THE HANKEL TRANSFORMATION
AND SPACES OF TYPE $W$

by
S.J.L. van Eijndhoven
and
M.J. Kerkhof
THE HANKEL TRANSFORMATION AND SPACES OF TYPE $\mathcal{W}$

by

S.J.L. van Eijndhoven and M.J. Kerkhof

In their celebrated monographs on generalized functions, [GS 2-3], Gelfand and Shilov have introduced spaces of type $\mathcal{S}$ and of type $\mathcal{W}$. Through functional analytic methods it has been proved in the papers [EG 1-2] that the spaces $\mathcal{S}_\alpha^\alpha$ give rise to Hankel invariant test function spaces. In [Pa], Pathak suggested a considerable extension of certain results in [EG 1-2]. Unfortunately, the proofs of some results in [Pa] are incorrect, cf. Appendix. In this paper, we present correct proofs and add some new results.

1. The Hankel transformation

Let $J_\mu$ denote the first type Bessel function of order $\mu$. The corresponding Hankel transformation of a function $\varphi$ on $(0, \infty)$ is defined by

$$ (\mathcal{H}_\mu \varphi)(x) = \int_0^\infty (x\xi)^{\mu/2} \varphi(\xi) J_\mu(x\xi) d\xi. $$

In this paper we consider a modified version of this transformation, viz. the transformation $\widetilde{\mathcal{H}}_\mu$ defined by

$$ (\widetilde{\mathcal{H}}_\mu \varphi)(x) = \int_0^\infty (x\xi)^{-\mu/2} J_\mu(x\xi) \varphi(\xi) \xi^{2\mu+1} d\xi. $$

Observe that $\widetilde{\mathcal{H}}_\mu = x^{-(\mu+\frac{1}{2})} \mathcal{H}_\mu x^{\mu+\frac{1}{2}}$. Abusing terminology we call also $\widetilde{\mathcal{H}}_\mu$ a Hankel transformation.

Here we study the case $\mu \geq -\frac{1}{2}$.

2. The Young inequality

We consider the following collection of functions on $[0, \infty)$:

$$ K = \{ M \in C^2([0, \infty)) \mid M(0) = M'(0) = 0, M'(\infty) = \infty \text{ and } M''(x) > 0, x > 0 \}. $$

The functions which belong to $K$ are subadditive,

$$ M(x) + M(x') \leq M(x + x'), \quad x, x' > 0. $$
For each $M \in K$ we define its Young dual $M^\vee$ as follows: Let $m = M'$ and let $m^*$ denote reciprocal of $m$. We put

\begin{equation}
M^\vee(x) = \int_0^x m^*(\xi)d\xi, \quad x \geq 0.
\end{equation}

Then clearly $M^\vee \in K$ and $M = (M^\vee)^\vee$. The proof of the following result is left to the reader.

(2.4) Lemma.

Let $M \in K$. Then for all $x, y \geq 0$

$$xy \leq M(x) + M^\vee(y)$$

with equality if and only if $y = M(x) = M'(x)$.

So we have

$$\inf_{y \geq 0} [-xy + M^\vee(y)] = -M(x).$$

We mention the following classical example. Let $M_a(x) = ax^{1/a}$ with $0 < a < 1$.

Then we have

$$M_a^\vee(y) = (1 - a)y^{1/(1-a)}$$

and Lemma 2.4 yields the Hölder inequality

$$xy \leq ax^{1/a} + (1 - a)y^{1/(1-a)}.$$

3. Spaces of type $W$

Fix $M, \Omega \in K$. In [GS 3] the following function spaces are defined.

(3.1) Definition.

Let $a > 0$. The space $W_{M,a}$ consists of all $C^\infty$-functions $\varphi$ on $\mathbb{R}$ which satisfy:

$$\forall 0 < a' < a \ \forall \xi \in \mathbb{N} \ \exists c_{\xi,a'}, \forall x \in \mathbb{R}:$$

$$|\varphi^{(\xi)}(x)| \leq c_{\xi,a'}, \exp[-M(a'|x|)].$$

By $W_{0,M,a}$ we denote the space of all even functions on $W_{M,a}$. 
(3.2) Definition.
Let $b > 0$. The space $W^b_{o, b}$ consists of all entire functions $\varphi$ which satisfy
\[ \forall b' > b \forall k \in \mathbb{N} \exists c_{k,b,b'} > 0 \forall z \in \mathbb{C} : \]
\[ |z^k \varphi(z)| \leq c_{k,b,b'} \exp[\Omega(b'|\text{Im } z)] . \]
By $W^b_{o, b}$ we denote all even functions in $W^b_{o, b}$.

(3.3) Definition.
Let $a > 0$, $b > 0$. The space $W^b_{a, b}$ consists of all entire functions $\varphi$ for which
\[ \forall 0 < a' < a \forall b' > b \exists c_{a', b, b'} > 0 \forall z \in \mathbb{C} : \]
\[ |\varphi(z)| \leq c_{a', b, b'} \exp\{-M(a'|\text{Re } z) + \Omega(b'|\text{Im } z)\} . \]
By $W^b_{a, b}$ we denote the space of all even functions in $W^b_{a, b}$.

(3.4) Definition.
The spaces $W^b_M$, $W^o_M$ and $W^o_M$ are defined by
\[ W^b_M = \bigcup_{a > 0} W^b_{o, b} , \quad W^o_M = \bigcup_{b > 0} W^o_{o, b} , \quad W^o_M = \bigcup_{a > 0} W^o_{o, a} , \quad W^o_M . \]
and correspondingly the spaces $W^b_M$, $W^o_M$ and $W^o_M$.

4. Spaces of type $S$

Let $S$ denote the space of rapidly decreasing functions, i.e.
\[ \varphi \in S : \iff \forall k \in \mathbb{N} \forall \ell \in \mathbb{N} \sup_{x \in \mathbb{R}} |x^k \varphi^{(\ell)}(x)| < \infty . \]
By $S_e$ we denote the space of all even functions in $S$. In [GS 2] the following subspaces of $S$ are introduced.

(4.1) Definition.
Let $a > 0$, $A > 0$. The space $S^a_A$ consists of all functions $\varphi \in S$ with the property that
\[ \forall 0 < A < \tilde{A} \forall \ell \in \mathbb{N} \exists c_{\tilde{A}, A, \ell} > 0 \forall x \in \mathbb{R} : \]
The space $S_{a,A}$ consists of all even functions in $S_{a,A}$.

(4.2) Lemma.
For $0 < a < 1$, put $M_a : x \mapsto \alpha^{-1/a}, x > 0$. Then we have

$$S_{a,A} = \omega_{M_a,a} \quad \text{and} \quad S_{e,a,A} = \omega_{e_{M_a,A}}$$

with $a = (A \exp(a))^{-1}$.

Proof. Cf. [GS], p. 172.

(4.3) Definition.
Let $\beta > 0$, $B > 0$. The space $S_{\beta,B}$ consists of all $\varphi \in S$ with the property that

$$\forall 0 < B < B \quad \forall k \in \mathbb{N} \quad \exists C_{B,k} > 0 \quad \forall x \in \mathbb{R} :$$

$$|x^k \varphi (x)| \leq C_{B,k} \beta^k \varphi^B$$

The space $S_{e,\beta,B}$ consists of all even functions in $S_{\beta,B}$.

(4.4) Lemma.
For $0 < \beta < 1$, put $\Omega_{\beta} : y + (1 - \beta)y^{1-1/\beta}, y > 0$. Then

$$S_{\beta,B} = \omega_{\Omega_{\beta},b}, \quad S_{e,\beta,B} = \omega_{e_{\Omega_{\beta},b}}$$

where $b = B \exp(\beta)$.


(4.5) Definition.
Let $a, \beta > 0$ and $A, B > 0$. The space $S_{a,\beta,A}$ consists of all $\varphi \in S$ with the property that

$$\forall 0 < A < A \quad \forall 0 \leq B < B \quad \exists C_{A,B} > 0 \quad \forall k \in \mathbb{N} \quad \forall x \in \mathbb{R} :$$

$$|x^k \varphi (x)| \leq C_{A,B} \alpha^A \beta^B \varphi^A \beta^B$$

The space $S_{e,\beta,A}$ consists of all even functions in $S_{a,\beta,A}$. 
(4.6) Lemma.
Let $0 < \alpha < 1$, $0 < \beta < 1$ and $A, B > 0$. Then

$$ S^\alpha_B = \omega^\alpha_{\alpha, A} \, S^\alpha_B \, \omega^\alpha_{\alpha, A} \quad \text{and} \quad S^\alpha_B = \omega^\alpha_{\alpha, A} \, S^\alpha_B \, \omega^\alpha_{\alpha, A} $$

where $a = (A \exp(\alpha))^{-1}$ and $b = B \exp(\beta)$.

Proof. Cf. [GS 2], p. 212.

(4.7) Definition.
Let $\alpha > 0$, $\beta > 0$. The spaces $S_\alpha$, $S^\beta$ and $S^\beta_\alpha$ are defined by

$$ S_\alpha = \cup_{A>0} S^\alpha_A \quad \text{and} \quad S^\beta = \cup_{B>0} S^\beta_B \quad \text{and} \quad S^\beta_\alpha = \cup_{A>0, B>0} S^\beta_{\alpha, A} $$

and correspondingly $S^\beta_\alpha$, $S^\beta_\beta$ and $S^\beta_\alpha$.

5. Some properties of Bessel functions

In this section we recall some relevant properties of the Bessel functions $J_\nu$ and the Hankel functions $H^{(1)}_\nu$ and $H^{(2)}_\nu$. As a general reference we use the book of Magnus et al. [MOS], p. 60 ff.

(5.1) (Recurrence) relations

Let $C_\nu$ denote one of the functions $J_\nu$, $H^{(1)}_\nu$ and $H^{(2)}_\nu$.

(5.1.a) $\left( \frac{1}{z} \frac{d}{dz} \right)^m [z^\nu C_\nu(z)] = z^{\nu-m} C_{\nu-m}(z)$,

(5.1.b) $\left( \frac{1}{z} \frac{d}{dz} \right)^m [-z^{-\nu} C_\nu(z)] = (-1)^m z^{-\nu-m} C_{\nu+m}(z)$, $m = 0, 1, 2, \ldots$.

Further we have

(5.1.c) $H^{(1)}_\nu(-x) = e^{-(\nu+1)\pi i} H^{(2)}_\nu(x)$, $x > 0$, $\text{Re } \nu > -\frac{1}{2}$,

(5.1.d) $J_\nu = \frac{1}{2}(H^{(1)}_\nu + H^{(2)}_\nu)$.
(5.2) Integral representations

(5.2.a) \( z^{-\nu} J_\nu(z) = \frac{2^{-\nu+1-\frac{1}{2}-\frac{\lambda}{2}}}{\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{-\frac{\nu}{2}-\frac{\lambda}{2}} \cos zt \, dt, \quad \text{Re} \, \nu > -\frac{1}{2}, \, z \in \mathbb{C}. \)

(5.2.b) \( J_n(z) = \frac{-n}{\pi} \int_0^\pi \exp[iz \cos t] \cos nt \, dt, \quad n = 0, 1, 2, \ldots, \, z \in \mathbb{C}, \)

(5.2.c) \( z^{-\nu} J_{\nu+n}(z) = 2^{-\nu} \sqrt{\pi} (-i)^n \frac{\Gamma(2\nu) \Gamma(n+1)}{\Gamma(\nu+\frac{1}{2}) \Gamma(n+2\nu)} \cdot \int_0^\pi \exp[iz \cos t] C_n^{(\nu)}(\cos t) \sin^{2\nu}(t) \, dt \)

where \( \nu \neq 0, \, \nu > -\frac{1}{2}, \, n = 0, 1, 2, \ldots \) and where \( C_n^{(\nu)} \) denotes the \( n \)-th Gegenbauer polynomial.

(5.2.d) \( H_\nu^{(1)}(z) = \frac{(z\pi z)^{-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \exp[i(z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi)] \int_0^\infty e^{-t} t^{-\frac{\nu}{2}-\frac{1}{2}} (1 + \frac{it}{2t})^{-\frac{1}{2}} \, dt \)

where \( \nu > -\frac{1}{2}, \, |\delta| < \frac{1}{2}\pi \) and \( -\frac{1}{2}\pi + \delta < \arg(z) < \frac{3}{2}\pi + \delta. \)

On the basis of these integral representations the following estimations can be derived in a straightforward manner.

(5.3) Growth order estimations

(5.3.a) \( J_\nu(x) = O(x^\nu), \quad x \downarrow 0, \quad J_\nu(x) = O(x^{-\nu}), \quad x \to \infty. \)

(5.3.b) \( \forall \nu > -\frac{1}{2} \forall n \in \mathbb{N} \cup \{0\} \exists B_{\nu,n} > 0 \forall z \in \mathbb{C} : |z^{-\nu} J_{\nu+n}(z)| \leq B_{\nu,n} \exp|\text{Im} \, z|. \)

(5.3.c) Let \( \nu \geq \frac{1}{2}. \) Then

\[ \exists A_{\nu} > 0 \forall z \in \mathbb{C}, |z| > 1: |z^{\nu} H_{\nu}^{(1)}(z)| \leq A_{\nu} \exp[-|\text{Im} \, z|]. \]

The function \( z \mapsto z^{\nu} H_{\nu}^{(1)}(z) \) is bounded on \( |z| \leq 1. \)

(5.3.d) Let \( -\frac{1}{2} < \nu < \frac{1}{2}. \) Then

\[ \exists A_{\nu} > 0 \forall z \in \mathbb{C} : |z^{\nu} H_{\nu}^{(1)}(z)| \leq A_{\nu} \exp[-|\text{Im} \, z|]. \]

For the growth order estimates (5.3 c-d) we refer also to [MOS], p. 139.
6. Auxiliary results

Throughout this section we take $v > -\frac{1}{2}$.

(6.1) Lemma.

Let $\varphi \in \mathcal{S}e$ be analytic within a strip $T_c = \{z \in \mathbb{C} \mid \text{Im } z < c\}$, $0 < c \leq \infty$ ($T_\infty = \mathbb{C}$). Let there be a positive bounded continuous function $g$ on $[0, c)$ such that

$$\forall k \in \mathbb{N} \cup \{0\} \exists c_k > 0 \forall z \in T_c : |z^k \varphi(z)| \leq c_k g(|\text{Im } z|).$$

Then for all $\eta$, $0 < \eta < c$, and all $x > 1$

$$(H_{\eta} \varphi)(x) = \frac{1}{2} \int_{-\infty}^{\infty} \left( x(\xi + i\eta) \right)^{-v} H_{\eta}^{(1)}(x(\xi + i\eta)) \varphi(\xi + i\eta)(\xi + i\eta)^{2v+1} d\xi.$$  

Observe that for $c = \infty$ all $\eta > 0$ may be taken.

Proof.

Let $0 < \eta < c$. Consider the contour $C_{\eta, R, \varepsilon}$:

Let $x > 1$. Then the function $G_{x, v}$ defined by

$$G_{x, v}(w) = (xw)^{-v} H_{v}^{(1)}(xw) \varphi(w) w^{2v+1}$$

is analytic within the contour $C_{\eta, R, \varepsilon}$. So

$$\int_{C_{\eta, R, \varepsilon}} G_{x, v}(w) dw = 0.$$  

Because of the growth conditions imposed on $\varphi$, the estimations (5.3 c-d) and the regularity properties of $H_{v}^{(1)}$ at $z = 0$, the contributions of II, IV and VI tend to zero as $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$. Therefore with (5.1 c-d) we obtain
(6.2) Lemma.
Let \( \varphi \in \mathcal{S} \) satisfy the conditions stated in Lemma 6.1. Then for each \( \nu > -\frac{1}{2} \), \( \mathbb{I}_\nu \varphi \) is bounded on \( \mathbb{R} \). In addition, there exists \( D_\nu > 0 \) such that for all \( x \in \mathbb{R} \) with \( |x| > 1 \)

\[
\left| (\mathbb{I}_\nu \varphi)(x) \right| \leq D_\nu \inf_{0 < \eta < c} g(\eta) \exp[-|x|] .
\]

Proof.
In [EG 1] we have proved that \( \mathbb{I}_\nu \varphi \in \mathcal{S} \). Therefore \( \mathbb{I}_\nu \varphi \) is bounded on \( \mathbb{R} \) and we need only investigate its growth behaviour at \( \infty \). So let \( x > 1 \) and let \( 0 < \eta < c \).

First we consider the case \( -\frac{1}{2} < \nu < \frac{1}{2} \).

By (5.3.d) there exists \( A_\nu > 0 \) such that

\[
\left| (x(\xi + i\eta))^{\frac{1}{2}}(\mathbb{I}_\nu^{'\nu}(x(\xi + i\eta))) \right| \leq A_\nu \exp[-x\eta] .
\]

Further, there exists \( C_{\nu, \varphi} > 0 \) such that

\[
\left| (\xi + i\eta)^{1+\frac{1}{2}}\varphi(\xi + i\eta) \right| \leq C_{\nu, \varphi} \frac{g(\eta)}{1 + \xi^2} .
\]

Thus we obtain

\[
(*) \quad \frac{1}{2} \int_{-\infty}^{\infty} (x(\xi + i\eta))^{-\nu}(\mathbb{I}_\nu^{'\nu}(x(\xi + i\eta)))\varphi(\xi + i\eta)(\xi + i\eta)^{2\nu+1}d\xi \leq
\]

\[
\leq \frac{1}{2} A_\nu C_{\nu, \varphi} x^{-\nu-\frac{1}{2}} g(\eta) \exp[-x\eta] \int_{-\infty}^{\infty} \frac{1}{1 + \xi^2} d\xi .
\]
Next, we consider the case $v \geq \frac{1}{2}$.

We set $I_{x,n} = \{ \xi \in \mathbb{R} \mid |x(\xi + i\eta)| \geq 1 \}$. For $\xi \in I_{x,n}$ we have

$$| (x(\xi + i\eta))^{\frac{1}{2}} H_v(1) (x(\xi + i\eta)) | \leq A_v^{(1)} \exp[-x\eta].$$

On $|z| < 1$ the function $z H_v(1)(z)$ is bounded. So for some constant $A_v^{(2)} > 0$ indepent of $x$ and $\eta$, we have

$$| (x(\xi + i\eta))^{\frac{1}{2}} H_v(1) (x(\xi + i\eta)) | \leq A_v^{(2)} \exp[-x\eta], \quad \xi \notin I_{x,n}.$$

Further, there exist $C_{\nu, \varphi}^{(1)}, C_{\varphi}^{(2)} > 0$ such that

$$| (\xi + i\eta)^{\nu+\frac{1}{2}} \varphi(\xi + i\eta) | \leq C_{\nu, \varphi}^{(1)} \frac{g(\eta)}{1 + \xi^2},$$

$$| (\xi + i\eta)^{\nu} \varphi(\xi + i\eta) | \leq C_{\varphi}^{(2)} \frac{g(\eta)}{1 + \xi^2}.$$

Thus we obtain

$$(**) \quad \frac{1}{2} \int_{\mathbb{R}} (x(\xi + i\eta))^{-\nu} H_v(1) (x(\xi + i\eta)) \varphi(\xi + i\eta) (\xi + i\eta)^{2\nu+1} d\xi =$$

$$= \left| \frac{1}{2} \int_{I_{x,n}} + \int_{\mathbb{R} \setminus I_{x,n}} \right| \left( \ldots \right) d\xi \leq$$

$$\leq \frac{1}{2} A_v^{(1)} C_{\nu, \varphi}^{(1)} \nu - \frac{1}{2} g(\eta) \exp[-x\eta] \int_{I_{x,n}} \frac{1}{1 + \xi^2} d\xi +$$

$$+ \frac{1}{2} A_v^{(2)} C_{\varphi}^{(2)} \nu - 2 \nu \exp[-x\eta] \int_{\mathbb{R} \setminus I_{x,n}} \frac{1}{1 + \xi^2} d\xi.$$

From (*) and (**) and Lemma (6.1) we derive that for all $v > \frac{1}{2}$, there exists $D_{\nu, \varphi} > 0$ such that for all $x > 1$, $\nu, 0 < \eta < c$:

$$| (\mathbb{H}_v \varphi)(x) | \leq D_{\nu, \varphi} g(\eta) \exp[-x\eta].$$

Thus the result follows, because $\mathbb{H}_v \varphi$ is even. \qed
(6.3) Lemma.
Let \( \varphi \in \mathcal{S}_v \) be analytic within the strip \( T_c', \ 0 < c \leq \\infty \). Suppose there exists a function \( g \) on \([0,c)\) which is positive bounded and continuous such that

\[
\exists M_K \exists a > 0 \ \forall k \in \mathbb{N} \setminus \{0\} \ \exists C_k > 0 \ \forall \xi = \xi + in \in T_c' : \\
|\xi^k \varphi(\xi)| \leq C_k g(n) \exp[-M(a|\xi|)] .
\]

Then \( \mathcal{H}_v \varphi \) extends to an entire function. Moreover, there exists \( D_{v,\varphi} > 0 \) such that for all \( x, y \in \mathbb{R} \)

\[
|\mathcal{H}_v \varphi(x + iy)| \leq D_{v,\varphi} \exp[M(\frac{|y|}{a})]
\]

and, in particular, for \( x \in \mathbb{R} \) with \( |x| > 1 \)

\[
|\mathcal{H}_v \varphi(x + iy)| \leq D_{v,\varphi} \exp[M(\frac{|y|}{a})] \inf_{0 < n < c} g(n) \exp[-|x|n] .
\]

Proof.
The growth conditions imposed on \( \varphi \) ensure that \( \mathcal{H}_v \varphi \) can be extended to an even entire function. There exists a constant \( C_{v,\varphi}^{(1)} > 0 \) such that

\[
(1 + \xi^2) \xi^{2v+1} |\varphi(\xi)| \leq C_{v,\varphi}^{(1)} \exp[-M(a|\xi|)] , \ \xi \in \mathbb{R} .
\]

So with (5.3.b) we obtain a constant \( C_{v,\varphi}^{(1)} > 0 \) such that

\[\text{(*)} \quad |\mathcal{H}_v \varphi(z)| = \left| \int_0^\infty (z \xi)^{-v} J_v(z \xi) \varphi(\xi) \xi^{2v+1} d\xi \right| \leq \]

\[
\leq C_{v,\varphi}^{(1)} \int_0^\infty \exp[|\xi| \text{Im } z] - M(a\xi)](1 + \xi^2)^{-1} d\xi \leq \]

\[
\leq D_{v,\varphi}^{(1)} \exp[M(\frac{|\text{Im } z|}{a})]
\]

where \( z \in \mathbb{C} \) and

\[
D_{v,\varphi}^{(1)} = C_{v,\varphi}^{(1)} \int_0^\infty (1 + \xi^2)^{-1} d\xi .
\]
The function $w \mapsto w^{-\nu} H_{\nu}^{(1)}(w)$ is analytic on the complex $w$-plane cut along the negative imaginary axis. So for all $\xi \in \mathbb{R}$ and all $\eta$, $0 < \eta < \infty$ the function

$$z \mapsto (z(\xi + i\eta))^{-\nu} H_{\nu}^{(1)}(z(\xi + i\eta))$$

is analytic on the region \( \{ z \in \mathbb{C} \mid \text{Re} \, z > 0, \text{Im} \, z > 0 \} \).

Now by standard arguments we obtain (cf. Lemma (6.1))

$$\int_{-\infty}^{\infty} (z(\xi + i\eta))^{-\nu} H_{\nu}^{(1)}(z(\xi + i\eta)) \varphi(\xi + i\eta)(\xi + i\eta)^{2\nu+1} d\xi$$

for all $z \in \mathbb{C}$ with $\text{Re} \, z > 1$ and $\text{Im} \, z > 0$.

First we consider the case $-\frac{1}{2} < \nu < \frac{1}{2}$. Then we have the estimate

$$\int_{-\infty}^{\infty} (z(\xi + i\eta))^{-\nu} H_{\nu}^{(1)}(z(\xi + i\eta)) \varphi(\xi + i\eta)(\xi + i\eta)^{2\nu+1} d\xi \leq$$

$$\leq C_{\nu, \varphi}^{(2)} g(\eta) \exp[-x\eta] \int_{-\infty}^{\infty} \exp[-M(a\xi) - y\xi](1 + \xi^2)^{-1} d\xi \leq$$

$$\leq D_{\nu, \varphi}^{(2)} g(\eta) \exp[-x\eta + M(\frac{\nu}{a})]$$

where $z = x + iy$, $x > 1$ and $y > 0$ and where

$$D_{\nu, \varphi}^{(2)} = C_{\nu, \varphi}^{(2)} \int_{-\infty}^{\infty} (1 + \xi^2)^{-1} d\xi .$$

Next, we let $\nu \geq \frac{1}{2}$. We fix $z \in \mathbb{C}$ with $\text{Re} \, z > 1$ and $\text{Im} \, z > 0$. Also, let $0 < \eta < c$. We set

$$I_{z, \eta} = \{ \xi \in \mathbb{R} \mid |z(\xi + i\eta)| \geq 1 \} .$$

Then following (5.3.c) there exists a constant $A_{\nu} > 0$ such that

$$|z(\xi + i\eta))^{-\nu} H_{\nu}^{(1)}(z(\xi + i\eta))| \leq A_{\nu} \exp[-(x\eta + y\xi)] , \quad \xi \in I_{z, \eta}$$

and

$$|z(\xi + i\eta))^{-\nu} H_{\nu}^{(1)}(z(\xi + i\eta))| \leq A_{\nu} \exp[-(x\eta + y\xi)] , \quad \xi \notin I_{z, \eta} .$$

So we get the following estimation
\[ (***) \quad \left| \int_{I_{z,n}} (z(x + in))^{-\nu} H^{(1)}_{\nu} (z(x + in))\varphi(x + in) (x + in)^{2\nu + 1} d\xi \right| \leq A_\nu C^{(3)}_{\nu, \varphi} |z|^{-\nu - \frac{1}{2}} g(\eta) \exp[-\eta] \int_{-\infty}^{\infty} \exp[-y x - M(a|\xi|)](1 + \xi^2)^{-1} d\xi \]

\[ \leq D^{(1)}_{\nu, \varphi} g(\eta) \exp[-\eta + M \varphi^0] \]

and, similarly,

\[ (****) \quad \left| \int_{I_{z,n}} (z(x + in))^{-\nu} H^{(1)}_{\nu} (z(x + in))\varphi(x + in) (x + in)^{2\nu + 1} d\xi \right| \leq A_\nu C^{(2)}_{\nu, \varphi} |z|^{-2\nu} g(\eta) \exp[-\eta] \int_{-\infty}^{\infty} \exp[-y x - M(a|\xi|)](1 + \xi^2)^{-1} d\xi \leq D^{(2)}_{\nu, \varphi} g(\eta) \exp[-\eta + M \varphi^0] \]

We observe that the constants \( D^{(1)}_{\nu, \varphi} \) and \( D^{(2)}_{\nu, \varphi} \) do not depend on \( z \) and \( \eta \).

By (**)-(****) we derive that for all \( \nu > -\frac{1}{2} \) there exists a constant \( D_{\nu, \varphi} > 0 \) such that for all \( \eta, 0 < \eta < \eta_0 \) and all \( z \in \mathbb{C} \) with \( \text{Im} z > 0 \) and \( \text{Re} z > 1 \)

\[ (\ast) \quad |H^{\nu}_{\nu} \varphi(z)| \leq D_{\nu, \varphi} \exp[M \frac{\text{Im} z}{a}] \inf_{0 < \eta < \eta_0} g(\eta) \exp[-\eta \text{Re} z] \]

By (\ast) and (\ast) the wanted result follows.

7. The Hankel transformation and spaces of type \( \mathcal{W} \)

Let \( \mathcal{I} \) denotes the Fourier transformation

\[ (\mathcal{I} \varphi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(y) e^{-iyx} dy \]

In [GS 3], the following results have been proved

\[ \mathcal{I} \mathcal{F}(\mathcal{W}_{M,a}^x) = \mathcal{W}_{M, a}^{x^2}, \quad \mathcal{I} \mathcal{F}(\mathcal{W}_{\Omega, b}^x) = \mathcal{W}_{\Omega, b}^{x^2}, \quad \mathcal{I} \mathcal{F}(\mathcal{W}_{M, a}^x) = \mathcal{W}_{M, a}^{x^2} \]
Since $\mathcal{H}_{-\frac{1}{2}}$ equals the Fourier cosine transformation with the aid of the above relations we get

\[(7.1) \textbf{Corollary.} \]

\[
\mathcal{H}_{-\frac{1}{2}}(\omega e_{M,a}) = \omega e^{\frac{1}{2}}_a, \quad \mathcal{H}_{-\frac{1}{2}}(\omega e_{\Omega,b}) = \omega \Omega^{\frac{1}{2}}_b, \\
\mathcal{H}_{-\frac{1}{2}}(\omega e_{M,a}) = \omega e^{\frac{1}{2}}_a.
\]

In this section we prove similar results for the transformations $\mathcal{H}_v$, $v > -\frac{1}{2}$. First, we present the following auxiliary result

\[(7.2) \textbf{Lemma.} \]

Let $M \in K$, let $a > 0$ and let $\varphi \in \mathcal{S}_e$. Then we have

\[
\varphi \in \mathcal{W} e_{M,a} \iff \forall 0 < a' < a, \forall \xi \in \mathbb{N} \cup \{0\} \exists c_{\xi,a} > 0 \quad \forall x \in \mathbb{R}^n : \\
| ((x^{-1}D_{\xi})^k \varphi)(x) | \leq C_{\xi,a}, \exp[-M(a' |x|)]
\]

where we set $D = \frac{d}{dx}$.

\textbf{Proof.}

The proof is a consequence of the following relations

\[
(x^{-1}D)^n = \sum_{k=1}^{n} \frac{\xi!}{k! (n-k)!} x^{-1}(x^{-1}D)^n-k, \\
p^n = \sum_{k=0}^{\xi!} \frac{(-1)^k k! n-k}{k! (n-2k)!} x^{n-2k} (x^{-1}D)^n-k. 
\]

Our results are contained in the following lemmas.

\[(7.3) \textbf{Lemma.} \]

Let $v > -\frac{1}{2}$. Let $M \in K$ and let $a > 0$. Then we have

\[
\mathcal{H}_v(\omega e_{M,a}) \subset \omega e^{M,a}_{-1}. 
\]
Proof.
Take a fixed $\phi \in \mathcal{W}_{M,a}$. Following the preceding lemma, for each $a'$ with $0 < a' < a$ and each $l \in \mathbb{N}$, there exists $C_{l,a'} > 0$ such that

$$|((x^{-1}D)^l \phi)(x)| \leq C_{l,a'} \exp[-M(a'|x|)].$$

So the function $\mathbb{H}_v((x^{-1}D) \phi)$ extends to an entire even function for all $l \in \mathbb{N}$.

Now let $0 < a' < a'' < a$, let $k \in \mathbb{N}$ and $z \in \mathbb{C}$. Then we have

$$z^k (\mathbb{H}_v \phi)(z) = \int_0^\infty (z\xi)^{-\nu} J_{\nu+k} (z\xi) \phi(\xi) \xi^{2\nu+1} \, d\xi$$

$$= (-1)^k \int_0^\infty (z\xi)^{-\nu} J_{\nu+k} (z\xi) ((x^{-1}D)^k \phi)(\xi) \xi^{2\nu+k+1} \, d\xi$$

where we inserted the recurrence relations (5.2.a). With (5.3.b) we get the estimate

$$|z^k (\mathbb{H}_v \phi)(z)| \leq \int_0^\infty |z\xi|^{-\nu} |J_{\nu+k} (z\xi)| |((x^{-1}D)^k \phi)(\xi)| \xi^{2\nu+k+1} \, d\xi$$

$$\leq C_{k,a''} B_\nu k \int_0^\infty \exp[|\text{Im } z| - M(a'' \xi)] \xi^{2\nu+k+1} \, d\xi.$$

Now the inequality $-M(a'' \xi) \leq -M(a' \xi) - M((a'' - a') \xi)$ together with Young's inequality yield

$$|z^k (\mathbb{H}_v \phi)(z)| \leq D_{k,a'} \exp[M \frac{\text{Im } z}{a''}]$$

where

$$D_{k,a'} = C_{k,a''} B_\nu,k \int_0^\infty \exp[-M((a'' - a') \xi)] \xi^{2\nu+k+1} \, d\xi.$$

Since $k \in \mathbb{N}$ and $0 < a' < a$ are both arbitrarily taken, we get $\phi \in \mathcal{W}_{M,a}^\infty$.
(7.4) Lemma. Let $v \succ \frac{1}{2}$. Let $\Omega \in K$ and let $b > 0$. Then we have

$$\mathbb{I}^\nu(\Omega, b) = \mathbb{W}_\nu^{\Omega, b^{-1}}.$$

Proof. Let $\varphi \in \mathbb{W}_\nu^{\Omega, b}$. Then $\varphi$ is an entire function with the property that

$$\forall b' > b \quad \forall k \epsilon \mathbb{N} \cup \{0\} \quad \exists c_{k, b', 0} > 0 \quad \forall z \epsilon \mathbb{C}:
$$

$$|z^k \varphi(z)| \leq c_{k, b', 0} \exp[\Omega(b'|\Im z|)].$$

Let $b' > b$ and $k \epsilon \mathbb{N} \cup \{0\}$. The recurrence relations (5.1.a) induce the equality

$$(x^{-1}D_x)^k \mathbb{I}^\nu \varphi = (-1)^k \mathbb{I}^{\nu + k} \varphi.$$ 

Now we can apply Lemma (6.2) where we take $c = \infty$ and $g(n) = \exp[\Omega(b'n)]$ with $n > 0$. It follows that there exists a constant $D_{k, b'} > 0$ such that for all $x > 1$

$$|((x^{-1}D_x)^k \mathbb{I}^\nu \varphi)(x)| \leq D_{k, b'} \exp[\Omega(b'n) - xn].$$

Since for all $x > 1$

$$\inf_{n > 0} [\Omega(b'n) - xn] = -\Omega^x \left(\frac{1}{b'}\right)$$

it follows that

$$|((x^{-1}D_x)^k \mathbb{I}^\nu \varphi)(x)| \leq D_{k, b'} \exp[-\Omega^x \left(\frac{1}{b'}\right)].$$

Hence $\mathbb{I}^\nu \varphi \in \mathbb{W}_\nu^{\Omega, \frac{1}{b'}}$ because of Lemma (7.2).

(7.5) Lemma. Let $\Omega, \Omega' \in K$ and let $a, b > 0$. Then we have

$$\mathbb{I}^\nu(\Omega, b) = \mathbb{W}_\nu^{\Omega, \frac{1}{a}, b}.$$
Proof.
Let $\varphi \in \mathcal{W}_{M,a}$ and let $0 < a' < a$ and $b' > b$. Then for each $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all $\xi, \eta \in \mathbb{R}$

$$\left| (\xi + i\eta)^k \varphi(\xi + i\eta) \right| \leq C_k \exp\left[\Omega(b' | \eta|) \right] \exp\left[-M(a' | \xi|) \right].$$

So we can Lemma 6.3 with $g(\eta) = \exp[\Omega(b' \eta)]$, $\eta > 0$. It follows that $\mathcal{I}_\varphi$ extends to an entire function and there exists a constant $C_{\varphi,a',b'} > 0$ such that

(*) \quad \left| (\mathcal{I}_\varphi \varphi)(x + iy) \right| \leq C_{\varphi,a',b'} \exp\left[M^x \left(\frac{\sqrt{y}}{a'}\right)\right] \inf_{\eta > 0} \exp\left[ -x |\eta| + \Omega(b' \eta) \right]

for all $x, y \in \mathbb{R}$ and

(**) \quad \left| (\mathcal{I}_\varphi \varphi)(x + iy) \right| \leq C_{\varphi,a',b'} \exp\left[M^x \left(\frac{\sqrt{y}}{a'}\right)\right] \exp\left[ -M^x \left(\frac{|x|}{b'}\right) + M^x \left(\frac{\sqrt{y}}{a'}\right)\right]

for all $x, y \in \mathbb{R}$ with $|x| > 1$.

Now (*) and (**) yield: $\exists C_{\varphi,a',b'} \forall x \in \mathbb{R} \forall y \in \mathbb{R}$:

$$\left| (\mathcal{I}_\varphi \varphi)(x + iy) \right| \leq C_{\varphi,a',b'} \exp\left[ -M^x \left(\frac{|x|}{b'}\right) + M^x \left(\frac{\sqrt{y}}{a'}\right)\right].$$

Summarizing we have

\textbf{(7.6) Theorem.}
Let $\nu \geq -\frac{1}{4}$. Let $M, \Omega \in K$ and let $a, b > 0$.

$$\mathcal{I}_\varphi (\mathcal{W}_{M,a}) = \mathcal{W}_{\frac{M^x}{a'}, a'}, \quad \mathcal{I}_\varphi (\mathcal{W}_{\Omega}) = \mathcal{W}_{\frac{M^x}{\Omega}, \Omega},$$

$$\mathcal{I}_\varphi (\mathcal{W}_{\Omega, b}) = \mathcal{W}_{\frac{M^x}{\Omega}, \frac{1}{b'}, b'}, \quad \mathcal{I}_\varphi (\mathcal{W}_{M, a}) = \mathcal{W}_{\frac{M^x}{a'}, a'}.$$

Proof.
We observe that $\mathcal{I}^2 = I$. \qed
8. The Hankel transformation and spaces of type $S$

As a consequence of Theorem (7.6) and Lemmas (4.2), (4.4) and (4.6) we have

(8.1) Theorem.
Let $v \geq -\frac{1}{2}$. Let $A, B > 0$ and let $0 < \alpha, \beta < 1$

\[ \mathcal{H}_v(Se_{\alpha,A}) = Se_{\alpha,A}, \quad \mathcal{H}_v(Se_\alpha) = Se_\alpha \]

\[ \mathcal{H}_v(Se_{\beta,B}) = Se_{\beta,B}, \quad \mathcal{H}_v(Se_\beta) = Se_\beta \]

\[ \mathcal{H}_v(Se_{\alpha,A}) = Se_{\alpha,A}, \quad \mathcal{H}_v(Se_\beta) = Se_\beta. \]

In this final section we prove some results for the limiting case $\alpha = 0,1$ or $\beta = 0,1$. We have the following Paley-Wiener type of result.

(8.2) Theorem.
Let $v > -\frac{1}{2}$ and let $A > 1$. We have

\[ \mathcal{H}_v(Se_{0,A}) = Se_{0,A}, \quad \mathcal{H}_v(Se_0) = Se_0. \]

Proof.

Remark: $Se_{0,A}$ consists of all even $C^\infty$-functions with support contained in $[-A,A]$. $Se_{0,A}$ consists of all even entire functions with the property that for each $k \in \mathbb{N}$ there exists $C_{k,A} > 0$ such that

\[ |(x + iy)^k \varphi(x + iy)| \leq C_{k,A} \exp[A|y|], \quad x + iy \in \mathbb{C}. \]

Let $\varphi \in Se_{0,A}$. Then $\mathcal{H}_v$ extends to an even entire function; for all $k \in \mathbb{N} \cup \{0\}$ and all $z \in \mathbb{C}$

\[ z^k(\mathcal{H}_v \varphi)(z) = \int_0^A (z\xi)^{-v} \mathcal{H}_{v+k}(z\xi)((\xi^{-1}D_\xi)^k \varphi)(\xi) \xi^{2v+k+1} \, d\xi. \]

A straightforward estimation yields

\[ |z^k(\mathcal{H}_v \varphi)(z)| \leq \exp[A|\text{Im } z|] \int_0^A |((\xi^{-1}D_\xi)^k \varphi)(\xi)| \xi^{2v+k+1} \, d\xi. \]
whence $\Pi_t \in \mathcal{S}_0, A$.

Let $\varphi \in \mathcal{S}_0, A$. Then $\varphi$ satisfies the conditions of Lemma (6.1) with $c = \infty$ and $g: \eta \mapsto \exp[A \eta], \eta > 0$. So there exists a constant $D_\varphi > 0$ such that for all $|x| > 1$

$$|(\Pi_t \varphi)(x)| \leq D_\varphi \inf_{\eta > 0} (\exp[A \eta - |x| \eta])$$

$$= D_\varphi \begin{cases} 
1 & \text{for } |x| \leq A \\
0 & \text{for } |x| > A .
\end{cases}$$

So the support of the $C^\infty$-function $\Pi_t \varphi$ is contained in $[-\tilde{A}, \tilde{A}]$ where $
\tilde{A} = \max\{1, A\}$.

(8.3) Theorem.

Let $\nu > -\frac{1}{a}$ and let $0 < \alpha \leq 1$. Then we have

$$\Pi_t (\mathcal{S}_1) = \mathcal{S}_1^\alpha .$$

Proof.

Remark: $\mathcal{S}_1^\alpha, 0 < \alpha \leq 1$, consists of all functions $\varphi \in \mathcal{S}_0$ which are analytic on a strip $T_c, 0 < c < \infty$ and for which there is a $a > 0$ such that

$$\sup_{x+iy \in T_c} |\varphi(x + iy)| \exp[-a \alpha |x|^{1/\alpha}] < \infty .$$

Here $a$ and $c$ depend on the choice of $\varphi$.

$\mathcal{S}_1^\alpha, 0 < \alpha < 1$, consists of all even entire functions with the property that $\exists a > 0 \exists b > 0 \exists c_{a,b} > 0 \forall x,y \in \mathbb{R}$:

$$|\varphi(x + iy)| \leq c \exp[-a |x| + b(1 - \alpha) |y|^{1/1-\alpha}] .$$

In [EG 2] it has been proved that $\Pi_t (\mathcal{S}_1) = \mathcal{S}_1^1$. So we only have to consider $0 < \alpha < 1$.

C: Let $\varphi \in \mathcal{S}_1^1$. Then $\varphi$ is analytic on a strip $T_c, c > 0$. On this strip $\varphi$ satisfies:

$$\exists a > 0 \forall x + iy \in T_c^\nu \exists c_{k,a} > 0 \forall x + iy \in T_c$$

$$|(x + iy)^k \varphi(x + iy)| \leq c_{k,a} \exp[-a |x|^{1/\alpha}] .$$
So we can apply Lemma (6.3) where we take \( g \equiv 1 \) and \( M(x) = ax^{1/\alpha}, \ x > 0 \). Then we get for \( |\text{Re} \ z| > 1 \)

\[
| (\mathbb{H}_\psi \varphi)(z) | \leq D_{a,c} \exp \left( -\frac{|\text{Im} \ z|}{a} \right) \inf_{0<n<c} \exp[-\eta \text{Re} \ z] 
\]

so that

\[
| (\mathbb{H}_\psi \varphi)(z) | \leq D_{a,c} \exp[M \left( -\frac{|\text{Im} \ z|}{a} \right) - c |\text{Re} \ z|] .
\]

Also, following Lemma (6.3) there exists \( \widetilde{D}_{a,c} > 0 \) such that for all \( z \in \mathcal{C} \)

\[
| (\mathbb{H}_\psi \varphi)(z) | \leq \widetilde{D}_{a,c} \exp[M \left( -\frac{|\text{Im} \ z|}{a} \right)] .
\]

Since \( M^\psi(y) = (1 - \alpha)y^{1/1-\alpha} \) we obtain \( \mathbb{H}_\psi \varphi \in \mathcal{S}^\alpha_1 \).

Let \( \varphi \in \mathcal{S}^\alpha_1 \). Then there exists \( a, b > 0 \) such that

\[
\forall \ k \in \mathbb{N} \cup \{0\} \exists C_k > 0 \ \forall \ z = \xi + i\eta \in \mathcal{C}:

| \varphi(\xi + i\eta) | \leq C_k \exp[-a|\xi| + b(1 - \alpha)|\eta|^{1/1-\alpha}] .
\]

Due to the growth and regularity properties of \( J_\psi \) it follows that \( \mathbb{H}_\psi \varphi \) can be extended to an analytic function within the strip \( T_a = \{ z \in \mathcal{C} | |\text{Im} \ z| < a \} \). In \( T_a \), the function \( \mathbb{H}_\psi \varphi \) satisfies

\[
(\ast) \quad | (\mathbb{H}_\psi \varphi)(z) | \leq D_\psi (a - |\text{Im} \ z|)^{-1}
\]

for some \( D_\psi > 0 \). Now let \( z \in T_a \) with \( \text{Re} \ z > 1 \). Then for all \( \eta > 0 \) we have

\[
(\mathbb{H}_\psi \varphi)(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} (z(\xi + i\eta))^\psi H^\psi_\psi(1)(z(\xi + i\eta))\varphi(\xi + i\eta)(\xi + i\eta)^{2\nu + 1}d\xi
\]

With the same techniques as used in the proof of Lemma (6.3) we find a constant \( C_{b,\psi} > 0 \) (independent of \( z \) and \( \eta \)) such that

\[
| (\mathbb{H}_\psi \varphi)(z) | \leq C_{b,\psi} \exp[-\eta \text{Re} \ z + b(1 - \alpha)\eta^{1/1-\alpha}] .
\]

Taking the infimum over \( \eta > 0 \) we get

\[
(\ast\ast) \quad | (\mathbb{H}_\psi \varphi)(z) | \leq C_{b,\psi} \exp[- \frac{\alpha}{b^{1-\alpha}}(\text{Re} \ z)^{1/\alpha}] .
\]

Since \( \mathbb{H}_\psi \varphi \) is even, \( (\ast) \) and \( (\ast\ast) \) yield \( \mathbb{H}_\psi \varphi \in \mathcal{S}^\alpha_1 \). \( \square \)
Remark. Theorems (8.2) and (8.3) are not stated as such in [Pa].

Appendix.
In this appendix we list some errata in Pathak's paper [Pa]. First, we note that instead of the spaces $\mathcal{W}_{M,a}$, $\mathcal{W}_{n,b}$ and $\mathcal{W}_{n,b}$, Pathak uses the spaces $\mathcal{W}_{\mu,M,a}$, $\mathcal{W}_{\mu,n,b}$ and $\mathcal{W}_{\mu,n,b}$.

Correspondingly, instead of $\hat{W}_{\mu}$, the transformation $\hat{W}'_{\mu}$ is used, cf. (1.1).

The major errata are the following:

- On p. 92 of [Pa], it is stated that for $M(x) = \Omega(x) = \frac{1}{x^2}$ the space $\mathcal{W}_{\mu,M,a}$ equals the space $\mathcal{T}(X,A)$ introduced in [EG 1]. This observation is incorrect, because $\mathcal{T}(X,A)$ consists of all functions $\varphi$ with the property that $\varphi(x) = x^{-(\mu+\frac{1}{2})} \psi(x)$ where $\psi$ extends to an entire function satisfying:

$$\forall 0 < a < b, \exists C > 0$$

$$|\psi(x + iy)| \leq C a \exp[-ax^2 + \frac{1}{a} y^2].$$

- In Theorem (5.2) of [Pa], p. 94, it is stated that $\hat{W}'_{\mu}[\mathcal{U}_{\mu,n,b}] \subseteq \mathcal{U}_{\mu,n,b}$, which corresponds to the statement in Theorem (7.2) of the present paper. The proof of this statement in [Pa] is incorrect. Indeed, from the inequality

$$|u^{-1}D_u q_u^{-1} \varphi(u)| \leq A \sum_{n=0}^{x-1} \frac{C^{r+1}}{n!} \exp[-\Omega((b + \rho)|y|) - u|y|]$$

where $y \in \mathbb{R}$ is arbitrary, it cannot be concluded that

$$|u^{-1}D_u q_u^{-1} \varphi(u)| \leq C q^{\delta} \exp[-\Omega(\frac{1}{b} - \delta)u]$$

where $(b + \rho)^{-1} = \frac{1}{b} - \delta$, on the basis of Young's inequality. To this end, observe that Young's inequality yields
\[ u|y| \leq \Omega^X(u/(b + \rho)) + \Omega((b + \rho)|y|) \]

and hence
\[ -u|y| + \Omega((b + \rho)|y|) \geq -\Omega^X(u/(b + \rho)) . \]

(See [Pa], p. 95.)

- In [Pa], p. 96, the following formula is given

\[ (*) \quad \psi(u + it) = \int_0^\infty \varphi(x + iy)((x + iy)(y + it))^{1/2} \gamma^x((x + iy)(u + it))dx \]

where
\[ \psi(u) = \int_0^\infty \varphi(x)(xu)^{1/2} \gamma^x(xu)dx . \]

Formula (*) is false, in general. E.g. take \( \mu = -\frac{1}{2} \) and \( \varphi(x) = e^{-\frac{1}{2}x^2} \). (To this end observe that \( \sqrt{t} \gamma_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi}} \cos t \). Hence, the proof of Theorem (5.9) in [Pa] is incorrect.

- Finally, with respect to p. 97 in [Pa] we observe that using Young's inequality as indicated yields

\[ |s^{\mu-\frac{1}{2}}\psi(s)| \leq C_{\delta, \rho}^\mu \exp[\Omega([b + \rho]y) + M((u + 1)/\rho) + \\
+ \Omega(\rho|y|)] \sum_{r=0}^n x^{2(n-r)} \exp\{-M((a - \delta)x) + M(\gamma x) + \Omega(\gamma^{-1}|t|)\} \]

which is not in agreement with the corresponding formula in [Pa].


