Solution to problem 94-18* : A definite integral

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Proof. For large enough $\operatorname{Re} \lambda$, $\operatorname{Re} \mu$, $\operatorname{Re} \nu$ the formulas (2) and (3) yield—specialized to the case of $x \in \mathbb{R}^2$, i.e., $n = 2$ —

$$\begin{aligned} I(\lambda, \mu, \nu, \alpha) &= \frac{1}{2\pi^2} \left[\left(\frac{J_\lambda(\alpha|x|)}{(\alpha|x|)^\lambda} * \frac{J_\mu(|x|)}{|x|^\mu} \right) * \frac{J_\nu(|x|)}{|x|^\nu} \right] \quad (0) \\ &= \frac{1}{2\pi^2} \left[\frac{J_\lambda(\alpha|x|)}{(\alpha|x|)^\lambda} * \left(\frac{J_\mu(|x|)}{|x|^\mu} * \frac{J_\nu(|x|)}{|x|^\nu} \right) \right] \quad (0) \\ &= \frac{\Gamma(\mu + \nu - 1)}{\pi \Gamma(\mu) \Gamma(\nu)} \left[\frac{J_\lambda(\alpha|x|)}{(\alpha|x|)^\lambda} * \frac{J_{\mu+\nu-1}(|x|)}{|x|^{\mu+\nu-1}} \right] \quad (0) \\ &= \frac{2\Gamma(\mu + \nu - 1)}{\Gamma(\mu) \Gamma(\nu) \alpha^\lambda} \int_0^\infty J_\lambda(\alpha t) J_{\mu+\nu-1}(t) t^{2-\lambda-\mu-\nu} dt. \end{aligned}$$

From formula 2.12.31.1 in [3, p. 209, 210], we infer that

$$I(\lambda, \mu, \nu, \alpha) = \frac{1}{2^{\lambda+\mu+\nu-3} \Gamma(\lambda+1) \Gamma(\mu) \Gamma(\nu)} {}_2F_1(1, 2-\mu-\nu, \lambda+1; \alpha^2).$$

The general case is thus deduced by analytic continuation, since $I(\lambda, \mu, \nu, \alpha)$ is absolutely convergent under the stated conditions.

For $k \in \mathbb{N}$, we obtain from [4, formula 7.3.1.123, p. 462] the following corollary which contains the second equation of the problem as a special case (i.e., $\mu = \nu = m$, $k = 1$).

COROLLARY 3. Let $k \in \mathbb{N}$, $\mu, \nu \in \mathbb{C}$, $\operatorname{Re} \mu > 1 - k$, $\operatorname{Re} \nu > 1 - k$, $\operatorname{Re}(\mu + \nu) > 1$, $\operatorname{Re}(\mu + \nu) > \frac{5}{2} - k$. Then

$$\begin{aligned} I(k, \mu, \nu, \alpha) &= \frac{\Gamma(\mu + \nu + k - 1)}{2^{k+\mu+\nu-3} \Gamma(\mu) \Gamma(\nu) \Gamma(\mu + \nu - 1) (-\alpha^2)^k} \left[(1 - \alpha^2)^{k+\mu+\nu-2} \right. \\ &\quad \left. - \sum_{l=0}^{k-1} \frac{(2 - \mu - \nu - l) \cdots (2 - \mu - \nu - 1)}{l!} \alpha^{2l} \right]. \end{aligned}$$

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A Definite Integral

*Problem 94-18**, by M. L. GLASSER (Clarkson University), J. BOERSMA, AND P. J. DE DOELDER (Technical University, Eindhoven, the Netherlands).

Evaluate the integral

$$F_\nu(x) = \int_0^\infty \frac{1 - J_0(x\sqrt{2t})(\sqrt{1+t^2} - t)^\nu}{t\sqrt{1+t^2}} dt \quad (\nu \geq -1, x \geq 0).$$

This integral arises in studying the amplitude-amplitude correlation function for oscillations of a stack of lipid membranes [1].

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Dedicated to the memory of P. J. de Doelder.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, the Netherlands) and M. L. GLASSER (Clarkson University, Potsdam, NY).

We present a closed-form expression for $F_\nu(x)$ in case ν is an odd integer. For general $\nu > -1$, $F_\nu(x)$ is expressed in terms of Meijer's G -functions.

First we evaluate $F_\nu(0)$. Via the substitution $t = \sinh(s/2)$ we find, by [1, form. 1.7.2(14)],

$$\begin{aligned} F_\nu(0) &= \int_0^\infty \frac{1 - (\sqrt{1+t^2} - t)^\nu}{t\sqrt{1+t^2}} dt = \frac{1}{2} \int_0^\infty \frac{1 - e^{-\nu s/2}}{\sinh(s/2)} ds \\ (1) \quad &= \int_0^\infty \frac{e^{-s/2} - e^{-(\nu+1)s/2}}{1 - e^{-s}} ds = \psi\left(\frac{1}{2}\nu + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right), \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Using standard recurrence relations for the Bessel function J , we determine the derivative

$$(2) \quad \frac{d}{dx}(xF'_\nu(x)) = 2x \int_0^\infty J_0(x\sqrt{2t}) \frac{(\sqrt{1+t^2} - t)^\nu}{\sqrt{1+t^2}} dt.$$

In the latter integral we set [2, form. 4.14(1)]

$$\frac{(\sqrt{1+t^2} - t)^\nu}{\sqrt{1+t^2}} = \int_0^\infty J_\nu(y) e^{-ty} dy$$

and the order of integration is interchanged. The resulting inner integral is evaluated by [2, form. 4.14(25)], viz.

$$\int_0^\infty J_0(x\sqrt{2t}) e^{-ty} dt = y^{-1} \exp[-\tfrac{1}{2}x^2/y].$$

Consequently we find, by [2, form. 4.15(8)],

$$\begin{aligned} (3) \quad \frac{d}{dx}(xF'_\nu(x)) &= 2x \int_0^\infty J_\nu(y) y^{-1} \exp[-\tfrac{1}{2}x^2/y] dy \\ &= 2x \int_0^\infty J_\nu(t^{-1}) t^{-1} \exp[-\tfrac{1}{2}x^2 t] dt = 4x J_\nu(x) K_\nu(x). \end{aligned}$$

By integration of (3) starting from the initial value (1), we are led to the representation

$$\begin{aligned} (4) \quad F_\nu(x) &= \psi\left(\frac{1}{2}\nu + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) + 4 \int_0^x s^{-1} ds \int_0^s t J_\nu(t) K_\nu(t) dt \\ &= \psi\left(\frac{1}{2}\nu + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) + 4 \int_0^x t J_\nu(t) K_\nu(t) \log(x/t) dt, \quad \nu > -1. \end{aligned}$$

From the Mellin transform [2, form. 6.8(41), corrected]

$$(5) \quad \mathcal{M}[J_\nu(t)K_\nu(t)] = \int_0^\infty J_\nu(t)K_\nu(t)t^{s-1}dt = \frac{2^{s-3}\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{4}s + \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{4}s + \frac{1}{2}\nu)},$$

valid for $\operatorname{Re} s > \max(0, -2\nu)$, we deduce the special values

$$\begin{aligned} \int_0^\infty t J_\nu(t)K_\nu(t)dt &= \frac{1}{2}, \\ \int_0^\infty t J_\nu(t)K_\nu(t)\log t dt &= \frac{1}{2}[\log 2 + \frac{1}{2}\psi(1) + \frac{1}{2}\psi(\frac{1}{2}\nu + \frac{1}{2})] \\ &= \frac{1}{4}\psi(\frac{1}{2}\nu + \frac{1}{2}) - \frac{1}{4}\psi(\frac{1}{2}) - \frac{1}{2}\gamma, \end{aligned}$$

where γ is Euler's constant. These values are now used to reduce the representation (4) to

$$(6) \quad F_\nu(x) = 2\log x + 2\gamma + 4 \int_x^\infty t J_\nu(t)K_\nu(t)\log(t/x)dt.$$

The latter form is suitable to determine the asymptotic expansion of $F_\nu(x)$ as $x \rightarrow \infty$; the leading terms are given by

$$F_\nu(x) = 2\log x + 2\gamma - 2x^{-1}e^{-x}[\sin(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + O(x^{-1})], \quad (x \rightarrow \infty).$$

Starting from the differential equations for the Bessel functions J_ν and K_ν , we determine the primitive function

$$2 \int t J_\nu(t)K_\nu(t)dt = t[J_\nu(t)K'_\nu(t) - J'_\nu(t)K_\nu(t)] + C,$$

which is used to integrate by parts in (4) and (6). As a result we obtain the representations

$$(7) \quad F_\nu(x) = \psi(\frac{1}{2}\nu + \frac{1}{2}) - \psi(\frac{1}{2}) + 2 \int_0^x \left[J_\nu(t)K'_\nu(t) - J'_\nu(t)K_\nu(t) + \frac{1}{t} \right] dt, \quad \nu > -1,$$

$$\begin{aligned} (8) \quad F_\nu(x) &= 2\log x + 2\gamma - 2 \int_x^\infty [J_\nu(t)K'_\nu(t) - J'_\nu(t)K_\nu(t)] dt \\ &= 2\log x + 2\gamma + 2J_\nu(x)K_\nu(x) + 4 \int_x^\infty J'_\nu(t)K_\nu(t) dt \\ &= 2\log x + 2\gamma - 2J_\nu(x)K_\nu(x) - 4 \int_x^\infty J_\nu(t)K'_\nu(t) dt. \end{aligned}$$

Notice that the representations (6) and (8) may serve to define $F_\nu(x)$ for any $\nu \in \mathbb{R}$ and $x > 0$.

It does not seem possible to evaluate any of the integrals (7) and (8) in closed form, except if ν is an odd integer. Setting $\nu = \pm 1$ in (8), we find

$$\begin{aligned} (9) \quad F_1(x) &= 2\log x + 2\gamma - 2 \int_x^\infty [J_1(t)K'_1(t) - J'_1(t)K_1(t)] dt \\ &= 2\log x + 2\gamma + 2 \int_x^\infty [J_1(t)K_0(t) + J_0(t)K_1(t)] dt \\ &= 2\log x + 2\gamma + 2J_0(x)K_0(x), \end{aligned}$$

$$\begin{aligned} (10) \quad F_{-1}(x) &= 2\log x + 2\gamma - 2 \int_x^\infty [-J_1(t)K'_1(t) + J'_1(t)K_1(t)] dt \\ &= 2\log x + 2\gamma - 2J_0(x)K_0(x). \end{aligned}$$

Consider next the difference

$$\begin{aligned} F_{v+1}(x) - F_{v-1}(x) &= \int_0^\infty J_0(x\sqrt{2t}) \frac{(\sqrt{1+t^2}-t)^{v-1}}{t\sqrt{1+t^2}} \left[1 - (\sqrt{1+t^2}-t)^2 \right] dt \\ &= 2 \int_0^\infty J_0(x\sqrt{2t}) \frac{(\sqrt{1+t^2}-t)^v}{\sqrt{1+t^2}} dt, \end{aligned}$$

which is identical to the integral in (2). Thus, in view of (3), we arrive at the recurrence relation

$$(11) \quad F_{v+1}(x) - F_{v-1}(x) = 4J_v(x)K_v(x).$$

By starting from (9) or (10), the relation (11) permits the closed-form evaluation of $F_v(x)$ when v is an odd integer.

The Mellin transform of $F_v(x)$ is easily determined from (3) and (5). By taking the inverse transform, we obtain the following representations by Mellin-Barnes integrals:

$$(12) \quad F_v(x) = \psi\left(\frac{1}{2}v + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma(\frac{1}{2}s) \Gamma(\frac{1}{2}v + \frac{1}{2} + \frac{1}{4}s)}{s \Gamma(\frac{1}{2}v + \frac{1}{2} - \frac{1}{4}s)} x^{-s} ds,$$

$$\max(-2, -2v-2) < c < 0, \quad v > -1;$$

$$(13) \quad F_v(x) = 2 \log x + 2\gamma + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma(\frac{1}{2}s) \Gamma(\frac{1}{2}v + \frac{1}{2} + \frac{1}{4}s)}{s \Gamma(\frac{1}{2}v + \frac{1}{2} - \frac{1}{4}s)} x^{-s} ds,$$

$$c > 0, \quad v > -1.$$

The two integrals differ by the residue at the double pole $s = 0$ of the integrand. By closing the integration contour in (12) to the left, the integral may be evaluated by residue calculus. As a result we find a power-series expansion of $F_v(x)$ containing powers x^{2k} and $x^{2v+2+4k}$, and, if v is an integer, logarithmic terms $x^{2v+2+4k} \log x$, where $k = 0, 1, 2, \dots$. The Mellin-Barnes integrals in (12) and (13) can be identified with certain Meijer's G -functions, as defined in [1, form. 5.3(1)]. Thus, (12) and (13) can be expressed as

$$F_v(x) = \psi\left(\frac{1}{2}v + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) + \frac{1}{2\sqrt{\pi}} G_{26}^{32} \left(\frac{x^4}{64} \left| \begin{matrix} 1, 1 \\ \frac{1}{2}, 1, \frac{1}{2} + \frac{1}{2}v, 0, 0, \frac{1}{2} - \frac{1}{2}v \end{matrix} \right. \right),$$

$$F_v(x) = 2 \log x + 2\gamma + \frac{1}{2\sqrt{\pi}} G_{15}^{40} \left(\frac{x^4}{64} \left| \begin{matrix} 1 \\ 0, 0, \frac{1}{2}, \frac{1}{2} + \frac{1}{2}v, \frac{1}{2} - \frac{1}{2}v \end{matrix} \right. \right).$$

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