Disequilibria in a macro-economic model

Heuvel, van den, P.J.

Published: 01/01/1979

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 09. Jan. 2019
Memorandum COSOR 79-02

Disequilibria in a macro-economic model

by

Paul van den Heuvel

Eindhoven, May 1979

The Netherlands
Abstract

This paper discusses a model of an economy with a consumption sector and a production sector, which is similar to the models, developed by Barro and Grossman (1971), Malinvaud (1977), Böhm (1976) and others. These models are characterized by the fact that demand and supply are not equal.

This disequilibrium situation is studied under the assumption that demand and supply of the consumption sector are based on a utility function. In particular it is investigated, which conditions are to be imposed on the utility function in order to obtain the same results as the literature mentioned above.

In order to arrive at a realizable trade in the model, two approaches have been followed. The economy is described by a game and a possible adjustment mechanism is suggested.
1. Introduction

The classical economists assumed, that prices adjusted rapidly enough to equalize demand and supply. A typical example of a classical model of the economy is the model of Walras. The equality of demand and supply in such a model is called an "equilibrium". In the thirties of this century the equilibrium model seemed to be not realistic anymore. Demand and supply (for instance of labour) were not in equilibrium. To explain this Keynes presented in his "General Theory" (1936) a model in which rigid prices give rise to a "disequilibrium".

Keynes noticed, that too low wages cause a low consumption demand, which leads to a small production, accompanied by unemployment. This conclusion contradicts the one of the classical economists. The latter contended that lowering the wage rate would induce an increase of labour demand and a decrease of labour supply. This would result in a smaller unemployment.

Keynes' theory had an important influence on government policies. Nevertheless most economic theorists after Keynes addressed themselves to the Walrasian equilibrium models. Just recently "Neokeynesians", like Clower (1965) and Leyonhufoud (1968) gave a new impulse to Keynesian theory. They emphasized, just like Keynes, the impact of the interaction of several markets. Following them, micro-economic (individual agents) as well as macro-economic (aggregated sectors of agents) studies on the subject of disequilibrium emerged. Micro-economic disequilibrium models were given by Drèze (1975), Benassy (1975) and others. Barro and Grossman (1971), Malinvaud (1977) and Böhm (1970) developed macro-economic models, as did Gepts (1971) and Weddepohl (1978).

An important concept in the models of Gepts and Weddepohl is the "set of acceptable trades" for both the consumption sector and the production sector.

An "acceptable trade" in this context is a combination of an amount of labour and an amount of consumption goods, that the one sector is willing to exchange with the other sector, in situations, where the other sector imposes upper bounds on these amounts. The formal definitions of Gepts
and Weddepohl differ slightly. Weddepohl calls a trade "acceptable" if it is optimal for some pair of upper bounds. He makes some assumptions with regard to the supply and demand functions. He proves that his set of acceptable trades of the consumption sector is identical to the one of Gepts.

In this paper it is investigated if the same results can be obtained, when assumptions are made with regard to the utility function, instead of with regard to the supply and demand functions.

For the production sector the "set of acceptable trades" can be defined analogously.

The "sets of acceptable trades" are the results of the maximization problems of the sectors separately. The next question is, which is the result of the maximization problem when solved simultaneously.

This problem is approached in two ways. In the first place the economy is described as a game and a "Nash equilibrium" is defined. In the second place a possible quantity adjustment mechanism is suggested. This mechanism generates sequences of solutions. The relation between the limits of these sequences and the "Nash equilibria" of the game defined will be investigated.

Finally a classification of market situations is given, which is the same as in Malinvaud (1977).

The subdivision of this report is roughly as follows. In Chapter 2 the model is formulated. Chapter 3 contains the analysis of the maximization problems of the sectors for exogenous behaviour of the other sector. In Chapter 4 the interaction between the sectors is investigated and a classification of transactions is given.

An appendix is added with a proof of a theorem stated in Chapter 3.
2. Description of the basic model

We consider an economy with two sectors: a consumption sector and a production sector. There are three commodities: money, labour and a single consumption good.

The behaviour of the sectors can be described as follows.

The consumption sector exchanges with the production labour against money and utilizes this money together with its initial money resources to buy a certain amount of the consumption good from the production sector. The consumption sector tries to maximize its utility, which is given by a utility function.

The production sector uses the labour, supplied by the consumption sector to produce the consumption good. The process of production is determined by a production function. The production sector tries to maximize its profit.

In the sequel the following concepts will be used. A market is an institution of exchange of a commodity against money. The volume of this exchange is the result of the exchange of information of suppliers and demanders. In this paper a model is considered with a labour market and a consumption goods market.

Rationing is the imposing of upper bounds on demand and supply in order to equalize them. In the context of this paper these upper bounds are not meant to be imposed exogenously, but they are the results of a process of information exchange of suppliers and demanders.

An (Walras) equilibrium is the equality of demand and supply without rationing. A disequilibrium is the equality of supply and demand after rationing.

In order to describe the economy in mathematical terms, we will make use of the following notation.
p : price of one unit of labour
ω : wage rate, i.e. the price of one unit of labour
l : number of units of labour
x : number of units of the consumption good
m : amount of money in the consumption sector after the exchange on the markets
m₀ : initial amount of money in the consumption sector
l max : maximum number of units of labour, that can be supplied by the consumption sector.

The model will be of the short run type. This implies, that the price p and the wage rate ω are fixed (and capital is neglected). Also the initial money stocks m₀ and the maximum labour supply l max are fixed. Throughout this report the following assumption will hold.

Assumption 2.1.
p > 0, ω > 0, m₀ ≥ 0 and l max > 0.

The variables l, x and m will be restricted to nonnegative values.
The actual exchange of labour, goods and money takes place simultaneously on the two markets. A pair (l, x) is called a trade. The amount of money in the consumption sector, that would be present after the realization of the trade, is defined by m := m₀ + ωl − px.

This expression will be called the budget equation. Rationing in this model means putting upper bounds on l and x. Money stocks in are not rationed.
The transaction is the actual trade, that takes place between the sectors.

The following assumptions are made with respect to the utility function and the production function.
Assumption 2.2.

The utility function of the consumption sector $U$ is defined on the domain $\{(l,x,m) \in \mathbb{R}_+^3 | l \leq l_{\text{max}}\}$ and satisfies:

1) $U$ is three times continuously differentiable.
2) $U$ has a negative definite Hessian.
3) $U$ is monotonously increasing in $x$ and $m$ and decreasing in $l$.

Assumption 2.3.

The production function of the production sector $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ has the following properties.

1) $F(0) \leq 0$
2) $F$ is twice differentiable
3) $F$ is strictly concave
4) $F$ is increasing
5) $\lim_{l \to \infty} F(l) = \infty$
6) $\lim_{l \to \infty} F'(l) = 0$.

The production function presents the relationship between the maximum output of consumption goods and the level of labour input.

The optimal trades of the consumption sector and the production sector can independently be derived as the solutions of the following maximization problems.

(1) \[
\max\{U(l,x,m) | 0 \leq l \leq l_{\text{max}}, \ x \geq 0, \ m = m_0 + \omega l - px \geq 0\}
\]

(2) \[
\max\{px - \omega l | l \geq 0, \ x \leq F(l)\}.
\]

The solution of (1) will be denoted by $(l_c^*, x_c^*, m_c^*)$. The solution $l_c^*$ will be called notional labour supply, while $x_c^*$ will be called the notional demand of goods.
The solution of (2) will be denoted by \((\ell^*_p, x^*_p)\), where \(\ell^*_p\) is called the **notional supply of goods**. It is easy to verify that

\[
\ell^*_p = 0 \text{ if } F'(0) \leq \frac{\omega}{p}.
\]

and

\[
F'(\ell^*_p) = \frac{\omega}{p} \text{ if } F'(0) > \frac{\omega}{p}.
\]
3. The set of acceptable trades

3.1. Acceptable trades

If the optimal trades, that are derived as the solutions of the maximization problems (1) and (2), do not coincide, there must be rationing. In this chapter attention is given to trades that are candidates to be realized. To that end the concept of acceptable trades is defined following Weddepohl (1978).

If \((\bar{\ell}, \bar{x}) \in \mathbb{R}^2_+\) is a pair of upper bounds on labour supply and goods demand, the maximization problem of the consumption sector becomes

\[
\max \{ V(\ell, x, m) | 0 \leq \ell \leq \min \{ \ell_{\max}^c, \bar{\ell} \}, \ 0 \leq x \leq \bar{x}, \ m = m_0 + \omega \ell - px \geq 0 \}
\]

Similarly, the maximization problem of the production sector becomes

\[
\max \{ px - \omega \ell | 0 \leq \ell \leq \bar{\ell}, x \leq \min \{ F(\ell), \bar{x} \} \}.
\]

The \(\ell\)- and \(x\)-components of the solutions of (3) and (4) are denoted as follows.

\[
(L_c(\bar{\ell}, \bar{x}), X_c(\bar{\ell}, \bar{x})) \quad \text{solution of (3) for } \ell \text{ and } x,
\]

\[
(L_p(\bar{\ell}, \bar{x}), X_p(\bar{\ell}, \bar{x})) \quad \text{solution of (4)}.
\]

A trade is called acceptable if it is the optimal solution of the problems (3) and (4) respectively, for some pair of upper bounds \((\bar{\ell}, \bar{x})\).

**Definition 3.1.**

The set of acceptable trades of the consumption sector is

\[
H_c := \{(\ell, x) \in \mathbb{R}^2_+ | \exists (\bar{\ell}, \bar{x}) \in \mathbb{R}^2_+ : \ell = L_c(\bar{\ell}, \bar{x}), \ x = X_c(\bar{\ell}, \bar{x}) \}.
\]

The set of acceptable trades of the production sector is

\[
H_p := \{(\ell, x) \in \mathbb{R}^2_+ | \exists (\bar{\ell}, \bar{x}) \in \mathbb{R}^2_+ : \ell = L_p(\bar{\ell}, \bar{x}), \ x = X_p(\bar{\ell}, \bar{x}) \}.
\]
The set of acceptable trades of the economy is

\[ H := H_c \cap H_p \]

It is convenient to introduce the following definition.

**Definition 3.2.**

The restricted utility function, is the function defined on the domain \( \{ (\bar{l}, x) \in \mathbb{R}_+^2 | \bar{l} \leq \bar{l}_{\text{max}} \} \) and into \( \mathbb{R}_+ \), given by

\[ \tilde{U}(\bar{l}, x) := U(\bar{l}, x, m_0 + \omega \bar{\ell} - px) \]

where \( U \) is the utility function of the consumption sector.

**Proposition 3.1.**

Under the Assumptions 2.1 and 2.2, the \( \bar{l} \)- and \( x \)-component of the solution \( (L_c(\bar{l}, \bar{x}), X_c(\bar{l}, \bar{x})) \) of the maximization problem (3), is the solution of the following maximization problem:

\[ \max \{ \tilde{U}(\bar{l}, x) | 0 \leq \bar{l} \leq \min[\bar{l}, \bar{l}_{\text{max}}], \quad 0 \leq x \leq \bar{x}, \quad m_0 + \omega \bar{\ell} - px \geq 0 \} \cdot \]

In the sequel the following lemma will be used (cf. Weddepohl 1978, Assumption A4).

**Lemma 3.1.**

Under the assumption 2.1 and 2.2, for every \( (\bar{l}, \bar{x}) \in \mathbb{R}_+^2 \):

\[ L_c(\bar{l}, \bar{x}) = \min[\bar{l}, L_C(\infty, \bar{x})] \]

\[ X_c(\bar{l}, \bar{x}) = \min[\bar{x}, X_C(\bar{l}, \infty)] \cdot \]

**Proof**

To begin, it may be checked that the Hessian of the restricted utility function satisfies
The Hessian of $\tilde{U}(\ell,x)$ is thus also negative definite by Assumption 2.2. 2). Hence $\tilde{U}(\ell,x)$ is strictly concave.

The pair $(L_c(\ell,x), X_c(\ell,x))$ is the solution of (5). The solution $(L_c(\omega,\omega), X_c(\omega,\omega))$ exists, because the objective function $\tilde{U}(\ell,x)$ is continuous and the feasible set is compact.

Furthermore $(\ell^*, x^*)_c = (L_c(\omega,\omega), X_c(\omega,\omega))$ is unique, since $\tilde{U}(\ell,x)$ is strictly concave.

It will be proved first, that, the following implication holds

$$\ell < L_c(\omega,x) \Rightarrow L_c(\ell,x) = \ell.$$  

Let $\tilde{\ell} \leq L_c(\omega,x)$. Suppose $L_c(\ell,x) < \tilde{\ell}$. Next note that the following inequality will always be true.

$$\tilde{U}(L_c(\ell,x), X_c(\ell,x)) < \tilde{U}(L_c(\omega,x), X_c(\omega,x))$$

Consider the convex combination

$$(\ell_1, x_1) := \lambda_1 (L_c(\ell,x), X_c(\ell,x)) + (1 - \lambda_1) (L_c(\omega,x), X_c(\omega,x))$$

and note that there will be real numbers $\lambda_1 \in (0,1)$ such that $L_c(\ell,x) < \lambda_1 < \ell_1$.

Since furthermore $x_1 \leq x$ and $m_o + \omega_1 \geq 0$, the convex combination $(\ell_1, x_1)$ is a point in the feasible region of the maximization problem (5). From inequality (6) and strict concavity of $\tilde{U}(\ell,x)$ it follows that

$$\tilde{U}(L_c(\ell,x), X_c(\ell,x)) < \tilde{U}(\ell_1,x_1)$$

This is a contradiction to the fact that $(L_c(\ell,x), X_c(\ell,x))$ is the optimal solution of (5). Therefore $L_c(\ell,x) \geq \ell$. 

\[ \begin{bmatrix} \tilde{U}_{\ell \ell} & \tilde{U}_{\ell x} \\ \tilde{U}_{x \ell} & \tilde{U}_{xx} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{x x} & U_{x m} & U_{x n} \\ U_{x x} & U_{x m} & U_{x n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
The other implication

\[ \tilde{l} > L_C (\omega, \check{x}) \Rightarrow L_C (\tilde{l}, \check{x}) = L_C (\omega, \check{x}) \]

follows immediately from the fact that the hypothesis \( \tilde{l} > L_C (\omega, \check{x}) \)
implies that the point \((L_C (\omega, \check{x}), X_C (\omega, \check{x}))\) lies in the feasible region of the maximization problem (5) and therefore will be the solution of (5). The first equality of Lemma 3.1 is proved now.

The proof of the second equality is analogous. Thus Lemma 3.1 is proved.

With Lemma 3.1 the following corollaries are straightforward.

**Corollary 3.1.**

Under the Assumptions 2.1 and 2.2 the following equalities hold

\[ L_C (\tilde{l}, \omega) = \min[\tilde{l}, \lambda^*_c] \]
\[ X_C (\omega, \check{x}) = \min[\check{x}, \lambda^*_c] . \]

**Corollary 3.2.**

Under the assumptions 2.1 and 2.2 the following relations hold

\[ \tilde{l} \leq L_C (\omega, \check{x}) \land \check{x} \leq X_C (\tilde{l}, \omega) \Rightarrow L_C (\tilde{l}, \check{x}) = \tilde{l} \land X_C (\tilde{l}, \check{x}) = \check{x} \]
\[ \tilde{l} > L_C (\omega, \check{x}) \land \check{x} \leq \lambda^*_c \Rightarrow L_C (\tilde{l}, \check{x}) = L_C (\omega, \check{x}) \land X_C (\tilde{l}, \check{x}) = \check{x} \]
\[ \tilde{l} \leq \lambda^*_c \land \check{x} > X_C (\tilde{l}, \omega) \Rightarrow L_C (\tilde{l}, \check{x}) = \tilde{l} \land X_C (\tilde{l}, \check{x}) = X_C (\tilde{l}, \omega) \]
\[ \tilde{l} > \lambda^*_c \land \check{x} > \lambda^*_c \Rightarrow L_C (\tilde{l}, \check{x}) = \lambda^*_c \land X_C (\tilde{l}, \check{x}) = \lambda^*_c . \]
Corollary 3.3.

Under the Assumptions 2.1 and 2.2, it is true that if $\overline{x} \leq x^*_c$, then $L_c(\overline{\omega}, \overline{x})$ is equal to the solution of

$$\max_{\lambda} \{ \tilde{U}(\lambda, \overline{x}) \mid 0 \leq \lambda \leq \lambda_{\text{max}} \}.$$ 

Analogously, if $\lambda \leq \lambda^*_c$, then $X_c(\lambda, \omega)$ equals the solution of

$$\max_{x} \{ \tilde{U}(\lambda, x) \mid 0 \leq x \leq \frac{1}{\omega}(m_0 + \omega \lambda) \}.$$ 

With Lemma 3.1 an alternative expression can be obtained for the set of acceptable trades of the consumption sector.

Theorem 3.1.

Under the Assumptions 2.1 and 2.2, the set of acceptable trades

$$H_c := \{ (\lambda, x) \in \mathbb{R}_+^2 \mid \exists (\overline{\lambda}, \overline{x}) \in \mathbb{R}_+^2 : \lambda = L_c(\overline{\lambda}, \overline{x}), x = X_c(\overline{\lambda}, \overline{x}) \}$$

is identical to the set

$$\tilde{H}_c := \{ (\lambda, x) \in \mathbb{R}_+^2 \mid \lambda \leq L_c(\omega, x), x \leq X_c(\lambda, \omega) \}.$$ 

Proof

Let $(\lambda, x) \in H_c$. Then by Lemma 3.1 it follows that $L_c(\lambda, x) = \lambda$ and $H_c(\lambda, x) = x$. Therefore $(\lambda, x) \in H_c$.

Let $(\lambda, x) \notin \tilde{H}_c$. Then either $\lambda > L_c(\omega, x)$ or $x > X_c(\lambda, \omega)$ or both.

By definition of $L_c(\lambda, x)$ and $X_c(\lambda, x)$ the inequality

$$\tilde{U}(\lambda, x) \leq \tilde{U}(L_c(\lambda, x), X_c(\lambda, x))$$

holds. It is always true, that

$$\tilde{U}(L_c(\lambda, x), X_c(\lambda, x)) \leq \tilde{U}(L_c(\omega, x), X_c(\omega, x))$$

hence

$$\tilde{U}(\lambda, x) \leq \tilde{U}(L_c(\omega, x), X_c(\omega, x))$$

In case $\lambda > L_c(\omega, x)$, then for the convex combination

$$(\lambda_2, x_2) := \lambda_2 (L_c(\omega, x), X_c(\omega, x)) + (1 - \lambda_2) (\lambda, x)$$

we have for any real number $\lambda_2 \in (0,1)$
As a result of these inequalities, strict concavity and the inequality 
\[ \tilde{U}(\ell, x) \leq \tilde{U}(L_c(\omega, x), X_c(\omega, x)) \] 
we have
\[ \tilde{U}(\ell_2, x_2) > \tilde{U}(\ell, x) . \]

This implies that the point \((\ell, x)\) with \(\ell > L_c(\omega, x)\) can not be a solution of (5) for any pair \((\bar{\ell}, \bar{x})\) with \(\bar{\ell} \geq \ell\) and \(\bar{x} \geq x\). Thus there is no pair \((\bar{\ell}, \bar{x})\) such that \((\ell, x) = (L_c(\bar{\ell}, \bar{x}), X_c(\bar{\ell}, \bar{x}))\).

Hence \((\ell, x) \notin H_c\). In case \(x > X_c(\ell, \omega)\) a similar argument can be applied. Therefore the relation \((\ell, x) \notin H_c \Rightarrow (\ell, x) \notin H_c\) is true and the equality \(H_c = \tilde{H}_c\) is proved.

It may be noted that the assumption of Weddepohl that \(X_c(\bar{\ell}, \bar{x})\) is a non-decreasing function of \(\bar{\ell}\) and \(L_c(\bar{\ell}, \bar{x})\) is a non-decreasing function of \(\bar{x}\), is not necessary for the result of Theorem 3.1 to hold.

For the production sector the following result can be obtained.

**Lemma 3.2.**

Under the Assumptions 2.1 and 2.3, for every \((\bar{\ell}, \bar{x}) \in \mathbb{R}_+^2\), the solution \((L_p(\bar{\ell}, \bar{x}), X_p(\bar{\ell}, \bar{x}))\) of (4) has the following properties. The component \(L_p(\bar{\ell}, \bar{x})\) is the solution of

\[ \max\{pF(\ell) - w|\ell| \leq \ell \leq \min[\bar{\ell}, F^{-1}(\bar{x})]\} \]

\[ X_p(\bar{\ell}, \bar{x}) = F(L_p(\bar{\ell}, \bar{x})) . \]

**Proof**

It will be proved first, that the equality (8) holds. For any pair \((\bar{\ell}, \bar{x}) \in \mathbb{R}_+^2\) we have
\[ X_p(\bar{\ell}, \bar{x}) \leq F(L_p(\bar{\ell}, \bar{x})) . \]

Since the function \( F \) is continuous and increasing, the inverse function \( F^{-1} : \mathbb{R} \to \mathbb{R}_+ \), can be defined and is continuous and increasing. It follows that

\[ F^{-1}[X_p(\bar{\ell}, \bar{x})] \leq L_p(\bar{\ell}, \bar{x}) . \]

Hence

\[ pX_p(\bar{\ell}, \bar{x}) - \omega F^{-1}[X_p(\bar{\ell}, \bar{x})] \geq pX_p(\bar{\ell}, \bar{x}) - \omega L_p(\bar{\ell}, \bar{x}) . \]

It also follows by definition of \( L_p \) and \( X_p \) and the fact, that the point \((F^{-1}[X_p(\bar{\ell}, \bar{x})], X_p(\bar{\ell}, \bar{x}))\) lies in the feasible region of the maximization problem (4), that

\[ pX_p(\bar{\ell}, \bar{x}) - \omega F^{-1}[X_p(\bar{\ell}, \bar{x})] \leq pX_p(\bar{\ell}, \bar{x}) - \omega L_p(\bar{\ell}, \bar{x}) . \]

Hence

\[ L_p(\bar{\ell}, \bar{x}) = F^{-1}[X_p(\bar{\ell}, \bar{x})] \]

and so

\[ X_p(\bar{\ell}, \bar{x}) = F[L_p(\bar{\ell}, \bar{x})] . \]

Therefore the \( \ell \)-component \( L_p(\bar{\ell}, \bar{x}) \) of the solution of (4) is also the solution of the maximization problem (7).

Before proceeding we may show, that the solution of (4) always exists. From the fact that \( L_p(\bar{\ell}, \bar{x}) \) is the solution of (7) it is evident, that \( L_p(\omega, \omega) \) is the solution of

\[ \max\{pF(\ell) - \omega \ell \mid \ell \geq 0\} . \]

If there is a value of \( \ell \) for which \( F'(\ell) = \frac{\omega}{p} \), then the differentiable and concave objective function reaches a maximum.

If \( F'(0) \leq \frac{\omega}{p} \), then \( L_p(\omega, \omega) = 0 \). If \( F'(0) > \frac{\omega}{p} \), then the facts that the function \( F'(\ell) \) is continuous and decreasing and that \( \lim_{\ell \to \infty} F'(\ell) = 0 \),

\[ \ell \to \infty \]

imply that there is a finite value of \( \ell \), for which the equality \( F'(\ell) = \frac{\omega}{p}(\ell > 0) \) holds. Hence \( \ell^*_p : L_p(\omega, \omega) < \infty \) and \( x^*_p : X_p(\omega, \omega) = F[L_p(\omega, \omega)] < \infty \).
The solution \((\ell_p(\tilde{\ell},\tilde{x}), \Lambda_p(\tilde{\ell},\tilde{x}))\) of (4) can be expressed alternatively.

**Lemma 3.3.**

Under the Assumption 2.1 and 2.3, for every \((\tilde{\ell},\tilde{x}) \in \mathbb{R}^2_+\)

\[
\begin{align*}
\ell_p(\tilde{\ell},\tilde{x}) &= \min\{\tilde{\ell},\mu^{-1}(\tilde{x}),\ell^*_p\} \\
\Lambda_p(\tilde{\ell},\tilde{x}) &= \min\{\mu(\tilde{\ell}),\tilde{x},\ell^*_p\}
\end{align*}
\]

**Proof**

Since the objective function \(pF(\ell) - \omega \ell\) in the maximization problem (7) is increasing for \(\ell \in [0,\ell_p]\), we can write

\[
\ell_p(\tilde{\ell},\tilde{x}) = \max\{0 \leq \ell \leq \min\{\tilde{\ell},\mu^{-1}(\tilde{x}),\ell^*_p\}\}
\]

and hence

\[
\ell_p(\tilde{\ell},\tilde{x}) = \min\{\tilde{\ell},\mu^{-1}(\tilde{x}),\ell^*_p\}
\]

The second equality follows from equality (8) of Lemma 3.2.

**Lemma 3.3 implies the following corollary.**

**Corollary 3.4.**

Under the Assumptions 2.1 and 2.3, the following equalities hold

\[
\begin{align*}
\ell_p(\omega,\tilde{x}) &= \min\{\omega,\ell^*_p\} \\
\Lambda_p(\omega,\tilde{x}) &= \min\{\omega,\tilde{x},\ell^*_p\} \\
\ell_p(\omega,\tilde{x}) &= \min\{\mu^{-1}(\tilde{x}),\ell^*_p\} \\
\Lambda_p(\omega,\tilde{x}) &= \min\{\mu(\tilde{\ell}),\tilde{x},\ell^*_p\}
\end{align*}
\]

With Lemma 3.3 alternative expressions can be obtained for the set of acceptable trades of the production sector.
Theorem 3.2.

Under the Assumptions 2.1 and 2.3, the set of acceptable trades

\[
H_p := \{ (\lambda, x) \in \mathbb{R}_+^2 \mid \exists (\bar{\lambda}, \bar{x}) \in \mathbb{R}_+^2 : \lambda = L_p(\bar{\lambda}, \bar{x}), x = X_p(\bar{\lambda}, \bar{x}) \}
\]

is identical to the sets

\[
\tilde{H}_p := \{ (\lambda, x) \in \mathbb{R}_+^2 : \lambda \leq L_p(\infty, x), x \leq X_p(\infty, \infty) \}
\]

and

\[
\tilde{\tilde{H}}_p := \{ (\lambda, x) \in \mathbb{R}_+^2 : \lambda \leq \lambda^*_p, x = F(\lambda) \}.
\]

Proof

It will be proved first that \(H_p = \tilde{\tilde{H}}_p\). The condition

\[
\exists (\bar{\lambda}, \bar{x}) \in \mathbb{R}_+^2 : \lambda = L_p(\bar{\lambda}, \bar{x}), x = X_p(\bar{\lambda}, \bar{x})
\]

is by Lemma 3.3 equivalent with

\[
\exists (\bar{\lambda}, \bar{x}) \in \mathbb{R}_+^2 : \lambda = \min[\bar{\lambda}, F^{-1}(\bar{x})], x = F(\lambda).
\]

The latter condition is equivalent with \(\lambda \leq \lambda^*_p, x = F(\lambda)\), hence \(H_p = \tilde{\tilde{H}}_p\).

The equation \(\tilde{H}_p = \tilde{\tilde{H}}_p\) follows immediately from Corollary 3.4.

3.2. The Effective Demand and Supply Functions

Following Theorem 3.1 and Theorem 3.2, the sets of acceptable trades of both sectors can be expressed using functions of only one argument. This is a quite attractive result. An investigation of the sets of acceptable trades will involve an investigation of the properties of these functions. The relevant functions have the following names.
Definition 3.3.

The function $L_C(\omega,x)$ is called the effective labour supply function.

The function $X_C(\ell,\omega)$ is called the effective goods demand function.

The function $L_P(\omega,x)$ is called the effective labour demand function.

The function $X_P(\ell,\omega)$ is called the effective goods supply function.

The word "effective", which has its origin in Keynesian literature, is justified by the fact that the particular demands and supplies play an effective role in the process of generating a transaction (see Chapter 4), this in contrast to the "notional" quantities.

In Figure 3.1, the functions $L_C(\omega,x)$ and $X_C(\ell,\omega)$ are represented in one picture. The set of acceptable trades of the consumption sector is indicated by the shaded area (see Theorem 2.1). Figure 3.2 depicts $L_P(\omega,x)$ and $X_P(\ell,\omega)$. The acceptable trades of the production sector are the elements of the intersection of the graphs of these functions (cf. Corollary 3.4).

**Figure 3.1.**
acceptable trades of the consumption sector

**Figure 3.2.**
acceptable trades of the production sector
3.3. Some Properties of the Sets of Acceptable Trades

Böhm (1976) assumed the functions \( L_c(\omega, x) \) and \( X_c(\ell, \omega) \) of the consumption sector to be increasing and concave in their arguments. The first question of this section to be considered is which conditions are to be imposed on the utility function to warrant this result. In the process of answering that question we will have use for the following theorem.

**Theorem 3.3.**

Let the Assumptions 2.1 and 2.2 hold. If for every triple \((\ell, x, m)\), which satisfies \(0 \leq \ell \leq \ell_c, \quad x = X_c(\ell, \omega)\) and \(m = m_0 + \omega \ell - px\), the partial derivative

\[
\frac{\partial}{\partial \ell} U_{xl}(\ell, x) = U_{xl}(\ell, x, m) - pU_{m\ell}(\ell, x, m) + \omega[U_{xm}(\ell, x, m) - pU_{mm}(\ell, x, m)]
\]

is positive, then \(X_c(\ell, \omega)\) is an increasing function of \(\ell\) on \([0, \ell_c]\).

If the same inequality holds for every triple \((\ell, x, m)\), which satisfies \(0 \leq x \leq x_c^*, \quad \ell = L_c(\omega, x), \quad m = m_0 + \omega \ell - px\), then \(L_c(\omega, x)\) is an increasing function of \(x\) on \([0, x_c^*]\).

**Proof.**

See Appendix.

Conditions for the concavity of the functions \(L_c(\omega, x)\) and \(X_c(\ell, \omega)\) can also be given (see Remark 2 under the proof of Theorem 3.3 in de Appendix).

These conditions turn out to be very complicated.

If the functions \(L_c(\omega, x)\) and \(X_c(\ell, \omega)\) are increasing and concave, then the set of acceptable trades \(H_c\) is convex.

In the Figures 3.3 - 3.6 a few cases of monotonous effective demand and supply functions are depicted. The figures show contour curves of equal utility. For a labour volume \(\bar{\ell} \in [0, \ell_c^*]\) the effective demand \(X_c(\bar{\ell}, \omega)\) is the volume of \(x\), where the line \(\bar{\ell} = \bar{x}\) is tangent to the contour curve. Analogously, for a volume of consumption \(\bar{x} \in [0, x_c^*]\), the effective labour supply \(L_c(\omega, \bar{x})\) is the volume of \(\ell\), where the line \(x = \bar{x}\) is tangent to the contour curve.
Figure 3.3.
increasing effective demand and supply functions ($\tilde{U}_{xx} > 0$)

Figure 3.4.
decreasing effective demand and supply functions ($\tilde{U}_{xx} < 0$)

Figure 3.5.
constant effective demand and supply functions ($\tilde{U}_{xx} = 0$, $\tilde{U}_{xx} < \tilde{U}_{kk}$)

Figure 3.6.
constant effective demand and supply functions ($\tilde{U}_{xx} = 0$, $\tilde{U}_{xx} > \tilde{U}_{kk}$)
Having discussed the situation with respect to the consumption sector, some remarks can be made about the sets of acceptable trades of the production sector $H_p$ and of the set of acceptable trades of the economy $H$.

It follows from Theorem 3.2, that $H_p$ is a compact subset of the graph of an increasing and concave function. Hence $H = H_c \cap H_p$ is a compact subset of the graph of an increasing and concave function (which might be empty or non-connected). This property implies the following proposition.

**Proposition 3.2.**

Under the assumptions 2.1 - 2.3, there is a unique trade $(\hat{\ell}, \hat{x})$ satisfying

$$\hat{\ell} = \max \{ \ell \mid (\ell, x) \in H \}$$
$$\hat{x} = \max \{ x \mid (\ell, x) \in H \}$$

if $H$ is not empty.

The trade $(\hat{\ell}, \hat{x})$ in Proposition 3.2 is the best trade out of the set of acceptable trades of the economy $H$. This will be proved in the following theorem.

**Theorem 3.4.**

Under the assumptions 2.1 - 2.3, the trade $(\hat{\ell}, \hat{x})$ in Proposition 3.2 satisfies

$$\tilde{U}(\hat{\ell}, \hat{x}) = \max \{ \tilde{U}(\ell, x) \mid (\ell, x) \in H \}$$
$$p\hat{x} - \omega \hat{\ell} = \max \{ p\ell - \omega \ell \mid (\ell, x) \in H \}.$$  

**Proof**

For all $(\ell, x) \in H$ the inequalities $0 \leq \ell \leq \hat{\ell}$, $0 \leq x \leq \hat{x}$ and $m_\ell + \omega \ell - px \geq 0$ hold. By definition of $L_c(\ell, x)$ and $X_c(\ell, x)$ we thus have
\[ V(\ell, x) \in \mathbb{H} : \tilde{U}(\ell, x) \leq \tilde{U}(L_C(\ell, \bar{x}), X_C(\ell, \bar{x})) \]

Lemma 3.1 implies \( L_C(\ell, \bar{x}) = \min[\ell, L_C(\omega, x)] \).

Since \((\ell, \bar{x}) \in \mathbb{H} \), \( \ell \leq L_C(\omega, \bar{x}) \).

Hence \( L_C(\ell, \bar{x}) = \ell \) and analogously \( X_C(\ell, \bar{x}) = \bar{x} \).

Now, (10) follows immediately.

The relation (11) follows from the facts that for any \((\ell, x) \in \mathbb{H} \) the equality \( px - \omega \ell = pF(\ell) - \omega \ell \) holds (cf. Lemma 3.2) and that the function \( pF(\ell) - \omega \ell \) is increasing for \( \ell \leq \ell^* \).

Theorem 3.4 justifies the following definition.

**Definition 3.4.**

The trade \((\ell, \bar{x})\) in Proposition 3.2 is the **optimal trade** of the economy.

The role of the optimal trade in the interaction of both sectors will be investigated in the next chapter.
4. The interactions between the sectors

4.1. A Game Theoretic Approach

In this section the interaction between the sectors will be approached in a game theoretic way, using the concept of a Nash equilibrium. We start by considering an n-person game following the ideas of Makarov and Rubinov (1977).

Let \( \{1, \ldots, n\} \) be the set of players. Each player \( i \) has a set \( \tilde{Y}_i \subset \mathbb{R}^m \) at his disposal, from which he can select a strategy \( y_i \). The set \( \tilde{Y}_i \) is called the strategy set of player \( i \), \( i = 1, \ldots, n \).

By choosing a strategy \( y_i \), the player \( i \) receives a pay-off. The value of this pay-off is determined by an object function

\[
V_i : \tilde{Y}_i \to \mathbb{R}_+, \quad i = 1, \ldots, n.
\]

However, not all combinations of strategies are allowed. If \( \prod_{j=1}^n \tilde{Y}_j \) is the Cartesian product of the strategy sets and \( \Omega(\tilde{Y}_i) \) is the class of subsets of the strategy set \( \tilde{Y}_i \), then for each player \( i \) there exists a multifunction,

\[
\gamma_i : \prod_{j=1}^n \tilde{Y}_j \to \Omega(\tilde{Y}_i).
\]

This multifunction will be called the choice map of the player \( i \). The choice map determines the allowed strategies of the player for given strategies of the other players.

A Nash equilibrium is a combination of strategies, such that the strategy of each player maximizes the objective for given strategies of all other players. Formally we may define this as follows:

**Definition 4.1.**

A Nash equilibrium of the game \( \{ (\tilde{Y}_i)_{i=1}^n, (V_i)_{i=1}^n, (Y_i)_{i=1}^n \} \) is an allocation \( \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n) \), such that for \( i = 1, 2, \ldots, n \)

1) \( \tilde{y}_i \in Y_i(\tilde{y}) \);

2) \( V_i(\tilde{y}_i) = \max \{ V_i(y_i) \mid y_i \in Y_i(\tilde{y}) \} \).
In the economy, considered so far, we may assume, that there are 2 players: the consumption sector $C$ and the production sector $P$. Their strategy sets are the sets of acceptable trades (see Section 3.1).

$$\tilde{Y}_C := H_C = \{(l, x) \in \mathbb{R}^2 \mid l \leq L_C(\infty, x), \ x \leq X_C(l, \infty)\},$$

$$\tilde{Y}_P := H_P = \{(l, x) \in \mathbb{R}^2 \mid 0 \leq l \leq L_P(\infty, x), \ x \leq X_P(l, \infty)\}.$$

The objectives of the players are given by the functions

$$V_C(l, x) := \tilde{U}(l, x)$$

$$V_P(l, x) := px - \omega l.$$

The choice maps are defined by

$$Y_{C}(l, x, x) := \{(l, x) \in H_C(l, x) \mid \ x \leq X_C(l, x)\},$$

$$Y_{P}(l, x, x) := \{(l, x) \in H_P(l, x) \mid \ x \leq X_P(l, x)\}.$$

The formulation of the game may give rise to some comments. It may be noted, that the utility function plays a threefold role for the consumption sector, i.e.

1) in the determination of the boundary of the strategy set,

2) in the object function,

3) in the choice function, in the determination of the boundary of the set of possible strategies, if the strategy of the other sector is given.

A similar observation may be made with respect to the role profit plays for the production sector.

Another observation is that the choice map $Y_P$ can be written as

$$Y_{P}(l, x, x) := \{(l, x) \in H_P(l, x) \mid \ x \leq X_P(l, x)\}.$$

It will be proved, that the optimal trade of the economy, which was defined in Section 3.3 as
\( \hat{l} := \max\{l \mid (l,x) \in H\} \)

\( \hat{x} := \max\{x \mid (l,x) \in H\} \)

is a Nash equilibrium.

Theorem 4.1.

Under the Assumptions 2.1 - 2.3, the quadruple \((\hat{l},\hat{x},\hat{l},\hat{x})\) is a Nash equilibrium of the game \(G, G, G, G, G, G, G, G\).

Proof

It will be proved first, that the optimal trade \((\hat{l},\hat{x})\) is an element of both sets \(Y_c(\hat{l},\hat{x},\hat{l},\hat{x})\) and \(Y_p(\hat{l},\hat{x},\hat{l},\hat{x})\).

In order to prove this both sets will be determined. By definition the trade \((\hat{l},\hat{x})\) is an element of \(H_c\). This implies the following inequalities:

\[ \hat{l} \leq L_c(\infty, \hat{x}) \]

\[ \hat{x} \leq X_c(\hat{l}, \infty) \]

Also by definition

\[ \forall (l,x) \in H : l \leq L_c(\infty, \hat{x}) \land x \leq X_c(\hat{l}, \infty) . \]

Therefore we have

\[ Y_c(\hat{l},\hat{x},\hat{l},\hat{x}) = \{(l,x) \in H \mid l \leq L_c(\infty, \hat{x}) \land x \leq X_c(\hat{l}, \infty)\} = H . \]

Analogously

\[ Y_p(\hat{l},\hat{x},\hat{l},\hat{x}) = \{(l,x) \in H \mid l \leq \hat{l} \land x \leq \hat{x}\} = H . \]

Hence the trade \((\hat{l},\hat{x})\) is an element of both sets.

Theorem 3.4. implies that the optimal trade maximizes both utility and profit on \(H\), hence the second condition of a Nash equilibrium is fulfilled. \(\square\)
One might ask if the Nash equilibrium is unique for the type of game defined. In general the answer is negative. Figure 4.1 shows a situation with more than one Nash equilibrium. In the figure the production function $F(\ell)$ and the effective demand function $X_c(\ell, \infty)$ have two points, $K_1$ and $K_2$, in common. If the coordinates of the points are denoted by $(\ell_1, x_1)$ and $(\ell_2, x_2)$ respectively, then

\[
x_1 = F(\ell_1) = X_c(\ell_1, \infty) < \min\{x_c^*, x_p^*\}
\]

\[
x_2 = F(\ell_2) = X_c(\ell_2, \infty) < x_1.
\]

**Figure 4.1.**

A case of more than one Nash equilibrium

Since we have

\[
Y_p(\ell_1, x_1, \ell_1, x_1) = Y_c(\ell_1, x_1, \ell_1, x_1) = \{(\ell, x) \in \mathbb{R}_+^2 | x = F(\ell) \}
\]

\[
\ell \leq \ell_2 \cup \{(\ell_1, x_1)\}
\]
and \((l_1, x_1)\) is maximal on this set for both sectors, the combination 
\((l_1, x_1', l_1, x_1)\) is a Nash equilibrium. Similarly the quadruples 
\((l_2, x_2', l_2, x_2)\) and \((l_2, x_2, l_2, x_2)\) are Nash equilibria.

4.2. An Adjustment Mechanism

In the literature (See Barro and Grossman (1971), Malinvaud (1977), Böhm
(1976), for example), it is usually tacitly assumed, that there are rapid
quantity adjustments in an economy with fixed prices. These adjustments
lead to an agreement between the sectors and a transaction will be the
result. A possible example of such an adjustment mechanism is given in
the following definition.

Definition 4.2.

The Adjustment Mechanism AM is defined by the following properties:

1) The production sector and the consumption sector announce, one by one,
offers of \((l, x)\)-combinations, that they intend to trade, taking into
account the preceding offer of the other sector.

2) The offers will be the solutions of the maximization problems of the
corresponding sectors, in which the preceding offer of the other
sector is treated as a pair of upper bounds.

The adjustment mechanism AM generates sequences of trades. For every
starting point there is a sequence. We are especially interested in two
of these sequences, to wit the arrays, that start off in the notional
solutions of the two sectors.

Formally we define these sequences as follows.

Definition 4.3.

The sequence \(\{(l^s_s, x^s_s)_{s=1}^\infty\}\) is the array, in which the elements are
solutions of the maximization problems (3) and (4) in the following way

\((l_1^1, x_1^1) = (l_c^*, x_c^*)\)
and for $s = 1, 2, \ldots$

$$ (l^{2s}, x^{2s}) = (L_p(l^{2s-1}, x^{2s-1}), X_p(l^{2s-1}, x^{2s-1})) $$

$$ (l^{2s+1}, x^{2s+1}) = (L_c(l^{2s}, x^{2s}), X_c(l^{2s}, x^{2s})) . $$

Analogously, the sequence $\{(l_s^S, x_s^S)_{s=1}^{\infty}\}$ is the array with the elements

$$(l^1, x^1) = (l_p^*, x_p^*)$$

and for $s = 1, 2, \ldots$

$$ (l^{2s}, x^{2s}) = (L_c(l^{2s-1}, x^{2s-1}), X_c(l^{2s-1}, x^{2s-1})) $$

$$ (l^{2s+1}, x^{2s+1}) = (L_p(l^{2s}, x^{2s}), X_p(l^{2s}, x^{2s})) . $$

**Theorem 4.2.**

If the Assumptions 2.1 - 2.3 hold and the functions $L_c(\omega, x)$ and $X_c(l, \omega)$ are increasing, the optimal trade of the economy $(\hat{l}, \hat{x})$ is the unique limit of the sequences $\{(l_s^S, x_s^S)_{s=1}^{\infty}\}$ and $\{(l_s^S, x_s^S)_{s=1}^{\infty}\}$.

**Proof**

The argumentation is identical for both sequences.

In order to show that the pair $(\hat{l}, \hat{x})$ is a pair of lower bounds of the sequence, it will be proved that

$$ (l^3, x^3) \geq (\hat{l}, \hat{x}) \Rightarrow (l^{s+1}, x^{s+1}) \geq (\hat{l}, \hat{x}) . $$

Let $(l^S, x^S) \geq (\hat{l}, \hat{x})$. First consider the case where the trade $(l^S, x^S)$ is an offer of the production sector. Then $(l^{S+1}, x^{S+1}) = (L_c(l^S, x^S), X_c(l^S, x^S))$ and relation (12) will be proved with the help of Corollary 3.2. Just as in the corollary four cases may be distinguished. We will prove relation (12) for the second case, i.e. we assume that $l^S > L_c(\omega, x^3)$ and $x^S \leq x^*_c$. The corollary implies $L_c(l^S, x^3) = L_c(\omega, x^S)$. Since by assumption $x^S \geq \hat{x}$ and the function $L_c(\omega, x)$ is increasing, we have
\[ L_c(\infty, x^s) \geq L_c(\infty, \tilde{x}) \]

Furthermore, \((\lambda, \tilde{x}) \in H_0\) implies \(L_c(\lambda, \tilde{x}) = \lambda\).

Therefore the following relation holds

\[ \lambda^{s+1} = L_c(\lambda^s, x^s) = L_c(\infty, x^s) \geq L_c(\infty, \tilde{x}) \geq L_c(\lambda, \tilde{x}) \geq \lambda. \]

For the component \(x^{s+1}\) similarly

\[ x^{s+1} = X_c(\lambda^s, x^s) = x^s \geq \tilde{x}. \]

In either one of the following cases

\[ \lambda^s \leq L_c(\infty, x^s) \wedge x^s \leq X_c(\lambda^s, \infty) \]

\[ \lambda^s \leq \lambda^* \wedge x^s > X_c(\lambda^s, \infty) \]

\[ \lambda^s > \lambda^* \wedge x^s > x^* \]

The relation (12) can be proved in an analogous way. Thus relation (12) is proved for the case that \((\lambda^s, x^s)\) is an offer of the producers.

Next, let the trade \((\lambda^s, x^s)\) be an offer of the consumption sector. Then \((\lambda^{s+1}, x^{s+1}) = (L_c(\lambda^s, x^s), X_c(\lambda^s, x^s))\) and by lemma 3.3, this is equal to one of the pairs \((\lambda^s, F(\lambda^s)), (F^{-1}(x^s), x^s)\) and \((\lambda^*, x^*)\). In all of these three cases it is easy to verify relation (12).

Hence relation (12) is proved.

We have \((\lambda^*, x^*) \geq (\lambda, \tilde{x})\) and if the functions \(L_c(\infty, x)\) and \(X_c(\lambda, \infty)\) are increasing, then \((\lambda^*_c, x^*_c) \geq (\lambda, \tilde{x})\).

Hence

\[ (\lambda^1, x^1) \geq (\lambda, \tilde{x}). \]

The relations (12) and (13) imply, that the pair \((\lambda, \tilde{x})\) is a pair of lower bounds of the sequence. Since it follows from the definition, that the sequence is non-increasing, there is a limit \((\lambda, \xi)\) of the sequence with \((\lambda, \xi) \geq (\lambda, \tilde{x}). \)
The trade \((\bar{l}, \bar{x})\) is unique and therefore it is the only limit of the sequence.

The conclusion is, that if one of the notional solutions is taken as a starting point, the adjustment mechanism AM leads to the optimal trade of the economy, and that is one of the Nash equilibria of the game considered before. As soon as an agreement is reached, a transaction can be realized. If adjustment mechanism AM is followed, the transaction is the optimal trade of the economy.

4.3. Characterization of Market Situations

In this section the Nash equilibria of section 4.1 are classified. The Nash equilibria will be considered as possible transactions. Following Malinvaud (1977) these transactions get the names of the table below (See also the Figures 4.2 - 4.7).

<table>
<thead>
<tr>
<th>Type of transaction</th>
<th>characterization</th>
<th>market situation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walras equilibrium</td>
<td>( l = \bar{\ell}^* \leq \ell^<em>_c \leq \ell^</em>_p )</td>
<td>equilibrium in both markets</td>
</tr>
<tr>
<td></td>
<td>( x = \bar{x}^* \leq x^<em>_c \leq x^</em>_p )</td>
<td></td>
</tr>
<tr>
<td>Repressed Inflation</td>
<td>( l = L_c(\omega,x) \leq \ell^*_p )</td>
<td>both demand sides rations</td>
</tr>
<tr>
<td></td>
<td>( x &lt; x^*_c \leq X_c(\ell,\omega) )</td>
<td></td>
</tr>
<tr>
<td>Keynesian Unemployment</td>
<td>( l &lt; \ell^*_p &lt; L_c(\omega,x) )</td>
<td>both supply sides rations</td>
</tr>
<tr>
<td></td>
<td>( x = X_c(\ell,\omega) \leq x^*_c )</td>
<td></td>
</tr>
<tr>
<td>Classical Unemployment</td>
<td>( l \leq \ell^*_p &lt; L_c(\omega,x) )</td>
<td>consumption sector rations</td>
</tr>
<tr>
<td></td>
<td>( x = x^*_p \leq X_c(\ell,\omega) )</td>
<td></td>
</tr>
<tr>
<td>Only production sector</td>
<td>( l \leq \ell^<em>_p &lt; \ell^</em>_c \leq \ell^*_p )</td>
<td>production sector rations</td>
</tr>
<tr>
<td>rationed</td>
<td>( x = x^<em>_c \leq F(\ell^</em>_c) \leq x^*_p )</td>
<td></td>
</tr>
<tr>
<td>Unemployment, union-strained producers</td>
<td>( l \leq \ell^<em>_p &lt; L_c(\omega,x^</em>_p) )</td>
<td>labour supply rationed</td>
</tr>
<tr>
<td></td>
<td>( x = x^<em>_p = X_c(\ell^</em>_p,\omega) )</td>
<td>equilibrium goods</td>
</tr>
<tr>
<td>Constrained consumers on the goods market only</td>
<td>( l = \ell^<em>_p \leq L_c(\omega,x^</em>_p) )</td>
<td>equilibrium labour</td>
</tr>
<tr>
<td></td>
<td>( x = x^<em>_p \leq X_c(\ell^</em>_p,\omega) )</td>
<td>goods demand rationed</td>
</tr>
</tbody>
</table>
Figure 4.2.
Walras equilibrium

Figure 4.3.
Repressed Inflation

Figure 4.4.
Keynesian Unemployment

Figure 4.5.
Classical Unemployment

Figure 4.6.
Only production sector rationed

Figure 4.7.
Unemployment, unconstrained producers
A Walras equilibrium is a transaction \((l, x)\) with \((l^*, x^*) = (l^*_P, x^*_P)\) (see Figure 4.2). In a Walras equilibrium there is no rationing. Both sectors are satisfied. A Walras equilibrium always is an optimal trade. If there is rationing of labour demand for the producers and consumption demand for the consumers, the transaction will be called a "Repressed Inflation"-transaction (see Figure 4.3). Then there is an excess demand on both markets. The fact that this feature is normally accompanied by inflation, while prices are fixed in this model, explains the name.

In a "Keynesian Unemployment"-transaction both supply sides are rationed (see Figure 4.4). This situation is called "Keynesian Unemployment", because it is associated with the explanation of unemployment by Keynes (see Chapter 1).

If the consumption sector is rationed on both markets, the transaction is of the "Classical Unemployment"-type (see Figure 4.5). The production sector is completely satisfied. This is the only case of transaction, which is interior in \(H_C\). "Classical Unemployment" is the context in which classical economists used to place unemployment (see Chapter 1). Intermediate cases are represented in the figures 4.6 - 4.8.

The Nash equilibria \(K_1\) and \(K_2\) in Figure 4.1 are Keynesian Unemployment-disequilibria. These two trades can not be reached with the adjustment mechanism \(AM\), if one of the notional solutions is the first offer.
6. Conclusions

The model, put forward in the preceding chapters, indicates what kind of disequilibrium states can occur if demand and supply, as the results of maximization problems, do not coincide.

In this report an "acceptable trade" for a sector is defined as a combination of the variables labour and consumption goods, which is optimal for some pair of upper bounds. A realisation of a trade is called a "transaction". A transaction must be an acceptable trade for both sectors.

It is proved, that trades are acceptable, if they do not exceed "effective goods demand/supply" and "effective labour supply/demand" (cf. Weddepohl, 1978). The latter concepts are defined as the solution of the maximization problems for fixed upper bounds on the commodities of the other market. Böhm (1976) assumes these functions for the consumption sector to be increasing and concave. In this paper conditions were derived which are to be imposed on the utility function to get these properties as a result. It was also shown, that the "effective demand/supply functions" do not need to be increasing.

The interaction between the sectors has been approached in two ways. The first was a game theoretic approach, in which the transaction, i.e. the resulting exchange, is regarded as a Nash equilibrium. The second approach considered the sequence, generated by some specified adjustment mechanism. The limit of this sequence was the "optimal trade". The optimal trade is the unique trade in the set of acceptable trades of the economy, that has both maximum utility for the consumption sector and maximum profit for the production sector. This optimal trade is a Nash equilibrium. However there can be more Nash equilibria.

Nash equilibria are considered to be possible transactions. A classification of these transactions was given, which was similar to the one of Malinvaud (1977).
Acknowledgement

The author is very grateful to Jan de Jong for many remarks and numerous discussions.
Appendix

Proof of Theorem 3.3.

The theorem can be proved by showing that the derivative of $X_c(l,\infty)$ with respect to $l$ for $l \in [0,l_c^*]$, respectively the derivative of $L_c(\infty,x)$ with respect to $x$ for $x \in [0,x_c^*]$, is positive.

Let $l \in [0,l_c^*]$. In Corollary 3.3 it is stated, that if the inequality $l \leq l_c^*$ holds, then $X_c(l,\infty)$ is the solution of

$$\max \{ \widetilde{U}(l,x) | 0 \leq x \leq \frac{1}{p}(m_0 + \omega l) \} ,$$

where $\widetilde{U}(l,x)$ is the restricted utility function (cf. Definition 3.1). Since the pair $(L_c(l,\infty), X_c(l,\infty))$ is the solution of (A1) and since $L_c(l,\infty) = l$ (cf. Corollary 3.1) we have

$$\widetilde{U}_x(L_c(l,\infty), X_c(l,\infty)) = \widetilde{U}_x(l, X_c(l,\infty)) = 0 .$$

If we define for all $l \in [0,l_c^*]$

$$x(l) := X_c(l,\infty) ,$$

then we may determine the derivatives of $x(l)$ with respect to $l$ by differentiating $\widetilde{U}_x(x(l),l) = 0$ with respect to $l$ and solving the result for $\frac{dx(l)}{dl}$ . Thus we obtain

$$\frac{dx(l)}{dl} = - \frac{\widetilde{U}_{x \ell}(l,x(l))}{\widetilde{U}_{xx}(l,x(l))}$$

This expression is positive if $\widetilde{U}_{x \ell}(l,x(l)) > 0$. Working out this inequality leads to

$$\widetilde{U}_{x \ell}(l,x(l)) - p\widetilde{U}_{m \ell}(l,x(l)) + \omega[\widetilde{U}_{xm}(l,x(l)) - p\widetilde{U}_{mm}(l,x(l))] > 0 .$$

A similar proof can be applied to show that the derivative of $L_c(\infty,x)$ with respect to $x$ is positive.
Remark 1

The expression (A3) can be obtained in terms of the "bordered Hessian" (cf. Takayama, 1974), which is defined by

\[
\begin{bmatrix}
0 & -\omega & p & 1 \\
-\omega & U_{xx} & U_{x} & U_{xm} \\
p & U_{x} & U_{xx} & U_{xm} \\
1 & U_{m} & U_{mx} & U_{mm}
\end{bmatrix}
\]

The denominator in (A2) is the cofactor of $U_{xx}$ in the bordered Hessian. The numerator is the cofactor of $U_{x}$.  

Remark 2

From expression (A2) the second derivative $\frac{d^2 x(t)}{dt^2}$ can be obtained. The function $X_{c}(t, \infty)$ is concave in $t$, if this second derivative is non-positive. Concavity results if the following inequality holds.

\[
(A4) \quad \frac{d^{2}U_{x}(t, x(t))}{dt^{2}} U_{x}(t, x(t)) - \frac{dU_{x}(t, x(t))}{dt} U_{xx}(t, x(t)) \leq 0.
\]

Working out (A4) leads to a very complicated expression in second and third derivatives of the utility function $U(t, x, m)$. Concavity of the function $L_{c}(\infty, x)$ is guaranteed by a similar condition.
References


<table>
<thead>
<tr>
<th>Authors</th>
<th>Title and Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Makarov, V.L. and</td>
<td>Mathematical Theory of Economic Dynamics and Equilibria, Springer-Verlag, New</td>
</tr>
<tr>
<td>Weddepohl, H.N.</td>
<td>Equilibria with Rationing in an Economy with Increasing Returns, Tilburg University, Dept. of Econometrics (1978)</td>
</tr>
</tbody>
</table>