f = [c, d]$ and $g = [d, b]$ (again, the cases where $f = -[c, d]$ and $g = -[d, b]$ lead to isomorphic oriented partial linear spaces). The Jacobi identity on $a, b$ and $d$ then requires that $[a, g] = f$. This is the last line of the Fano plane. No more lines can be added, since each point is already collinear with every other point. This plane, depicted in Figure 4.1(b), also satisfies the Jacobi identity. The orientation on the lines is the same as is often used to define the octonions, see e.g. [Bae02].

**Proposition 4.4.** Let $(P, L, \sigma)$ be an oriented partial linear space. Then $\sigma$ is a Lie orientation if and only if for every plane $(P', L')$ of $(P, L)$, the restriction $\sigma|_{L'}$ of $\sigma$ to $L'$ is a Lie orientation.

**Proof.** Let $(P', L')$ be a plane of $(P, L)$. Clearly $\mathcal{L}_\mathbb{F}(P', L', \sigma|_{L'})$ is a subalgebra of $\mathcal{L}$. Hence, it is certainly necessary for $\mathcal{L}$ to be a Lie algebra that $\mathcal{L}_\mathbb{F}(P', L', \sigma|_{L'})$ be a Lie algebra. To show sufficiency, suppose that every $\mathcal{L}_\mathbb{F}(P', L', \sigma|_{L'})$ is a Lie algebra for planes $(P', L')$. Let $p, q, r \in P$. We will show that the Jacobi identity holds for $p, q$ and $r$. If there are no two intersecting lines containing $p, q$ and $r$, then the Jacobi identity certainly holds. Otherwise, they are contained in a plane; then the Jacobi identity holds because of the assumption. So the Jacobi identity holds on all triples of basis elements. Since the Jacobi identity is linear, we are done.

**Corollary 4.5.** If $(P, L, \sigma)$ is a Lie oriented partial linear space, then all of its planes are dual affine planes of order 2 or Fano planes.
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(a) The dual affine plane of order 2.

(b) The Fano plane.

Figure 4.1: The two Lie orientable planes. The arrows indicate a Lie orientation.

4.3. Geometrical considerations

Corollary 4.5 gives us a powerful tool to analyze the partial linear spaces that can be Lie oriented. It induces the question whether it is possible to find all geometries satisfying its property. A hopeful sign is the following lemma.

**Lemma 4.6.** Let \((P, L)\) be a finite partial linear space. If all of its planes are dual affine planes of order 2 or Fano planes, then its diameter is at most 2.

**Proof.** Take a path \((p, q, r, s)\). Since \(\{p, q, r\}\) generates a dual affine plane of order 2 or a Fano plane, \(p\) is collinear with at least two of the three points on the line containing \(q\) and \(r\). So is \(s\), because of the plane containing \(\{q, r, s\}\). Hence there is a point collinear with both \(p\) and \(s\). Thus the shortest path between any two points cannot consist of four or more points, so the diameter is at most 2. \(\Box\)

In this section, we will see a number of partial linear spaces that satisfy the property of Corollary 4.5, and we will review the geometrical classification.

We start with another small example. Let \(V = \mathbb{F}_2^4\) and let \(Q\) be the nondegenerate quadratic form of Witt defect 1. Then we can coordinatize \(V\) such that

\[Q(x, y, z, w) = x^2 + xy + y^2 + zw.\]

Let \(P\) be the set of vectors of \(Q\)-norm 1 and let \(L\) be the set of triples \(\{u, v, u + v\}\) of points that all have \(Q\)-norm 1. This is clearly a partial linear space with line length 3. If \(\{u, v, u + v\}\) and \(\{u, w, u + w\}\) are (distinct) intersecting lines, then the restriction of \(Q\) to the subspace \(\langle u, v, w \rangle_F\) is almost determined by that fact alone. It is easy to see that the subgeometry corresponding to this subspace is a dual affine plane of order 2. (A proof of a similar fact will be given as part of the proof of Lemma 4.19.) The geometry is depicted in Figure 4.2 with an orientation \(\sigma\). Simple checking reveals that the Jacobi identity is satisfied in all five planes. Hence, \(L_F(P, L, \sigma)\) is a Lie algebra. In Theorem 4.38, we will see that it is of type \(C_2\).

Now let \((V, Q)\) be an arbitrary orthogonal space over \(\mathbb{F}_2\). Let \(P\) be the set of points \(p\) such that \(Q(p) = 1\) and \(p\) is not in the radical of the bilinear form associated to \(Q\). Let \(L\) be the set of projective lines all of whose points are in \(P\). Then the same arguments as above show that \((P, L)\) is a partial linear space with line length 3, and that all of its planes are dual affine planes of order 2. We will denote it by \(O(V, Q)\).
We can make a similar construction starting from a symplectic form. Let $B$ be a symplectic form on a vector space $V$ over $\mathbb{F}_2$ and let the point set $P$ consist of the projective points of $V$ outside the radical of $B$. Let $L$ be the set of hyperbolic lines of $V$; that is, $L$ consists of the projective lines $\ell$ of $V$ such that $B$ restricted to $\ell$ is not the zero map. The partial linear space $(P, L)$ is denoted $Sp(V, B)$. We will show that all planes in $Sp(V, B)$ are dual affine planes of order 2. Two intersecting lines span a subspace, restricted to which the symplectic form has this matrix:

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
$$

It is easily seen that the vector $(1, 0, 1)$ is perpendicular to all vectors in this subspace. (It need not be in the radical of the form over $V$, though.) The six remaining projective points form a dual affine plane of order 2.

Another construction is not so clearly related to forms. We can construct the oriented partial linear space in Figure 4.2 in a different way: let $\Omega$ be the set $\{1, 2, 3, 4, 5\}$ and let $P$ be the set of subsets of $\Omega$ of size two. Let $L$ consist of the triples of points such that the symmetric difference between the three is empty. Then $(P, L)$ is a partial linear space, and again, all of its planes are dual affine planes of order 2.

This construction can be generalized as follows. Let $\Omega$ and $\Omega'$ be disjoint sets. Define the point set $P$ as the set of subsets $p$ of $\Omega \cup \Omega'$ such that $|p \cap \Omega| = 2$. The line set $L$ consists of the triples of points such that the symmetric difference between the three is empty. The partial linear space $(P, L)$ is then denoted by $T(\Omega, \Omega')$.

In [Hal89], J.I. Hall proved that these three families are the only partial linear spaces in which all planes are dual affine planes of order 2.

**Theorem 4.7.** Let $(P, L)$ be a partial linear space. If all planes in $(P, L)$ are dual affine of order 2, then $(P, L)$ is isomorphic to $Sp(V, B)$ or $O(V, Q)$ or $T(\Omega, \Omega')$, for some symplectic space $(V, B)$ over $\mathbb{F}_2$ or orthogonal space $(V, Q)$ over $\mathbb{F}_2$ or disjoint sets $\Omega$ and $\Omega'$.
What remains is to classify the partial linear spaces containing Fano planes. Let us define $V = \mathbb{F}_2^n$ for some $n \geq 3$, then clearly all planes in its projective geometry $PG(V)$ are Fano planes. If we subsequently remove all projective points in a subspace $U \equiv \mathbb{F}_2^k$, and all lines incident with them, then what happens to the planes depends on their intersection with $U$:

- planes disjoint from $U$ clearly stay Fano planes;
- a plane intersecting $U$ in a 1-dimensional subspace is turned into a dual affine plane of order 2;
- a plane intersecting $U$ in a 2-dimensional subspace is turned into 4 noncollinear points;
- a plane contained in $U$ disappears.

Since no new planes are introduced, we see that this partial linear space is also a plane that satisfies the property of Corollary 4.5. We denote this space by $PG(\mathbb{F}_2^n - \mathbb{F}_2^k)$. If $k = 0$, we will also write $PG(\mathbb{F}_2^n - \mathbb{F}_2^k)$. If $k \geq n - 2$, then there is no subspace of dimension 3 disjoint from $U$, so $PG(\mathbb{F}_2^n - \mathbb{F}_2^k)$ will not contain a Fano plane.

The following proposition is Theorem 1 from M.P. Hale, Jr. [Hal77]. It states that these are the only partial linear spaces satisfying the condition of Corollary 4.5, other than those of Theorem 4.7.

**Theorem 4.8.** Let $(P, L)$ be a partial linear space. If $(P, L)$ contains at least one Fano plane and all of its planes are either the Fano plane or dual affine of order 2, then $(P, L)$ is isomorphic to $PG(\mathbb{F}_2^n - \mathbb{F}_2^k)$, for some $n \geq 3$ and $0 \leq k \leq n - 3$.

We have now proven Theorem 4.2.

In Section 4.5, we will study the geometries from Theorem 4.7, and in Section 4.6 we will study those from Theorem 4.8. We will see that all of these partial linear spaces can be given a Lie orientation, except $PG(\mathbb{F}_2^n - \mathbb{F}_2^k)$ for $k \leq n - 4$.

### 4.4. Two utility constructions

For some of the proofs in the following sections, it will be useful to examine two constructions of an oriented partial linear space from another oriented partial linear space.

Take any particular algebra $L_\mathbb{F}(P, L, \sigma)$. Let $p \in P$. Define $\sigma_p$ by

$$
\sigma_p(\ell) = \begin{cases} 
\sigma(\ell) & \text{if } p \notin \ell, \\
(\sigma(\ell))^{-1} & \text{if } p \in \ell.
\end{cases}
$$

(4.1)

Then $L_\mathbb{F}(P, L, \sigma)$ and $L_\mathbb{F}(P, L, \sigma_p)$ are isomorphic, by the algebra isomorphism that maps $p$ to $-p$ and fixes all other basis elements. So if we simultaneously reverse the orientation of all lines through one point, we end up with an isomorphic algebra. This operation will be called flipping the sign of $p$. Note that these automorphisms of the algebra generate an elementary Abelian 2-group. It is now easy to check the following lemma.

**Lemma 4.9.** If one assigns arbitrary orientations to an arbitrary triple of lines in the dual affine plane of order 2, then this assignment can be uniquely extended to a Lie orientation of the dual affine plane.

For the second construction, take for each point $p \in P$ two copies, $p \in P$ and $p' \in P'$; let $2P = P \cup P'$ and let $\pi : 2P \to P$ be the natural projection. For each line $\ell = \{p, q, r\}$, take the four triples $\ell'$ such that $|\ell' \cap P'|$ is even, $\pi(\ell') = \ell$ and $|\ell'| = 3$ (note that one of these triples is $\ell$ itself). The resulting partial linear space is called the double of $(P, L)$ and denoted $2(P, L)$. If an orientation $\sigma$ is given for the lines of $L$, then we define $2(P, L, \sigma)$ by...
letting the lines of $2(P, L)$ inherit the orientation from $(P, L, \sigma)$ in the natural way: such that $\pi(p^{\sigma(\ell)}) = (\pi(p))^{\sigma(\pi(\ell))}$ for all incident point-line pairs $(p, \ell)$. This is illustrated in Figure 4.3. A more complex example can be seen in Figures 4.4 and 4.5.

**Lemma 4.10.** If $L_F(P, L, \sigma)$ satisfies the Jacobi identity, then so does $L_F(2(P, L, \sigma))$.

**Proof.** Consider a plane $\Pi$ in $2(P, L, \sigma)$. If the two generating lines project down to the same line in $(P, L, \sigma)$, then this holds for all lines in the plane. So $\Pi$ is isomorphic to the plane in Figure 4.3 and a simple check shows that in this case the Jacobi identity holds. Thus we assume that the two generating lines project to different lines in $(P, L, \sigma)$. Let $\Xi$ be the projection of $\Pi$ in $(P, L, \sigma)$. For each line in $\Xi$ there is exactly one corresponding line in $\Pi$, and it is oriented in the same way. So if the Jacobi identity is satisfied in $\Xi$, then it is satisfied in $\Pi$ as well. \qed

**Lemma 4.11.** If $\text{char } F = 2$, then $L_F(2(P, L, \sigma))$ contains an Abelian ideal $I$ of dimension $\dim L_F(P, L, \sigma)$, and $L_F(2(P, L, \sigma))/I = L_F(P, L, \sigma)$. If $\text{char } F \neq 2$, then $L_F(2(P, L, \sigma)) = L_F(P, L, \sigma) \oplus L_F(P, L, \sigma)$ over an extension of $F$.

**Proof.** If $\ell = \{p, q, r\} \in L$ and $p^{\sigma(\ell)} = q$, then for all $\alpha \in F$, we have $[p + \alpha p', q] = r + \alpha r'$ and $[p + \alpha p', q'] = r' + \alpha r$. Let $I_{\alpha} = \langle p + \alpha p' \mid p \in P \rangle_F$ for $\alpha = \pm 1$, then $I_{\alpha}$ are ideals. If $\text{char } F \neq 2$, then extend the field to include $\sqrt{2}$; then $p \mapsto \frac{1}{2} \sqrt{2}(p + \alpha p')$ provides an isomorphism from $L_F(P, L, \sigma)$ to $I_{\alpha}$. Otherwise, the two $I_{\alpha}$’s coincide and form an Abelian ideal. The map $x \mapsto x + I_{\alpha}$ forms an isomorphism between $L_F(P, L, \sigma)$ and $L_F(2(P, L, \sigma))/I_{\alpha}$. \qed

### 4.5. Partial linear spaces all of whose planes are dual affine

In this section, we will examine in detail the three families of partial linear spaces all of whose planes are dual affine planes of order 2, culminating in the proof of the following theorem:

**Theorem 4.12.** Every partial linear space such that all of its planes are dual affine of order 2 is Lie orientable.
The main tool for this study will be so-called $E_2$-groups, a generalization of extraspecial 2-groups due to J.I. Hall [Hal88] (who defined $E_p$-groups as a generalization of extraspecial $p$-groups for every prime $p$). In Section 4.5.1, we start by giving Hall’s definition and some of his results. This will show the relation between an $E_2$-group and an orthogonal space. In Section 4.5.2, we use $E_2$-groups to find a Lie orientation for $O(V,Q)$. This gives rise to a Lie algebra. We can also construct the Lie algebra directly from the $E_2$-group; we will see this in Section 4.5.2 as well. Then in Section 4.5.3, we will concern ourselves with the other two families of partial linear spaces all of whose planes are dual affine planes of order 2, $Sp(V,B)$ and $T(\Omega,\Omega')$. We will see that $T(\Omega,\Omega')$ embeds into $Sp(V,B)$ and $Sp(V,B)$ embeds into $O(V,Q)$. This will lead to a Lie orientation for $T(\Omega,\Omega')$ and $Sp(V,B)$, finishing the proof of Theorem 4.12. The different constructions and maps are presented somewhat informally in the diagram below; the labels at the arrows indicate the section number in which the corresponding map or
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construction is discussed.

\[ \begin{align*}
E_2\text{-group} & \quad \xleftarrow{4.5.1} O(V, Q) \quad \xleftarrow{4.5.3} \text{Sp}(V, B) \quad \xleftarrow{4.5.3} \mathcal{T}(\Omega, \Omega') \\
\mathcal{L}_E(O(V, Q), \sigma) & \quad \xleftarrow{4.5.2} \sigma \text{ for } O(V, Q) \quad \xleftarrow{4.5.3} \sigma \text{ for } \text{Sp}(V, B) \quad \xleftarrow{4.5.3} \mathcal{T}(\Omega, \Omega')
\end{align*} \]

4.5.1. The definition of \(E_2\)-groups and Hall’s results. The definition and results in this section are all from J.I. Hall [Hal88].

**Definition 4.13.** A group \(E\) is called an \(E_2\)-group if it is a 2-group and it has a normal subgroup \(Z\) of order 2 such that \(E/Z\) is elementary Abelian. \(Z\) is called the scalar subgroup of \(E\).

We will denote the nontrivial element of \(Z\) by \(z\). Note that \(z\) is a central element. By working in the quotient \(E/Z\), we see many properties of \(E_2\)-group elements. As is customary, we denote the coset in \(E/Z\) containing \(e\) by \(eZ\). For all \(e\) in an \(E_2\)-group, \((eZ)^2 = Z\), so \(e^2 \in Z\) and \(e^4 = 1\). If \(e\) has order 4, then \(e^{-1}Z = eZ\), but \(e^{-1}\) and \(e\) are different, so \(e^{-1} = ze\). Furthermore, for any \(f\), the commutator \([eZ, fZ] = Z\), so \([e, f] \in Z\).

**Proposition 4.14.** Let \(E\) be an \(E_2\)-group with scalar subgroup \(Z = \langle z \rangle_{Gp}\) and \(V = E/Z\) an elementary Abelian \(p\)-group. Let \(B\) and \(Q\) be maps determined by

\[
\begin{align*}
Q &: V \to \mathbb{F}_2, \\
B &: V \times V \to \mathbb{F}_2,
\end{align*}
\]

\(e^2 = z^Q(eZ), \quad [e, f] = z^B(eZ, fZ).\)

Then \(Q\) is a quadratic form and \(B\) is the associated bilinear form.

We will denote such a group \(E\) by \(E(V, Q)\).

**Theorem 4.15.** Let \(V\) be a vector space over \(\mathbb{F}_2\). Let \(Q\) be a quadratic form on \(V\). Then there exists an \(E_2\)-group \(E = E(V, Q)\).

**Theorem 4.16.** Two \(E_2\)-groups \(E(V, Q)\) and \(E(V', Q')\) are isomorphic if and only if there is an isomorphism \(\phi: V \to V'\) with \(Q' = Q \circ \phi\).

4.5.2. Constructing an orientation from an \(E_2\)-group. Given an \(E_2\)-group \(E\), we can construct a related oriented partial linear space. The orientation will turn out to be a Lie orientation. The resulting Lie algebra can also be constructed directly from \(E\). In this section we will explore both constructions, starting with the one where we construct an oriented partial linear space.

Let \(E\) be an \(E_2\)-group and let \(V\) and \(Q\) be such that \(E = E(V, Q)\). We set \((P, L) = O(V, Q)\). Then \(P\) consists of vectors where \(Q\) is nonzero, outside the radical of \(Q\); that is, cosets \(eZ\) where \(e\) is noncentral and has order 4. If \(eZ\) is such a coset, then \(e^{-1} = ez\) and \(eZ = \{e, e^{-1}\}\). The lines are triples of vectors where each is the sum of the others, and the form is nontrivial; that is, triples of cosets where each is the product of the others and where representatives do not commute. So if \(eZ, fZ \in P\) and \([e, f] = z\), then they form a line together with \(efZ\). Let

\[
\begin{align*}
e f &=: g, \\
f g &= fef = f[e, f]fe = z^2e = e \\
ge e &= [e, f]f e^2 = z^2f = f.
\end{align*}
\]
On the other hand,

\[ eg = e^2 f = zf = f^{-1}, \]
\[ gf = ef^2 = ze = e^{-1}, \]
\[ fe = [e, f]^{-1} ef = zg = g^{-1}. \]

This leads to an orientation on this partial linear space as follows. Choose a representative \( e \) in every coset \( eZ \in P \). Let \( R \) denote this set of representatives. Take a line \([eZ, fZ, gZ] \) where the chosen representatives of the points are \( e, f \) and \( g \). Then \( ef = g \) or \( ef = g^{-1} \); in the last case, \( fe = g \). Now \( \sigma([eZ, fZ, gZ]) \) is the permutation \((eZ, fZ, gZ) \) in the first case and \((fZ, eZ, gZ) \) in the second.

**Definition 4.17.** Let \( E \) be an \( E_2 \)-group. If \( P, L \) and \( \sigma \) are as above, then \((P, L, \sigma) \) is called the oriented partial linear space of \( E \) and \( R \).

We will see below that the set of representatives chosen does not affect the isomorphism type of the Kaplansky algebra of the oriented partial linear space. Hence we will sometimes omit \( R \), although \( R \) does affect the orientation of the partial linear space.

**Lemma 4.18.** Let \( E \) be an \( E_2 \)-group. Let \( P \) and \( L \) be the point and line sets of the associated partial linear space. If \( R \) and \( R' \) are different sets of representatives for \( P \) and \( \sigma \) and \( \sigma' \) are the associated orientations, then \( \mathcal{L}_E(P, L, \sigma) \cong \mathcal{L}_E(P, L, \sigma') \).

**Proof.** If \( R \) and \( R' \) differ for only one point \( eZ \), then \( \sigma \) and \( \sigma' \) differ on lines containing \( eZ \) and they are the same on the other lines. This is the operation of flipping the sign of a point. This leads to the following isomorphism.

Denote the Lie algebra element corresponding to \( eZ \) in \( \mathcal{L}_E(P, L, \sigma) \) by \( \hat{e} \) and the element in \( \mathcal{L}_E(P, L, \sigma') \) by \( \hat{e}' \). An isomorphism between \( \mathcal{L}_E(P, L, \sigma) \) and \( \mathcal{L}_E(P, L, \sigma') \) is then given by mapping \( \hat{e} \) to \( \hat{e}' \) if \( R \) and \( R' \) have the same representative in \( eZ \) and mapping it to \( -\hat{e}' \) otherwise.

**Lemma 4.19.** The oriented partial linear space of an \( E_2 \)-group is Lie oriented.

**Proof.** Let \( E \) be an \( E_2 \)-group and let \( P, L \) and \( \sigma \) be the associated point set, the associated line set, a set of representatives and the associated orientation, respectively. Let \( e, f \) and \( g \) be pairwise distinct representatives in \( R \) and denote the elements of the Kaplansky algebra corresponding to them by \( \hat{e}, \hat{f} \) and \( \hat{g} \), respectively. We will show that the Jacobi identity holds on \( \hat{e}, \hat{f} \) and \( \hat{g} \).

If \( eZ, fZ \) and \( gZ \) are linearly dependent in the \( F_2 \)-vector space \( E/Z \), then they form a line and \((\hat{e}, \hat{f}, \hat{g}) \) is the first example (related to the quaternion algebra) of Section 4.2. Then the Jacobi identity holds. So we assume that \( eZ, fZ \) and \( gZ \) span a three-dimensional subspace of \( E/Z \). If none of \( eZ, fZ \) and \( gZ \) are collinear, then the Jacobi identity holds trivially, so we assume that, say, \( eZ \) and \( fZ \) are both on a line \( \ell \). Then if \( gZ \) is collinear to no point on \( \ell \), the Jacobi identity still holds trivially, so we assume that \( gZ \) is collinear to a point on \( \ell \). The quadratic form on the ordered basis \([eZ, fZ, gZ] \) now looks like this:

\[
\begin{pmatrix}
1 & 1 & \ast \\
0 & 1 & \ast \\
0 & 0 & 1
\end{pmatrix},
\]

where the stars are not both zero. If there is a single 1 in \((\ast)\), then we can, if necessary, switch \( eZ \) and \( fZ \) to get the situation where \( gZ \) is collinear to \( eZ \), but not to \( fZ \). By, if
necessary, changing the representative for some of the points, which does not change
the isomorphism class of the algebra and thus also not the validity of the Jacobi identity,
we obtain the points, lines and orientations as in Figure 4.5.2. The only orientation that
is not trivial to derive is that
\[ f \cdot e f g = e f g = z^2. \]
Then
\[ [\hat{e}, [\hat{f}, \hat{g}]] + [\hat{f}, [\hat{g}, \hat{e}]] + [\hat{g}, [\hat{e}, \hat{f}]] = [\hat{e}, \hat{f} g] + [\hat{f}, -\hat{g} e] + [\hat{g}, e f] = e f g - e f g = 0. \]

On the other hand, if \((e) = (1)\), then, again by if necessary changing the representative for
some of the points, we obtain the points, lines and orientations as in Figure 4.5.2; again,
there is only one orientation that is not trivial to see: \[ e g \cdot f g = e g[f, g] g f = z^2 e f = e f. \] In
this case,
\[ [\hat{e}, [\hat{f}, \hat{g}]] + [\hat{f}, [\hat{g}, \hat{e}]] + [\hat{g}, [\hat{e}, \hat{f}]] = [\hat{e}, \hat{f} g] + [\hat{f}, -\hat{g} e] + [\hat{g}, e f] = 0. \]

So we see that starting from an \(E_2\)-group, we can find a Lie oriented partial linear
space. But in fact, we can do this step directly by means of a construction in [CT]. We
reproduce this construction here.

**Definition 4.20.** Let \(G\) be a finite group. For \(g \in G\), denote the element \(g - g^{-1}\) of the
group algebra \(F[G]\) by \(\hat{g}\). We call
\[ \mathcal{L}_F(G) := \langle \hat{g} \mid g \in G \rangle_{\text{Lie}} \subseteq \text{Lie}(F[G]) \]
the Plesken Lie algebra of \(G\) over \(F\).

The multiplication expands to
\[ [\hat{g}, \hat{h}] = gh - gh^{-1} \hat{g}^{-1} h + \hat{g}^{-1} h^{-1}, \]
so \(\mathcal{L}_F(G)\) is linearly spanned by the elements \(\hat{g}\). Clearly \(\hat{g} = 0\) if and only if \(g = g^{-1}\).
Furthermore, \(\hat{g} = -\hat{g}^{-1}\). If \(S\) is a set of algebra elements \(\hat{g}\) where \(S\) does not contain \(g^{-1}\)
for any \(\hat{g} \in S\), then its elements are linearly independent: \(F[G]\) can be decomposed into
subspaces \(\langle g, g^{-1} \rangle_F\) that are all linearly independent, and the elements of \(S\) are nonzero
and in different such subspaces.

We turn our attention to the Plesken Lie algebra of an \(E_2\)-group \(E\). The nonzero
generators are those \(\hat{g}\) where \(g\) has order 4. If \(e\) is an element of order 4, then \(\hat{e} = (1 - z)e\).
Note that $1 - z$ is a central element of the group algebra, and

$$(1 - z)^2 = 1 - 2z + z^2 = 2(1 - z).$$

Since $ze = e^{-1}$, we also have $z\hat{e} = e^{-1} - e = -\hat{e}$. Furthermore, we can obtain a basis of $L_F(E)$ by taking the algebra elements corresponding to a set of representatives for the points. Let $R$ be such a set of representatives. If $e, f$ and $g$ are three elements of $R$ with $ef = z^n g$, then

$$[\hat{e}, \hat{f}] = (1 - z)^2(ef - fe) = (1 - z)^2(z^n g - z^{1+n} g) = z^n(1 - z^3)\hat{g} = 4 \cdot (-1)^3\hat{g}.$$  

Since the dimensions of $L_F(E)$ and the Kaplansky algebra of the oriented partial linear space of $E$ are both equal to half the number of elements of $E$ of order 4, we have proven the following proposition:

**Proposition 4.21.** Let $E$ be an $E_2$-group. Choose a set $R$ of representatives for the points. The map from the Plesken Lie algebra of $E$, to the Kaplansky algebra of the oriented partial linear space of $E$, given by $\hat{e} \mapsto 4\hat{e}$ for $e$ in $R$, where $\hat{e}$ is the element of the Kaplansky algebra corresponding to $e$, is an isomorphism if $\text{char } F \neq 2$.

If $\text{char } F = 2$ and we take $E$ as in the proposition, then the regular multiplication on the subalgebra $\langle eZ \mid e \in R \rangle_F$ of $F[E/Z]$ turns out to satisfy the Jacobi identity. This subalgebra is then isomorphic to the Kaplansky algebra of the oriented partial linear space of $E$. This is easily checked by hand.

### 4.5.3. Finding an orientation for all spaces where all planes are dual affine.

In this section we will use the $E_2$-group construction to provide an orientation for all partial linear spaces consisting of dual affine planes of order 2. First we will embed $T'(\Omega, \Omega')$ into $Sp(F_2^{2n}, B)$ for some $n$, and $Sp(F_2^{2n}, B)$ into $O(F_2^{2n+1}, Q)$ (this last embedding will turn out to be an isomorphism). These will be embeddings as a subspace; so points are injectively mapped to points and lines to lines, and the relation of noncollinearity is also preserved. These embeddings are adapted from [Tay92]. We first consider the case where $\Omega' = \emptyset$ and the forms are nondegenerate.

**Lemma 4.22.** For all finite sets $\Omega$, the partial linear space $T'(\Omega, \emptyset)$ embeds as a subspace into $Sp(F_2^{2n}, B)$ for some nondegenerate symplectic space $(F_2^{2n}, B)$.

**Proof.** Let $\Omega = \{1, \ldots, 2n + 2\}$ for $n \geq 2$. We will embed $T'(\Omega, \emptyset)$ into $Sp(F_2^{2n}, B)$ as a subspace. Since $T'((1, \ldots, k), \emptyset)$ clearly embeds into $T'((1, \ldots, k + 1), \emptyset)$ as a subspace, this will be sufficient.

Let $V$ be the set of (unordered) partitions $\{\Gamma_1, \Gamma_2\}$ of $\Omega$ with $|\Gamma_1| \equiv |\Gamma_2| \equiv 0 \pmod{2}$. There is a bijection between even subsets of $\Omega$ and subsets of $\{1, \ldots, 2n + 1\}$, given by omitting $2n + 2$ if present in one direction and adding $2n + 2$ to odd sets in the other direction. Hence there are $2^{2n+1}$ even subsets of $\Omega$. For every even subset $\Gamma$, the sets $\Gamma$ and $\Omega \setminus \Gamma$ are different and define the same partition of $\Omega$, and no other subset defines that partition, so there are $2^{2n}$ partitions of $\Omega$ into even subsets.

We define an addition on $V$ by letting

$$\{\Gamma_1, \Gamma_2\} + \{\Gamma'_1, \Gamma'_2\} = \{\Gamma_1 \Delta \Gamma'_1, \Gamma_1 \Delta \Gamma'_2\},$$

where $\Delta$ is the symmetric difference operator. This is clearly well-defined, associative and commutative, the unit element is $\{\emptyset, \Omega\}$ and every element is its own inverse. Hence $V$ is a $2n$-dimensional vector space over $F_2$.

We will show that the partitions $E_i := \{[i, i + 1], \Omega \setminus [i, i + 1]\},$ for $1 \leq i \leq 2n$, form a basis. Take $\{\Gamma_1, \Gamma_2\} \in V$, where we have labelled the subsets such that $\Gamma_2$ is the one containing $2n + 2$. Let $\Theta_1 = \Gamma_1$. Then iteratively, for $1 \leq i \leq 2n$, define $\Theta_{i+1}$ as follows.
If $\Theta_i$ contains $i$, then set $\Theta_{i+1} = \Theta_i \Delta [i, i + 1]$, otherwise set $\Theta_{i+1} = \Theta_i$. By induction we see that each $\Theta_i$ contains no values less than $i$ or greater than $2n + 1$, and is even. Hence $\Theta_{n+1}$ is empty. This shows that $\Gamma_1$ is the iterated symmetric difference of sets $[i, i + 1]$ for $1 \leq i \leq 2n$; if we take the vector sum of the corresponding partitions $E_i$, we obtain $[\Gamma_1, \Gamma_2]$. This shows that the $2n$ vectors above span all of $V$; since dim $V = 2n$, it follows that they form a basis.

Let us define the bilinear form $B$ on $V$ by

$$B([\Gamma_1, \Gamma_2], [\Gamma_1', \Gamma_2']) \equiv [\Gamma_1 \cap \Gamma_2] \pmod{2}.$$ 

This form is nondegenerate, since if for some $\Gamma$, we have $\emptyset \neq \Gamma \neq \Omega$, then we can take $a \in \Gamma \neq \beta$, set $\Gamma_1 = \{a, \beta\}$ and obtain that $B((\Gamma, \Omega \setminus \Gamma), (\Gamma_1, \Omega \setminus \Gamma_1)) = 1$.

Note that the symmetric group on $\Omega$ has a faithful linear action on $V$. This action preserves $B$. Consider the transposition $(i, j)$ for, say, $1 \leq i < j \leq 2n + 2$. It fixes those partitions where $i$ and $j$ are in the same part and changes all other partitions. So the kernel of the map $\phi_{ij}$ that sends $v \in V$ to $\delta_{ij} - v$ consists of all partitions having $i$ and $j$ in the same part. This is a proper subspace of $V$. Since all transpositions are conjugate, its dimension does not depend on $i$ and $j$, and because we can see that for $\operatorname{Ker} \phi_{1,2}$ contains every $E_i$ except $E_2$, the codimension is one. Since $B$ is nondegenerate, there is exactly one nonzero vector perpendicular to $\operatorname{Ker} \phi_{ij}$. Since $E_{[i, j]} := \{[i, j], \Omega \setminus [i, j]\}$ is indeed perpendicular to all vectors with $i$ and $j$ in the same part, that is the vector.

The points $E_{[i, j]}$ and $E_{[i', j']}$ are collinear in $\operatorname{Sp}(V, B)$ if and only if $|[i, j] \cap [i', j']| = 1$, and in that case, the third point on the line is $E_{[i, j \Delta [i', j']]}$. This corresponds exactly to the line set in $\mathcal{T}(\Omega, \emptyset)$.

**Lemma 4.23.** For every nondegenerate symplectic space $(F_2^{2n}, B)$, the partial linear space $\operatorname{Sp}(F_2^{2n}, B)$ is isomorphic to $O(F_2^{2n+1}, Q)$, where $Q$ defines the nondegenerate orthogonal space over $F_2^{2n+1}$.

**Proof.** Let $B_Q$ be the bilinear form associated to $Q$ and let $u \in F_2^{2n+1}$ be the nonzero vector in the radical of $B_Q$. Note that over $F_2$, the bilinear form associated to a quadratic form is alternating. Since $B_Q$ is nondegenerate on $F_2^{2n+1}/\langle u \rangle_F$, we can identify $\operatorname{Sp}(F_2^{2n}, B)$ with $\operatorname{Sp}(F_2^{2n+1}/\langle u \rangle_F, B_Q)$.

Because $Q$ is nondegenerate and $u \in \operatorname{Rad} B_Q$, we have $Q(u) = 1$, and for every $v \in F_2^{2n+1}$, we have $Q(v + \lambda u) = Q(v) + \lambda^2$, so either $v$ or $v + u$ has $Q$-value equal to one. The embedding is given by sending $v + \langle u \rangle_F$ to either $v$ or $v + u$, whichever of the two has $Q$-value equal to one. The embedding is surjective: every element with $Q$-value equal to one has a preimage, except $u$ itself; but $u$ is in $\operatorname{Rad} B_Q$, so it is not a point of $O(F_2^{2n+1}, Q)$.

To see that this map preserves lines, take any two collinear points $v + \langle u \rangle_F, w + \langle u \rangle_F$ in $\operatorname{Sp}(F_2^{2n+1}/\langle u \rangle_F, B_Q)$ such that $Q(v) = Q(w) = 1$. Since $v + \langle u \rangle_F$ and $w + \langle u \rangle_F$ are collinear, $B_Q(v, w) = Q(v) + Q(w) + Q(v + w) = Q(v + w) = 1$. Hence $v$ and $w$ are also collinear in $O(F_2^{2n+1}, Q)$. On the other hand, if $v + \langle u \rangle_F$ and $w + \langle u \rangle_F$ are not collinear, then $Q(v + w) = 0$, so $v$ and $w$ are also not collinear.

To extend these embeddings to the cases where $\Omega'$ is not empty and where $B$ and $Q$ are degenerate, we use the following three lemmas.

**Lemma 4.24.** Let $\Omega$ and $\Omega'$ be disjoint, and $\omega \notin \Omega \cup \Omega'$. Then $\mathcal{T}(\Omega, \Omega' \cup \{\omega\}) = 2\mathcal{T}(\Omega, \Omega')$.

**Proof.** For every point $p$ in $\mathcal{T}(\Omega, \Omega')$, there are two points, $p$ and $p' := p \cup \{\omega\}$, in $\mathcal{T}(\Omega, \Omega' \cup \{\omega\})$. The set of lines of $\mathcal{T}(\Omega, \Omega' \cup \{\omega\})$ consists of the lines of $\mathcal{T}(\Omega, \Omega')$, together
with three copies for every line \( \ell \), in which two of the points have been replaced by their copies with \( \omega \) added. This is exactly the construction of doubling. \( \square \)

**Lemma 4.25.** Let \( V \) be a vector space over \( \mathbb{F}_2 \) with an alternating form \( B \) and a quadratic form \( Q \) (not necessarily related). If \( u \) is a nonzero vector in the radical of \( B \) or of \( Q \), respectively, and \( U = \langle u \rangle \), then \( \text{Sp}(V, B) = 2\text{Sp}(V/U, B) \) or \( \mathcal{O}(V, Q) = 2\mathcal{O}(V/U, Q) \), respectively.

**Proof.** Take a basis of \( V \) containing \( u \). Let \( V' \) be the subspace of \( V \) spanned by all basis elements other than \( u \). Then for \( v, w \in V' \), we see that \( B(v + U, w + U) = B(v, w) \) and \( Q(v + U) = Q(v) \), so \( \text{Sp}(V/U, B) \equiv \text{Sp}(V', B) \) and \( \mathcal{O}(V/U, Q) = \mathcal{O}(V', Q) \).

Clearly for every point \( v \) of \( \text{Sp}(V', B) \) or \( \mathcal{O}(V', Q) \), there are two points \( v \) and \( v + u \) in \( \text{Sp}(V, B) \) or \( \mathcal{O}(V, Q) \). For the orthogonal space we need that \( Q(v + u) = Q(v) \) for this, which follows from \( u \) being in the radical of \( Q \).

Let \( \{ v, w, v + w \} \) be a line in the smaller partial linear space. Any line in the larger linear space that maps to this line can be obtained by adding \( u \) to an even number of the points. This is exactly the construction of doubling. \( \square \)

**Corollary 4.26.** Let \( V \) be a vector space over \( \mathbb{F}_2 \) and \( B \) a symplectic form on \( V \). Then there exist a vector space \( W \) over \( \mathbb{F}_2 \) and a quadratic form \( Q \) on \( W \) such that \( \text{Sp}(V, B) \) is isomorphic to \( \mathcal{O}(W, Q) \).

**Lemma 4.27.** If a partial linear space \( (P, L) \) embeds as a subspace into \( (Q, M) \), then \( 2(P, L) \) embeds as a subspace into \( 2(Q, M) \).

**Proof.** Denote the copy of \( p \in P \) in \( 2P \) by \( p' \), and similarly for \( q \in Q \). Let \( \pi_P \) be the projection of \( 2P \) onto \( P \) and \( \pi_Q \) the projection of \( 2Q \) onto \( Q \). Let \( \rho_1 \) be the embedding of \( (P, L) \) into \( (Q, M) \). Define \( \rho_2 : 2(P, L) \to 2(Q, M) \) as an extension of \( \rho_1 \) by letting \( \rho_2 \) map \( p' \) to \( \rho_1(p)p' \). Then \( \rho_2 \) is clearly injective and it maps lines to lines. A commutative diagram detailing the different maps is depicted below.

\[
\begin{array}{ccc}
2P & \xrightarrow{\rho_2} & 2Q \\
\pi_P \downarrow & & \downarrow \pi_Q \\
P & \xrightarrow{\rho_1} & Q
\end{array}
\]

Suppose that \( p_1 \) and \( p_2 \) are two points in \( 2P \) (in \( P \) or \( P' \)). If there is a line \( m \) connecting \( \rho_2(p_1) \) and \( \rho_2(p_2) \), then \( \pi_Q(m) \) connects \( \pi_Q(\rho_2(p_1)) = \rho_1(\pi_P(p_1)) \) and \( \pi_Q(\rho_2(p_2)) = \rho_1(\pi_P(p_2)) \). Since \( \rho_1 \) is an embedding that preserves noncollinearity, \( \pi_P(p_1) \) and \( \pi_P(p_2) \) are collinear, so \( p_1 \) and \( p_2 \) are collinear as well. \( \square \)

We are now ready to prove Theorem 4.12.

**Theorem 4.12.** Let \( (P, L) \) be a partial linear space where every plane is dual affine of order 2. By Theorem 4.7, it is of the form \( \text{Sp}(V, B) \) or of the form \( \mathcal{O}(V, Q) \) or of the form \( \mathcal{T}(\Omega, \Omega') \). By Lemmas 4.22, 4.23, 4.24 and 4.27 and Corollary 4.26, \( (P, L) \) embeds as a subspace into a partial linear space of the form \( \mathcal{O}(V, Q) \). \( \square \)

### 4.6. Partial linear spaces containing the Fano plane

In Theorem 4.8 we saw that a Lie oriented partial linear space containing a Fano plane is of the form \( \text{PG}(\mathbb{F}_2^2) - \text{PG}(\mathbb{F}_2^2) \) with \( k \leq n - 3 \). In this section we will see that such a partial linear space necessarily has \( k = n - 3 \), and that there is indeed a Lie orientation on \( \text{PG}(\mathbb{F}_2^2) - \text{PG}(\mathbb{F}_2^{n-3}) \). This is stated in Theorem 4.30.
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Lemma 4.28. There is no Lie orientation on $PG(F_2^3)$.

Proof. Let $(P, L) = PG(F_2^3)$. Suppose that $\sigma$ is a Lie orientation on $(P, L)$. Let $\ell = \{a, b, c\}$ and $m = \{x, y, z\}$ be two disjoint lines in $PG(F_2^3)$ with $\sigma(\ell) = (a, b, c)$ and $\sigma(m) = (x, y, z)$. Let $ax$ denote the third point on the line on $a$ and $x$, and similarly for the other pairs of points in $\ell \times m$. We have now named all 15 points. By potentially flipping the sign of $ax$, we may assume that $\sigma([x, ax, a]) = (x, ax, a)$, and similarly for the 8 other lines $k$ connecting $m$ to $\ell$: we assume the orientation is such that $\sigma(k)$ maps $k \cap \ell$ to $k \cap m$.

We define projections $\pi_\ell: P \setminus m \to \ell$ and $\pi_m: P \setminus \ell \to m$, mapping the point $pq$, with $p \in \ell$ and $q \in m$, to $p$ or to $q$ respectively (and fixing $\ell$ and $m$, respectively). If a plane $\Pi$ contains $m$ or $\ell$, then we will see that the restriction of $\sigma$ to $\Pi$ is fully determined by the choices that we have made, as follows. Take for $\Pi$ the plane containing $\ell$ and $x$. It is depicted in Figure 4.6. The Jacobi identity on $a, c$ and $x$ tells us that

$$-bx + [cx, a] + [ax, c] = 0;$$

since all three of these terms are $\pm bx$, we find $[cx, a] = [ax, c] = -bx$. This determines the value of $\sigma$ on the lines $\{cx, bx, a\}$ and $\{c, bx, ax\}$. The Jacobi identity on $a, b$ and $x$ additionally gives the value on $\{cx, b, ax\}$. These are depicted in Figure 4.6. We see that $\pi_\ell$ is orientation-reversing on these lines.

The same is true if we take for $\Pi$ a different plane containing $\ell$, and if we take for $\Pi$ a plane containing $m$, then $\pi_m$ is orientation-reversing on the lines in there as well.

Now take a plane containing only one point of both $\ell$ and $m$, e.g. the plane containing $a, x$ and $by$, as depicted in Figure 4.8. The Jacobi identity on these three points cannot be satisfied anymore:

$$[[a, x], by] + [[x, by], a] + [[by, a], x] = [ax, by] - [bz, a] + [cy, x] = [ax, by] = \pm cz. \quad \square$$

Lemma 4.29. If $n \geq 3$, then $PG(F_2^{n+1}) - PG(F_2^{n-2})$ is the double of $PG(F_2^n) - PG(F_2^{n-3})$.

Proof. Let $V$ and $U$ be vector spaces such that the point set of $PG(F_2^{n+1}) - PG(F_2^{n-2})$ is $V - U$ and let $V' \subseteq V$ be such that $V' - (V' \cap U) = PG(F_2^{n}) - PG(F_2^{n-3})$. Let $U' = V' \cap U$. Lines in $PG(F_2^{n+1}) - PG(F_2^{n-2})$ are two-dimensional subspaces of $V$ that do not intersect.
U. Choose a nonzero vector \( u \) in \( U \setminus V' \); then

\[
V = V' \oplus \mathbb{F}_u.
\] (4.2)

A diagram containing all these spaces and \( u \) can be found in Figure 4.13. All points of \( V - U \) can be obtained by taking the points of \( V' - U' \) and for each such point \( p \), adding the point \( p + u \). The lines of \( V - U \) are two-dimensional subspaces not intersecting \( U \); the three points on it sum to zero. Decomposing the vectors corresponding to these points according to Eq. (4.2), we see that an even number of the points is not in \( V' \) and the components in \( V' \) (which are nonzero) sum to zero; hence they also form a line. This is exactly the construction of the double point-line space.

□

Theorem 4.30. Let \((P, L)\) be a partial linear space containing a Fano plane. Then there exists a Lie orientation \(\sigma\) on \((P, L)\) if and only if \((P, L)\) is isomorphic to \(\text{PG}(F_2^n) - \text{PG}(F_2^{n-3})\).

Proof. Suppose that \((P, L, \sigma)\) is a Lie oriented partial linear space and \((P, L)\) contains a Fano plane. By Theorem 4.8, the partial linear space \((P, L)\) is of the form \(\text{PG}(F_2^k) - \text{PG}(F_2^k)\) with \(k \leq n - 3\). If \(k < n - 3\), then \((P, L)\) contains a copy of \(\text{PG}(F_2^k)\). This is a contradiction with Lemma 4.28. Hence \(k = n - 3\).

Now let us find an orientation \(\sigma\) such that \(\text{PG}(F_2^n) - \text{PG}(F_2^k), \sigma)\) is Lie oriented. Lemma 4.3 provides an orientation for \(n = 3\). Lemmas 4.10 and 4.29 then provide the induction step.

□

4.7. Simple Lie algebras

In the previous sections we have explored what orientations on the different partial linear spaces lead to Lie algebras. In this section we will examine the Lie algebras themselves, and in particular find a geometrical criterion to decide if a given partial linear space allows an orientation such that its Kaplansky algebra is simple. Showing sufficiency of the criterion is fairly simple and is done in Lemma 4.32; necessity is a bit more involved and is done in Lemma 4.33. First let us introduce the criterion.

Definition 4.31. A partial linear space \((P, L)\) is called reducible if there are distinct \(p, q \in P\) with \(p^+ = q^+\). If \((P, L)\) is not reducible, it is called irreducible.

Lemma 4.32. If \((P, L)\) is irreducible, then \(L_F(P, L, \sigma)\) is a simple algebra.
Proof. Let $J$ be a nonzero ideal in $L$. We will show that $J$ contains some $p \in P$. Then $J$ contains all of $P$, and thus $J = L$. Let $v$ be a nonzero element of $J$. Number the elements of $P$ as $p_1, p_2, \ldots$. Let $I_v$ be the set of indices where $v$ has a nonzero coefficient. Write $v = \sum_{i \in I_v} \alpha_i p_i$ for some set of nonzero $\alpha_i \in \mathbb{F}$. If $|I_v| = 1$, then we are done. Hence assume that $1, 2 \in I_v$.

There is a point $p$ in the symmetric difference of $p_1^\perp$ and $p_2^\perp$, say in $p_1^\perp \setminus p_2^\perp$. Then $[p, v] = \sum_{i \in I_v} \alpha_i [p, p_i]$. Each product $[p, p_i]$ is either 0 or plus or minus a standard basis element; in particular, $[p, p_1] = 0$ and $[p, p_2] \neq 0$. So $[p, v]$ will have strictly fewer nonzero coefficients than $v$, but it will still be nonzero itself. By repeating this procedure, we obtain an element in $P \cap J$ again, leading to $J = L$. □

Lemma 4.33. If $(P, L)$ is a reducible partial linear space, then there is a Lie orientation $\sigma$ such that $L_F(P, L, \sigma)$ is not simple.

Proof. Let $(P, L)$ be a reducible partial linear space. Suppose there is a Fano plane in $(P, L)$. Then $(P, L)$ is more than a single Fano plane, because in the Fano plane $p^\perp = \{p\}$. Then Lemmas 4.11 and 4.29 provide a Lie orientation such that $L_F(P, L, \sigma)$ is not simple.

Hence we may assume that $(P, L)$ contains only dual affine planes of order 2. Suppose that $(P, L) = \mathcal{T}(\Omega, \Omega')$. Suppose furthermore that $\Omega' = \emptyset$. Then $p = \{i, j\}$ for some $i, j \in \Omega$, and

$$p^\perp = \{(k, \ell) \mid |(i, j) \cap (k, \ell)| = 0, 2\}.$$  

Hence there can be no different $q$ with $q^\perp = p^\perp$, so $(P, L)$ is irreducible, which is a contradiction to the assumptions from the Lemma. So if $(P, L) = \mathcal{T}(\Omega, \Omega')$, then $\Omega'$ is nonempty, whence, by Lemmas 4.11 and 4.24, we find a Lie orientation such that $L_F(P, L, \sigma)$ is not simple.

So we may assume that $(P, L) = Sp(V, B)$ or that it is $O(V, Q)$. We will show that $B$ is degenerate or that $Q$ is singular, respectively. Let $v \in V$.

- For $Sp(V, B)$, we see that $v^\perp$ is the set of nonzero vectors that are perpendicular to $v$ with respect to $B$ and are not in the radical of $B$. The subspace spanned by these vectors is the set of vectors perpendicular to $v$. If $B$ is nondegenerate, this uniquely identifies $v$, so $B$ is apparently degenerate.

- For $O(V, Q)$, the elements of $v^\perp$ lie in the subspace $S$ of vectors perpendicular to $v$ with respect to the bilinear form $B_Q$ associated to $Q$. In particular, $v^\perp$ consists of the vectors $w \in S \setminus Rad(B_Q)$ for which $Q(w) = 1$. Clearly $v \in v^\perp$. Suppose $\langle v^\perp \rangle_F$ is a proper subspace $R$ of $S$; then $Q$ is zero outside $R$. Apparently $v \in R$. Let $u \in S \setminus R$, then $Q(u) = B(u, v) = 0$. But then $u + v \in S \setminus R$, and $Q(u + v) = Q(u) + Q(v) + B(u, v) = 1$.

So $\langle v^\perp \rangle_F = S$. If $Q$ is nondegenerate, this uniquely identifies $v$ modulo the radical of $B_Q$. So either $Q$ is degenerate, or there are two vectors $v, w \in V$ with $Q(v) = Q(w) = 1$ and $v + w \in Rad(B_Q)$. Then $Q(v + w) = Q(v) + Q(w) + B(v, w) = B(v + w, w) + B(w, w) = 0$. So either $v = w$ or $Q = 0$ is nonempty. Hence $Q$ is degenerate.

Then, by Lemmas 4.11 and 4.25, we find a Lie orientation $\sigma$ such that $L_F(P, L, \sigma)$ is not simple. □

Corollary 4.34. The partial linear spaces $(P, L)$ such that $L_F(P, L, \sigma)$ is simple for all Lie orientations $\sigma$ are the following:

- $PG(F_2^2)$ (the single Fano plane);
- $Sp(V, B)$, where the symplectic form is nondegenerate;
- $O(V, Q)$, where the quadratic form is nondegenerate;
• $\mathcal{T}(\Omega, \emptyset)$.

In the following section, we will see what the corresponding Lie algebras are.

### 4.8. The isomorphism types of the Lie algebras

In this section, we will determine the isomorphism types of the Lie algebras that we have constructed earlier in this chapter.

**Lemma 4.35.** Any subspace of $\mathcal{L}_E(P, L, \sigma)$ spanned by an inclusionwise maximal set of basis elements corresponding to noncollinear points is a Cartan subalgebra.

**Proof.** Let $H$ be such a subspace. Since the multiplication restricted to $H$ is zero, $H$ is certainly an Abelian subalgebra. Denote the set of points corresponding to the generators of $H$ by $U$, and write $\tilde{p}$ for the Lie algebra element corresponding to the point $p$.

Let $u \in U$ and suppose $[\sum \alpha_i \tilde{v}_i, \tilde{u}] \in H$. Suppose $v_i \not\in U$ and $v_i \sim u$. Denote the third point on the line connecting $v_i$ and $u$ by $w$. Note that $w \not\in U$, since $u$ is already an element of $U$. Then $[\tilde{v}_i, \tilde{u}] = \pm \tilde{w} \not\in H$. Furthermore, different basis vectors $\tilde{v}_i$ cannot “compensate” this element outside $H$, since for $i \neq j$, the product $[\tilde{v}_j, \tilde{u}]$ is a multiple of a different standard basis vector. So $\alpha_i = 0$. Since every $v_i \not\in U$ is collinear to some point in $U$ (otherwise $U$ would not be maximal), $\sum \alpha_i \tilde{v}_i \in H$. Hence $H$ is self-normalizing. Thus it is a Cartan subalgebra.

**Theorem 4.36.** $\mathcal{L}_E(\mathcal{T}, \emptyset)$ is of Chevalley type $B_k$ if $n = 2k + 1$ and of type $D_k$ if $n = 2k$.

**Proof.** Take the subalgebra $H$ spanned by the elements $h_\ell := [2\ell - 1, 2\ell]$ where $\ell \leq k$, and where, like in the proof of Lemma 4.35, the tilde turns a point into the corresponding algebra element. This spanning set satisfies the requirements of Lemma 4.35, so $H$ is a Cartan subalgebra. If $n$ is even, then the rank, dimension and simplicity are sufficient to decide that this is a Lie algebra of type $D_k$, but for $B_k$ we need to analyze the root system. This analysis also works for $D_k$, so we continue for both odd and even $n$. Thus for $\ell < m \leq k$ and for $\alpha, \beta \in \{\pm 1\}$, let us define

$$r_{\alpha, \beta}^{\ell, m} = [2\ell - 1, 2m] + \alpha \sqrt{-1}[2\ell - 1, 2m] + \beta \sqrt{-1}[2\ell, 2m] - \alpha \beta[2\ell, 2m].$$

Furthermore, if $n$ is odd, then for $\ell \leq k$ and $\alpha \in \{\pm 1\}$, we additionally define

$$r_{\ell}^{\alpha} = [2\ell - 1, n] + \alpha \sqrt{-1}[2\ell, n].$$

Now

$$[h_\ell, r_{\alpha, \beta}^{\ell, m}] = -\beta \sqrt{-1}r_{\alpha, \beta}^{\ell, m}, \quad [h_m, r_{\alpha, \beta}^{\ell, m}] = -\alpha \sqrt{-1}r_{\alpha, \beta}^{\ell, m}, \quad [h_\ell, r_{\ell}^{\alpha}] = -\alpha \sqrt{-1}r_{\ell}^{\alpha},$$

and all other products between elements of the form $h_\ell$ on the one hand, and elements of the form $r_\ell$, on the other hand, are zero. Thus we define $\{e_\ell\}$ to be the ordered basis of $H^*$ dual to the basis of $H$ consisting of the elements $\sqrt{-1}h_\ell$; then the projective point containing $r_{\alpha, \beta}^{\ell, m}$ acts as $\beta e_\ell + \alpha e_m$ and $r_{\ell}^{\alpha}$ if present, acts as $\alpha e_\ell$. These root systems are of type $D_k$ and $B_k$ if $n$ is even or odd, respectively.

**Theorem 4.37.** Let $B$ be a nondegenerate symplectic form over $\mathbb{F}_2^{2k}$. Let $\sigma$ be the Lie orientation arising from the $E_2$-group. Then $\mathcal{L}_E(\text{Sp}(\mathbb{F}_2^{2k}, B), \sigma)$ is of Chevalley type $A_{2k-1}$. 

4.9. Uniqueness of Lie orientations

In the sections so far, we have seen which partial linear spaces can be Lie oriented. We have seen that for each Lie orientation, flipping the sign of points gives a different Lie orientation that leads to the same Lie algebra. However, what we haven’t seen is Lie orientations on the same partial linear space that lead to different Lie algebras. We conjecture that this cannot happen.

**Conjecture 4.41.** Let $(P, L)$ be a partial linear space and let $\sigma$ and $\tau$ be two Lie orientations on $(P, L)$. Then $\mathcal{L}_R(P, L, \sigma)$ is isomorphic to $\mathcal{L}_R(P, L, \tau)$.

This conjecture is supported by the following. Firstly, in Section 4.9.1 we present a proof of the conjecture for the case where $(P, L)$ is of the form $\mathcal{T}(Q, \Omega')$. Furthermore, we verified the conjecture by a computer program implemented in GAP [GAP06] for the nonsingular orthogonal spaces up to dimension 11 and for the spaces of the form $PG(F_2^2) - PG(F_2^{8-k})$ for $n$ up to 5.

In Section 4.9.2, we provide an attempt at showing that the conjecture holds for all orthogonal spaces, and thus also for all symplectic spaces. Section 4.9.3 considers the possible extension of this method to the spaces of the form $PG(F_2^2) - PG(F_2^{10})$. 

---

**Proof.** A Cartan subalgebra $H$ is spanned by elements $v$ where $v$ ranges over the nonzero vectors in a maximal totally isotropic subspace $U$, by Lemma 4.35. We have a simple Lie algebra of dimension $2k - 1$ and of rank $2^{k} - 1$. The only Chevalley algebra with such parameters is $\mathfrak{s}(\mathbb{C}^2)$. □

**Theorem 4.38.** Let $\mathcal{Q}$ be a nondegenerate quadratic form over $\mathbb{F}_2^k$. Let $\sigma$ be the Lie orientation arising from the $E_2$-group. If $\mathcal{Q}$ has Witt defect 1, then $\mathcal{L} = L_{\mathcal{C}}(O(\mathbb{F}_2^2), \mathcal{Q}, \sigma)$ is of Chevalley type $C_{2^{k-1}}$, and if $\mathcal{Q}$ has Witt defect 0, then $\mathcal{L}$ is of Chevalley type $D_{2^{k-1}}$.

**Proof.** In [CT], Cohen and Taylor found the Plesken Lie algebras over the complex field of the extraspecial groups. By Proposition 4.21, $\mathcal{L}$ is the Plesken algebra of the $E_2$-group $E(V, \mathcal{Q})$. Since $\mathcal{Q}$ is nondegenerate, $E$ is the extraspecial 2-group $2^{1+2k}$ with the same sign as $\mathcal{Q}$. Theorem 6.2 of Cohen and Taylor [CT] then states that $\mathcal{L}$ is $\mathfrak{o}(\mathbb{C}^2)$ if $E = 2^{1+2k}$ or $\mathfrak{sp}(\mathbb{C}^2)$ if $E = 2^{1+2k}$. □

**Theorem 4.39.** Consider the partial linear space $(P, L) = PG(F_2^2) - PG(F_2^{8-3})$. Let $\sigma$ be the Lie orientation on $(P, L)$ arising from starting with a Lie orientation on the Fano plane and repeatedly performing the doubling construction of Lemma 4.10 until we obtain a partial linear space isomorphic to $(P, L)$. Furthermore, let $\mathbb{F}$ be an extension field of $\mathbb{F}_3$. Then $\mathcal{L}_E(P, L, \sigma)$ is isomorphic to $\oplus_{k=1}^{2n-3} A_2(\mathbb{F})'$, where $A_2(\mathbb{F})'$ means the commutator subalgebra of the Lie algebra of type $A_2$ over $\mathbb{F}$. 

**Proof.** If $(P, L, \sigma)$ is a single oriented Fano plane, then we can take any vertex $p$ as a generator for a Cartan subalgebra; if $[p, q, r] \in L$ and $p^{q(r)} = q$, then $q + \sqrt{-1}r$ and $q - \sqrt{-1}r$ span eigenspaces of $p$. These eigenspaces form the images of a set of root spaces under modding out the centre. Lemma 4.11 finishes the proof. □

**Theorem 4.40.** For the oriented partial linear spaces $(P, L, \sigma)$ referred to in Theorems 4.36, 4.37 and 4.38, the algebras $\mathcal{L}_R(P, L, \sigma)$ are real compact forms.

**Proof.** Consider the matrix of the Killing form of a Kaplansky algebra with respect to the standard basis. Any off-diagonal entry is 0 and the diagonal entry at a point $p$ is minus the number of points collinear to $p$ (other than $p$ itself). Hence the Killing form is negative definite. Since all the algebras in the statement are semisimple, they are the real compact forms. □
4.9.1. Uniqueness of the orientation for \( \mathcal{T}(\Omega, \Omega') \). In this section we will prove that the orientation found in the previous section is the unique orientation leading to a Lie algebra for partial linear spaces of the form \( \mathcal{T}(\Omega, \Omega') \). Recall that a point of this partial linear space is a subset \( p \) of \( \Omega \cup \Omega' \) such that \( |p \cap \Omega'| = 2 \). If this intersection is \( \{a, b\} \) and \( X = p \cap \Omega' \), then we will denote \( p \) by \( abX \) or \( ab \) if \( X = \emptyset \). A typical line of \( \mathcal{T}(\Omega, \Omega') \) will then consist of the points \( abX, acY \) and \( bc(X \triangle Y) \), where \( \triangle \) denotes the symmetric difference.

**Lemma 4.42.** Let \( (P, L) \) be of the form \( \mathcal{T}(\Omega, \Omega') \). Let \( \sigma \) be a Lie orientation on \( (P, L) \). Then \( \mathcal{L}_F(P, L, \sigma) \) is isomorphic to \( \mathcal{L}_F(P, L, \tau) \), where \( \tau \) is derived from a total order \( < \) on \( \Omega \) by

\[
\tau(\ell) : abX \mapsto acY \mapsto bc(X \triangle Y) \mapsto abX \quad \text{if} \quad a < b < c \quad \text{and} \quad \ell = [abX, acY, bc(X \triangle Y)].
\]

**Proof.** We may assume that \( \Omega = \{1, \ldots, n\} \). If \( n = 2 \), then \( \mathcal{T}(\Omega, \Omega') \) consists of only one point or it is disconnected. We consider the case where \( n = 3 \). Let \( |\Omega'| = k \). We show that the lemma holds for \( n = 3 \) by induction on \( k \).

If \( k = 0 \), then \( (P, L) \) is a single line, so by potentially flipping the sign of an arbitrary point, \( \sigma \) and \( \tau \) agree. Hence we assume that \( k > 0 \) and that \( \sigma \) and \( \tau \) agree on lines not containing \( w \in \Omega' \). During the course of the proof, we will modify \( \sigma \) by flipping the signs of some points containing \( w \). This clearly does not violate the property that \( \sigma \) and \( \tau \) agree on lines not involving \( w \), and it does not modify the isomorphism class of \( \mathcal{L}_F(P, L, \sigma) \) either.

Let \( \omega = \{w\} \) and let \( \Omega^- = \Omega' \setminus \omega \). We first flip the signs of a subset of \( \{12\omega, 13\omega, 23\omega\} \) to make \( \sigma \) and \( \tau \) agree on the four lines involving these three points and \( \{12, 13, 23\} \). We will refer to this action as step 1. Then let \( p = \overline{12\omega} \). For all \( X \subseteq \Omega^- \), consider the lines \( \ell = \overline{12\omega}, \overline{13(X \cup \omega)}, \overline{23X} \) and \( m = \{12\omega, 13X, 23(X \cup \omega)\} \). We flip the signs of \( 13(X \cup \omega) \) and \( 23(X \cup \omega) \), if necessary, to make \( \sigma \) and \( \tau \) agree on \( \ell \) and \( m \), respectively. All actions for \( \ell \) will be referred to as in step 2 and the actions for \( m \) as step 3. Now \( \sigma \) and \( \tau \) agree on all lines containing \( \overline{12\omega} \). Similarly, we flip the sign of \( 12(X \cup \omega) \) if necessary to make \( \sigma \) and \( \tau \) agree on \( \overline{12(X \cup \omega)}, \overline{13\omega}, \overline{23X} \). This will be referred to as step 4.

Now consider a line \( \ell = \overline{13\omega}, \overline{12\omega} \) with again \( X \subseteq \Omega^- \). In the plane containing \( \ell \) and \( \overline{12\omega} \), the three lines other than \( \ell \) have had their orientation already determined as being in agreement with \( \tau \), as can be seen in Figure 4.9: the orientation of \( \{13\omega, 23\} \) has been determined in step 1, the orientation of \( \{12\omega, 13X, 23(X \cup \omega)\} \) was determined in step 3, and the orientation on \( [23, 13X, 12X] \) is known by the induction hypothesis. If three of the four orientations of lines in a dual affine plane are given, then by Lemma 4.9, only one orientation for the remaining line leads to a Lie algebra. Since both \( \sigma \) and \( \tau \) lead to a Lie algebra, they must be equal on \( \ell \).

We propose to reapply the reasoning of the previous paragraph to different vertices: in each vertex above, replace 1 by 2 and vice versa, if they occur (and replace “step 3” by “step 4”). This leads to different orientations for the lines, but still three of the four lines would be given the same orientation by \( \sigma \) and \( \tau \), and hence so would the fourth. Thus we still obtain the conclusion that \( \ell = \overline{23\omega}, \overline{12X}, \overline{13(X \cup \omega)} \) is similarly oriented by \( \sigma \) and \( \tau \). If instead we would switch the numbers 2 and 3, we would get again the same conclusion (now using step 4 instead of step 3). So \( \sigma \) and \( \tau \) agree on all lines connecting \( 13\omega \) or \( 23\omega \) to a point of the form \( 12X \), and lines connecting \( 23\omega \) to \( 13X \), where \( w \not\in X \). Since lines connecting \( 13\omega \) to \( 23X \) were already taken care of in step 4,
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and lines containing $\overline{12}\omega$ in steps 2 and 3, we now have agreement between $\sigma$ and $\tau$ on all lines containing $12\omega$, $13\omega$ or $23\omega$.

Now let us consider an arbitrary undecided line $\ell$. Two of its points will contain $w$; say $\ell = \{12(X \cup \omega), 13(Y \cup \omega), 23(X \setminus Y)\}$, with $X$ and $Y$ contained in $\Omega^-$. Then $12(X \cup \omega)$ and $13(Y \cup \omega)$ are collinear to $23\omega$. Consider the plane containing $\ell$ and $23\omega$: two of its lines contain $23\omega$ (so $\sigma$ and $\tau$ agree on these two) and for another line, the orientation is decided by the induction hypothesis (so $\sigma$ and $\tau$ agree on that line as well). Hence $\sigma$ and $\tau$ agree on the orientation of $\ell$. This solves the case for $n = 3$.

We will proceed by induction on $n$. So let $n > 3$ and assume that $\sigma$ is equal to $\tau$ on lines where none of the points contain $n \in \Omega$. This time, we will flip the sign of points containing $n$.

Let $p = \overline{1n}$. All lines $\ell$ containing $p$ have point set $\{1aX, p, anX\}$ for some $a$ with $1 < a < n$ and some $X \subseteq \Omega'$. For all these lines, flip the sign of $anX$ if necessary to make $\sigma$ equal to $\tau$ on this line. Let $m$ be an arbitrary line where the orientation has not yet been decided. Then $m$ involves points of the form $anX$; say $m = \{anX, bnY, ab(X \setminus Y)\}$ for some $a < b < n$ and $X, Y \subseteq \Omega'$.

- If $a > 1$, three of the lines in the plane containing $m$ and $p$ have the same orientation under $\sigma$ and $\tau$: two of them by our potentially flipping the sign of points and one by the induction hypothesis. The plane can be seen in Figure 4.11. The last line of the plane is $m$, and since $\sigma$ is a Lie orientation, it is necessarily equal to $\tau(m)$ on $m$.
- If $a = 1$, then we may assume that $X$ is nonempty, otherwise $p \in m$ and the orientation of $m$ was already equal under $\sigma$ and $\tau$ after potentially flipping the sign of $bnY$. So let $X$ be nonempty. We are considering $m = \{1nX, bnY, 1b(X \setminus Y)\}$. We may need to flip the sign of $1nX$ to make $\sigma(m)$ equal to $\tau(m)$ in this case. But this could introduce ambiguities if one line of this form requires the sign of $1nX$ to be flipped, whereas another line requires that the sign not be flipped. So we need to show that such a situation does not occur.

  We consider two arbitrary neighbours of $1nX$, say $1bY$ and $1cZ$ with $b \leq c < n$. If $b < c$, the plane containing these three is as in Figure 4.12. Since $\sigma$ is a Lie orientation, the orientations on the lines $\{1bY, 1nX, bn(X \setminus Y)\}$ and
Figure 4.11: The plane containing $\overline{anX}$, $\overline{bnY}$ and $\overline{1n}$.

Figure 4.12: The plane containing $\overline{1nX}$, $\overline{1bY}$ and $\overline{1cZ}$.

\[
\{1cZ, \overline{1nX}, \overline{cn(X \triangle Z)}\} \text{ are either both such that the sign of } \overline{1nX} \text{ should be flipped, or both such that it should not be flipped. So two lines containing } \overline{1nX} \text{ where the third point in } \Omega \text{ is different, agree on whether the sign of } \overline{1nX} \text{ should be flipped. Hence if } b = c, \text{ then we choose } d \text{ such that } 1 < d < n \text{ and } b \neq d \text{ (this is possible because } n > 3). \text{ Now the lines through } \overline{1bY} \text{ and } \overline{1cZ} \text{ both agree with the line through } \overline{1d}, \text{ so they agree with one another. Thus all lines agree on whether the sign of } \overline{1nX} \text{ should be flipped.} \]

\[\square\]

4.9.2. Constructing an $E_2$-group from an oriented partial linear space. In Section 4.5.2, we have seen a way to construct a Lie orientation from an $E_2$-group. In the resulting Lie orientation, we will find sufficient structure to recover the $E_2$-group. It might be possible to extend this to recovering an $E_2$-group from every Lie oriented partial linear space of the form $O(V, Q)$. This would ensure that all Lie orientations come from an $E_2$-group, which by Theorem 4.16 implies that the $E_2$-groups are isomorphic, whence by Proposition 4.21, the Kaplansky algebras are isomorphic. Because of Corollary 4.26 and Lemma 4.42, this would prove Conjecture 4.41 for the three families of partial linear spaces all of whose planes are dual affine.

The following definition shows how to recover the $E_2$-group from the Lie orientation.

**Definition 4.43.** Let $(P, L, \sigma)$ be a Lie oriented partial linear space where all planes are dual affine. Let

\[
E(P, L, \sigma) = \left\{ z, \tilde{p} \mid p \in P, \begin{array}{l}
z^2 = [\tilde{p}, z] = \tilde{p}^2 z = 1, \\
[\tilde{p}, q] = 1 \quad \text{if } p \perp q, \\
\tilde{p}q = \begin{cases} 
\tilde{q} & \text{if } \ell = [p, q, r] \in L \text{ and } p^{\sigma(\ell)} = q, \\
z\tilde{p} & \text{if } \ell = [p, q, r] \in L \text{ and } q^{\sigma(\ell)} = p.
\end{cases}
\end{array}\right\}_{GP}
\]

We call $E(P, L, \sigma)$ the $E_2$-group of $(P, L, \sigma)$.

**Lemma 4.44.** If $E$ is an $E_2$-group, then the $E_2$-group of the oriented partial linear space of $E$ is $E$.

**Proof.** Let $(P, L, \sigma)$ be the oriented partial linear space of $E$ and let $F = E(P, L, \sigma)$. We first show that $F$ is an $E_2$-group. The scalar subgroup of $F$ will be $Z_F = \langle z \rangle_{GP}$. Since $z$ commutes with all other generators, $Z_F$ is normal.
4.9. 4.4.3. Code loops. In this section, we will present a construction for Lie oriented partial linear spaces containing a Fano plane, which has parallels to the $E_2$-group for the Lie oriented partial linear spaces where all planes are dual affine. At present, the construction does not give us any new results; however, if it would be possible to use $E_2$-groups to prove Conjecture 4.41 for Lie oriented partial linear spaces all of whose planes are dual affine, then such a proof might be extensible to a unified proof of the conjecture for all Lie oriented partial linear spaces using these code loops.

The equivalents of $E_2$-groups in this construction are so-called code loops. We start the discussion by recalling basic definitions from loop theory. Then we continue with the definition of a code loop by Griess [Gri86], and state some results from that paper and from Chein and Goodaire [CG90]. Finally, we demonstrate the construction.

Definition 4.45. Recall that a loop is a set $M$ with a binary operation, denoted by $\cdot$ or by juxtaposition, such that there exists a unit element $1$ and the equation $ab = c$ has a unique solution whenever two of the three variables are fixed. (By convention, juxtaposition has higher priority or “binding power” than $\cdot$)
A loop is said to be Moufang or a Moufang loop if any of these three equivalent identities hold for all \(a, b, c \in M\):

\[
ab \cdot ca = (a \cdot bc)a; \\
(ab \cdot c)b = a(b \cdot cb); \\
a(b \cdot ac) = (ab \cdot a)c.
\]

For any loop elements \(a, b, c\), one defines the commutator \([a, b]\) and the associator \([a, b, c]\) by the identities

\[
ab = ba \cdot [a, b] \\
ab \cdot c = (a \cdot bc)[a, b, c].
\]

The commutator subloop \(M'\) (resp., associator subloop \(A(M)\)) of a loop \(M\) is its subloop generated by the commutators of all pairs (resp., associators of all triples) of its elements. For any loop \(M\), let \(M^2 = \{a^2 \mid a \in M\}\) be the set of its squares.

Moufang loops are di-associative; that is, any two elements generate an associative subloop. Furthermore, Moufang loops have the inverse property: for all Moufang loop elements \(a\), the left and right inverse of \(a\) are equal.

**Definition 4.46.** We will identify vectors in \(\mathbb{F}_2^k\) with the set of coordinates at which they are nonzero. The cardinality function \(|\cdot| : \mathbb{F}_2^k \to \mathbb{N}_0\) is then equal to the weight function. We write \(|u \cap v|\) to denote the number of coordinates at which \(u\) and \(v\) are both nonzero. A doubly even code or doubly even subspace is a subspace \(V\) of \(\mathbb{F}_2^k\) such that every \(v \in V\) has \(|v|\) divisible by four, and for all \(u, v \in V\), the number \(|u \cap v|\) is even.

Let \(\phi : V \times V \to \mathbb{F}_2\) satisfy, for all \(u, v, w \in V\),

\[
\phi(u, u) \equiv \frac{|u|}{4} \pmod{2}; \\
\phi(u, v) + \phi(v, u) \equiv \frac{|u \cap v|}{2} \pmod{2}; \\
\phi(u, v) + \phi(u + v, w) + \phi(v, u + v) + \phi(u, v + w) \equiv |u \cap v \cap w| \pmod{2}.
\]

Then \(\phi\) is called a factor set on \(V\). Two factor sets are equivalent if their difference, evaluated at \((u, v)\), is expressible as \(\alpha(u) + \alpha(v) + \alpha(u + v)\) for some function \(\alpha : V \to \mathbb{F}_2\) with \(\alpha(0) = 0\).

Let \(V\) be a doubly even code and \(\phi\) a factor set on \(V\). Then the set \(\mathbb{F}_2 \times V\) with product

\[
(a, u)(b, v) = (a + b + \phi(u, v), u + v)
\]

is called a code loop.

The following two theorems are from Griess [Gri86].

**Theorem 4.47.** For any doubly even code \(V\), there exist factor sets on \(V\), and any two of them are equivalent. Equivalent factor sets on \(V\) define isomorphic loops on \(V\).

**Theorem 4.48.** Code loops are Moufang loops.

In [CG90], Chein and Goodaire prove Lemma 4.49 and Theorem 4.50.

**Lemma 4.49.** If a Moufang loop \(M\) satisfies \(M^2 = \{1, z\}\), then its associator subloop \(A(M)\) and commutator subloop \(M'\) are both contained in \(\{1, z\}\).

**Theorem 4.50.** A finite loop \(M\) of cardinality at least 2 is isomorphic to a code loop if and only if \(M\) is a Moufang loop with \(|M^2| \leq 2\).
We give a sketch of the proof of this theorem, because we will use certain aspects of it to prove Lemma 4.52.

Proof sketch of Theorem 4.50. The “only if”-part of the proof follows from Theorem 4.48 and from the fact that every code loop element \((a, u)\) satisfies \((a, u)^2 = (\phi(u, u), 0)\).

The converse is proven as follows. The case \(|M^2| = 1\) is solved by deriving that this only occurs for elementary Abelian 2-groups, which are code loops if their size is at least 2. So we may assume \(M^2 = \{1, z\}\). Let \(x_1\) be an element with \(x_1^2 = z\), then there is a minimal set of generators containing \(x_1\), say \(\{x_1, \ldots, x_{n+1}\}\). The proof proceeds by induction on \(n\), the base case of \(n = 0\) being solved by the code \(\langle (1, 1, 1) \rangle_F\).

So let \(n > 0\) and \(H = \langle x_1, \ldots, x_n \rangle_{\text{loop}}\). By the induction hypothesis, there is a doubly even code \(U\) (in, say, \(F'_2\)), such that \(H\) is isomorphic to a code loop with code \(U\). We proceed to construct a doubly even code \(V\) such that \(M\) is isomorphic to a code loop with code \(V\).

Let \(r = s + 4(\binom{n}{2}) + 8n + 7\); the code \(V\) is constructed in \(F'_2\), as follows. View each vector \(v \in F'_2^r\) as consisting of consecutive blocks \(B_0, \ldots, B_{\binom{n}{2} + n + 1}\) of \(F'_2\)-elements, block \(B_0\) being of length \(s\), the blocks \(B_1, \ldots, B_{\binom{n}{2}}\) of length four, the blocks \(B_{\binom{n}{2} + 1}, \ldots, B_{\binom{n}{2} + n}\) of length eight and the final block \(B_{\binom{n}{2} + n + 1}\) being of length seven. We define the \(n + 1\) spanning vectors of \(V\) as follows. Let \(\pi\) be a bijection between \(\{1, \ldots, \binom{n}{2}\}\) and the unordered 2-subsets of \([1, \ldots, n]\). For \(1 \leq i \leq n\), let \(u_i \in U\) be the codeword such that \((0, u_i)\) corresponds to \(x_i \in H\). We define \(v_i \in F'_2\) as follows.

- \(B_0\) is the zero vector.
- \(B_0(i, j) = (1, 1, 1, 1)\) for all \(j \neq i\).
- \(B_{\binom{n}{2} + 1}\) is a block of four 1’s followed by four 0’s if \(n\) is even and a block of eight 1’s if \(n\) is odd.
- The other blocks \(B_j\) consist of 0’s.

\(v_{n+1}\) has the following blocks:

- \(B_0\) is the zero vector.
- \(B_0(i, j) = (1, 0, 0, 0)\) if the associator \([x_i, x_j, x_{n+1}] = z\); it is \((0, 0, 0, 0)\) if the associator is 1. By Lemma 4.49, there are no other possibilities.
- To determine \(B_{\binom{n}{2} + 1}\), let \(m_i\) be the number of common 1’s between \(v_i\) and \(v_{n+1}\) in blocks \(B_0, \ldots, B_{\binom{n}{2}}\), let \(q_i\) be defined by the commutator \([x_i, x_{n+1}]\) being equal to \(e^q\), and let \(p_i\) be the least nonnegative solution of the congruence \(m_i + p_i \equiv 2q_i \pmod{4}\). Now \(B_{\binom{n}{2} + j}\) is such that its first \(p_i\) entries are 1’s and the rest 0’s.
- \(B_{\binom{n}{2} + n + 1}\) consists of \(p\) 1’s followed by \(0\)’s, where \(p\) is such that \(|v_{n+1}| \equiv 4 \pmod{8}\) if \(x_{n+1}^2 = e\) and is a multiple of \(8\) otherwise.

The proof then goes on to show that \(M\) is isomorphic to the code loop with code \(V\). □

Corollary 4.51. \(E_2\)-groups are code loops.

Proof. Every group is a Moufang loop. Since an \(E_2\)-group has a single nontrivial square, it is a code loop. □
Consider the proof of Theorem 4.50. We are going to construct the code $V$ for both $M_1$ and $M_2$, as prescribed by the proof. Clearly the vectors $v_i$ with $i \leq n$ are equal. For $v_{n+1}$, we look at the different groups of blocks separately.

- Block $B_0$ is clearly the same in both codes.
- Block $B_{n(i,j)}$ is equal in both codes because of Eq. (4.3).
- For $i \leq n$, block $B_{(j)}$ is equal in both codes because all blocks $B_j$ with $0 \leq j \leq \binom{n}{2}$ are the same and because of Eq. (4.4).
- Block $B_{(j)+1}$ is the same in both codes because all other blocks are the same.

Since $M_1$ and $M_2$ are code loops with the same code, Theorem 4.47 shows that they are isomorphic. 

**Definition 4.53.** Let $n \geq 3$. Let $V = \mathbb{F}^n_2$ and let $U = \mathbb{F}^{n-3}_2$, and let $\sigma$ be a Lie orientation on $PG(V) - PG(U)$. Let $M(PG(V) - PG(U), \sigma)$ be the set $\mathbb{F}^n_2 \times V$ with a binary operation $\cdot$ given by $(\alpha, v) \cdot (\beta, v') = (\alpha + \beta + f(v, v'), v + v')$, where $f$ is defined as follows:

1. If $u \in U$, then $f(u, v) = f(v, u) = 0$ for all $v$.
2. If $v, v' \in V \setminus U$ and $v + v' \in U$, then $f(v, v') = 1$.
3. If $v, v', v + v' \in V \setminus U$, then
   \[
   f(v, v') = \begin{cases} 
   0, & \text{if } \sigma((v, v')_F) : v \mapsto v', \\
   1, & \text{if } \sigma((v, v')_F) : v' \mapsto v.
   \end{cases}
   \]

We call $M(PG(V) - PG(U), \sigma)$ the loop of $(PG(V) - PG(U), \sigma)$.

The loop $M$ of any oriented partial linear space of the form $PG(V) - PG(U)$ can easily be seen to satisfy $|M|^2 = 2$. So in order to see if $M$ is a code loop, we need only check the Moufang identity.

**Lemma 4.54.** The loop of the Lie oriented Fano plane is a code loop.

**Proof.** This can easily be checked by hand. It is a well-known loop; for example, it is isomorphic to the loop consisting of plus and minus the unit octonions. It is the code loop of the extended Hamming $(8,4)$-code. 

**Lemma 4.55.** If $M(PG(\mathbb{F}^n_2) - PG(\mathbb{F}^{n-3}_2), \sigma)$ is a code loop, then $M(2(PG(\mathbb{F}^n_2) - PG(\mathbb{F}^{n-3}_2), \sigma))$ is also a code loop.

**Proof.** In order to verify the Moufang identity on a triple of elements $(\alpha_1, v_1), (\alpha_2, v_2)$ and $(\alpha_3, v_3)$, one needs to check the equality of

\[
(\alpha_1, v_1)(\alpha_2, v_2) \cdot (\alpha_3, v_3)(\alpha_1, v_1) = (f(v_1, v_2) + \alpha_1 + \alpha_2, v_1 + v_2)(f(v_3, v_1) + \alpha_1 + \alpha_3, v_1 + v_3) = (f(v_1, v_2) + f(v_3, v_1) + f(v_1 + v_2, v_1 + v_3) + \alpha_2 + \alpha_3, v_2 + v_3)
\]

and

\[
((\alpha_1, v_1) \cdot (\alpha_2, v_2)(\alpha_3, v_3))(\alpha_1, v_1) = (\alpha_1, v_1)(f(v_2, v_3) + \alpha_2 + \alpha_3, v_2 + v_3) \cdot (\alpha_1, v_1) = (f(v_1, v_2 + v_3) + f(v_2, v_3) + \alpha_1 + \alpha_3, v_1 + v_2 + v_3)(\alpha_1, v_1) = (f(v_1 + v_2 + v_3, v_1) + f(v_1, v_2 + v_3) + f(v_2, v_3) + \alpha_2 + \alpha_3, v_2 + v_3).
\]
Hence we need to show that
\[ f(v_1, v_2) + f(v_3, v_1) + f(v_1 + v_2, v_1 + v_3) = f(v_1 + v_2 + v_3, v_1) + f(v_1, v_2 + v_3) + f(v_2, v_3). \]

Let \( V = \mathbb{F}_2^{n+1} \) and \( U = \mathbb{F}_2^{n-2} \subset V \). Choose \( V' \subset V \) of dimension \( n \) such that \( U' := V' \cap U \) has dimension \( n - 3 \). Let \( u \in U \setminus U' \). Now \( PG(V) - PG(U) = 2(PG(V') - PG(U')) \), and the extra copy of a point \( v \) in \( PG(V') - PG(U') \) that is added to obtain \( PG(V) - PG(U) \) can be taken to be \( u + v \). We saw this earlier in the proof of Lemma 4.29. A diagram of the different spaces involved is shown in Figure 4.13.

Let \( \pi \) be the projection from \( V \) to \( V' \) with \( u \) in its kernel. A copy of a line \( \ell = \{v, w, v + w\} \subset V' \) that is added in the process of doubling will typically be \( \{v + u, w + u, v + w\} =: \ell' \). By the construction of doubling, the orientation of \( \ell' \) is such that for all \( x \in \ell' \), we have \( \pi(x^{\ell}(v)) = \pi(x)^{\ell'(v)} \).

Let \( f^- \) be the map \( f \) from Definition 4.53 for \( PG(V') - PG(U') \) and let \( f^+ \) be the map for \( PG(V) - PG(U) \). It is clear that if \( v, w \in V' \), we have \( f^-(v, w) = f^+(v, w) \). We will show that \( f^+(u + v, u + w) = f^+(v, v + w) = f^+(v, u + w) = f^+(v, w) \). Then, since we know by assumption that the Moufang identity holds for \( PG(V') - PG(U') \), we see that it holds in \( PG(V) - PG(U) \) as well.

Clearly adding \( u \) to a vector \( v \) does not change whether \( v \) is in \( U \). Hence the number of the rule in Definition 4.53 determining the value of \( f^+ \) is the same for \( f^+(v, w) \) and for its variants where \( u \) is added to one or both arguments. The only possibility for disagreement is then if the rule is rule number 3, because that is the only rule where both 0 and 1 are possible outcomes. But since \( \pi(u + v) = v \) and \( \pi(u + w) = w \), and since \( \pi \) commutes with the orientation, these disagreements do not occur.

We have seen that the Moufang identity holds, which is sufficient to show that \( M \) is a code loop.

**Corollary 4.56.** For all \( n \geq 3 \), there is a Lie orientation \( \sigma \) for \( PG(\mathbb{F}_2^n) - PG(\mathbb{F}_2^{n-3}) \) such that \( M(PG(\mathbb{F}_2^n) - PG(\mathbb{F}_2^{n-3}), \sigma) \) is a code loop.
Note, however, that the above construction does not necessarily work after flipping the signs of a set of points.

4.10. A generalization

In this section, we study a generalization of the construction of Definition 4.1 by allowing the factor $±1$ in the multiplication to become an arbitrary field element. We will see that the partial linear spaces giving rise to Lie algebras include the same four families found in Section 4.3. We saw in Lemma 4.28 that some of the partial linear spaces in these families cannot be Lie oriented. Theorem 4.64 shows that the construction of this section does give rise to a Lie algebra for all partial linear spaces in these four families. Lemmas 4.59 and 4.60 show that some other partial linear spaces give rise to a Lie algebra as well.

**Definition 4.57.** Let $(P, L)$ be a partial linear space with lines of length 3. Let $\tau : P \times P \to F$ be a map satisfying for all lines $[p, q, r]$ the property that

\[
\tau(p, q) = \tau(q, r) = \tau(r, p) = -\tau(q, p) = -\tau(p, r) = -\tau(r, q),
\]

and with $\tau(p, q) = 0$ if $p \perp q$. We call $(P, L, \tau)$ a *generalized scaled partial linear space* and $\tau$ a *generalized scaling*. If $\tau(p, q) = 0$ only if $p \perp q$, we call $\tau$ a *scaling* and $(P, L, \tau)$ a *scaled partial linear space*. One can obtain a scaled partial linear space from a generalized scaled partial linear space by omitting the lines where $\tau = 0$; this will be called the scaled partial linear space associated to that generalized scaled partial linear space.

Let $(P, L, \tau)$ be a scaled partial linear space. Let $M$ be the algebra over $F$, linearly spanned by the formal basis $P$, with the bilinear multiplication denoted by brackets and determined by

\[
[p, q] = \begin{cases} 
0, & \text{if } \tau(p, q) = 0, \\
\tau(p, q), & \text{if } [p, q, r] \in L.
\end{cases}
\]

We call $M$ the *scaled Kaplansky algebra* of the scaled partial linear space $(P, L, \tau)$ over $F$, and denote it by $M(P, L, \tau)$. If $M(P, L, \tau)$ is a Lie algebra, then $(P, L, \tau)$ is called a *Lie scaled partial linear space* and $\tau$ is called a *Lie scaling* on $(P, L)$.

Note that Lie scalings are scalings, not just generalized scalings. Furthermore, in order to avoid confusion, we will from now on refer to the Kaplansky algebras defined in Definition 4.1 as *unscaled Kaplansky algebras*.

We will say that a generalized scaled partial linear space $(P, L, \tau)$ is given by a diagram such as Figure 4.15 if the following two conditions hold:

- the point and line sets of the diagram are $P$ and $L$, respectively, and
- every line $[p, q, r] \in L$ is oriented by an arrow in the diagram, which orientation we may assume to be alphabetical, and the label along that line is $\tau(p, q)$.

Furthermore, if we plan to define the scaled Kaplansky algebra of the associated scaled partial linear space over a field of characteristic 2, we may omit the arrows in the diagram.

Clearly if $\tau(p, q) \in \{±1\}$ for all $p, q \in P$ with $p \perp q$, then we can define an orientation $\sigma$ for all lines $\ell = [p, q, r]$ as $\sigma(\ell) = [p, q, r]$ whenever $\tau(p, q) = 1$; we then obtain that $M(P, L, \tau) = L_{\sigma}(P, L, \sigma)$. Reversing this procedure, we can obtain a Lie scaled partial linear space from every Lie oriented partial linear space. Thus, we find Lie scalings on

---

1The reason for this convention is that it allows us to meaningfully discuss the class of partial linear spaces that can be given a Lie scaling: if we would allow generalized scalings, then one could arbitrarily add lines with associated $\tau$-value zero to such a partial linear space. On the other hand, since generalized scalings make proofs such as that of Lemma 4.60 more elegant, we do not omit them completely.
the four families of partial linear spaces on which we have found Lie orientations in sections 4.5 and 4.6.

Let \( \lambda \in \mathbb{F}^* \) and for all \( p, q \in P \), let \( \tau'(p, q) = \lambda \tau(p, q) \), then \( M(P, L, \tau) \) and \( M(P, L, \tau') \) are isomorphic: the map sending each \( p \in P \) to \( \lambda p \) provides an isomorphism. Hence every scaled Kaplansky algebra of the partial linear space consisting of a single line of length three is isomorphic; in particular, they are all isomorphic to the scaled Kaplansky algebra where \( \tau(p, q) = \pm 1 \). This is the unscaled Kaplansky algebra. If more than one line is involved, however, there need not be an isomorphic unscaled Kaplansky algebra.

The construction of flipping the sign of a point \( p \) also works for the scaled Kaplansky algebra: it results in the sign of \( \tau(p, q) \) being changed for all \( q \), and thus also the sign of e.g. \( \tau(q, r) \) whenever \( \{ p, q, r \} \in L \). The construction of doubling also has a natural analogue, which we explore below. Let \( (P, L, \tau) \) be a scaled partial linear space. We define a scaling on \( 2(P, L) \) as follows. Consider the projection \( \pi: 2P \to P \), fixing each element of \( P \) and sending each copy \( p' \) of \( p \) to its original, then we define \( \tau(p, q) = \tau(p', q) \) and \( 2(P, L, \tau) = (2(P, L, \tau) \circ \pi). \) The proof of the following lemma is then essentially the same as that of Lemma 4.10.

**Lemma 4.58.** If \( (P, L, \tau) \) is a Lie scaled partial linear space, then so is \( 2(P, L, \tau). \)

We will investigate the planes that can be given a Lie scaling in Lemma 4.59. It will give rise to one plane in addition to the planes of Lemma 4.3. Subsequently, we analyze the resulting Lie algebras.

**Lemma 4.59.** If \( (P, L, \tau) \) is a Lie scaled partial linear space that is a plane, then \( (P, L) \) is either the dual affine plane of order 2, or it is the Fano plane with one line omitted, or it is the Fano plane.

**Proof.** Let \( \{ a, b, c \} \) and \( \{ a, d, e \} \) be two intersecting lines. Examine the Jacobi identity on \( b, c \) and \( d \):

\[
[[b, c], d] + [[c, d], b] + [[d, b], c] = \tau(b, c)\tau(a, d)e + [[c, d], b] + [[d, b], c].
\]

Since \( [[c, d], b] \) and \( [[d, b], c] \) are zero or a multiple of a standard basis element, they are a multiple of \( e \) if nonzero; and at least one of them is a nonzero multiple of \( e \). So we may assume that \( [[c, d], b] \) is a nonzero multiple of \( e \), and we have the full dual affine plane structure of Figure 4.1(a). We see that every pair of intersecting lines is contained in a dual affine plane of order 2 (though this inclusion is not necessarily as a subspace).

If there are more lines in this plane, then there is a line connecting \( c, e \) and a new point, say, \( g \), and/or a line connecting \( b \) and \( d \) to \( g \), and/or a line connecting \( a \) and \( f \) to \( g \). Since the sets \( \{ c, e, g \} \), \( \{ b, d \} \) and \( \{ a, f \} \) are in the same orbit under automorphisms of the dual affine plane of order 2, we may assume that there is a line \( \{ c, e, g \} \). The lines \( \{ a, b, c \} \) and \( \{ c, e, g \} \) are contained in a dual affine plane of order 2, so \( g \) is collinear to \( a \) and/or \( b \); furthermore, the line connecting \( g \) to \( a \) or \( b \) intersects the line connecting \( e \) to the other of those two points. So there is a line \( \{ a, f, g \} \) or a line \( \{ b, d, g \} \). The resulting partial linear spaces are isomorphic, so we assume that it is \( \{ b, d, g \} \). This is the Fano plane with one line missing. It can be seen in Figure 4.14.

At most one line can be added to the resulting partial linear space while remaining in the space generated by \( \{ a, b, c \} \) and \( \{ a, d, e \} \), which results in a Fano plane. 

**Lemma 4.60.** Let \( (P, L, \tau) \) be a Lie scaled partial linear space that is a plane. Then \( M(P, L, \tau) \) is isomorphic to an unscaled Kaplansky algebra, or \( (P, L, \tau) \) is given by Figure 4.16 and \( \text{char } \mathbb{F} = 2 \).

**Proof.** We have seen in Lemma 4.59 that \( (P, L) \) is one of three planes of that lemma, all of which can be embedded into the Fano plane by potentially adding a point and/or
a number of lines. If \((P, L)\) is the dual affine plane of order 2, we add a disconnected point to enable us to consider the different possible planes using the same diagram; we now have a plane on 7 points regardless of the plane generated by the two lines. We will detect the addition of the point later, if it happened, and will remove the point again. So there is a generalized scaling on the Fano plane, such that the associated scaled partial linear space is \((P, L, \tau)\), potentially after removing one point. We may assume that this generalized scaled partial linear space is given by Figure 4.15.

The Jacobi identity now translates to this set of equations:

\[
\begin{align*}
\alpha + \beta \zeta + \gamma e &= 0, \\
\beta + \alpha \zeta + \gamma \delta &= 0, \\
\gamma + \alpha e + \beta \delta &= 0, \\
\delta + \beta \gamma + e \zeta &= 0, \\
\epsilon + \alpha \gamma + \delta \zeta &= 0, \\
\zeta + a \beta + \delta e &= 0, \\
\alpha \delta + \beta e + \gamma \zeta &= 0.
\end{align*}
\]

Note that this is symmetric in all variables. If we suppose that one of the parameters is zero, say \(\zeta\), then we find \(\gamma^2 = 1\); after possibly flipping the sign of the topmost point to obtain \(\gamma = -1\), this leads to the following set of equations:

\[
\zeta = \gamma + 1 = e - \alpha = \delta - \beta = 2a\beta = a^2 + \beta^2 - 1 = 0.
\]

If \(\text{char } F \neq 2\), then one of \(\alpha\) and \(\beta\) is zero. Since the situation is symmetric, suppose \(\alpha = 0\). We have found the dual affine plane of order 2, with all nonzero values of \(\tau\) equal to plus or minus one. Hence \(M(P, L, \tau)\) is isomorphic to an unscaled Kaplansky algebra and we are done. If, on the other hand, we work in a field of characteristic 2, then the equation \(2a\beta = 0\) is vacuously true. We see that there is one degree of freedom; we can choose, say, \(\alpha\), freely; this determines \(\beta\) by \(\alpha^2 + \beta^2 = 1\) (remember that squaring is a field automorphism if \(\text{char } F = 2\)). If \(F = \mathbb{F}_2\), then either \(\alpha\) or \(\beta\) is zero and we are in the same situation as before, but for extensions of \(\mathbb{F}_2\) they can both be nonzero. Then \(e = \alpha\) and \(\delta = \beta\) show that we have only one line where the values of \(\tau\) are zero, so \((P, L)\) is the
Fano plane with one line missing. We will see a family of scaled partial linear spaces later where this situation appears as a special case.

We have exhausted the possibilities where one of the parameters is zero, so let us assume that all parameters are nonzero. Suppose that one of the parameters is plus or minus one; say $\zeta^2 = 1$. Then we may flip the sign of the topmost point in Figure 4.15 to obtain $\zeta = 1$. Eqs. (4.5a) and (4.5b) imply that

$$\alpha(1 - \zeta^2) = \gamma(\zeta \delta - \epsilon), \quad (4.6a)$$
$$\beta(1 - \zeta^2) = \gamma(\zeta \epsilon - \delta); \quad (4.6b)$$

thus $\delta = \epsilon$. Hence, Eqs. (4.5d) and (4.5e) show that $\alpha = \beta$. By Eq. (4.5c), we may substitute $-2\alpha \delta$ for $\gamma$. The remaining equations then say that

$$\alpha(1 - \delta^2) = 0,$$
$$\delta(1 - \alpha^2) = 0,$$
$$1 + \alpha^2 + \delta^2 = 0.$$  

This can only happen in characteristic 3. Then all parameters are plus or minus one, so we obtain an algebra isomorphic to one resulting from the full Fano plane and characteristic 3 in Lemma 4.3. We see that $M(P, L, \tau)$ is isomorphic to an unscaled Kaplansky algebra, so this case is finished.

This leaves the case where none of the parameters is plus or minus one, or zero. Then Eqs. (4.6) and their equivalents for $\delta$ and $\epsilon$ give us that

$$\alpha = \gamma(\zeta \delta - \epsilon) \quad \text{and} \quad \delta = \gamma(\zeta \epsilon - \zeta \alpha), \quad (4.7a)$$
$$\beta = \gamma(\zeta \epsilon - \delta) \quad \text{and} \quad \epsilon = \gamma(\zeta \delta - \zeta \alpha). \quad (4.7b)$$

Substituting the expressions for $\alpha$ and $\beta$ into those for $\delta$ and $\epsilon$ and vice versa, we obtain

$$\alpha = \frac{\gamma^2(\alpha(1 + \zeta^2) - 2\zeta \beta)}{(1 - \zeta^2)^2}, \quad \delta = \frac{\gamma^2(\delta(1 + \zeta^2) - 2\zeta \epsilon)}{(1 - \zeta^2)^2}, \quad (4.8a)$$
$$\beta = \frac{\gamma^2(\beta(1 + \zeta^2) - 2\zeta \alpha)}{(1 - \zeta^2)^2}, \quad \epsilon = \frac{\gamma^2(\epsilon(1 + \zeta^2) - 2\zeta \delta)}{(1 - \zeta^2)^2}. \quad (4.8b)$$

Expanding further, we find that either the characteristic is 2 and $\gamma^2 = (1 - \zeta^2)^2/(1 + \zeta^2)$, or $2\zeta \gamma^2 = \pm((1 - \zeta^2)^2 - \gamma^2(1 + \zeta^2))$.

Suppose we are in the last case. Possibly flipping the sign of one of the two points on the circle not on the bottom line, in order to make the $\pm$ in the equation above into a $+$ by adapting $\zeta$, we obtain, after some rearranging, $\gamma^2 = (\zeta - 1)^2$. If necessary, we flip the sign of the topmost point for $\gamma$ to obtain $\gamma = \zeta - 1$. Substituting this into Eqs. (4.8), we find $\beta = -\alpha$ and $\epsilon = -\delta$; then Eq. (4.5a) shows that $\delta = -\alpha$. With all of these substitutions, Eqs. (4.5c), (4.5f) and (4.5g) look like this:

$$\zeta - 1 + 2\alpha^2 = 0,$$
$$\zeta - 2\alpha^2 = 0,$$
$$\zeta(\zeta - 1) - 2\alpha^2 = 0.$$  

Thus $\zeta = \zeta(\zeta - 1) = 1 - \zeta$. This cannot be satisfied.
Hence we may assume that we are in the only case left, where the characteristic is 2 and \( \gamma^2 = (1 - \zeta^2)^2/(1 + \zeta^2) = (1 + \zeta^2)^2/(1 + \zeta^2) = 1 + \zeta^2 \). Then \( \gamma = \sqrt{1 + \zeta^2} = 1 + \zeta \). From Eqs. (4.7), we find \( a^2 + \beta^2 + \zeta + 1 = 0 \), so

\[
\zeta = 1 + \alpha + \beta, \quad \gamma = \alpha + \beta.
\]

Together with

\[
\delta = \frac{\alpha(1 + \alpha + \beta) + \beta}{\alpha + \beta} = 1 + \alpha, \\
\epsilon = \frac{\beta(1 + \alpha + \beta) + \alpha}{\alpha + \beta} = 1 + \beta,
\]

this suffices for satisfying Eqs. (4.5). This is the situation of Figure 4.16.

Both the situation earlier where exactly one of the parameters was zero, and the dual affine plane of order 2 (when it occurs with \( \text{char } F = 2 \)) can be obtained as special cases of this situation. We obtain the situation where one parameter is zero if the set \( \{1, \alpha, \beta\} \) spans a vector space of dimension 2 over \( F_2 \), the field with two elements; we obtain the dual affine plane if the dimension is 1. □

**Lemma 4.61.** Let \( \text{char } F = 2 \) but \( |F| > 2 \). Let \( (P, L, \tau) \) be the scaled partial linear space associated to the generalized scaled partial linear space given by Figure 4.16. If \( \alpha \notin F_2 \) or \( \beta \notin F_2 \), then \( M(P, L, \tau) \) is simple.

As indicated in the proof of Lemma 4.60, the conditions of the present lemma imply that at most one of the lines in the figure has a zero parameter attached to it.

**Proof.** Let \( x \) be a nonzero element of an ideal \( I \) (so \( I \neq 0 \)). Write \( x = \sum_{p \in P} \lambda_p p \). First suppose that Figure 4.16 defines a scaling (i.e., that none of the indicated values of \( \tau \) are zero). Let \( y = [p, [p, x]] \) and write \( y = \sum_{p \in P} \mu_p p \). Then for all \( q \in P \), we see that \( \mu_q = -\tau(p, q)^2 \lambda_p p \), so in particular, \( \mu_q = 0 \) if and only if \( p = q \) or \( \lambda_q = 0 \). Hence we can clear the coefficients of the points one by one. We do this as long as there is more than one nonzero coefficient. Then we see that \( I \) contains a point in \( P \). So \( I = M(P, L, \tau) \).

Now suppose that Figure 4.16 does not define a scaling, only a generalized scaling; say the parameter on line \( \ell \) is zero. If a point \( q \) is on \( \ell \), then \( (\text{ad } q)^2 \) clears the coefficients
4.10. A GENERALIZATION

of all points on \( \ell \); otherwise it only clears the coefficients of \( q \). So if \( x \) has a nonzero coefficient at one of the basis elements \( p \notin \ell \), then by the procedure above we find that \( p \in I \) and \( I = M(P, L, \tau) \). On the other hand, if \( x \) is a nonzero linear combination of points in \( \ell \), then left multiplication by a point \( q \) not on \( \ell \) yields a point with nonzero coefficients only at points not on \( \ell \), whence we proceed as before to obtain an element in \( I \cap P \). We see that \( I = M(P, L, \tau) \).

We have not determined the isomorphism type of this simple 7-dimensional Lie algebra over a field of characteristic 2 for all parameter values. Recently, M. Vaughan-Lee [VL06] classified all simple Lie algebras over \( \mathbb{F}_2 \) of dimension at most 9. There are two of dimension 7: one newly discovered by Vaughan-Lee and the Witt algebra \( W(1; 3') \). The last occurs for example if we set \( a = 0 \) and \( \beta = z \), the generator of \( \mathbb{F}_4 \). An isomorphism from \( W(1; 3') \) to \( M \) is then given by the following map, where we have labelled the vertices of Figure 4.16 clockwise from 1 up to 6, starting in the lower left, and the 7 in the middle:

\[
\begin{align*}
&x^6 \partial_1 \quad \mapsto \quad v_2 + v_3 + v_4 + v_7 \\
&x^5 \partial_1 \quad \mapsto \quad v_1 + zv_5 + z^2v_6 \\
&x^4 \partial_1 \quad \mapsto \quad z^2v_2 + zv_3 + v_4 \\
&x^3 \partial_1 \quad \mapsto \quad v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 \\
&x^2 \partial_1 \quad \mapsto \quad v_1 + zv_2 + z^2v_3 + v_4 + zv_5 + z^2v_6 \\
&x \partial_1 \quad \mapsto \quad v_1 + z^2v_2 + zv_3 + v_4 + z^2v_5 + zv_6 \\
&\partial_1 \quad \mapsto \quad v_1 + v_5 + v_6 + v_7.
\end{align*}
\]

It would be interesting to know if the algebra by Vaughan-Lee can also occur as a scaled Kaplansky algebra.

Let \( (P, L, \tau) \) be given by Figure 4.16. The Lie algebra \( M(P, L, \tau) \) can also be obtained from a different construction, which provides insight allowing us to find a Lie scaling for all partial linear spaces of the form \( PG(\mathbb{F}_2) - PG(\mathbb{F}_2^2) \). In order to find this alternative construction, label each point in the Fano plane \( (P, L) \) by the vector in \( \mathbb{F}_2^3 \) corresponding to that projective point. Then \( \tau \) defines an alternating \( \mathbb{F}_2 \)-bilinear map from \( \mathbb{F}_2^3 \) into \( \mathbb{F} \), the codomain of \( \tau \). Consider the following definition.

**Definition 4.62.** Let \( V \) be a vector space over a field \( \mathbb{K} \) and let \( B \) be a \( \mathbb{K} \)-bilinear alternating map from \( V \times V \) to an extension field \( \mathbb{F} \) of \( \mathbb{K} \). Let \( M \) be the \( \mathbb{F} \)-algebra linearly spanned by the formal basis \( \{ e_v \mid v \in V, v \neq 0 \} \). Let the bilinear multiplication map on \( M \) be determined by

\[
[e_u, e_v] = B(u, v)e_{u+v}.
\]

We call \( M \) the **Block algebra** of \( V \) and \( B \) over \( \mathbb{F} \), and denote it by \( M(V, B) \).

This is a generalization of one of Kaplansky’s families of Lie algebras, found (in even greater generality) by Block in [Bi65]. Now \( M(P, L, \tau) \) corresponds to the case where \( \mathbb{K} = \mathbb{F}_2 \). But the following lemma holds for all characteristics.

**Lemma 4.63.** Let \( V \) be a vector space over \( \mathbb{K} \), let \( \mathbb{F} \) be an extension field of \( \mathbb{K} \) and let \( B: V \times V \to \mathbb{F} \) be a \( \mathbb{K} \)-linear alternating map. Then \( M(V, B) \) is a Lie algebra.

**Proof.** Clearly \( [e_u, e_v] = B(u, v)e_{u+v} = -B(v, u)e_{u+v} = -[e_v, e_u] \) for all \( u, v \in V \). We prove the Jacobi identity for \( u, v, w \in V \).

\[
[e_u, [e_v, e_w]] = B(u, v + w)B(v, w)e_{u+v+w}.
\]
the cyclic rotations of these expressions sum to
\[ B(u, v)B(v, w) + B(u, w)B(v, w) + B(v, w)B(w, u) + B(v, u)B(w, u) + B(w, u)B(u, v) + B(w, v)B(u, v) = 0. \]

\[ \square \]

**Theorem 4.64.** Let \((P, L)\) be one of the partial linear spaces of Theorem 4.2. Then there exists a Lie scaling on \((P, L)\).

**Proof.** If \((P, L)\) is one of the partial linear spaces in Theorem 4.7, then there exists a Lie orientation on \((P, L)\) by Theorem 4.12, so there is also a Lie scaling. Hence we take \((P, L) = PG(F_2^n) - PG(F_2^k)\).

We will prove the case where \(k = 0\) below. If \(k > 0\), then \((P, L) = 2(PG(F_2^{n-1}) - PG(F_2^{k-1}))\). We may assume by induction on \(k\) that there exists a Lie scaling on \(PG(F_2^{n-1}) - PG(F_2^{k-1})\). Lemma 4.58 then shows that there is a scaling on \((P, L)\).

Let \(k = 0\). We will construct a Lie scaling over the field \(F\) of order \(2^\binom{n}{2}\). Let \(V = F_2^n\) with basis \(\{e_1, \ldots, e_n\}\). Take a set of \(\binom{n}{2}\) elements of \(F\) that are linearly independent over \(F_2\), and label them \(\alpha_{i,j}\) with \(i < j \leq n\). Let \(B\) be the alternating \(F_2\)-bilinear map from \(V \times V\) to \(F\) determined by \(B(e_i, e_j) = \alpha_{i,j}\) if \(i < j\). We will show that \(B\) is a Lie scaling on \((P, L)\).

Let \(u, v \in V\). Since \(B\) is alternating, \(B(u, v) = B(u, u + v)\). This shows that \(B\) is a generalized scaling. In order to show that it is a scaling, let \(u = \sum \lambda_i e_i\) and \(v = \sum \mu_i e_i\). Then
\[ B(u, v) = \sum_{i < j} (\lambda_i \mu_j - \lambda_j \mu_i) \alpha_{i,j}. \]

Because all \(\alpha_{i,j}\) are linearly independent over \(F_2\), \(B(u, v) = 0\) if and only if \(\lambda_i \mu_j = \lambda_j \mu_i\) for all \(i\) and \(j\). Hence distinct nonzero vectors are not perpendicular with respect to \(B\). We have shown that \(B\) is a scaling.

It is clear that \(M(P, L, B)\) and \(M(V, B)\) are isomorphic. Since \(M(V, B)\) is a Lie algebra, so is \(M(P, L, B)\). Thus \(B\) is a Lie scaling. \(\square\)
A

# This utility function returns the list of indices at which a certain
# predicate holds.
positionsProperty := function (l, func)
    return Filtered ([1 .. Length (l)], i -> func (l [i]));
end;

#

A.1. Definition of the Melikyan algebra
#

# Indexing of \(\mathbb{W}\)-type algebras after ideas from the code by Willem de Graaf.
SimpleLieAlgebraTypeM := function (n, F)
    local n1, n2,   # The parameters.
        one, zero, # Shortcuts to the field elements.
        dimO, dimW, # Dimensions of the \(O\) and \(\mathbb{W}\) spaces.
        OBasis,   # A representation of a basis for \(O\).
        posO,     # Function to find the position of a given
                   # \(OBasis\) element in the basis.
        OProduct, # The regular product of two elements of \(OBasis\).
        WBasis,   # A representation of a basis for \(\mathbb{W}\).
        div,      # The divergence function for elements of \(WBasis\).
        posW,     # Function to find the position of a \(WBasis\)
                   # element in the basis.
        WOProduct, # The action of \(\mathbb{W}\) on \(O\).
        WProduct, # The regular product of two elements of \(WBasis\).
        WBracket, # The commutator of two elements of \(WBasis\)
                   # w.r.t. \(WProduct\).
        tildify, clean, # Utility functions.
        table, i, w1, j, w2, result, term, prod, x2, x1, d;  # Temporary results and counters.

    n1 := n [1];
    n2 := n [2];
    dimO := 5^(n1 + n2);
    dimW := 2*dimO;
one := One (F);
zero := Zero (F);

# The element \([a, b]\) of OBasis represents the element
# \(x^a y^b\) of 0.
OBasis := Cartesian ([0 .. 5^n1 - 1], [0 .. 5^n2 - 1]);

# The position of an OBasis element in the basis.
posO := function (o)
    return o [2] + 5^n2 * o [1] + 1;
end;

# Given two OBasis elements \(x_1\) and \(x_2\), returns a list with a
# coefficient \(coeff\) and the position \(pos\) of a basis element, such
# that \(x_1 \cdot x_2 = coeff \cdot OBasis [pos]\)
OProduct := function (x1, x2)
    local pow;
    pow := ShallowCopy (x1 + x2);
    if pow [1] < 5^n1 and pow [2] < 5^n2 then
        return [Binomial (pow [1], x1 [1]) * (Binomial (pow [2], x1 [2]) * one),
                posO (pow)];
    else
        return [zero, 1];
    fi;
end;

# The element \([a, b], c]\) of WBasis represents the element \(f \partial_c\),
# where \(f\) is the element of OBasis represented by \([a, b]\).
WBasis := Cartesian (OBasis, [1, 2]);

# The divergence: \(f \partial_1 + g \partial_2 \mapsto \partial_1 (f) + \partial_2 (g)\), maps WBasis
# elements to OBasis elements. Note: if the result is 0, we return
# that instead of the OBasis element.
div := function (abc)
    local ab, pos;
    if abc [1] [abc [2]] = 0 then
        return 0;
    fi;
    pos := abc [2];
    ab := ShallowCopy (abc [1]);
    ab [pos] := ab [pos] - 1;
    return ab;
end;

# The position of the WBasis element \([OBasis (o), c]\) in the basis,
# where \(o\) is the number of an OBasis element.
posW := function (o, c)
    return 2 * o + c - 2;
end;

# Given a WBasis element \([a_1, b_1], c]\) and an OBasis element \([a_2,\n# b_2]\), representing the usual monomials, this function computes
# \(p = x_1^{a_1} x_2^{b_1} (\partial_1 x_1^{a_2} x_2^{b_2})\),
# and returns a list \([pos, coeff]\) with the position in OBasis of
# the basis element this is a multiple of, and its coefficient; so
# that
A.1. DEFINITION OF THE MELIKYAN ALGEBRA

\# p = coeff * OBasis [pos].

\texttt{WOProduct := function (w1, x2)
local pow, prod;
if x2 [w1 [2]] > 0 then
  pow := ShallowCopy (x2);
  pow [w1 [2]] := pow [w1 [2]] - 1;
  return OProduct (w1 [1], pow);
else
  return [zero, 1];
fi;
end;
}

\# Given two WBasis elements \([a1, b1, c1]\) and \([a2, b2, c2]\),
\# representing the usual monomials, this
\# function computes
\# \(p = x_{a1}^{\alpha_1} x_{b1}^{\beta_1} (\partial_{c1}(x_{a2}^{\alpha_2} x_{b2}^{\beta_2})) \partial_{c2}\),
\# and returns a list \([pos, coeff]\) with the position in WBasis of
\# the basis element this is a multiple of, and its coefficient; so
\# that \(p = coeff * WBasis [pos]\).

\texttt{WProduct := function (x1, x2)
local prod;
prod := WOProduct (x1, x2 [1]);
if prod [1] <> zero then
  return [prod [1], posW (prod [2], x2 [2])];
else
  return [zero, 1];
fi;
end;
}

\# The bracket on W is defined as mapping \(x1, x2\) to their
\# commutator, where the multiplication is as above. This function
\# returns a list \(ls\) of, alternatingly, coefficients and positions,
\# such that the bracket of \(x1\) and \(x2\) is equal to
\# However, if any coefficient is 0, the corresponding list
\# elements are omitted. So the list returned has length 4, 2 or 0.

\texttt{WBracket := function (x1, x2)
local result, prod;
prod := WProduct (x1, x2);
if prod [1] <> zero then
  result := prod;
else
  result := [];
fi;
prod := WProduct (x2, x1);
if prod [1] <> zero then
  Append (result, [- prod [1], prod [2]]);
fi;
return result;
end;
}

\# Vectors will be represented in the same format as for GAP
\# structure constants tables, that is, as lists with alternatingly
\# a coefficient and a position in an ordered basis. The order of
\# the basis elements is: first the basis elements of W, then of
\# \(W\), then of \(\tilde{W}\).

\# \texttt{tildify} adds cst to each even position in \(ls\). It is useful for
# mapping a result of WBracket from W to Wtilde, or an OBasis element to the correct position in the full basis.
tildify := function (ls, cst)
local i;
i := 2;
while IsBound (ls [i]) do
    ls [i] := ls [i] + cst;
i := i + 2;
end;
end;

# clean is a function that 'cleans' a list before submission to SetEntrySCTable. That is, if any positions occur twice, the # coefficients are added.
clean := function (ls)
local ps, i;
ps := rec ();
i := 2;
while IsBound (ls [i]) do
    if IsBound (ps.(ls [i])) then
        ls [ps.(ls [i]) - 1] := ls [ps.(ls [i]) - 1] + ls [i - 1];
        Unbind (ls [i - 1]);
        Unbind (ls [i]);
    else
        ps.(ls [i]) := i;
        i := i + 2;
    fi;
end;
return Compacted (ls);
end;

table := EmptySCTable (dimO + 2 * dimW, Zero (F), "antisymmetric");
for i in [1 .. dimW] do
    w1 := WBasis [i];
    for j in [1 .. dimW] do
        w2 := WBasis [j];
        if i < j then
            # Compute the product for w1 and w2 in W.
            # This is simply [w1, w2].
            SetEntrySCTable (table, i, j, clean (WBracket (w1, w2)));
        fi;
    fi;
    if i < j then
        # Compute the product for \tilde{w1} and \tilde{w2} in \tilde{W}.
        # This is f_1 g_2 - f_2 g_1 if w1 = f_1 \partial_1 + f_2 \partial_2, w2 = g_1 \partial_1 + g_2 \partial_2.
            prod := OProduct (w1 [1], w2 [1]);
            if prod [1] <> zero then
                SetEntrySCTable (table, i + dimW + dimO, j + dimW + dimO,
                    [(3 - 2 * w1 [2]) * # This is the coefficient # plus or minus one.
                        prod [1], prod [2] + dimW]);
            fi;
        fi;
    fi;
end;
# Compute the product for \( w_1 \) in \( W \), \( \tilde{w}_2 \) in \( \tilde{W} \).
# This is defined as \([w_1, \tilde{w}_2] + 2 \text{div}(w_1) \tilde{w}_2\).  
# First \([w_1, \tilde{w}_2]\):
result := WBracket (w1, w2);
tildify (result, dimW + dimO);
# then add \(2 \text{div}(w_1) \tilde{w}_2\):
if d <> 0 then
term := OProduct (d, w2 [1]);
if term [1] <> zero then
Append (result, [2 * term [1],
posW (term [2], w2 [2]) + dimW + dimO]);
fi;
fi;
tildify (result, dimW);
SetEntrySCTable (table, i, j + dimW + dimO, clean (result));

for j in [1 .. dimO] do
x2 := OBasis [j];

# Compute the product for \( w_1 \) in \( W \), \( x_2 \) in \( O \).
# This is \( w_1 (x_2) - 2 \text{div} (w_1) x_2 \).
# \( w_1 (x_2)\):
result := WOProduct (w1, x2);
# - \(2 \text{div} (w_1) x_2\):
d := div (w1);
if d <> 0 then
term := OProduct (d, x2);
if term [1] <> zero then
Append (result, [-2 * term [1], term [2]]);
fi;
fi;
tildify (result, dimW);
SetEntrySCTable (table, i, j + dimW, clean (result));

# Compute the product for \( \tilde{w}_1 \) in \( \tilde{W} \), \( x_2 \) in \( O \).
# This is \( -x_2 w_1 \) (note: without tilde).
# We put it into the table as the product of \( x_2 \) and \( w_1 \), so
# that we don’t have to bother with the minus sign.
result := OProduct (x2, w1 [1]);
SetEntrySCTable (table, j + dimW, i + dimW + dimO,
[result [1], posW (result [2], w1 [2])]);

od;

for i in [1 .. dimO] do
x1 := OBasis [i];
for j in [i + 1 .. dimO] do
x2 := OBasis [j];

# Compute the product for \( x_1 \) and \( x_2 \) in \( O \).
# This is \( 2 (x_2 \partial_2 (x_1) - x_1 \partial_2 (x_2)) \partial_1 + 
# 2 (x_1 \partial_1 (x_2) - x_2 \partial_1 (x_1)) \partial_2 \).
# First \( 2 x_2 \partial_2 (x_1) \partial_1 \):
result := WProduct ([x2, 2], [x1, 1]);
result [1] := 2 * result [1];
# Add \(-2 x_1 \partial_1 (x_2) \partial_1 \):
term := WProduct ([x1, 2], [x2, 1]);
Append (result, [- 2 * term [1], term [2]]);
# Add 2 x1 \( \partial_1(x_2) \partial_2 \):
term := WProduct ([x1, 1], [x2, 2]);
Append (result, [2 * term [1], term [2]]);
# Add - 2 x2 \( \partial_1(x_1) \partial_2 \):
term := WProduct ([x2, 1], [x1, 2]);
Append (result, [- 2 * term [1], term [2]]);
tildify (result, dimW + dimO);
SetEntrySCTable (table, i + dimW, j + dimW,
clean (result));
od;
od;
return LieAlgebraByStructureConstants (F, table);
end;

# Instantiation of the algebra. signature is the pair of parameters.
signature := [1, 1];
field := GF (5);
m := SimpleLieAlgebraTypeM (signature, field);
MakeImmutable (m);
mb := Basis (m);

# A.2. Grading and filtration of the Melikyan algebra

#
# genTags is a list describing the order of the basis of m: first
# the elements of W (signature), then those of O (signature), then
# those of W-tilde (signature). The element
# \( x^i_1 x^j_2 \partial_k \) of W is represented by \([i, j], k \]
# \( x^i_1 x^j_2 \) of O is represented by \([i, j], 0 \]
# \( \tilde{x}^i_1 x^j_2 \partial_k \) of \( \tilde{W} \) is represented by \([i, j], -k \]

ob := Cartesian ([0 .. 5^signature[1] - 1], [0 .. 5^signature[2] - 1]);
wb := Cartesian (ob, [1, 2]);
ob := Cartesian (ob, [0]);
wbt := List (wb, we -> [we [1], - we [2]]);
genTags := Concatenation (wb, ob, wbt);

# Maps a tag occurring in genTags to the corresponding standard basis
# element.
tagToElt := tag -> mb [Position (genTags, tag)];

# A list of degrees over \( \mathbb{Z}^2 \); z2grading [n] is the degree of mb [n] in
# the grading.
z2grading := List (genTags, function (tag)
  local mon;
  mon := tag [1] * [[2, 1], [1, 2]];
  if tag [2] = -2 then
    return mon + [0, -1];
  elif tag [2] = -1 then
    return mon + [-1, 0];
  fi;
  return mon + [1, 0];
);
elif tag [2] = 0 then
    return mon + [-1, -1];
elif tag [2] = 1 then
    return mon + [-2, -1];
elif tag [2] = 2 then
    return mon + [-1, -2];
fi;
Error ("Bad genTags member in z2grading");
end;
# zgrading [n] is the degree in the Z-grading.
zgrading := List (z2grading, Sum);
# gradingComponent is a function mapping natural numbers to the
# corresponding components in the Z-grading.
gradingComponent := n -> VectorSpace (field, mb {positionsProperty
    (zgrading, g -> g = n)});
# filtrationComponent is a function mapping natural numbers to the
# corresponding components in the invariant descending filtration.
filtrationComponent := n -> VectorSpace (field, mb {positionsProperty
    (zgrading, g -> g >= n)}, 0 * mb [1]);

# A.3. Automorphisms of the Melikyan algebra
#
# The following computations have only been proven to be valid if
Assert (1, signature [1] = signature [2]);
#
# Changes the first argument!
# Returns a list ind of indices such that
# Concatenation (basis, vectors {ind})
# is a basis for the space spanned by basis and vectors.
# basis should be a mutable basis. It is enlarged to a basis of the
# space returned by this function.

independentIndices := function (basis, vectors)
    local ind, i;
    ind := [];
    for i in [1 .. Size (vectors)] do
        if not IsContainedInSpan (basis, vectors [i]) then
            ind := [ind, i];
            CloseMutableBasis (basis, vectors [i]);
        fi;
    od;
    return Flat (ind);
end;

# Let phi be an automorphism. Let dxtIm and dytIm be the images of
# ˜∂_1 and ˜∂_2, and highd1Im and highd2Im the images of the
# highest-degree standard basis elements ending in ˜∂_1 and
# ˜∂_2, under phi, respectively. This function returns the matrix
# of phi.
toFullMatrix := function (dxtIm, dytIm, highd1Im, highd2Im)
    local dxt, dyt, images, originals, lastoriginals,
    lastimages, basis, neworiginals, newimages,
    indepposns, hom;
    dxt := tagToElt ([[9, 0], -1]);
    dyt := tagToElt ([[9, 0], -2]);
images := [highd1Im, highd2Im];
originals := mb {(Length (mb) - 1, Length (mb))};
lastoriginals := originals;
lastimages := images;
basis := MutableBasis (field, originals);

while Length (lastimages) > 0 do
    neworiginals := Concatenation (List ([dxt, dyt], elt -> List (lastoriginals, orig -> elt * orig)));
    newimages := Concatenation (List ([dxtIm, dytIm], elt -> List (lastimages, img -> elt * img)));
    indepposns := independentIndices (basis, neworiginals);
    # This extends basis to also span neworiginals.
    lastoriginals := neworiginals {indepposns};
    lastimages := newimages {indepposns};
    Append (originals, lastoriginals);
    Append (images, lastimages);
end;

hom := LeftModuleHomomorphismByImages (m, m, originals, images);
return List (mb, gen -> Coefficients (mb, gen ^ hom));
# Returns exp (ad v) (w), if v is of nilpotency index at most p.

result := result + term;
i := i + 1;
term := mat * term;
o;
return result;
end

# Returns exp (ad v) (w), if v is of nilpotency index at most p.

exp := function (v, w)
local zero, result, term, i;
zero := Zero (w);
result := w;
term := v * w;
i := One (field);
while term <> zero and not IsZero (i) do
  # Invariants:
  # result = ∑_{j=0}^{i-1} (ad v)^j (w) / j!
  # term = (ad v)^i (w) / (i-1)!
  term := term / i;
  result := result + term;
  term := v * term;
i := i + 1;
o;
return result;
end

# The following few items have only been verified for signature [1, 1].
Assert (1, signature = [1, 1]);

# List of standard basis elements v such that exp (ad v) is well-defined and an automorphism.
expadMbs := mb {Filtered ([1 .. Length (mb)], t ->
zgrading [t] > 1 and
not genTags [t] in
[[[1, 1], 1], [[1, 1], 2], [[1, 1], 0]]);

# Returns whether the given function is an automorphism.
isAut := phi -> ForAll (Combinations (mb, 2), pair ->
phi (pair [1] * pair [2]) =
phi (pair [1]) * phi (pair [2]));

# Check for this thesis, referred to from section 3.3.3, page 86:
check_1 := function ()
  return expadMbs = Filtered (mb, be -> isAut (v -> exp (be, v)));
end;

# List of matrices of automorphisms.
autmats := function ()
local x1, x2, td1, td2, high1, high2, result;
x1 := tagToElt ([[1, 0], 0]);
x2 := tagToElt ([[0, 1], 0]);
td1 := tagToElt ([[0, 0], -1]);
td2 := tagToElt ([[0, 0], -2]);
high1 := mb [Length (mb) - 1];
high2 := mb [Length (mb)];
result := Concatenation (
List (Concatenation (expadMbs,
    List ([0, 1, 2], i -> Sum ([[2, 0], i], # $x_1^2 \partial_i$ +
        [[1, 1], i], # $x_1 x_2 \partial_i$ +
        [[0, 2], i], # $x_2^2 \partial_i$;
        tagToElt))),
    be -> matExp (TransposedMat (AdjointMatrix (mb, be))))),
List (GeneratorsOfGroup (GL (2, field)),
    matrixToFullMatrix),
[toFullMatrix (exp (x2, td1)+2* tagToElt ([[0, 3], 1]),
    exp (x2, td2) + 2 * tagToElt ([[0, 3], 2]),
    high1, high2),
toFullMatrix (exp (x1, td1)+3* tagToElt ([[3, 0], 1]),
    exp (x1, td2) + 3 * tagToElt ([[3, 0], 2]),
    high1, high2));
# perform caching for this expensive function
MakeImmutable (result);
autmats := function ()
    return result;
end;
return result;
end;

# Action with which these matrices act on the algebra.
onAlgebraElements := function (e, g)
    return LinearCombination (mb, Coefficients (mb, e) * g);
end;

# Check for this thesis, referred to in section 3.3.3, page 86:
# “It turns out that these two maps can be extended to automorphisms,
# which can be verified in GAP.”
# This follows from the check below, which shows that all matrices
# produced by autmats () represent automorphisms.
# check_2 := function ()
#     return ForAll (autmats (), mat ->
#         isAut (v -> onAlgebraElements (v, mat)));
# end;

# A.4. The sandwich algebra and the sandwich elements

# This section assumes that n1 = n2 = 1.
Assert (1, signature = [1, 1]);

# Check for this thesis, referred to in section 3.5, page 99:
# check_3 := function ()
#     local xid2, phi;
#     xid2 := tagToElt ([[1, 0], 2]);
#     phi := List ([1 .. Length (mb)], function (n)
#         if genTags [n] [1] = [0, 4] and genTags [n] [2] in [-1, 1] then
#             return Coefficients (mb,
A.4. THE SANDWICH ALGEBRA AND THE SANDWICH ELEMENTS

exp (x1d2, mb [n]) -
tagToElt ([[4, 0], 2 * genTags [n] [2]]);
else
    return Coefficients (mb, exp (x1d2, mb [n]));
fi;
end;
return isAut (v -> onAlgebraElements (v, phi));
end;

# The subalgebra S, defined in Theorem 3.31.
sandwichAlgebra := Subalgebra (m, Concatenation (  
    Basis (filtrationComponent (7)),  
    [tagToElt ([[0, 3], 1])],  
    tagToElt ([[3, 0], 2]]),  
    List ([[1 .. 3], i ->  
        tagToElt ([[i, 3 - i], 1]) -  
        tagToElt ([[i - 1, 4 - i], 2]])));

# The subspace U, referred to in the proof of Lemma 3.38.
subspaceU := Subspace (sandwichAlgebra,  
    mb {Filtered ([1 .. Size (z2grading)], function (i)  
        local x, y;  
        x := z2grading [i] [1];  
        y := z2grading [i] [2];  
        return x + y > 6  
            and 2 * x - y <= 8  
            and 2 * y - x <= 8  
            and x + y < 14;  
    end)});

# Basis for the subspace V, referred to in the proof of Lemma 3.38.
basisV := Basis (Intersection (gradingComponent (6),  
    sandwichAlgebra));

# Check for this thesis, referred to in section 3.5, page 100:
# check_4 := function ()
#    local fc9, vxProducts;
#    fc9 := filtrationComponent (9);
#    vxProducts := Filtered (List (Cartesian (basisV,  
#        Basis (filtrationComponent (1))),  
#        Product),  
#        vx -> vx in subspaceU and vx in fc9));
#    return Intersection (fc9, subspaceU) =  
#        Subspace (subspaceU, vxProducts);
# end;

# Check for this thesis, referred to in section 3.5, page 100:
# check_5 := function ()
#    local checked, matrices, n, extra;
#    checked := Basis (Intersection (sandwichAlgebra,  
#        filtrationComponent (9)));
#    matrices := autmats ();
#    n := Size (matrices);
#    matrices := matrices [[n - 1, n]]; # These are the matrices in Aut1M.
#    extra := List (Cartesian (basisV, matrices), pair ->

onAlgebraElements (pair [1], pair [2]));
extra := List (Cartesian (extra, matrices), pair ->
onAlgebraElements (pair [1], pair [2]));
return Subspace (sandwichAlgebra, Concatenation (checked, basisV, extra)) = sandwichAlgebra;
end;

# If mat is a list of at least one vector space element, and its first
# element is nonzero, then it returns a linearly independent subset
# spanning the space spanned by all elements of mat.
filterDependent := function (mat)
local result, i;
result := [mat [1]];
for i in [2 .. Size (mat)] do
  if not (mat [i] in VectorSpace (field, result)) then
    Add (result, mat [i]);
  fi;
od;
return result;
end;

# This function returns the conductor in a given algebra from one
# subspace into another.
conductor := function (algebra, from, to)
local ab, rest, result, felt, newbasis;
if 0 in [Dimension (algebra) - Dimension (to), Dimension (from)] then
  return algebra;
fi;
# A basis of the full algebra containing a basis of "to":
ab := BasisNC (algebra, filterDependent (Concatenation (Basis (to),
  Basis (algebra))));
# The set of indices into ab corresponding to basis elements
# outside "to":
rest := [Dimension (to) + 1 .. Dimension (algebra)];
result := Basis (algebra);
for felt in Basis (from) do
  newbasis := NullspaceMatDestructive (List (result, relt ->
    Coefficients (ab, felt * relt) {rest}));
  if newbasis = [] then
    result := [];
    break;
  fi;
  result := newbasis * result;
od;
result := SubspaceNC (algebra, result, "basis");
MakeImmutable (result);
return result;
end;

# Given a list [s1, ..., sn] and an integer 0 <= k <= n, this
# function creates a new list [s(n + 1), ..., sm] of subspaces not yet
# occurring in [s1, ..., sn] that can be obtained by calling mapInto
# with two si's, such that at least one of the i's is greater than
# k and both are at most n. It will return the new list [s1, ..., sm]
# of subspaces and update table: when calling, this should be a k^k
# matrix where table [i] [j] holds the index in [s] where mapInto (si,
# sj) is.
# The subspaces are given as two lists, spaces and bases. bases [i] is
# an echelonized list of basis elements for spaces [i].
# This function returns its results by changing its arguments!
expandSubspaceList := function (algebra, spaces, k, table)
    local n, i, jmin, j, newspace, pos;
    n := Size (spaces);
    for i in [1 .. n] do
        if not IsBound (table [i]) then
            table [i] := [];
        fi;
        if i <= k then
            jmin := k + 1;
        else
            jmin := 1;
        fi;
        for j in [jmin .. n] do
            newspace := conductor (algebra, spaces [i], spaces [j]);
            pos := Position (spaces, newspace);
            if pos = fail then
                Add (spaces, newspace);
                pos := Size (spaces);
                if ValueOption ("verbose") = true then
                    Print ("Found subspace nr. ", pos, ".\n");
                fi;
            fi;
            table [i] [j] := pos;
        od;
    od;
end;

# This function iterates the previous one to obtain a set closed under
# taking conductors. Given an algebra and a list of subspaces, it
# returns a two-argument list, the first of which is the closure of
# the given list of subspaces under taking conductors and the second a
# table the [i][j] entry of which contains k if the conductor of
# spaces [i] into spaces [j] is spaces [k]. (This table can be used to
# find how a space in the result can be expressed as a conductor in
# the given starting spaces.)
iterateESL := function (algebra, spaces)
    local table, k, n;
    spaces := ShallowCopy (spaces); # Do not damage the argument.
    table := [];
    k := 0;
    n := 3;
    repeat
        expandSubspaceList (algebra, spaces, k, table);
        k := n;
        n := Size (spaces);
        if ValueOption ("verbose") = true then
            Print ("Ended an ESL iteration; n = ", n, ".\n");
        fi;
    until n = k;
    return [spaces, table];
end:

# First argument should be a vector space, the other arguments should
# be compatible vector space elements or lists of such
# elements. Returns the extension of the first argument by all vectors
# in the other arguments.
extension := function (arg)
local space, elements, i;
space := arg [1];
elements := [];
for i in [2 .. Length (arg)] do
  if Zero (space) = Zero (arg [1]) then
    Add (elements, arg [1]);
  else
    Append (elements, arg [1]);
  fi;
od;
return VectorSpace (LeftActingDomain (space),
Concatenation (Basis (space), elements));
end

# The list of 68 subspaces that can be obtained as repeated conductors
# using

invariantSubspaces :=
Concatenation (List ([[-3 .. 24], filtrationComponent]),
List ([0 .. 3], i -> extension (filtrationComponent (3 * i + 1),
tagToElt ([[i + 1, 0], 2]),
tagToElt ([[0, i + 1], 1]),
List ([0 .. 1], j ->
tagToElt ([[j + 1, i - j], 1]) -
tagToElt ([[j, i - j + 1], 2]))),
List ([0 .. 3], i -> extension (filtrationComponent (3 * i + 1),
List ([0 .. 1], j ->
(j + 1) * tagToElt ([[j + 1, i - j], 1]) +
(i - j + 1) * tagToElt ([[j, i - j + 1], 2]))),
List ([0 .. 3], i -> extension (filtrationComponent (3 * i + 3),
tagToElt ([[i + 1, 0], -2]),
tagToElt ([[0, i + 1], -1]),
List ([0 .. 1], j ->
tagToElt ([[j + 1, i - j], -1]) -
tagToElt ([[j, i - j + 1], -2]))),
List ([0 .. 3], i -> extension (filtrationComponent (3 * i + 3),
List ([0 .. 1], j ->
(j + 1) * tagToElt ([[j + 1, i - j], -1]) +
(i - j + 1) * tagToElt ([[j, i - j + 1], -2]))),
List ([4 .. 6], i -> extension (filtrationComponent (3 * i + 1),
tagToElt ([[4, i - 3], 2]),
tagToElt ([[i - 3, 4], 1]),
List ([i - 3 .. 3], j ->
(j + 1) * tagToElt ([[j + 1, i - j], 1]) +
(i - j + 1) * tagToElt ([[j, i - j + 1], 2]))),
List ([4 .. 6], i -> extension (filtrationComponent (3 * i + 1),
List ([i - 3 .. 3], j ->
tagToElt ([[j + 1, i - j], 1]) -
tagToElt ([[j, i - j + 1], 2]))),
List ([4 .. 6], i -> extension (filtrationComponent (3 * i + 3),
tagToElt ([[4, i - 3], -2]),
tagToElt ([[i - 3, 4], -1])).
\begin{verbatim}
List ([i - 3 .. 3], j ->
  (j + 1) * tagToElt ([[j + 1, i - j], -1]) +
  (i - j + 1) * tagToElt ([[j, i - j + 1], -2]))),
List ([4 .. 6], i -> extension (
  filtrationComponent (3 * i + 3),
  List ([i - 3 .. 3], j ->
    tagToElt ([[j + 1, i - j], -1]) -
    tagToElt ([[j, i - j + 1], -2])))),

[extension (filtrationComponent (3),
  tagToElt ([[0, 1], 0]), tagToElt ([[1, 0], 0]),
  tagToElt ([[0, 1], -2]) + tagToElt ([[1, 0], -1])),
extension (filtrationComponent (4),
  tagToElt ([[0, 2], 2]) + 3 * tagToElt ([[1, 1], 1]),
  tagToElt ([[1, 1], 2]) + 2 * tagToElt ([[2, 0], 1]),
  tagToElt ([[0, 1], -2]) + tagToElt ([[1, 0], -1])),
extension (filtrationComponent (5),
  tagToElt ([[0, 2], 2]) + 3 * tagToElt ([[1, 1], 1]),
  tagToElt ([[1, 1], 2]) + 2 * tagToElt ([[2, 0], 1])),
extension (filtrationComponent (6),
  tagToElt ([[0, 3], 2]) + 2 * tagToElt ([[1, 2], 1]),
  tagToElt ([[1, 2], 2]) + tagToElt ([[2, 1], 1]),
  tagToElt ([[2, 1], 2]) + 3 * tagToElt ([[3, 0], 1]),
  tagToElt ([[0, 2], -2]) + 3 * tagToElt ([[1, 1], -1]),
  tagToElt ([[1, 1], -2]) + 2 * tagToElt ([[2, 0], -1])),
extension (filtrationComponent (7),
  tagToElt ([[0, 3], 2]) - 2 * tagToElt ([[1, 2], 1]),
  tagToElt ([[1, 2], 2]) + tagToElt ([[2, 1], 1]),
  tagToElt ([[1, 1], -2]) + 2 * tagToElt ([[2, 0], -1])),
extension (filtrationComponent (8),
  tagToElt ([[0, 4], 1]),
  tagToElt ([[0, 4], 2]) - tagToElt ([[1, 3], 1]),
  tagToElt ([[1, 3], 2]) - tagToElt ([[2, 2], 1]),
  tagToElt ([[2, 2], 2]) - tagToElt ([[3, 1], 1]),
  tagToElt ([[3, 1], 2]) - tagToElt ([[4, 0], 1]),
  tagToElt ([[4, 0], 2]),
  tagToElt ([[0, 3], -2]) + 2 * tagToElt ([[1, 2], -1]),
  tagToElt ([[1, 2], -2]) + tagToElt ([[2, 1], -1]),
  tagToElt ([[2, 1], -2]) + 3 * tagToElt ([[3, 0], -1])),
extension (filtrationComponent (10),
  tagToElt ([[0, 4], 2]) - tagToElt ([[1, 3], 1]),
  tagToElt ([[1, 3], 2]) - tagToElt ([[2, 2], 1]),
  tagToElt ([[2, 2], 2]) - tagToElt ([[3, 1], 1]),
  tagToElt ([[3, 1], 2]) - tagToElt ([[4, 0], 1]),
  tagToElt ([[4, 0], 2]),
  tagToElt ([[0, 3], -2]) + 2 * tagToElt ([[1, 2], -1]),
  tagToElt ([[1, 2], -2]) + tagToElt ([[2, 1], -1]),
  tagToElt ([[2, 1], -2]) + 3 * tagToElt ([[3, 0], -1])),
extension (filtrationComponent (13),
  tagToElt ([[1, 4], 1]),
  tagToElt ([[1, 4], 2]) + 3 * tagToElt ([[2, 3], 1]),
  tagToElt ([[2, 3], 2]) + tagToElt ([[3, 2], 1]),
  tagToElt ([[3, 2], 2]) + 2 * tagToElt ([[4, 1], 1]),
  tagToElt ([[4, 1], 2]),
  tagToElt ([[0, 4], -2]) - tagToElt ([[1, 3], -1]),
  tagToElt ([[1, 3], -2]) - tagToElt ([[2, 2], -1]),
  tagToElt ([[2, 2], -2]) - tagToElt ([[3, 1], -1]),
  tagToElt ([[3, 1], -2]) - tagToElt ([[4, 0], -1]),
  tagToElt ([[4, 0], -2])),
extension (filtrationComponent (13),
  tagToElt ([[1, 4], 1]),
  tagToElt ([[1, 4], 2]) + 3 * tagToElt ([[2, 3], 1]),
  tagToElt ([[2, 3], 2]) + tagToElt ([[3, 2], 1]),
  tagToElt ([[3, 2], 2]) + 2 * tagToElt ([[4, 1], 1]),
  tagToElt ([[4, 1], 2]),
  tagToElt ([[0, 4], -2]))
\end{verbatim}
tagToElt ([[0, 4], -2]) - tagToElt ([[1, 3], -1]),
tagToElt ([[1, 3], -2]) - tagToElt ([[2, 2], -1]),
tagToElt ([[2, 2], -2]) - tagToElt ([[3, 1], -1]),
tagToElt ([[3, 1], -2]) - tagToElt ([[4, 0], -1])),

extension (filtrationComponent (16),
tagToElt ([[2, 4], 1]),
tagToElt ([[2, 4], 2]) + 2 * tagToElt ([[3, 3], 1]),
tagToElt ([[3, 3], 2]) + 3 * tagToElt ([[4, 2], 1]),
tagToElt ([[4, 2], 2]), tagToElt ([[1, 4], -1]),
tagToElt ([[1, 4], -2]) + 3 * tagToElt ([[2, 3], -1]),
tagToElt ([[2, 3], -2]) + tagToElt ([[3, 2], -1]),
tagToElt ([[3, 2], -2]) + 2 * tagToElt ([[4, 1], -1]),
tagToElt ([[4, 1], -2])),

extension (filtrationComponent (18),
tagToElt ([[2, 4], 0]), tagToElt ([[3, 3], 0]),
tagToElt ([[4, 2], 0]), tagToElt ([[2, 4], -1]),
tagToElt ([[2, 4], -2]) + 2 * tagToElt ([[3, 3], -1]),
tagToElt ([[3, 3], -2]) + 3 * tagToElt ([[4, 2], -1]),
tagToElt ([[4, 2], -2])),

extension (filtrationComponent (19),
tagToElt ([[3, 4], 1]),
tagToElt ([[3, 4], 2]) + tagToElt ([[4, 3], 1]),
tagToElt ([[4, 3], 2]), tagToElt ([[2, 4], -1]),
tagToElt ([[2, 4], -2]) + 2 * tagToElt ([[3, 3], -1]),
tagToElt ([[3, 3], -2]) + 3 * tagToElt ([[4, 2], -1]),
tagToElt ([[4, 2], -2])),

extension (filtrationComponent (20),
tagToElt ([[3, 4], 1]),
tagToElt ([[3, 4], 2]) + tagToElt ([[4, 3], 1]),
tagToElt ([[4, 3], 2])));

check_6 := function ()
  local iesl;
  iesl := iterateESL (m, [m, sandwichAlgebra, Subspace (m, [ ])]);
  return ForAll (iesl , sp -> sp in invariantSubspaces) and ForAll (invariantSubspaces , sp -> sp in iesl);
end;

check_7 := function ()
  return Number (invariantSubspaces , s ->
    s = Subalgebra (m, Basis (s))) = 66;
end;
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Samenvatting

Meetkundige structuren in Lie algebra’s vormen het onderwerp van dit proefschrift, in het bijzonder structuren in eindigdimensionale enkelvoudige Lie algebra’s. Lie algebra’s zijn niet-associatieve algebra’s die erg nuttig zijn bij het bestuderen van continue groepen. De classificatie van eindig-dimensionale enkelvoudige Lie algebra’s over het complexe lichaam is meer dan een eeuw oud, is kort en wordt gebruikt in verschillende delen van de wiskunde. De classificatie over lichamen van positieve karakteristiek daarentegen is nog niet voltooid, ondanks meer dan 75 jaar werk en vele artikelen. Een classificatie voor karakteristiek \( p \geq 5 \) is onlangs voltooid; behalve equivalenten van de algebra’s over \( \mathbb{C} \) zijn er vier families van algebra’s voor alle positieve karakteristieken en één familie die alleen voor \( p = 5 \) voorkomt, de Melikyan algebra’s genoemd. Deze families worden veel minder goed begrepen dan degene die ook in \( \mathbb{C} \) voorkomen. In dit proefschrift proberen we deze en andere algebra’s te begrijpen middels meetkunde. In deze aanpak gebruiken we een discrete incidentiestructuren met een mooie automorfismegroep.

In dit proefschrift komen we eerst een geval tegen waar we meetkundige informatie uit een Lie algebra extraheren: we bestuderen een constructie van klassieke Lie algebra’s \( L \) over een lichaam \( F \) door middel van hun extremale elementen. Dit zijn elementen \( x \) waarvoor \( [x, [x, L]] \subseteq Fx \); ze vormen de punten van een interessante incidentiemeetkunde. We vinden een presentatie van deze Lie algebra’s middels hun extremale elementen, geparametriseerd door een gewogen graaf op een aantal knopen dat ongeveer gelijk is aan de rang van \( L \). Dit alles is het resultaat van een samenwerking met voornamelijk Jos in ’t panhuis en Dan Roozemond.

Vervolgens bestuderen we de Melikyan algebra’s. We onderzoeken de structuur van deze Lie algebra’s met in het achterhoofd de onderliggende meetkunde. We geven een expliciete constructie van de automorfismegroep op het niveau van lineaire transformaties. We construeren de algebra voortgebracht door de sandwich-elementen (vergelijkbaar met extremale elementen) expliciet in het kleinste, 125-dimensionale geval, hetgeen leidt tot een lijst van deelruimten van de algebra die invariant zijn onder de automorfismen.

Tot slot vinden we een situatie waar een meetkundige structuur leidt tot een Lie algebra: we demonstreren een constructie van een algebra uit een partiële lineaire ruimte en een oriëntatie voor elke lijn. Dit geheel noemen we een georiënteerde partiële lineaire ruimte. De punten van de partiële lineaire ruimte vormen een basis van de algebra en de georiënteerde lijnen bepalen de vermenigvuldiging. We vinden de georiënteerde partiële lineaire ruimtes waarvoor de resulterende algebra een Lie algebra is en de voorwaarden waaronder deze Lie algebra enkelvoudig is, door de vlakken in de partiële lineaire ruimte te bestuderen; dat wil zeggen, de deelruimten voortgebracht door een tweetal snijdende lijnen. We klassificeren de resulterende Lie algebra’s.
Abstract

The subject of this thesis is geometrical structures in Lie algebras, in particular finite-dimensional simple Lie algebras. Lie algebras are nonassociative algebras that are very useful for the study of continuous groups. The classification of finite-dimensional simple Lie algebras over the complex field is more than a century old, is concise and is used in many parts of mathematics. On the other hand, the classification over fields of positive characteristic has not yet been completed, despite more than 75 years of work and many papers. A classification for characteristic $p \geq 5$ has recently been achieved; besides equivalents of the algebras over $\mathbb{C}$, there are four families of algebras for all positive characteristics and one family occurring only for $p = 5$, called the Melikyan algebras. These families are much less well understood than those also occurring in $\mathbb{C}$. In this thesis we try to understand these and other algebras using geometry. In our approach, we use a discrete incidence structure with a nice automorphism group.

In the first case that we encounter, we extract geometrical information from Lie algebras: we study a construction of classical Lie algebras $L$ over a field $F$ by means of their extremal elements. Extremal elements are elements $x$ for which $[x, [x, L]] \subseteq Fx$; they form the points of an interesting incidence geometry. We find a presentation of these Lie algebras by their extremal elements, parametrized by a weighted graph on a number of vertices approximately equal to the rank of $L$. All of this is joint work, mainly with Jos in ‘t panhuis and Dan Roozemond.

Subsequently we study the Melikyan algebras. We provide information on the structure of these Lie algebras with an eye towards the underlying geometry, often using the computer algebra system GAP. We give an explicit construction of the automorphism group at the level of linear transformations. We construct the algebra generated by sandwich elements (similar to extremal elements) explicitly in the smallest, 125-dimensional case, which leads to a list of subspaces of the algebra that are invariant under all automorphisms.

Finally, we find a situation where a geometrical structure leads to a Lie algebra: we present a construction of an algebra from a partial linear space and an orientation for each line. This aggregate is called an oriented partial linear space. The points of the partial linear space form a basis of the algebra and the oriented lines determine the multiplication. We find the conditions on the oriented partial linear space for the resulting algebra to be a Lie algebra and for it to be simple, by studying the planes of the partial linear space; that is, the subspaces generated by an intersecting pair of lines. We classify the resulting Lie algebras.
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From 2003 till 2007, he was a Ph.D. student under the supervision of Prof. Dr. Arjeh M. Cohen. The present thesis is the result of his work in this period.

His research interests are at the crossroads of mathematics and computer science: besides pure algebra, he is also interested in computer algebra, discrete optimization algorithms, cryptography and coding theory.