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Squeezing a sponge: a three-dimensional solution in poroelasticity

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A three-dimensional solution is derived for the fluid flow in a deformable porous medium. It is assumed that the deformation of the medium is described by Hooke's law and the flow of the fluid by Darcy's law, i.e. the theory of poroelasticity applies. The governing equations are completed by suitable boundary conditions such that a compressed saturated cubic sponge is modelled. The solution is a three-dimensional generalisation of the one-dimensional solution of Terzaghi. A two-dimensional plane strain solution is derived also. Both solutions give excellent possibilities to test numerical codes.

Keywords: Darcy, Hooke, poroelasticity

AMS subject classification: 35C10, 35M10, 74B05, 74F10

1. Introduction

The theory of poroelasticity describes fluid flow in a deformable porous medium. The flow of the fluid and the deformation of the solid matrix are coupled. Poroelasticity has applications in soil science, filtration and biology (see, e.g., references in [1]). In soil science the theory of poroelasticity describes the fundamental problem of consolidation, e.g., [3,10]. Numerical codes have been developed to solve the poroelastic equations for various geometries [6]. Exact solutions can be used to test the accuracy of these codes. However, the known solutions are one-dimensional [8,9], two-dimensional on semi-infinite regions as half-spaces and infinite layers [7], or three-dimensional on spheres [5]. The majority of numerical codes for poroelastic problems is based on finite differences or finite elements. Therefore, a finite computational domain is used. In order to test these codes exact solutions on finite domains are needed. As noted by computational poromechanicians [6], simple exact solutions in poroelasticity on finite domains are rare and too often lack realistic physical interpretations. Recently, Barry and Mercer [1] have derived some unusual solutions on finite domains. However, an immediate physical interpretation is lacking. We set ourselves the task of resolving this problem and provide a three-dimensional solution in poroelasticity with a concise and distinct

interpretation. The derived solution will be an useful tool for researchers in this area to test their numerical procedures.

In this paper a three-dimensional and a corresponding two-dimensional plane strain solution are derived that can be easily compared with numerically computed approximations. As usual in poroelasticity [4], it is assumed that the porous medium is saturated with a Newtonian fluid. The deformation of the medium can be described by the theory of linear elasticity, i.e. Hooke's law applies, and the flow of the fluid by Darcy's law. The set of governing equations is completed with suitably chosen boundary conditions.

2. Governing equations

Mixture theory assumes that the different phases are mixed and exist simultaneously at each point in space. Here only two phases are considered, a single fluid and a solid, respectively denoted by the indices f and s.

The true density of the l -phase, where $l = f, s$, is denoted by $\rho_T^l = dm^l/dV^l$, where dm^l is the mass of the l -phase in a small elementary volume dV^l containing only this phase. The porous medium is assumed to be initially homogeneous. Both the fluid as the solid phase are considered intrinsically incompressible. The assumptions imply that ρ_T^l is constant and uniform for both phases. Compression of the medium arises only because of a redistribution of the fluid and solid components.

The apparent density of the l -phase in the mixture is defined as $\rho^l = dm^l/dV$, where dV is a representative elementary volume [2] of the medium. Let the volume fraction of the l -phase be defined as $\phi^l = dV^l/dV$, then $\rho^l = \phi^l \rho_T^l$. For a binary porous medium $\phi = \phi^f$ is the porosity and thus $\phi^s = 1 - \phi$. Since the medium is assumed to be initially homogeneous, ϕ is initially uniform.

Conservation of mass for each phase implies

$$\frac{\partial \phi^l}{\partial t} + \nabla \cdot (\phi^l \mathbf{v}^l) = 0, \quad l = f, s,$$

where it is assumed that no sources or sinks exist. Here $\mathbf{v}^l$ is the average velocity [2] of the l -phase. Addition of these two equations gives

$$\nabla \cdot (\mathbf{q} + \mathbf{v}^s) = 0, \quad (1)$$

where the specific discharge relative to the solid matrix is defined as

$$\mathbf{q} = \phi(\mathbf{v}^f - \mathbf{v}^s). \quad (2)$$

If no body forces such as gravity are assumed and the inertial terms are neglected, the momentum balance for each phase is

$$\nabla \cdot \mathbf{T}^l + \boldsymbol{\pi}^l = \mathbf{0}, \quad l = f, s, \quad (3)$$

where $\mathbf{T}^l$ is the stress tensor and $\boldsymbol{\pi}^l$ is the momentum supply for each phase. Of course, $\boldsymbol{\pi}^f + \boldsymbol{\pi}^s = \mathbf{0}$.

It is assumed that the solid matrix is entirely elastic and initially isotropic. The deformations are considered to be small enough such that the infinitesimal elastic assumption is valid. These assumptions lead to the linear stress–strain relation

$$\mathbf{T}^s = -\phi^s p \mathbf{I} + 2\mu \boldsymbol{\epsilon} + \lambda \text{trace}(\boldsymbol{\epsilon}) \mathbf{I}, \quad (4)$$

where p is the fluid pressure, μ and λ are the Lamé stress constants and $\mathbf{I}$ is the identity tensor. The strain tensor is given by

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad (5)$$

where $\mathbf{u}$ is the displacement vector of the solid matrix. Note that $\text{trace}(\boldsymbol{\epsilon}) = \nabla \cdot \mathbf{u}$. From the infinitesimal elastic assumption it follows that the boundary conditions are applied at the original position of the material.

The fluid phase is assumed to be a Newtonian viscous fluid. This assumption leads to

$$\mathbf{T}^f = -\phi p \mathbf{I} + 2\mu^f \left(\mathbf{d} - \frac{1}{3} \text{trace}(\mathbf{d}) \mathbf{I} \right), \quad (6)$$

where μ^f is the viscous stress constant. The rate of strain tensor is given by

$$\mathbf{d} = \frac{1}{2}(\nabla \mathbf{v}^f + (\nabla \mathbf{v}^f)^T).$$

It is now assumed that the internal fluid viscosity is negligible compared with the momentum transfer, hence $\mu^f = 0$ is set in (6). With this assumption, the summation of (3), (4) and (6) lead to

$$\nabla \cdot \boldsymbol{\sigma} = \nabla p, \quad (7)$$

where the effective stress tensor is defined by

$$\boldsymbol{\sigma} = 2\mu \boldsymbol{\epsilon} + \lambda \text{trace}(\boldsymbol{\epsilon}) \mathbf{I}. \quad (8)$$

Note that $\text{trace}(\boldsymbol{\sigma}) = (2\mu + 3\lambda) \text{trace}(\boldsymbol{\epsilon})$.

Equation (7) is equivalent to the equilibrium equation of linear elasticity, i.e. Hooke's law applies, with a forcing term given by the gradient of the fluid pressure.

The momentum transfer in (3) is modelled by the linear relation

$$\boldsymbol{\pi}^s = -\boldsymbol{\pi}^f = k(\mathbf{v}^f - \mathbf{v}^s),$$

where k is assumed to be a constant and uniform scalar. Again, with the assumption that the internal viscosity is negligible compared with the momentum transfer, (2), (3) for $l = f$ and (6) lead to

$$\mathbf{q} = -K \nabla p, \quad (9)$$

where the permeability is defined by $K = \phi^2/k$. This equation can be interpreted as Darcy's law relative to the motion of the solid matrix.

The velocity and displacement of the solid phase are related by

$$\mathbf{v}^s = \frac{\partial \mathbf{u}}{\partial t}. \quad (10)$$

While there are several ways of formulating the poroelastic equations, here (1), (5) and (7)–(10) are used as the governing set. This set consists of 22 coupled equations with the unknowns p , $\mathbf{q}$, $\mathbf{u}$, $\mathbf{v}^s$, ϵ and σ .

3. A cubic sponge

Let a rubber sponge being saturated with water. A load applied to it will produce a compression. It is assumed that all assumptions of the previous section are fulfilled.

Here a cubic sponge is considered with edges of length $2L$. For elegance of calculus the origin is placed in the centre of the sponge, and the coordinate axes parallel to its edges. The sponge is reduced to a smaller cube by an instantaneous compression. Water is squeezed out of the pores. After this instantaneous compression the faces of the sponge are fixed in their compressed position and are sealed. The water in the pores and the stresses of the solid matrix redistribute to a stationary situation. Although it seems impossible to set up an experiment for this situation, it is very easy to implement it in a numerical code.

In order to model the situation at hand the governing set of equations (1), (5) and (7)–(10) is supplemented with suitable boundary conditions. For each face $\mathcal{F}$ of the cube, $\mathbf{v}$ is defined to be the outward unit normal vector and $\boldsymbol{\tau}$ symbolizes the tangential unit vectors. Consider the following boundary conditions:

- $\mathbf{u} \cdot \mathbf{v} = u_0 < 0$, where u_0 is a given constant value,
- $\boldsymbol{\tau} \cdot \sigma \cdot \mathbf{v} = 0$, i.e. perfect slip applies,
- $\mathbf{q} \cdot \mathbf{v} = 0$, i.e. an impervious boundary is assumed.

For infinite time the set of equations reduces to

$$\begin{aligned} \nabla \cdot \mathbf{q} &= 0, \\ \epsilon &= \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \\ \nabla \cdot \sigma &= \nabla p, \\ \sigma &= 2\mu\epsilon + \lambda \text{trace}(\epsilon)\mathbf{I}, \\ \mathbf{q} &= -K\nabla p, \\ \mathbf{v}^s &= \mathbf{0}. \end{aligned}$$

From the boundary conditions it follows that the solution of the governing set of equations is $p = 0$ (in fact, p is equal to some arbitrary constant value), $\mathbf{q} = \mathbf{0}$ and

$$\begin{aligned}\mathbf{u} &= \frac{u_0}{L} \mathbf{x}, \\ \boldsymbol{\epsilon} &= \frac{u_0}{L} \mathbf{I}, \\ \boldsymbol{\sigma} &= (2\mu + 3\lambda) \boldsymbol{\epsilon} = (2\mu + 3\lambda) \frac{u_0}{L} \mathbf{I}.\end{aligned}\tag{11}$$

For finite time the derivation of the solution is more elaborated. Two assumptions will be made that lead to the solution. It will be checked that this solution fulfills all necessary properties. As the initial condition it is assumed that all unknowns are equal to zero.

The first assumption is that the displacement fulfils

$$\mathbf{u}(\mathbf{x}, t) = \frac{u_0}{L} \mathbf{x} + \sum_{k=1}^{\infty} \gamma_k \alpha_k(t) (\beta'_k(x_1), \beta'_k(x_2), \beta'_k(x_3))^T, \tag{12}$$

where α_k and β_k are smooth scalar functions such that $\beta'_k(-x) = -\beta'_k(x)$ and $\beta'_k(L) = 0$, which are reasonable consequences of the symmetry and boundary conditions. By β'_k the derivative of β_k is denoted. From (12) it follows that $\mathbf{u} \cdot \boldsymbol{\nu} = u_0$ at each boundary face. It follows directly from (10) that

$$\mathbf{v}^s(\mathbf{x}, t) = \sum_{k=1}^{\infty} \gamma_k \alpha'_k(t) (\beta'_k(x_1), \beta'_k(x_2), \beta'_k(x_3))^T,$$

where α'_k denotes the derivative of α_k . Now the specific discharge needs to be chosen such that (1) holds. The second assumption is $\mathbf{q}(\mathbf{x}, t) = -\mathbf{v}^s(\mathbf{x}, t)$, i.e.

$$\mathbf{q}(\mathbf{x}, t) = - \sum_{k=1}^{\infty} \gamma_k \alpha'_k(t) (\beta'_k(x_1), \beta'_k(x_2), \beta'_k(x_3))^T.$$

It follows that $\mathbf{q} \cdot \boldsymbol{\nu} = 0$ at each boundary face. From (9) it follows that

$$p(\mathbf{x}, t) = \frac{1}{K} \sum_{k=1}^{\infty} \gamma_k \alpha'_k(t) (\beta_k(x_1) + \beta_k(x_2) + \beta_k(x_3)).$$

The strain tensor is given by (5), i.e.

$$\boldsymbol{\epsilon}(\mathbf{x}, t) = \frac{u_0}{L} \mathbf{I} + \sum_{k=1}^{\infty} \gamma_k \alpha_k(t) \text{diag}(\beta''_k(x_1), \beta''_k(x_2), \beta''_k(x_3)),$$

and the effective stress tensor by (8), i.e.

$$\begin{aligned}\boldsymbol{\sigma}(\mathbf{x}, t) = & (2\mu + 3\lambda) \frac{u_0}{L} \mathbf{I} + 2\mu \sum_{k=1}^{\infty} \gamma_k \alpha_k(t) \text{diag}(\beta_k''(x_1), \beta_k''(x_2), \beta_k''(x_3)) \\ & + \lambda \sum_{k=1}^{\infty} \gamma_k \alpha_k(t) (\beta_k''(x_1) + \beta_k''(x_2) + \beta_k''(x_3)) \mathbf{I}.\end{aligned}$$

It follows that $\boldsymbol{\tau} \cdot \boldsymbol{\sigma} \cdot \mathbf{v} = 0$ at each boundary face. The equilibrium equation (7) leads to

$$(2\mu + \lambda) \alpha_k(t) \beta_k'''(x) - \frac{1}{K} \alpha_k'(t) \beta_k'(x) = 0.$$

If α_k' and β_k''' are not equal to zero, this equation results into

$$(2\mu + \lambda) K \frac{\alpha_k(t)}{\alpha_k'(t)} = \frac{\beta_k'(x)}{\beta_k'''(x)}. \quad (13)$$

Define the constant C by $(2\mu + \lambda) K = L^2/C$, then the dimension of C is time. The following collection of solutions is obtained from (13) and the conditions for β_k' :

$$\begin{aligned}\alpha_k(t) &= \exp\left(-\frac{k^2 \pi^2 t}{C}\right), \\ \beta_k'(x) &= L \sin\left(\frac{k \pi x}{L}\right),\end{aligned} \quad k = 1, 2, \dots$$

The functions α_k are chosen such that the solution (11) is obtained for infinite time.

3.1. Displacement

The displacement is given by

$$u_i(\mathbf{x}, t) = \frac{u_0}{L} x_i + L \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{k \pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right). \quad (14)$$

The initial condition leads to

$$0 = u_i(\mathbf{x}, 0) = \frac{u_0}{L} x_i + L \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{k \pi x_i}{L}\right). \quad (15)$$

Integration from 0 to L gives

$$\frac{u_0}{L} \int_0^L x \sin\left(\frac{k \pi x}{L}\right) dx + \frac{L^2}{2} \gamma_k = 0.$$

Partial integration results into

$$\gamma_k = \frac{u_0}{L} \frac{2}{k \pi} (-1)^k.$$

If $x_i = L$, then $u_i = u_0$. In figure 1 we have plotted u_i for $x_i = L/2$.

Figure 1. The displacement u_i for $x_i = L/2$.

3.2. Velocity of the solid phase

It follows directly from (10) and (14) that

$$v_i^s(\mathbf{x}, t) = -L \sum_{k=1}^{\infty} \gamma_k \frac{k^2 \pi^2}{C} \sin\left(\frac{k\pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right).$$

Double differentiation of (15) leads to the conclusion that

$$v_i^s(\mathbf{x}, 0) = -L \sum_{k=1}^{\infty} \gamma_k \frac{k^2 \pi^2}{C} \sin\left(\frac{k\pi x_i}{L}\right) = 0.$$

3.3. Specific discharge

It is assumed that $\mathbf{q}(\mathbf{x}, t) = -\mathbf{v}^s(\mathbf{x}, t)$. Indeed, if $x_i = L$, then $q_i = 0$. In figure 2 we have plotted q_i for $x_i = L/4, L/2, 3L/4$.

It can be observed that for fixed x_i the function $q_i(x_i, \cdot)$ attains its maximum for finite positive time. Moreover, for a larger value of x_i the specific discharge q_i attains a larger maximum for a smaller time. In figure 3 we have plotted $\arg \max_t q_i(\mathbf{x}, t)$, i.e. the time for which q_i attains its maximum, for $0 < x_i < L$.

3.4. Fluid pressure

From (9) it follows that

$$p(\mathbf{x}, t) = (2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k k \pi \sum_{i=1}^3 \cos\left(\frac{k\pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right).$$

Figure 2. The specific discharge q_i for $x_i = L/4, L/2, 3L/4$. Figure 3. Times for which the specific discharge q_i attains its maximum.

Differentiation of (15) leads to the conclusion that

$$p(\mathbf{x}, 0) = (2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k k \pi \sum_{i=1}^3 \cos\left(\frac{k\pi x_i}{L}\right) = -3(2\mu + \lambda) \frac{u_0}{L}.$$

Furthermore,

$$\begin{aligned} p(\mathbf{0}, t) &= 3(2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k k \pi \exp\left(-\frac{k^2 \pi^2 t}{C}\right) \\ &= 6(2\mu + \lambda) \frac{u_0}{L} \sum_{k=1}^{\infty} (-1)^k \exp\left(-\frac{k^2 \pi^2 t}{C}\right). \end{aligned}$$

Figure 4. The pressure $p(\mathbf{0}, t)$.

This function is plotted in figure 4. In particular,

$$\lim_{t \downarrow 0} p(\mathbf{0}, t) = -3(2\mu + \lambda) \frac{u_0}{L}.$$

3.5. Strain tensor

The strain tensor is given by (5), i.e.

$$\begin{aligned} \epsilon(\mathbf{x}, t) = & \frac{u_0}{L} \mathbf{I} + \sum_{k=1}^{\infty} \gamma_k k \pi \exp\left(-\frac{k^2 \pi^2 t}{C}\right) \\ & \times \text{diag}\left(\cos\left(\frac{k\pi x_1}{L}\right), \cos\left(\frac{k\pi x_2}{L}\right), \cos\left(\frac{k\pi x_3}{L}\right)\right). \end{aligned}$$

In particular, $\lim_{t \downarrow 0} \epsilon(\mathbf{0}, t) = \mathbf{0}$. If $x_i = L$, then $\epsilon_{ii} = u_0/L$.

3.6. Effective stress tensor

The effective stress tensor is given by (8), i.e.

$$\begin{aligned} \sigma(\mathbf{x}, t) = & (2\mu + 3\lambda) \frac{u_0}{L} \mathbf{I} + 2\mu \sum_{k=1}^{\infty} \gamma_k k \pi \exp\left(-\frac{k^2 \pi^2 t}{C}\right) \\ & \times \text{diag}\left(\cos\left(\frac{k\pi x_1}{L}\right), \cos\left(\frac{k\pi x_2}{L}\right), \cos\left(\frac{k\pi x_3}{L}\right)\right) \\ & + \lambda \sum_{k=1}^{\infty} \gamma_k k \pi \sum_{i=1}^3 \cos\left(\frac{k\pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right) \mathbf{I}. \end{aligned}$$

Differentiation of (15) leads to the conclusion that $\sigma(\mathbf{x}, 0) = \mathbf{0}$.

4. Plane strain

The three-dimensional solution reduces to an essentially two-dimensional solution if $u_3 = 0$ and $q_3 = 0$. This implies $\epsilon_{3i} = \epsilon_{i3} = 0$, i.e. plane strain is dealt with. The essentially two-dimensional solution is obtained if the boundary conditions of the previous section hold only at the vertical faces of the cubic sponge, i.e. if $x_1 = \pm L$ or $x_2 = \pm L$. At the horizontal faces of the sponge, i.e. if $x_3 = \pm L$, homogeneous boundary conditions are supposed, i.e. $u_3 = 0$ and $q_3 = 0$. This means that the sponge is clamped by impervious horizontal plates.

For infinite time the solution of the governing set of equations (1), (5) and (7)–(10) is $p = 0$, $\mathbf{q} = \mathbf{0}$ and

$$\begin{aligned}\mathbf{u} &= \frac{u_0}{L} (x_1, x_2, 0)^T, \\ \boldsymbol{\epsilon} &= \frac{u_0}{L} \text{diag}(1, 1, 0), \\ \boldsymbol{\sigma} &= 2(\mu + \lambda) \frac{u_0}{L} \text{diag}\left(1, 1, \frac{\lambda}{\mu + \lambda}\right).\end{aligned}$$

For finite time the displacement is given by

$$u_i(\mathbf{x}, t) = \frac{u_0}{L} x_i + L \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{k\pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right), \quad i = 1, 2.$$

It follows directly from (10) that

$$v_i^s(\mathbf{x}, t) = -L \sum_{k=1}^{\infty} \gamma_k \frac{k^2 \pi^2}{C} \sin\left(\frac{k\pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right), \quad i = 1, 2.$$

It is assumed that $\mathbf{q}(\mathbf{x}, t) = -\mathbf{v}^s(\mathbf{x}, t)$. From (9) it follows that

$$p(\mathbf{x}, t) = (2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k k \pi \sum_{i=1}^2 \cos\left(\frac{k\pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right),$$

and therefore

$$p(\mathbf{x}, 0) = -2(2\mu + \lambda) \frac{u_0}{L}.$$

The strain tensor is given by (5), i.e.

$$\begin{aligned}\boldsymbol{\epsilon}(\mathbf{x}, t) &= \frac{u_0}{L} \text{diag}(1, 1, 0) + \sum_{k=1}^{\infty} \gamma_k k \pi \exp\left(-\frac{k^2 \pi^2 t}{C}\right) \\ &\quad \times \text{diag}\left(\cos\left(\frac{k\pi x_1}{L}\right), \cos\left(\frac{k\pi x_2}{L}\right), 0\right).\end{aligned}$$

The effective stress tensor is given by (8), i.e.

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x}, t) = & 2(\mu + \lambda) \frac{u_0}{L} \text{diag}\left(1, 1, \frac{\lambda}{\mu + \lambda}\right) + 2\mu \sum_{k=1}^{\infty} \gamma_k k \pi \exp\left(-\frac{k^2 \pi^2 t}{C}\right) \\ & \times \text{diag}\left(\cos\left(\frac{k\pi x_1}{L}\right), \cos\left(\frac{k\pi x_2}{L}\right), 0\right) \\ & + \lambda \sum_{k=1}^{\infty} \gamma_k k \pi \sum_{i=1}^2 \cos\left(\frac{k\pi x_i}{L}\right) \exp\left(-\frac{k^2 \pi^2 t}{C}\right) \mathbf{I}. \end{aligned}$$

Note that the plane strain problem is commonly described in terms of the set of equations (1), (5) and (7)–(10) in the two-dimensional space.

5. Conclusions

A saturated cubic rubber sponge is modelled. As boundary conditions the normal component of the displacement is given by some constant value at each boundary face. Furthermore, the boundary is assumed to be impervious. The water in the pores and the stresses of the solid matrix will redistribute to a stationary situation. The obtained solution can be considered as a three-dimensional generalisation of the famous one-dimensional solution of Terzaghi. A two-dimensional plane strain solution is derived also. Both solutions give excellent possibilities to test numerical codes.

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