1 Introduction

Since the work of van Glabbeek [9] there is a general agreement within computer science that bisimulation is the strongest notion of equivalence of interest on labelled transition systems. However, for example in the field of hybrid systems the need is felt for a stronger kind of equivalence than bisimulation. There, the problem of Zeno-behavior (an infinite number of events occurring in a finite time interval [16, 5, 3, 10], also called supertask in philosophy [15, 18]), gives rise to labelled transition systems that are considered different, but cannot be distinguished using bisimulation.

Bisimulation only regards a single transition at a time and is not capable of distinguishing between transfinitely long sequences. For example, the sequences shown in figure 1 are considered bisimilar. To be able to handle Zeno- and other kinds of transfinite behavior, we need to define to which (set of) states an infinitely long sequence of states leads. This is possible in a natural way if a topological structure on the state space of the labelled transition system is given. Topology is a field of mathematics in which general definitions of convergence and accumulation of sequences have been developed (see e.g. [7, 8]).

In this paper we define the notion of a labelled topological transition system, i.e., a labelled transition system where the state space is structured using a topology. Then, we define topological simulation and topological bisimulation. These
notions extend the traditional ones by considering not only single steps but arbitrary long (accumulating) sequences of steps in the transfer (zig-zag) conditions. We prove that these topological notions are a preorder and an equivalence respectively and that they are stronger than the non-topological notions. We also prove that they are topological notions (invariant under the application of continuous transition morphisms).

We study two specific topologies in more detail, viz. the indiscrete topology and the discrete topology. It turns out that for labelled topological transition systems with the indiscrete topology, (bi-)simulation and topological (bi-)simulation coincide under certain conditions. Also, for the discrete topology, the notions do not coincide unless the state spaces are finite.

2 Simulation and Bisimulation in Labelled Transition Systems

The semantics of many of the techniques used in computer science rely on labelled transition systems, structures containing a set of objects representing the physical state of a system (hence the objects are called states), and labelled transitions, representing the behavior that brings a system from one state into another.

Definition 1 (Labelled Transition System). A labelled transition system is a tuple \((X, \Sigma, \rightarrow)\), where \(X\) is the state space, \(\Sigma\) is the set of labels describing behaviors, and \(\rightarrow \subseteq \mathcal{X} \times \Sigma \times \mathcal{X}\) is the transition relation. As an abbreviation we write \(x \xrightarrow{\sigma} y\) for \((x, \sigma, y) \in \rightarrow\).

The transitions in a labelled transition system give rise to sequences of states and labels, called runs.

Definition 2 (Run). Let \(M = (X, \Sigma, \rightarrow)\) be a labelled transition system. A run of \(M\) is a pair \((\bar{x}, \bar{\sigma})\) of partial functions \(\bar{x} : \mathbb{N} \rightarrow X\) and \(\bar{\sigma} : \mathbb{N} \rightarrow \Sigma\) such that

1. either \(\text{dom}(\bar{x}) = \text{dom}(\bar{\sigma}) = \mathbb{N}\) (for infinite runs), or \(\text{dom}(\bar{x}) = [0, N + 1)\) and \(\text{dom}(\bar{\sigma}) = [0, N)\) for some \(N \in \mathbb{N}\) (for finite runs).
2. for all \(n \in \text{dom}(\bar{\sigma})\): \(\bar{x}(n) \xrightarrow{\bar{\sigma}(n)} \bar{x}(n + 1)\).

In the sequel, the functions \(\bar{x}\) and \(\bar{\sigma}\) of a run are also called sequences. The length of a run \((\bar{x}, \bar{\sigma})\) is the cardinality of the domain of \(\bar{\sigma}\).
An example of a labelled transition system is depicted in figure 2. One may see that, for example, the pair of sequences (1 2 3 4, a b c) is a run of length 3 in this system. However, not only the runs of a system, but also the branching structure is important. In order to take this branching structure into account, states from labelled transition systems may be compared using simulation and bisimulation. A state $x$ from some labelled transition system is said to be simulated by a state $y$ from another labelled transition system, if the branching structure and the behavior of $x$ can be mimicked by $y$. The inductive structure of the definition makes sure that all finite runs are considered, although only single steps are compared.

**Definition 3 (Simulation).** Let $M_1 = (X_1, \Sigma, \rightarrow)$ and $M_2 = (X_2, \Sigma, \longrightarrow)$ be labelled transition systems. A binary relation $R \subseteq X_1 \times X_2$ is a simulation if and only if for all $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1 R x_2$

1. if $x_1 \xrightarrow{\sigma} x'_1$ for some $\sigma \in \Sigma$ and $x'_1 \in X_1$, then there exists $x'_2 \in X_2$ such that $x_2 \xrightarrow{\sigma} x'_2$ and $x'_1 R x'_2$.

A state $x_1 \in X_1$ of $M_1$ is simulated by a state $x_2 \in X_2$ of $M_2$ (denoted $M_1, x_1 \preceq M_2, x_2$) if and only if there exists a simulation $R \subseteq X_1 \times X_2$ such that $x_1 R x_2$.

Simulation is a preorder on the states of a system (see [9]). Two states $x_1$ and $x_2$ are defined to be similar if and only if $x_1$ is simulated by $x_2$ and, vice versa, $x_2$ is simulated by $x_1$. Hence, similarity is the equivalence relation $\preceq \cap \preceq^{-1}$. It is not the case that the simulations witnessing $x_1 \preceq x_2$ and $x_2 \preceq x_1$ have to be the same one. If this is the case, the states are called bisimilar [14,12].

**Definition 4 (Bisimulation).** For given labelled transition systems $M_1 = (X_1, \Sigma, \rightarrow)$ and $M_2 = (X_2, \Sigma, \longrightarrow)$, a binary relation $R \subseteq X_1 \times X_2$ is a bisimulation if and only if for all $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1 R x_2$

1. if $x_1 \xrightarrow{\sigma} x'_1$ for some $\sigma \in \Sigma$ and $x'_1 \in X_1$, then there exists $x'_2 \in X_2$ such that $x_2 \xrightarrow{\sigma} x'_2$ and $x'_1 R x'_2$.
2. if \( x_2 \xrightarrow{\sigma} x'_2 \) for some \( \sigma \in \Sigma \) and \( x'_2 \in X_2 \), then there exists \( x'_1 \in X_1 \) such that \( x_1 \xrightarrow{\sigma} x'_1 \) and \( x'_1 \mathrel{R} x'_2 \).

Two states \( x_1 \in X_1 \) of \( M_1 \) and \( x_2 \in X_2 \) of \( M_2 \) are bisimilar (denoted \( M_1, x_1 \equiv M_2, x_2 \)) if and only if there exists a bisimulation \( R \subseteq X_1 \times X_2 \) such that \( x_1 \mathrel{R} x_2 \).

Bisimilarity is an equivalence that identifies less states than similarity does: \( \equiv \subseteq (\leq \cap \leq^{-1}) \) [9].

**Definition 5.** Let \( M = \langle X, \Sigma, \rightarrow \rangle \) and \( M' = \langle X', \Sigma', \rightarrow' \rangle \) be labelled transition systems. The labelled transition system \( M \) is simulated by the labelled transition system \( M' \) if and only if for any state \( x \in X \) there is a state \( x' \in X' \) such that \( M, x \equiv M', x' \). The labelled transition system \( M \) is bisimulated by the labelled transition system \( M' \) if and only if for any state \( x \in X \) there is a state \( x' \in X' \) such that \( M, x \equiv M', x' \).

Often, when comparing different systems, also sets of initial states \( I_1 \) and \( I_2 \) are given. In such a case, we say that \( M_1 \) is simulated by \( M_2 \) if and only if every initial state in \( I_1 \) is simulated by an initial state in \( I_2 \). In the remainder of this article, we do not consider initial states.

### 3 Topology

Given a set \( X \), a topology \( T \subseteq 2^X \) is a way of adding structure to this set. Roughly speaking, a topology defines which points \( U \subseteq X \) are in the neighborhood of a point \( x \in U \). In literature from the field of computer science, structure on sets is usually added by giving a metric. In [5, 11, 10], this metric is defined on the state space, while [2, 4] use a metric to define structure on the labels. Furthermore, this was, to our knowledge, never used with respect to bisimulation equivalence. Note that giving a metric on a set is only one way of inducing a topology. Alternatively, for example, a complete partial order gives rise to a topology as well [8, 13]. The following definitions are taken from [7].

**Definition 6 (Topology).** Let \( X \) be a set, then \( T \subseteq 2^X \) is a topology on \( X \) if and only if \( \emptyset \in T \), \( X \in T \), every finite intersection of elements of \( T \) is again an element of \( T \), and every arbitrary union of elements of \( T \) is again an element of \( T \).
The elements of $T$ are called open sets. An open set $U \in T$ containing $x \in U$ is called a neighborhood of $x$. The pair $(X, T)$ is called a topological space. Two special topologies are the indiscrete topology $T_0(X) = \{\emptyset, X\}$ and the discrete topology $T_D(X) = 2^X$. They will prove useful later on. As an example, the usual topology on the real numbers $\mathbb{R}$ is the arbitrary union of all the sets $\{x \in \mathbb{R} \mid x_- < x < x_+ \text{ with } x_-, x_+ \in \mathbb{R}\}$ (i.e. the arbitrary union of open intervals $(x_-, x_+)$).

**Definition 7 (Basis).** Let $(X, T)$ be a topological structure. A set $B \subseteq T$ is a basis for $T$ if and only if each non-empty element of $T$ is the union of elements of $B$.

![Fig. 3. Straight, Converging, and Accumulating Sequence](image)

As we have seen in the previous section, the behavior of a labelled transition system gives rise to sequences of states and labels. In figure 3, three possible sequences in a two dimensional state space are shown. The natural (Euclidean) topology in this case is the arbitrary union of all (open) areas around a point, in the same way as the topology of $\mathbb{R}$ is the union of all open intervals. According to this topology the first sequence, only partly visible, behaves in a straight line. The second sequence converges to a point and the third accumulates on a circle. Topologically speaking, a sequence that converges also accumulates. In section 4, we use the concept of accumulation to expand the notion of bisimulation with.

**Definition 8 (Convergence).** Let $(X, T)$ be a topological space, and $x$ a sequence over $X$. This sequence $x$ converges at $y \in X$ according to the topology $T$ if and only if for all neighborhoods $U$ of $y$ ($y \in U \in T$) there is $l \in \text{dom}(x)$ such that for all $m \in \text{dom}(x)$ with $l \leq m$ it holds that $x(m) \in U$.

**Definition 9 (Accumulation).** Let $(X, T)$ be a topological space, and $x$ a sequence over $X$. This sequence $x$ accumulates at $y \in X$ according to the topology $T$ (denoted $x \xrightarrow{T} y$) if and only if for all neighborhoods $U$ of $y$ ($y \in U \in T$) and all $l \in \text{dom}(x)$ there exists $m \in \text{dom}(x)$ such that $l \leq m$ and $x(m) \in U$. 
Note that a sequence may accumulate in multiple accumulation points. Furthermore, a finite sequence accumulates at least at its endpoint. One might want to check that, in the example in figure 3, the third line indeed accumulates on a circle, since every point on the circle (or rather every arbitrarily small open area around a point on the circle) is visited infinitely often in the sequence.

4 Topological Simulation

Recall that simulation is a way of comparing states of labelled transition systems by looking at the branching structure and the possible behavioral sequences. The formal definition of simulation regards two subsequent states and the label describing the behavior that accomplishes a transition from the first state into the second. Because this definition only compares single transitions at a time, finite sequences of labels and states are compared as well, but infinite sequences are not. In a topological space, an infinite sequence of states may accumulate. If we now extend the definition of simulation in order to compare the accumulation points of sequences of states, simulation can distinguish between transfinite behaviors. Note that also all intermediate states need to be related (see figure 4).

Definition 10 (Labelled Topological Transition System). A labelled topological transition system is a tuple \((X, T), \Sigma, \rightarrow\), where \((X, \Sigma, \rightarrow)\) is a labelled transition system and \((X, T)\) is a topological space.

On labelled topological transition systems also a run, simulation and bisimulation can be defined. The definitions are the same as for labelled transition systems. Hence, for these notions, the topology does not play a role. Topology was introduced as a structuring mechanism on the state space in order to define the states where an infinite run accumulates. Next, we present topological versions of simulation and bisimulation that require that also the infinite behavior of the transition systems is taken into account.

Definition 11 (Topological Simulation). Let \(M = \langle (X, T), \Sigma, \rightarrow \rangle\) and \(N = \langle (Y, U), \Sigma, \rightarrow \rangle\) be labelled topological transition systems. A binary relation \(R \subseteq X \times Y\) is a topological simulation if and only if for all \(x_0 \in X\) and \(y_0 \in Y\) such that \(x_0 \mathrel{R} y_0\)

1. for all runs \((x, \sigma)\) of \(M\) and for all \(x_\omega \in X\) such that \(x(0) = x_0\): if \(x \mathrel{\rightarrow} x_\omega\), then there exists a run \((y, \sigma)\) of \(N\) and there exists \(y_\omega \in Y\) such that \(y(0) = y_0\), \(y \mathrel{\rightarrow} y_\omega\), \(x_\omega \mathrel{R} y_\omega\), and \(x(n) \mathrel{R} y(n)\) for all \(n \in \text{dom}(\sigma)\).
A state $x \in X$ of $M$ is topologically simulated by a state $y \in Y$ of $N$ (denoted $M, x \preceq_{\text{top}} N, y$) if and only if there exists a topological simulation $R \subseteq X \times Y$ such that $x R y$.

Note that, since also runs of length 1 are considered, topological simulation is a stronger notion than simulation, which is proven in the next section. As in the case of simulation, the existence of a topological simulation gives rise to a preorder. Based on the notion of topological simulation, similarity $(\preceq_{\text{top}} \cap (\preceq_{\text{top}}))$ and bisimilarity can be defined. We are interested in the latter.

**Definition 12 (Topological Bisimulation).** Let $M = \langle (X, T), \Sigma, \rightarrow \rangle$ and $N = \langle (Y, U), \Sigma, \rightarrow \rangle$ be labelled topological transition systems. Two states $x \in X$ of $M$ and $y \in Y$ of $N$ are topologically bisimilar (denoted $M, x \models_{\text{top}} N, y$) if and only if there exists a binary relation $R \subseteq X \times Y$ such that $x R y$ and both $R$ and $R^{-1}$ are topological simulations.

As was the case in the non-topological setting, topological similarity and topological bisimilarity are different notions: $(\preceq_{\text{top}} \cap (\preceq_{\text{top}})) \neq \models_{\text{top}}$.

### 5 Properties of Topological (Bi)Simulation

In this section, we will give a number of properties of the notions of topological simulation and topological bisimulation. We start with proving that these notions are a preorder and an equivalence respectively. Then, we discuss the relation between the non-topological and topological notions. We show that the topological notions are stronger than the non-topological ones. Finally, we show that the notions are indeed topological [7], i.e., invariant under isomorphism.

**Theorem 1.** Topological simulation $(\preceq_{\text{top}})$ is a preorder.
Proof. We need to prove that the relation $\preceq_{\text{top}}$ is reflexive and transitive. Let $M = \langle (X, T), \Sigma, \rightarrow \rangle$ be a labelled topological transition system. It is trivial to see that the identity relation $\mathcal{I} = \{(x, x) \mid x \in X\}$ is a topological simulation. Hence $M, x \preceq_{\text{top}} M, x$.

Showing that $\preceq_{\text{top}}$ is transitive is more elaborate. Let $M = \langle (X, T), \Sigma, \rightarrow \rangle$, $N = \langle (Y, U), \Sigma, \rightarrow \rangle$, and $O = \langle (Z, V), \Sigma, \rightarrow \rangle$ be labelled topological transition systems. Let $x \in X$, $y \in Y$, and $z \in Z$. Suppose that $M, x \preceq_{\text{top}} N, y$ and $N, y \preceq_{\text{top}} O, z$. We must prove that $M, x \preceq_{\text{top}} O, z$. By definition there exist topological simulations $\mathcal{R}_1 \subseteq X \times Y$ and $\mathcal{R}_2 \subseteq Y \times Z$ such that $x \mathcal{R}_1 y$ and $y \mathcal{R}_2 z$. Let us now verify that the relation $(\mathcal{R}_1 \circ \mathcal{R}_2) \subseteq X \times Z$ such that $x \mathcal{R}_1 \mathcal{R}_2 z$ is also a topological simulation. Therefore, let $x_0 \in X$ and $z_0 \in Z$ such that $x_0 \mathcal{R}_1 y_0$ and $y_0 \mathcal{R}_2 z_0$. Let $(x, \sigma)$ be a run of $M$ and $x_\omega \in X$ such that $x_\omega = x_0$. Suppose that $x_\omega \mathcal{R}_1 x_\omega$. From $x_0 \mathcal{R}_1 y_0$ and $y_0 \mathcal{R}_2 z_0$, we obtain the existence of $y_0 \in Y$ and $y_0 \mathcal{R}_1 y_0$ and $x_0 \mathcal{R}_1 y_0$, implies that there exist a run $(y, \sigma)$ of $N$ and $y_\omega \mathcal{R}_2 z_0$ such that $y_\omega = x_\omega, x_\omega \mathcal{R}_1 x_\omega$ and $x_0 \mathcal{R}_1 y_0$, $y_0 \mathcal{R}_2 z_0$, implies that there exists a run $(x, \sigma)$ of $O$ and $z_\omega \in Z$ such that $y_\omega = x_\omega, x_\omega \mathcal{R}_2 z_\omega$ and $x_0 \mathcal{R}_1 x_\omega$. Therefore, $M, x_0 \preceq_{\text{top}} O, z_0$.

Let us continue by proving that topological bisimulation is indeed an equivalence.

**Theorem 2.** Topological bisimulation $(\leftrightarrow_{\text{top}})$ is an equivalence.

Proof. We need to prove that $\leftrightarrow_{\text{top}}$ is reflexive, symmetrical, and transitive. Let $M = \langle (X, T), \Sigma, \rightarrow \rangle$ be a labelled topological transition system. In the previous proof we found that the identity relation $\mathcal{I}$ is a topological simulation and $x \mathcal{I} x$. Obviously, as $\mathcal{I}^{-1} = \mathcal{I}$, then also $\mathcal{I}^{-1}$ is a topological simulation and $x \mathcal{I}^{-1} x$. Hence, $M, x \leftrightarrow_{\text{top}} M, x$.

Let $M = \langle (X, T), \Sigma, \rightarrow \rangle$ and $N = \langle (Y, U), \Sigma, \rightarrow \rangle$ be labelled topological transition systems. Let $x \in X$ and $y \in Y$. Suppose that $M, x \leftrightarrow_{\text{top}} N, y$. We must prove that $N, y \leftrightarrow_{\text{top}} M, x$. By definition there exists a relation $\mathcal{R} \subseteq X \times Y$ such that $\mathcal{R}$ is a topological simulation with $x \mathcal{R} y$ and $\mathcal{R}^{-1}$ is a topological simu-
lation with \( y \, R^{-1} \, x \). It follows immediately that \( R^{-1} \) witnesses the topological bisimilarity of \( y \) and \( x \), i.e. \( N, y \leftrightarrow_{\text{top}} M, x \).

Let \( M = \langle (X, T), \Sigma, \rightarrow \rangle \), \( N = \langle (Y, U), \Sigma, \rightarrow \rangle \), and \( O = \langle (Z, V), \Sigma, \sim \rangle \) be labelled topological transition systems. Let \( x \in X \), \( y \in Y \), and \( z \in Z \). Suppose that \( M, x \leftrightarrow_{\text{top}} N, y \) and \( N, y \leftrightarrow_{\text{top}} O, z \). We must prove that \( M, x \leftrightarrow_{\text{top}} O, z \). By definition there exist relations \( R_1 \subseteq X \times Y \) and \( R_2 \subseteq Y \times Z \) such that \( R_1 \) is a topological simulation with \( y R \, y \) and \( R_2 \) is a topological simulation with \( y R \, y \). In the previous proof we have found that the relation \( R_1 \circ R_2 \) is a topological simulation with \( x \circ R \circ z \). Similarly, we obtain that \( R_2 \circ R_1 \circ R \) is a topological simulation with \( z \circ R \circ x \). As \( (R_1 \circ R_2) \circ R = R_2 \circ R \circ R \), we thus have \( M, x \leftrightarrow_{\text{top}} O, z \).

Next, we will study the relations between the standard notions of simulation and bisimulation and their topological counterparts. As it turns out, the topological versions are stronger than the standard ones.

**Theorem 3.** Let \( M = \langle (X, T), \Sigma, \rightarrow \rangle \) and \( M' = \langle (X', T'), \Sigma, \rightarrow \rangle \) be labelled topological transition system. For any \( x \in X \) and \( x' \in X' \): if \( M, x \leftrightarrow_{\text{top}} M', x' \), then \( M, x \leftrightarrow_{\text{top}} M', x' \), and if \( M, x \leftrightarrow_{\text{top}} M', x' \), then \( M, x \leftrightarrow_{\text{top}} M', x' \).

**Proof.** This theorem follows immediately from the observation that any topological simulation \( R \subseteq X \times X' \) is also an ordinary simulation. Let \( x, y, x' \in X \) and \( \sigma \in \Sigma \) such that \( x R \, y \) and \( x \xrightarrow{\sigma} x' \). Then, by definition \( (xx', \sigma) \) is a run of \( M' \) of length 1. Furthermore, as \( xx' \) is a finite sequence, we have \( xx' \xrightarrow{T} x' \). Since \( R \) is a topological simulation, we have the existence of a run \( (yy', \sigma) \) of \( M' \) such that \( y \xrightarrow{\sigma} y' \). We conclude that \( R \) is a simulation. The proof of the second part of the lemma follows the same reasoning and is therefore omitted.

On topological spaces the notion of isomorphism (sometimes called homeomorphism) is defined in order to capture that the spaces have a corresponding structure. We will show that topological simulation and topological bisimulation are topologically invariant. A key ingredient in the definition of isomorphism is that of a continuous mapping.

**Definition 13 (Continuity).** Let \( (X, T) \) and \( (X', T') \) be topological spaces. A mapping \( f : X \rightarrow X' \) is continuous if and only if \( f^{-1}(U') \in T \) for each \( U' \in T' \). The inverse image \( f^{-1} : 2^{X'} \rightarrow 2^X \) of \( f \) is for all \( V' \in 2^{X'} \) defined as \( f^{-1}(V') = \{ v \in X \mid f(v) \in V' \} \).

In case \( f : X \rightarrow X' \) is injective, the inverse mapping of \( f \) is denoted in the same way: \( f^{-1} : X' \rightarrow X \).
Lemma 1. Let \((X, T)\) and \((X', T')\) be arbitrary topological spaces and let \(f : X \to X'\) be an arbitrary continuous mapping. For any sequence \(\tau\) over \(X\), and any \(x_\omega \in X\): if \(\tau x_\omega \), then \(f \circ \tau x_\omega \).

Proof. Suppose that \(\tau x_\omega \). We have to prove that \(f \circ \tau x_\omega \). Let \(U' \in T'\) be an arbitrary neighborhood of \(f(x_\omega)\) and let \(l \in \text{dom}(f \circ \tau)\). Since \(f\) is a continuous mapping between the topological spaces, we have the existence of a neighborhood \(f^{-1}(U') \in T\) of \(x_\omega\) (i.e. of \(f^{-1}(f(x_\omega))\)). Furthermore, by definition, we have that \(\text{dom}(\tau) = \text{dom}(f \circ \tau)\). From \(\tau x_\omega \) we then have that there exists \(m \in \text{dom}(\tau)\) such that \(l \leq m\) and \(\tau(m) \in f^{-1}(U')\). Then also there exists \(m \in \text{dom}(f \circ \tau)\) such that \(l \leq m\) and \((f \circ \tau)(m) \in f(f^{-1}(U'))\).

Definition 14 (Transition morphism). Let \(M = \langle (X, T), \Sigma, \to \rangle\) and \(M' = \langle (X', T'), \Sigma', \to' \rangle\) be labelled topological transition systems. A mapping \(f : X \to X'\) is a transition morphism if and only if for all \(x, y \in X\) and \(\sigma \in \Sigma\): if \(x \sigma y\), then \(f(x) \to' f(y)\).

Lemma 2. Let \(M = \langle (X, T), \Sigma, \to \rangle\) and \(M' = \langle (X', T'), \Sigma, \to' \rangle\) be arbitrary labelled topological transition systems and let \(f : X \to X'\) be an arbitrary transition morphism. For any run \((\tau, \sigma)\) of \(M\), \((f \circ \tau, \sigma)\) is a run of \(M'\).

Proof. This follows immediately from the definition of a run and the fact that \(f\) is a transition morphism.

Definition 15 (Isomorphism). The labelled topological transition systems \(M = \langle (X, T), \Sigma, \to \rangle\) and \(M' = \langle (X', T'), \Sigma, \to' \rangle\) are isomorphic if and only if there exists a bijective mapping \(f : X \to X'\) such that \(f\) is a continuous transition morphism from \(X\) to \(X'\) and \(f^{-1}\) is a continuous transition morphism from \(X'\) to \(X\).

Theorem 4 (Topological invariance of topological simulation). Let \(M = \langle (X, T), \Sigma, \to \rangle\) and \(M' = \langle (X', T'), \Sigma, \to' \rangle\) be labelled topological transition systems. Let \(f : X \to X'\) be a continuous transition morphism from \(X\) to \(X'\). Then, \(M, x \preceq_{\text{top}} M', f(x)\) for all \(x \in X\).

Proof. Define \(R = \{(x, f(x)) \mid x \in X\}\). We will prove that \(R\) is a topological simulation. Thereto, consider an arbitrary pair \((x, f(x)) \in R\). Let \((\tau, \sigma)\) be an arbitrary run of \(M\) such that \(\tau(0) = x\). Let \(x_\omega \in X\) such that \(\tau x_\omega \). From the fact that \((\tau, \sigma)\) is a run of \(M\) and the fact that \(f\) is a transition morphism,
we obtain, by Lemma 2, that \((f \circ \sigma)(0) = f(x(0)) = f(x)\). From the fact that \(x(0) \rightarrow x\) and the fact that \(f\) is continuous, we obtain, by Lemma 1, that \(f \circ \sigma(0) = f(x)\). Note that by definition \(x, R f(x)\) and \(\sigma(n) \rightarrow f(\sigma(n))\) for all \(n \in \text{dom}(\sigma)\). This proves that \(R\) is a topological simulation.

**Theorem 5 (Topological invariance of topological bisimulation).** For any two \(f\)-isomorphic labelled topological transition systems \(M = \langle (X, T), \Sigma, \rightarrow \rangle\) and \(M' = \langle (X', T'), \Sigma, \rightarrow' \rangle\) and any state \(x \in X\) we have \(M, x \simtop M', f(x)\).

**Proof.** As \(f\) is a continuous transition morphism from \(X\) to \(X'\) we have that \(R = \{ (x, f(x)) \mid x \in X \}\) is a topological simulation as was proven in the proof of the previous theorem. Similarly, as \(f^{-1}\) is a continuous transition morphism from \(X'\) to \(X\) we have that \(S = \{ (x', f^{-1}(x')) \mid x' \in X' \}\) is a topological simulation. As \(S = R^{-1}\) we have proven that \(R\) is a topological bisimulation.

A consequence of the previous theorem and the transitivity of both topological simulation and topological bisimulation is that applying different isomorphisms to related states from different labelled topological transition systems results in related states again (see figure 5).

![Fig. 5. Commuting Diagrams for Isomorphisms \(f\) and \(g\)](image)

## 6 Indiscrete and Discrete Topologies

Recall that \(T_1(X) = \{ \emptyset, X \}\) denotes the indiscrete topology for a set \(X\). We show that for labelled topological transition systems with indiscrete topologies, the topological and non-topological notions of (bi-)simulation coincide provided that, non-topologically speaking, each state has a (bi-)similar state in the other labelled transition system.

**Theorem 6.** Let \(M = \langle (X, T_1(X)), \Sigma, \rightarrow \rangle\) and \(M' = \langle (X', T_1(X')), \Sigma, \rightarrow' \rangle\) be labelled topological transition systems such that \(M\) is simulated by \(M'\). For any \(x \in X\) and \(x' \in X'\): if \(M, x \simtop M', x'\), then \(M, x \simtop M', x'\).
Proof. Suppose that \( M, x \not\sim M', x' \) for some \( x \in X \) and \( x' \in X' \). Then, there exists a simulation relation \( R \) with \( x \sim R x' \). We will prove that the relation \( R' = \{(y, y') \in X \times X' | M, y \not\sim M', y'\} \) is a topological simulation with \( x \sim R' x' \). Note that \( R' \) is a simulation. Now, consider arbitrary \( x_0 \in X \) and \( y_0 \in X' \) such that \( x_0 \sim R' y_0 \). Let \( (x, \sigma) \) be a run of \( M \) and \( x_\omega \in X \) such that \( \sigma(0) = x_0 \). Suppose that \( x \sim \rightarrow_{T_1(X)} x_\omega \). Now, we have to prove the existence of a run \( (y, \tau) \) of \( M' \) and \( y_\omega \in X' \) such that \( y(0) = y_0 \), \( y \sim \rightarrow_{T_1(X')} y_\omega \), and \( x(\omega) \sim R' y(\omega) \) for all \( \omega \in \text{dom}(\sigma) \).

From \( x_0 \sim R' y_0 \) and the fact that \( R' \) is a simulation, we obtain the existence of a run \( (y, \sigma) \) such that \( y(0) = y_0 \) and \( x(\omega) \sim R' y(\omega) \) for all \( \omega \in \text{dom}(\sigma) \). Furthermore, a special property of \( T_1(X) \) is that every sequence accumulates to every point in \( X \). Because \( M \) is simulated by \( M' \) we have the existence of a \( y_\omega \in X' \) such that \( x_\omega \sim R' y_\omega \). The indiscrete topology on \( X' \) then guarantees that \( y \sim \rightarrow_{T_1(X')} y_\omega \). This concludes the proof.

Theorem 7. Let \( M = \{(X, T_1(X)), \Sigma, \rightarrow\} \) and \( M' = \{(X', T_1(X')), \Sigma, \rightarrow\} \) be labelled topological transition systems such that \( M \) is bisimulated by \( M' \). For any \( x \in X \) and \( x' \in X' \): if \( M, x \sim M', x' \), then \( M, x \sim_{\text{top}} M', x' \).

Proof. This proof is omitted as it follows precisely the same lines as the proof of Theorem 6.

Corollary 1. Let \( M = \{(X, T_1(X)), \Sigma, \rightarrow\} \) and \( M' = \{(X', T_1(X')), \Sigma, \rightarrow\} \) be labelled topological transition systems. Then we have:

1. if \( M \) is simulated by \( M' \), then for any \( x \in X \) and \( x' \in X' \): \( M, x \sim M', x' \) if and only if \( M, x \sim_{\text{top}} M', x' \), and
2. if \( M \) is bisimulated by \( M' \), then for all \( x \in X \) and \( x' \in X' \): \( M, x \sim M', x' \) if and only if \( M, x \sim_{\text{top}} M', x' \).

As a direct consequence of the fact that simulation and bisimulation are reflexive, we have that the non-topological and topological notions coincide, when (different) states from one single labelled topological transition system with the indiscrete topology are compared.

Corollary 2. Let \( M = \{(X, T_1(X)), \Sigma, \rightarrow\} \) be a labelled topological transition system. For all \( x, x' \in X \): \( M, x \sim M', x' \) if and only if \( M, x \sim_{\text{top}} M', x' \) and \( M, x \sim M', x' \) if and only if \( M, x \sim_{\text{top}} M', x' \).

If we consider the discrete topology, we do not have that the topological and non-topological notions coincide! Consider the labelled transition system and the relation \( R \) on the states of the labelled transition system given in figure 6.
The relation $\mathcal{R} = \{(1, n) \mid n \in \mathbb{N} \land n > 1\}$, as depicted (suggestively) in the figure, is a witness for the following non-topological facts:

- state 1 is simulated by state 2, i.e., $1 \prec 2$;
- state 2 is simulated by state 1, i.e., $2 \prec 1$;
- the states 1 and 2 are bisimilar, i.e., $1 \leftrightarrow 2$.

Observe that we are now comparing states from the same labelled transition system. Hence, as there can be no misunderstanding about from which labelled transition system the states originate, we omit the labelled transition system from the notations.

Now, consider the topological notions under the assumption that the state space $X$ of this labelled transition system is structured by means of the discrete topology $T_D(X) = 2^X$. State 2 is still simulated by state 1: $2 \prec_{\text{top}} 1$. This is due to the following observations. State 2 has no infinite runs that accumulate. Hence, the infinite run does not have to be mimicked by such a run from state 1. In this setting, however, state 1 is not simulated by state 2: $1 \not\prec_{\text{top}} 2$. State 1 has an infinite run that accumulates in state 1. Hence, state 2 should also have such a run and moreover it should accumulate in a state related to state 1. However, the run by state 2 does not accumulate at all. The same observations lead to the conclusion that state 1 and state 2 are not topologically bisimilar: $1 \not\leftrightarrow_{\text{top}} 2$.

Traditionally, in computer science, systems are assumed to be discrete and finite. Above we have shown that the assumption that the state spaces are structured by means of the discrete topology is not sufficient for concluding that the topological and non-topological notions coincide. Based on this, the reader might be tempted to believe that for labelled topological transition systems with a finite state space and an arbitrary topology the non-topological and topological notions coincide. Again, this is not the case. Consider the labelled transition system depicted in figure 6. The state space of this labelled transition system is finite: $X = \{1, 2, 3, 4, 5, 6\}$. Considering the non-topological notions, we observe that the states 2 and 3 simulate each other and are bisimilar.
Assume that the topology on this state space is given by the basis

\[ B = \{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}\}. \]

The open sets from this basis with more than one element are clustered in the figure. Now, due to the topological structure imposed on the state space, there is an infinite run \((x, \sigma)\) with, for \(n \in \mathbb{N}\), \(x(n) = 2\) and \(\sigma(n) = a\) that accumulates in state 1. In order for state 2 to be topologically simulated by state 3, this must mean that there is also an infinite run \((y, \tau)\) with \(y(0) = 3\) that accumulates in a state that is related to state 1. The only candidates for this accumulation are the states 3 and 4. But, neither of these can be related to state 1, as state 1 can execute the action \(b\) and states 3 and 4 cannot. A similar reasoning shows that state 3 cannot be simulated by state 2. Therefore, we have \(2 \neq_{\text{top}} 3\) and \(3 \neq_{\text{top}} 2\). As a consequence, the states are also not topologically bisimilar.

If the state space of a labelled transition system is finite and is structured by the discrete topology, however, the notions of (bi-)simulation and topological (bi-)simulation coincide.

**Theorem 8.** For given labelled transition systems \(M = ((X, TD(X)), \Sigma, \rightarrow)\) and \(M' = ((X', T'), \Sigma, \rightarrow)\) with finite state space \(X'\), we have that for all \(x \in X\) and \(x' \in X'\): \(M, x \preceq_{\text{top}} M', x'\) if and only if \(M, x \preceq M', x'\).

**Proof.** The proof that the topological simulation implies the ordinary simulation follows from Theorem 3. It suffices to prove that ordinary simulation implies topological simulation.

Suppose that \(M, x \preceq M', x'\) is witnessed by the simulation \(R\). We will prove that \(R\) is also a topological simulation. Thereto, let \((x, \sigma)\) be a run of \(M\) with \(x(0) = x\) and let \(x_{\omega} \in X\). Suppose that \(x_{\omega} \rightarrow x_{\omega}\). Let, for all \(n \in \text{dom}(\sigma)\), \(\sigma_n : \mathbb{N} \rightarrow \Sigma\) be defined by \(\sigma_n(k) = \sigma(k)\) for all \(k < n\), and undefined otherwise. Hence, \(\text{dom}(\sigma_n) = [0, n)\).
First, we show, by induction on the natural number \(n\), that there exists a run \((y_n, \sigma_n)\) of \(M'\) of length \(n\) with \(y(0) = x'\) such that for all \(k \leq n\) we have \(\pi_0(n) \mathcal{R} y(n)\). For \(n = 0\), we need to prove \(\pi_0(0) \mathcal{R} y(0)\). Using \(\pi_0(0) = x, y(0) = x'\), and \(x \mathcal{R} x'\), this follows immediately. Now, suppose there exists a run \((y_n, \sigma_n)\) such that \(y(0) = x'\) and \(\pi_0(k) \mathcal{R} y_n(k)\) for all \(k \leq n\) (the induction hypothesis).

As \(\pi_0(n) \mathcal{R} y(n)\), \(\pi_0(n) \xrightarrow{\sigma(n)} \pi_0(n + 1)\) and \(\mathcal{R}\) is a simulation relation we have the existence of \(y' \in X'\) such that \(y(n) \xrightarrow{\sigma(n)} y'\) and \(\pi_0(n + 1) \mathcal{R} y'\). Define \(y_{n+1}\) by \(y_{n+1}(i) = y_n(i)\) for all \(i \leq n\), \(y_{n+1}(n + 1) = y'\), and undefined otherwise. Then we have the existence of a run \((y_{n+1}, \sigma_{n+1})\) of \(M'\) such that \(\pi_0(k) \mathcal{R} y_{n+1}(k)\) for all \(k \leq n + 1\).

All that remains to be proven is the existence of an accumulation point \(y_\omega \in X'\) such that \(y \xrightarrow{\omega} y_\omega\) and \(x_\omega \mathcal{R} y_\omega\). Obviously, if \(y\) is finite, the last element is the accumulation point. On the other hand, if \(y\) is infinite, then, using the facts that \(\pi_0(X) \xrightarrow{\omega} x_\omega\) and that \(T_D(X)\) is the discrete topology imply that \(x_\omega\) itself occurs infinitely often in \(\pi_0\) and each of those is bisimilar to the corresponding position in the sequence \(y\) and there can be only finitely many different states, at least one of them occurs infinitely often. Hence it is an accumulation point. Obviously, also \(x_\omega \mathcal{R} y_\omega\) in this case.

**Theorem 9.** For given labelled transition systems \(M = \langle (X, T_D(X)), \Sigma, \rightarrow \rangle\) and \(M' = \langle (X', T_D(X')), \Sigma, \rightarrow' \rangle\) with finite state spaces \(X\) and \(X'\), we have that for all \(x \in X\) and \(x' \in X'\): \(M, x \triangleleft_{\text{top}} M', x'\) if and only if \(M, x \triangleleft M', x'\), and \(M, x \leftrightarrow_{\text{top}} M', x'\) if and only if \(M, x \leftrightarrow M', x'\).

**Proof.** The theorem follows immediately from the previous theorem.

### 7 Conclusive remarks

We may conclude that the general agreement, that bisimulation is the strongest notion of equivalence of interest on labelled transition systems, common since the work of van Glabbeek [9], holds, as long as there is no topological structure on the state space. When phenomena like Zeno-behavior in hybrid systems are a reason to introduce and study accumulation points of sequences, a topological structure on the state space is a prerequisite. Choosing such a topology is a creative process, although it will often be guided by knowledge of the application domain.

In this paper, we have given definitions of topological simulation and bisimulation that answer to this need. Amongst others, we have shown that a discrete topology
results in normal bisimulation for finite state spaces, while other topologies make it possible to differentiate between transfinite behaviors, like Zeno-behavior.

The type of labelled transition systems considered in this paper is rather limited. In literature, labelled transition systems not only have a transition relation but also one or more predicates are defined on the state space to indicate for example initial and final states. We have indicated how the notion of initial state can be added to the definitions we have given in this paper. Future research may be concerned with how to deal with predicates on labelled transition systems in general.

Büchi automata and other types of automata on infinite words [17] are usually equipped with one or more acceptance sets and a more sophisticated notion of acceptance of infinite words. Recent research has lead to the assumption that a topology can be used to encode the acceptance set in Büchi automata. The Büchi acceptance set then forms the basis of the topology. In such a case, no extension with predicates is necessary. Topological bisimulation in itself captures the infinite aspects of Büchi automata. It is a stronger notion than language equivalence for infinite words. Further research is needed to further those claims.

Related Work In [4], both the state space and the label space are endowed with metrics. The purpose is in proving operational models defined in terms of labelled transition systems equal to denotational semantics.

In [6], bisimulation is characterized using a specific (Alexandroff) topology as continuity of the transition relation. In other words the author shows that the Alexandroff topology as a structure fits normal bisimulation. We, on the other hand, adapt the notion of bisimulation to take the topological structure of the state space into account.

In [1], the state space of a Kripke model for propositional modal logic is extended with a topology. This topology defines the accessibility relation between points in the model and hence defines the meaning of the modal operators. Consequently, bisimulation is also defined in terms of the open sets of this topology. These open sets play the role of our transition relations, rather than being an additional structure on the state space. The relation between their notion of bisimulation and our notion of topological bisimulation is not clear yet.

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